## Relativistic mass-energy relationship

The main purpose of this article is to explore the following formular,

$$E = mc^2 \tag{0.1}$$
$$E^2 - p^2c^2 = m_0^2c^4 \tag{0.2}$$

- The first equation is the mass-energy equation.
- The second one is the relativistic momentum-mass equation.

### Physical quantity

- relativistic mass

$$m = \frac{m_0}{\sqrt{1-(\frac{v}{c})^2}} = \gamma m_0 \tag{1}$$

where $m_0$ indicate rest mass.

- relativistic momentum

$$\vec{p} = m\vec{v} = \gamma m_0 \vec{v} = \frac{m_0\vec{v}}{\sqrt{1-(\frac{v}{c})^2}} \tag{2}$$

- force (theorem of momentum)

$$\begin{aligned} F &= \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} \\ &= \vec{v}\frac{dm}{dt} + m\frac{d\vec{v}}{dt} \end{aligned} \tag{3}$$

attention: *relativistic mass varies with velocity*

### Relativistic kinetic energy and mass-energy equation

In Newtonian mechanics,

- Translational kinetic energy

$$E_k = \frac{1}{2}mv^2 \tag{4}$$

- Rotational kinetic energy

$$E_k = \frac{1}{2}I\omega^2 \tag{5}$$

In relativistic mechanics, We still use the work done on the particle to represent the increment of the particle's kinetic energy (using eq.(3))

$$
\begin{aligned}
E_k &= \int_0^v \vec{F} \cdot d\vec{r} \\
&= \int_0^v \frac{d(m\vec{v})}{dt} \cdot d\vec{r} = \int_0^v d(m\vec{v}) \cdot \vec{v} \\
&= \int_0^v m\vec{v} \cdot d\vec{v} + \vec{v} \cdot \vec{v} dm \\
&= \int_0^v mvdv + v^2 dm.
\end{aligned} \tag{6}
$$

From eq.(1), we have

$$
\begin{aligned}
m^2(1 - \frac{v^2}{c^2}) = m_0^2 \\
\rightarrow m^2c^2 - m^2v^2 = m_0^2c^2
\end{aligned} \tag{7}
$$

differentiate both sides of the eq.(7), we can easily get

$$
\begin{aligned}
2mdm(c^2 - v^2) - 2m^2vdv = 0 \\
\rightarrow c^2dm = v^2dm + mvdv.
\end{aligned} \tag{8}
$$

Look back to eq.[6], we can substitute eq.(8) into it

$$
\begin{aligned}
E_k &= \int_{m_0}^{m} c^2 dm \\
&= (m - m_0)c^2,
\end{aligned} \tag{9}
$$

which means that when an object moves from rest to velocity $\vec{v}$, the energy it adds is the above eq. (9). Then, we can easily simplify eq.(9) to

$$
E = mc^2. \tag{0.1}
$$

Besides, we can get the speed of particle expressed by $E_k$ from eq.(9) and (1)

$$
v^2 = c^2[1 - (1 + \frac{E_k}{m_0c^2})^{-2}]. \tag{10}
$$

It indicates that when the kinetic energy of a particle increases due to a force doing work on it, the velocity also increases. However, the velocity could not increase all the time. The upper limitation of the speed is $c$.

When $v \ll c$, we can obtain

$$
\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots \approx 1 + \frac{1}{2}\frac{v^2}{c^2}. \tag{11}
$$

Then, we take it into eq.(9), we have

$$
\begin{aligned}
E_k &= [(1 + \frac{v^2}{c^2}) - 1]m_0c^2 \\
&= \frac{1}{2}m_0v^2,
\end{aligned} \tag{12}
$$

which turns back to the form of Newtonian mechanics

$$E_k = \frac{1}{2}mv^2. \tag{4}$$

**Relativistic momentum-mass relation**

From

$$\begin{cases} E = mc^2 \\ \vec{p} = m\vec{v} \end{cases}$$

we can get

$$\vec{v} = \frac{c^2}{E}\vec{p}. \tag{13}$$

Concerning eq.(1) and mass-energy equation, we can obtain

$$\begin{aligned} E &= \gamma m_0 c^2 \\ \rightarrow E^2(1 - \frac{v^2}{c^2}) &= m_0^2 c^4 \\ E^2(1 - \frac{c^2 p^2}{E^2}) &= m_0^2 c^4. \end{aligned} \tag{14}$$

which is

$$E^2 - p^2 c^2 = m_0^2 c^4. \tag{0.2}$$

Thus, we get the mass-momentum-energy equation. And that's how we define rest mass. We can see it from a more intuitive way

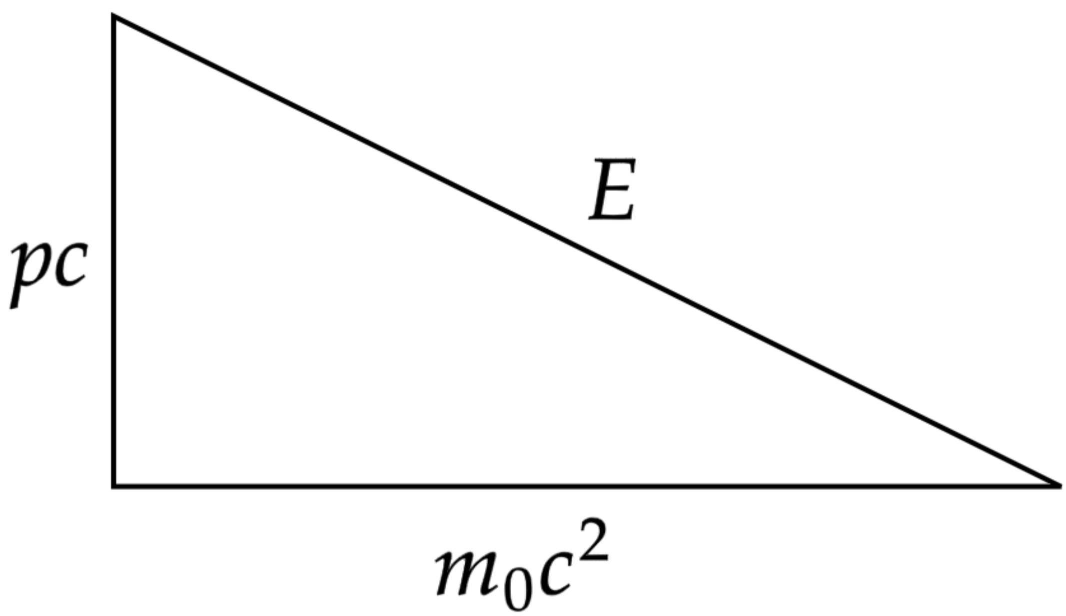


Figure 1: Right triangle of mass-momentum relation

For a particle of kinetic energy $E_k$, we have

$$E_k = E - E_0. \tag{15}$$

As we can see from the figure above (concerning eq.(15)),

$$\begin{aligned} E^2 &= p^2c^2 + m_0^2c^4 \\ \rightarrow E_k^2 + 2E_k m_0 c^2 &= p^2c^2. \end{aligned} \tag{16}$$

When $v \ll c$, which means $E_k \ll m_0c^2$, we can ignore $E_k^2$ in eq.(16). Thus, we can get

$$E_k = \frac{p^2}{2m_0}. \tag{17}$$

which turns back to Newtonian mechanics.