PART I: METHODOLOGY

Global optimization via α-dense curves

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Abstract

Purpose – To study constrained or unconstrained global optimization problems in a cube of $\mathbb{R}^d$ where d is a positive integer.

Design/methodology/approach – α-dense curves are initially used to transform this problem into a global optimization problem of a single variable. The optimization of the one variable is then treated by means of the Legendre-Fenchel Transform. This discrete convex envelope of the one variable function obtained previously, can then be computed.

Findings – Global optimization problems of this nature have already been extensively studied by the authors. In this paper they have coupled the Alienor method with Legendre-Fenchel Tranform to compute a discrete convex envelope of the function to minimize. A fast algorithm was successfully used to do this.

Research limitations/implications – This approach to global optimization is based on α-dense curves and numerical tests performed on a Pentium IV (1,700 MHz) computer used with Mathematica 4 software.

Practical implications – Gives the solutions illustrated in the numerous examples provided that show the practicality of the methodology.

Originality/value – A new approach based on extensive research into global optimization via α-dense curves.

Keywords Optimization techniques, Cybernetics, Transforms

Paper type Research paper

1. Introduction

Global optimization problems in $\mathbb{R}^d$ have been studied by Cherruault and colleagues (Cherruault, 1998, 1999; Ziadi and Cherruault, 1998) and transformed into global optimization problems of single variable function by using α-dense curves. Ziadi *et al.* (2001) have coupled the reducing transforms with other methods such as Evtushenko, Brent or Branch and Bound. The main difficulty described in Cherruault (1998, 1999) and Mora and Cherruault (1997, 1998) is the choice of the discretization step $\Delta\theta$. Indeed if $\Delta\theta$ is constant then the distance between the two points $M(\theta)$ and $M(\theta+\Delta\theta)$ on the α-dense curve increases with θ. To keep this distance constant (equal to the "α" density coefficient) it will be necessary to choose a variable step of discretization $\Delta\theta$, so that $\Delta\theta = \alpha/\alpha\theta$. $\Delta\theta$ is as little as $1/\theta$.

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In this paper, we shall couple the Alienor method with the Legendre-Fenchel Transform to compute a discrete convex envelope of the function to minimize. To do this, we use a fast algorithm proposed by Brenier (1989) and then developed by Lucet (1996).

This paper is subdivided as follows. Section 2 describes α-dense curves which is then followed by Legendre-Fenchel transform and convex envelope in Section 3 and finally in Section 4 Algorithm and numerical tests are given.

2. α-dense curves

2.1 Definitions

The original idea born in the 1980s consisted to find some global optimization methods in $\mathbb{R}^d$ which will permit us to transform multivariables problems into single variable problem. A method called Alienor (Cherruault and Guillez, 1980; Cherruault, 1994), was first invented by Cherruault and Guillez and allowed to approach a global minimization problem depending on d variables into a minimization problem according to a single variable. To do that reducing transformations (called Alienor) have been developed leading to α-dense curves in $\mathbb{R}^d$.

First consider the transformation in $\mathbb{R}^2$: Let (x_1, x_2) be a point of $\mathbb{R}^2$ and:

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta \tag{2.1}$$

in polar coordinates.

The restriction to Archimedes spiral defined by:

$$r = \alpha\theta, \quad \theta \geq 0$$

leads to:

$$x_1 = \alpha\theta\cos\theta, \quad x_2 = \alpha\theta\sin\theta \tag{2.2}$$

Definition. A subset S of $\mathbb{R}^2$ is α-dense in $\mathbb{R}^2$ if: $\forall M \in \mathbb{R}^2$, $\exists N \in S$ such that: $d(M,N) \leq \alpha$ where $d(M, N)$ is the Euclidean distance of M to N.

We can then prove the following lemma.

Lemma 2.1. The Archimedes spiral defined by: $r = \alpha\theta$ is $\pi\alpha$-dense in $\mathbb{R}^2$. In Cherruault (1999) it has been proved that the distance of two consecutive whorls is equal to $2\pi\alpha$. If M is a point of $\mathbb{R}^2$, it is located between two whorls of the Archimedes spiral and its distance to it is less than or equal to:

$$\frac{2\pi\alpha}{2} = \pi\alpha$$

Remark. The generalization of this result to $\mathbb{R}^d$ can be easily done by iterating this process. Consider three variables x_1, x_2, x_3.

First set:

$$x_1 = \alpha\theta_1\cos\theta_1, \quad x_2 = \alpha\theta_1\sin\theta_1, \quad x_3 = x_3,$$

Then connect θ_1 and x_3 by a second Archimedes spiral:

$$\theta_1 = \alpha\theta\cos\theta \quad \text{and} \quad x_3 = \alpha\theta\sin\theta.$$

which leads to:

$$x_1 = \alpha\theta\cos\theta\cos[\alpha\theta\cos\theta], \quad x_2 = \alpha\theta\cos\theta\sin[\alpha\theta\cos\theta]$$

and $x_3 = \alpha\theta\sin\theta$.

Theorem 1. Any point of $\mathbb{R}^d$ can be approximated by at least one point of the generalized Archimedes spiral (or the so-called reducing Alienor transformation).
For the proof see Cherruault (1998, 1999).

Remark. As for the case $d=2$ (we recall that d is the dimension) it is possible to calculate α for any dimension d (Cherruault, 1998). Some authors Cherruault (1999), Ammar and Cherruault (1993), Mora *et al.* (2002) and Ziadi and Cherruault (1998) have built other α-dense curves for which the α-density parameter can be easily calculated.

(1) Modified Alienor transform (Cherruault, 1998; Mora and Cherruault, 1997)

(2)
$$\begin{cases} x_i(\theta) = \alpha\theta\sin\alpha_i\theta, & i = 1, \ldots, d \\ \alpha_1 = 1, \ \dfrac{\alpha_i}{\alpha_{i+1}} = \rho, & i = 2, \ldots, d \end{cases}$$

where $\alpha = \sqrt{d-1}\pi\rho$.

(3)
$$\begin{cases} x_1(t) = t \\ x_i(t) = \dfrac{1}{2}(1 - \cos n^{i-1}t), & i = 2, \ldots, d \end{cases}$$

where n is a positive integer and $\alpha = \sqrt{d-1}/n$.

2.2 α-dense curves and global optimization

Let us consider the global minimization problem:

$$\operatorname*{Glob\,Min}_{x\in[0,1]^d} f(x) \tag{2.3}$$

where f is a continuous function defined on $[0,1]^d$ (d is a positive integer).

We introduce the α-dense curve $x(t)$, $t \in [0,1]$ defined by:

$$\begin{cases} x_1(t) = t \\ x_i(t) = \dfrac{1}{2}(1 - \cos n^{i-1}t), & i = 2, \ldots, d, \quad t \in [0,1] \end{cases}$$

We denote by g the function defined on $[0,1]$ by:

$$g(t) = f(x(t)), \quad t \in [0,1] \tag{2.4}$$

The global minimization problem (2.3) becomes a minimization problem according to a single variable t:

$$\operatorname*{Glob\,Min}_{t\in[0,1]} g(t) \tag{2.5}$$

We can then prove the following result.

Theorem 2. All the minima of f can be approximated by minima of g.

Proof. For the proof see Cherruault (1999). □

Remark. The converse is not true: the minima of g are not necessarily approximations of minima of f.

3. Legendre-Fenchel transform

3.1 Introduction

The use of the Legendre-Fenchel transform for computing the conjugate $u^{\times}$(defined below) of a given function u has been introduced briefly by Brenier (1989). Applying the same process to $u^{\times}$ permit us to define $u^{\times\times}$ the convex envelope of u or the so-called biconjugate of u. This idea has been then developed by Lucet (1996) to built a fast algorithm allowing us to calculate the conjugate and the bi-conjugate of functions depending on one or several variables. Corrias (1996) has also used the same idea to solve some Hamilton-Jacobi equations.

In this paper, we shall couple this algorithm based on the Legendre-Fenchel transform with α-dense curves in view to solve numerically constrained or unconstrained global optimization problems in several variables for continuous functions defined on $[0,1]^d$ or on $[-1,1]^d$.

3.2 Definitions

The following definitions arising in convex analysis will be used (Hiriart-Urruty and Lemarechal, 1993; Rockafellar, 1970).

For a given numerical continuous function u defined on $[a,b]$, we have the following definitions.

Definition 3.1. The function $u^{\times}$ defined on $\mathbb{R}$ by:

$$u^{\times}(s) = \sup_{x\in[a,b]} [sx - u(x)]$$

is called the Legendre-Fenchel transform of u (or the conjugate of u).

Definition 3.2. The function $u^{\times\times}$ defined on $[a, b]$ by:

$$u^{\times\times}(x) = \sup_{s\in\mathbb{R}} [sx - u^{\times}(s)]$$

is called the the bi-conjugate of u (or the convex envelope of u).

In view to compute $u^{\times}$ we use a discretization of the interval $[a, b]$.

Let

$$I_N = \left\{a + i\left(\frac{b-a}{N}\right), \quad i = 1, \ldots, N\right\}$$

where $N = 2^p$, for some positive integer p. We introduce then the discrete conjugate of u on I_N.

Definition 3.3. The discrete conjugate of u on I_N, noted $u_N^{\times}$, is defined by:

$$u_N^{\times}(s) = \sup_{x\in I_N} [sx - u(x)], \quad \forall s \in \mathbb{R}.$$

Remark. As we cannot compute $u_N^\times(s)$ for all s belonging to $\mathbb{R}$, we only do that on a finite number of points of a subinterval $J = [a^\times, b^\times]$ of $\mathbb{R}$.
That is to say, we need a discretization of J:
Let

$$J_M = \left\{ a + j\left(\frac{b^\times - a^\times}{M}\right), \quad j = 1, \ldots, M \right\}$$

where $M = 2^q$, for some positive integer q.

For any positive integer n one can see that:

$$I_{2n} = I_n \cup (I_n - \varepsilon) \tag{3.1}$$

where $\varepsilon = (b - a)/2n$. and then (Brenier, 1989) that:

$$u_{2n}^\times(s) = \mathrm{Max}\left[u_n^\times(s),\ \tilde{u}_n^\times(s) - \varepsilon s\right] \tag{3.2}$$

where

$$\tilde{u}_n^\times(s) = \sup_{x \in I_N}[sx - u(x - \varepsilon)] \tag{3.3}$$

Formulae (3.2) and (3.3) show that, to develop a recursive algorithm permitting us to compute $u_N^\times(s)$ for s belonging to J_M, we have to introduce a parameter τ and the following function:

$$u_N^\times(s, \tau) = \sup_{x \in I_N}[sx - u(x - \tau)], \quad s \in J_M. \tag{3.4}$$

We are then interested in computing $u_N^\times(s, 0)$ because

$$u_N^\times(s, 0) = u_N^\times(s), \quad s \in J_M.$$

Formula (3.2) becomes (Lucet, 1996):

$$u_{2n}^\times(s, \tau) = \mathrm{Max}\left[u_n^\times(s, \tau), u_n^\times\left(s, \tau + \frac{b - a}{2n}\right) - s\frac{b - a}{2n}\right] \tag{3.5}$$

Knowing

$$u_n^\times(s, \tau) \quad \text{and} \quad u_n^\times\left(s, \tau + \frac{b - a}{2n}\right)$$

at the step n, we can compute $u_{2n}^\times(s, \tau)$.

We need therefore to construct the following tree branch:

$$\left.\begin{array}{c} \tau \\ \tau + \dfrac{b - a}{2N} \end{array}\right\} \rightarrow \tau$$

To clarify these relationships we introduce (Lucet, 1996), the following notations:

$$\left.\begin{aligned} f_k^i &= \tau \\ f_{k+1}^i &= \tau + \frac{b-a}{2N} \end{aligned}\right\} \rightarrow f_k^{i+1} = \tau \tag{3.6}$$

$\left\{f_k^i\right\}_{k=1,\dots,2^i}$ stores the ith floor of the tree.

For example, if we take $N = 8$ we obtain:

$$\begin{array}{llll}
f_1^{p-3} = 0 & & & \\
f_2^{p-3} = \dfrac{b-a}{N/4} & f_1^{p-2} = 0 & & \\
f_3^{p-3} = \dfrac{b-a}{N/2} & & & \\
f_4^{p-3} = \dfrac{b-a}{N/2} + \dfrac{b-a}{N/4} & f_2^{p-2} = \dfrac{b-a}{N/2} & f_1^{p-1} = 0 & \\
f_5^{p-3} = \dfrac{b-a}{N} & & & \\
f_6^{p-3} = \dfrac{b-a}{N} + \dfrac{b-a}{N/4} & f_3^{p-2} = \dfrac{b-a}{N} & & \\
f_7^{p-3} = \dfrac{b-a}{N} + \dfrac{b-a}{N/2} & & & \\
f_8^{p-3} = \dfrac{b-a}{N} + \dfrac{b-a}{N/2} + \dfrac{b-a}{N/4} & f_4^{p-2} = \dfrac{b-a}{N} + \dfrac{b-a}{N/2} & f_2^{p-1} = \dfrac{b-a}{N} & f_1^{p} = 0
\end{array}$$

which leads, if $[a,b] = [0,1]$ to:

$$\begin{array}{lccccccccc}
n = 1 & \underset{0}{f_1^0} & \underset{\frac{1}{8}}{f_5^0} & \underset{\frac{1}{4}}{f_3^0} & \underset{\frac{3}{8}}{f_7^0} & \underset{\frac{1}{2}}{f_2^0} & \underset{\frac{5}{8}}{f_6^0} & \underset{\frac{3}{4}}{f_4^0} & \underset{\frac{7}{8}}{f_8^0} & 1 \\
n = 2 & \underset{0}{f_1^1} & \underset{\frac{1}{8}}{f_3^1} & \underset{\frac{1}{4}}{f_2^1} & \underset{\frac{3}{8}}{f_4^1} & & & & & \\
n = 4 & \underset{0}{f_1^2} & \underset{\frac{1}{8}}{f_2^2} & & & & & & & \\
n = 8 & \underset{0}{f_1^3} & & & & & & & &
\end{array}$$

Some convergence results have been proved in Lucet (1996, 1997) and Corrias (1996).

Proposition 1. If $u : [a,b] \rightarrow \mathbb{R}$ is upper semi-continuous on $[a, b]$ then $u_N^{\times}$ converges pointwise to $u_{[a,b]}^{\times}$.

Proposition 2. $u_{[-a,a]}^{\times}$ converges pointwise to $u^{\times}$ when a goes to infinity.

4. Global optimization

4.1 Unconstrained global optimization problems

Let f be a continuous numerical function:

$$f : [0,1]^d \to \mathbb{R}$$

We are interested in:

$$\operatorname*{Glob\,Min}_{[0,1]^d} f(x)$$

We consider the following α-dense curve Γ:

$$\begin{cases} x_1(t) = t \\ x_i(t) = \dfrac{1}{2}(\cos n^{i-1}\pi t), \quad 2 \le i \le d, \end{cases} t \in [0,1]$$

It has been proved in Ammar and Cherruault (1993) that Γ is $\sqrt{d-1}/n$-dense in $[0,1]^d$.

The restriction of f to Γ is noted by g as said previously:

$$g(t) = f(x_1(t), x_2(t), \ldots, x_d(t)), \quad t \in [0,1] \tag{4.1}$$

4.1.1 Numerical tests

Remarks.

(1) These numerical tests have been performed on Pentium IV (1,700 MHz) with Mathematica 4.

(2) We compute $g_N^{\times}$ on $[-1,2]$ for $N = 2^p$ and $M = 2^q$ and $g_{N_1}^{\times\times}$ on $[0,1]$ for $N_1 = 2^{p_1}$ and $M_1 = 2^{q_1}$

We give in this paragraph three examples in $\mathbb{R}^5$ and an example in $\mathbb{R}^{10}$.

Example 1.1. Let

$$f_1(x) = \sum_{i=1}^{5} \left(x_i - \frac{1}{2}\right)^2$$

which realizes its global minimum 0 at the point

$$x = (0.5, 0.5, 0.5, 0.5, 0.5)$$

1.1.1 The α-dense curve used is:

$$\gamma_s(t) = \begin{cases} t \\ \dfrac{1}{2}(1 - \sin n^{i-1}\pi t), \quad i = 2, \ldots, 5 \end{cases}, \quad t \in [0,1]$$

We compute the discrete conjugate on $[-1,2]$ for $p = 6$, $q = 8$ and the discrete convex envelope on [0,1] for $p_1 = q_1 = 9$. We obtain in 3 s for $n = 10$, the following results:

$$g^{\times\times}(0.5, 0.5, 0.5, 0.5, 0.5) = 0,$$

with the previous notations: $g^{\times\times}$ is the bi-conjugate or the discrete convex envelope of g, where g is given by:

$$g(t) = f(x(t)), \quad t \in [0,1].$$

1.1.2 For the same example let us use an other α-dense curve:

$$\gamma_c(t) = \begin{cases} t \\ \frac{1}{2}(1 - \cos n^{i-1}\pi t), \quad i = 2, \ldots, 5 \end{cases}, \quad t \in [0,1]$$

For the same parameters we have:

The minimizer $\bar{x}$ of $g^{\times\times}$ is: $(0.45448, 0.59946, 0.586237, 0.419057, 0.472402)$ and $g^{\times\times}(\bar{x}) = 0.195134$.

This calculus has been realized in 3 s.

Example 1.2. Let

$$f_2(x) = \exp[f_1(x)] = \exp\left(\sum_{i=1}^{5}\left(x_i - \frac{1}{2}\right)^2\right)$$

realizing its global minimum 1 at the point: $x = (0.5, 0.5, 0.5, 0.5, 0.5)$

1.2.1. For the same parameters used in the example (1.1.1) we obtain: $\bar{x} = (0.5, 0.5, 0.5, 0.5, 0.5)$ and $g^{\times\times}(\bar{x}) = 1$, which takes 3 s also.

1.2.2. Using the α-dense curve with the cosine, we obtain the following (Table I).

Example 1.3. Let

$$\begin{aligned} f_3(x) = {} & 4\left(x_1 - \frac{1}{2}\right)^2 + 3\left(x_2 - \frac{1}{2}\right)^2 + 2\left(x_3 - \frac{1}{2}\right)^2 \\ & + 2\left(x_4 - \frac{1}{2}\right)^2 + \left(x_5 - \frac{1}{2}\right)^2 - \left(x_1 - \frac{1}{2}\right)\left[-4\left(x_2 - \frac{1}{2}\right) - 2\left(x_3 - \frac{1}{2}\right)\right. \\ & \left. + 2\left(x_4 - \frac{1}{2}\right) + 2\left(x_5 - \frac{1}{2}\right)\right] - 2\left(x_2 - \frac{1}{2}\right)\left(x_4 - \frac{1}{2}\right) \\ & + 2\left(x_3 - \frac{1}{2}\right)\left(x_4 - \frac{1}{2}\right) + 2\left(x_4 - \frac{1}{2}\right)\left(x_5 - \frac{1}{2}\right) \end{aligned}$$

f realizes its minimum 0 at the point (0.5, 0.5, 0.5, 0.5, 0.5).

Table I.

	$p = 6, q = 8$ $p_1 = q_1 = 9, n = 10$	$p = 7, q = 9$ $p_1 = q_1 = 10, n = 20$
x_1	0.544547	0.523809
x_2	0.585239	0.462630
x_3	0.429063	0.462728
x_4	0.573365	0.460770
x_5	0.549009	0.5
$g^{\times\times}(\bar{x})$	1.02231	1.0049
Calculation time (s)	3	4

1.3.1. The parameters of example (1.1.1) lead to: $\bar{x} = (0.5, 0.5, 0.5, 0.5, 0.5)$, and $g^{\times\times}(\bar{x}) = 0$, in 3 s.

1.3.2. Using the α-dense curve with the cosine we obtain the following (Table II).

Example 1.4. Let

$$f_4(x) = \sum_{i=1}^{10} \left(x_i - \frac{1}{2} \right)^2$$

which realizes its global minimum 0 at the point: $x = (0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5)$.

1.4.1. The α-dense curve used is:

$$\gamma_s(t) = \begin{cases} t \\ \frac{1}{2}(1 - \sin n^{i-1} \pi t), \quad i = 2, \ldots, 9 \end{cases}, \quad t \in [0, 1]$$

For $p = 7$, $q = 9$, $p_1 = q_1 = 10$ and $n = 10$ we have: $\bar{x} = (0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5)$, and $g^{\times\times}(\bar{x}) = 0$ after 45 s.

1.4.2. Using the α-dense curve:

$$\gamma_c(t) = \begin{cases} t \\ \frac{1}{2}(1 - \cos n^{i-1} \pi t), \quad i = 2, \ldots, 9 \end{cases}, \quad t \in [0, 1]$$

we obtain the following (Table III).

4.2 Constrained optimization problems on $[1, 1]^d$

Let us consider the following problem:

$$\operatorname*{Glob\,Min}_{x \in K} f(x)$$

where K is a closed subset of $[-1, 1]^d$ defined by:

$$K = \{x \in \mathbb{R}^d / h(x) \leq 0\} \tag{4.2}$$

We suppose that h is a continuous function.

	$p = 6, q = 8$ $p_1 = q_1 = 9, n = 10$	$p = 7, q = 9$ $p_1 = q_1 = 10, n = 20$
x_1	0.54473	0.47087
x_2	0.582336	0.52821
x_3	0.45832	0.53169
x_4	0.53577	0.49913
x_5	0.53569	0.50865
$g^{\times\times}(\bar{x})$	0.26969	0.17807
Calculation time (s)	3	4

Table II.

	$p = 7, q = 9$ $p_1 = q_1 = 10, n = 10$	$p = 8, q = 10$ $p_1 = q_1 = 11, n = 20$
x_1	0.644545	0.578578
x_2	0.414725	0.488530
x_3	0.428703	0.570640
x_4	0.569759	0.56305
x_5	0.585106	0.53693
x_6	0.430400	0.454808
x_7	0.58669	0.518536
x_8	0.414519	0.462728
x_9	0.426635	0.460768
x_{10}	0.549009	0.499967
$g^{\times\times}(\bar{x})$	0.210632	0.132354
Calculation time (s)	45	59

Table III.

We use for this case the following α-dense curve:

$$\Gamma = \{(x_1(t), x_2(t), \ldots, x_d(t)\}$$

where

$$\begin{cases} x_1(t) = t \\ x_i(t) = \cos n^{i-1}(t), \quad i = 2, \ldots, d, \end{cases} t \in [-1, 1] \tag{4.3}$$

We recall that n is an integer which goes to infinity.

$\mathring{K}$ is the interior set of K.

We can suppose that 0 belongs to $\mathring{K}$.

We have to study two cases:

(1) $f(0) = \operatorname{Glob}\min_{x \in K} f(x)$

so we have to resolve the global optimization problem.

(2) If $f(0) > \operatorname{Glob}\min_{x \in K} f(x)$.

Let us introduce the function p defined by:

$$p(t) = \frac{1}{2}\left(1 - \frac{h(x(t)}{|h(x(t)|}\right) = \begin{cases} 1 & \text{if } x(t) \in \mathring{K} \cap \Gamma \\ 0 & \text{if } x(t) \notin \mathring{K} \cap \Gamma \end{cases} \tag{4.4}$$

and the function g defined by:

$$g(t) = f(p(t) \cdot x(t)), \quad t \in [-1, 1] \tag{4.5}$$

We have obviously:

$$g(t) = \begin{cases} f(x(t)), & \text{if } x(t) \in \mathring{K} \\ f(0), & \text{if } x(t) \notin K \end{cases}$$

By this way, we obtain:

$$\underset{\substack{t \in [-1,1] \\ x(t) \in K}}{\text{Glob Min}} f(x(t)) = \underset{t \in [-1,1]}{\text{Glob Min}}\, g(t)$$

because: if $x(t) \notin K$ then $g(x(t)) = f(0) \neq \underset{x \in K}{\text{Glob Min}} f(x)$.

The constrained global optimization problem becomes then an unconstrained global optimization problem of a single variable function which will be treated as in the previous paragraph.

4.1.2 Numerical tests

Remark. We compute $g_N^{\times}$ on $[-3,3]$ for $N = 2^p$ and $M = 2^q$ and $g_{N_1}^{\times\times}$ on $[-1,1]$ for $N_1 = 2^{p_1}$ and $M_1 = 2^{q_1}$. We give in this paragraph two examples in $\mathbb{R}^5$.

Example 2.1. Let

$$f_5(x) = \left(x_1 - \frac{1}{2}\right)^2 + \sum_{i=2}^{5} x_i^2$$

$$\mathbf{K} = \left\{ x \in \mathbb{R}^5 / h(x) = \left(\sum_{i=1}^{5} |x_i| \right) - 1 \leq 0 \right\}$$

We have:

$$\underset{x \in \mathbf{K}}{\text{Glob Min}} f_4(x) = f(0.5, 0, 0, 0, 0) = 0$$

Then we obtain the following (Table IV).

Example 2.2. Consider the function:

$$f_6(x) = x_1^2 + x_2^2 + x_3^2 + \exp\left[\left(x_4 - \frac{1}{2}\right)^2 + \left(x_5 - \frac{1}{2}\right)^2 \right]$$

	$p = 6, q = 8$ $p_1 = q_1 = 9, n = 20$	$p = 7, q = 9$ $p_1 = q_1 = 10, n = 30$
x_1	0.52869	0.51612
x_2	0.22981	-0.05064
x_3	0.07454	-0.05073
x_4	0.07845	-0.04808
x_5	2.210^{-11}	-0.12728
$g^{\times\times}(\bar{x})$	0.0653512	0.023912
Calculation time (s)	4	6

Table IV.

Table V.

	$p = 6, q = 8$ $p_1 = q_1 = 9, n = 10$	$p = 7, q = 9$ $p_1 = q_1 = 10, n = 20$
x_1	0.03542	0.02363
x_2	0.08421	0.04552
x_3	0.14214	−0.12138
x_4	0.54298	0.52404
x_5	0.57301	0.55078
$g^{\times\times}(\bar{x})$	1.09173	1.03062
Calculation time (s)	4	5

$$\mathbf{K} = \left\{ x \in \mathbb{R}^5 / h(x) = \left(\sum_{i=1}^{5} x_i^2 \right) - 1 \leq 0 \right\}$$

We have: $\operatorname{Glob}\operatorname*{Min}_{x \in \mathbf{K}} f_4(x) = f(0, 0, 0, 0.5, 0.5) = 1.$
The following results have been obtained (Table V).

Remark. For several constraints $h_i(x) \leq 0$, $i = 1, \ldots, k$ we can use the function

$$p(t) = \prod_{i=1}^{k} p_i(t),$$

where

$$p_i(t) = \frac{1}{2}\left(1 - \frac{h_i(x(t))}{|h_i(x(t))|}\right)$$

4.2 Conclusion

First note that we have obtained very good results in examples (1.1.1), (1.2.1), (1.3.1) and (1.4.1) because the point where f reaches its global minimum belongs to the α-dense curve used for these calculus. In the worst case where the curve does not contain the minimizer of the objective function, the error on the minimizer is less than the α-density coefficient and the error on the global minimum is less than $\varepsilon(\alpha)$ where $\varepsilon(\alpha)$ denotes the uniform-continuity parameter of the objective function.

These different examples show that the errors on the minimizer and on the global minimum for constrained or unconstrained global optimization problems decrease when the parameters p and n grow to infinity.

Finally, as in constrained problems, we can resolve any unconstrained global optimization problem on any closed subset Ω of $\mathbb{R}^d$: it suffices to define Ω by an equation as in (4.2) then to introduce the function p used in (4.4).

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