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COLLEGE OF ENGINEERING

Sorting

COMP 2210 – Dr. Hendrix

History



**Herman Hollerith's
tabulating machine and
“sorting box”**

1880 Census: 8 years to process

1890 Census: 1 year to process

“This apparatus works unerringly as the mills of the gods, but beats them hollow as to speed.” The Electrical Engineer, 11 Nov 1891.

<http://www.columbia.edu/cu/computinghistory/census-tabulator.html>

History

⑥

(g) We now formulate a set of instructions to effect this 4-way selection between $(x) = \delta$. We state again the contents of the short tanks already assigned:

- 1.) $V_{n_{(m)}} \rightarrow \bar{x}_1$ 5.) $V_{\delta} x_{(m)} \rightarrow \bar{x}_1, V_{\delta} x_{(m)}$
- 2.) $V_{\delta} x_{(m)} \rightarrow \bar{x}_1$ 6.) $V_{\delta} x_{(m+1)} \rightarrow \bar{x}_1$ 7.) $V_{(x)_{(m)}} \rightarrow \bar{x}_1, V_{(x)_{(m+1)}}$
- 3.) $V_{(x)_{(m)}} \rightarrow \bar{x}_1$ 8.) $V_{(x)_{(m+1)}} \rightarrow \bar{x}_1$ 9.) $V_{(x)_{(m+1)}} \rightarrow C$

Now let the instructions occupy the (long tank) words l_1, l_2, \dots :

- 1.) $T_1 - \bar{x}_1$ 6.) $V_{n_{(m)}} - n_{(m)}$
- 2.) $\bar{y}_1 + \bar{x}_1$ 7.) $V_{\delta} x_{(m)} \rightarrow$
- 3.) $\sigma \rightarrow \bar{l}_1$ 8.) $V_{\delta} x_{(m)} \rightarrow$ for $m' \geq m$
- 4.) $T_1 - \bar{x}_1$ 9.) $V_{\delta} x_{(m+1)} \rightarrow$ for $m' \geq m$
- 5.) $T_0 + \bar{x}_1$ 10.) $V_{(x)_{(m)}} \rightarrow$
- 6.) $O + \bar{l}_2$ 11.) $V_{(x)_{(m+1)}} \rightarrow$
- 7.) $\bar{x}_1 - \bar{x}_1$ 12.) $V_{(x)_{(m+1)}} \rightarrow$
- 8.) $\bar{l}_2 + \bar{l}_1$ 13.) $V_{(x)_{(m+1)}} \rightarrow$
- 9.) $\sigma \rightarrow \bar{l}_1$ 14.) $V_{(x)_{(m+1)}} \rightarrow$ for $m' < m$
- 10.) $\bar{l}_1 \rightarrow C$ 15.) $V_{(x)_{(m+1)}} \rightarrow$ for $m' < m$

Now

11.) $l_0, l_1, l_2, l_3 \rightarrow C$ for $(x), (y), (z), (0)$, respectively.

Thus at the end of this phase C is set l_0, l_1, l_2, l_3 , according to which case $\alpha_1(\delta)(y)(z)$ holds.

(h) We now pass to the case (x) . This has ~~2~~ 2 subcases $(x)_1$ and $(x)_2$, according to whether $x \geq 2$ or $x \leq 1$. According to which of the 2 subcases holds, C must be sent to the place where its instructions begin, i.e. the (long tank) words l_{11}, l_{12} . Their number must be ~~the same~~ in both cases.



John von Neumann's first program for the EDVAC

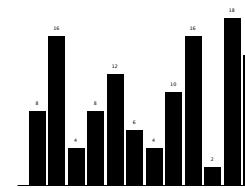
"I have also worked on sorting questions ... We wish to formulate code instructions for sorting ... and to see how much ... time they require. ... At any rate, the moral seems to be that the EDVAC ... is definitely faster than the [IBM sorters]. It is legitimate to conclude ... that the EDVAC is very nearly an "all purpose" machine." JvN, 1945

<http://dl.acm.org/citation.cfm?id=356581>

Sorting

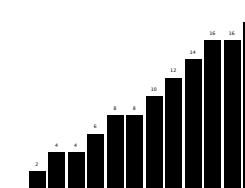
Input: List a of N elements (a_1, a_2, \dots, a_N)

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 8 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 2 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |



Output: A permutation of the elements in a such that $a_i \leq a_{i+1}$ for $1 \leq i \leq N - 1$

| | | | | | | | | | | | |
|---|---|---|---|---|---|----|----|----|----|----|----|
| 2 | 4 | 4 | 6 | 8 | 8 | 10 | 12 | 14 | 16 | 16 | 18 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |



Lots of algorithms ... selection sort, insertion sort, bubblesort, shaker sort, quicksort, merge sort, heapsort, samplesort, shellsort, solitaire sort, red-black sort, splaysort, psort, radix sort, counting sort, bucket sort, distribution sort, timsort, comb sort, ...

Inversions and exchanges

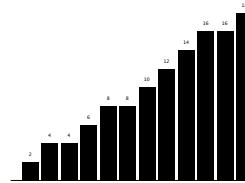
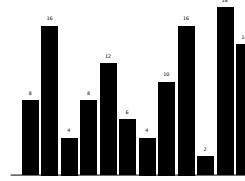
An **inversion** is a pair of elements that are out of order.

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 8 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 2 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

20 inversions: (8,4), (8,6), (8,4), (8,2), (16, 18), (4,2), (8,6), (8,4), (8,2), (12,6), (12,4), (12,10), (12,2), (6,4), (6,2), (4,2), (10,2), (16,2), (16,14), (18,14)

| | | | | | | | | | | | |
|---|---|---|---|---|---|----|----|----|----|----|----|
| 2 | 4 | 4 | 6 | 8 | 8 | 10 | 12 | 14 | 16 | 16 | 18 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

0 inversions



exchange = swap the locations of inverted elements

Sorting could be seen as a sequence of exchanges.

Inversions and exchanges

Randomly ordered data:

| | | | | | |
|---|----|---|---|----|---|
| 4 | 10 | 6 | 8 | 12 | 2 |
| 0 | 1 | 2 | 3 | 4 | 5 |

7 inversions

| | | | | | |
|----|---|---|---|---|----|
| 12 | 6 | 4 | 8 | 2 | 10 |
| 0 | 1 | 2 | 3 | 4 | 5 |

9 inversions

| | | | | | |
|---|----|----|---|---|---|
| 8 | 10 | 12 | 2 | 4 | 6 |
| 0 | 1 | 2 | 3 | 4 | 5 |

9 inversions

A list in reverse order (non increasing) would have the maximum number of inversions.

| list | # inversions | |
|------------------|-------------------------|----|
| 4, 3, 2, 1 | $0 + 1 + 2 + 3$ | 6 |
| 5, 4, 3, 2, 1 | $0 + 1 + 2 + 3 + 4$ | 10 |
| 6, 5, 4, 3, 2, 1 | $0 + 1 + 2 + 3 + 4 + 5$ | 15 |
| ... | | |

$$\sum_{i=0}^{N-1} i = \frac{(N-1)N}{2} = \frac{N^2 - N}{2}$$

In the worst case, there will be $O(N^2)$ inversions in the data.

Inversions and exchanges

Detecting an inversion:

```
private boolean less(Comparable x, Comparable y) {  
    return x.compareTo(y) < 0;  
}
```

| | | | | | |
|---|----|---|---|----|---|
| 4 | 10 | 6 | 8 | 12 | 2 |
| 0 | 1 | 2 | 3 | 4 | 5 |

less(a[5], a[0]) -> true

Correcting an inversion:

```
private void swap(Comparable[] a, int i, int j) {  
    Comparable temp;  
    temp = a[i];  
    a[i] = a[j];  
    a[j] = temp;  
}
```

| | | | | | |
|---|----|---|---|----|---|
| 4 | 10 | 6 | 8 | 12 | 2 |
| 0 | 1 | 2 | 3 | 4 | 5 |

swap(a, 5, 0)

| | | | | | |
|---|----|---|---|----|---|
| 2 | 10 | 6 | 8 | 12 | 4 |
| 0 | 1 | 2 | 3 | 4 | 5 |

Properties of sorts

Comparison sort: The only assumption about the data being sorted is that the data elements can be compared to each other

In-place: The list itself is rearranged and only a constant amount of extra space is required.

Adaptive: Running time is affected by initial state of input.

Stable: Equal elements maintain the same relative order.

all four are comparison sorts

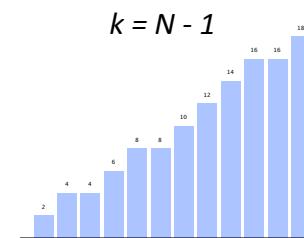
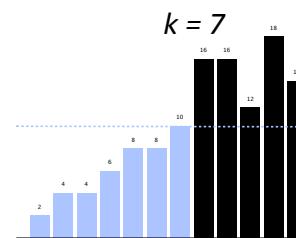
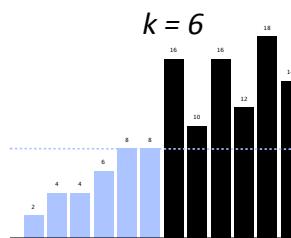
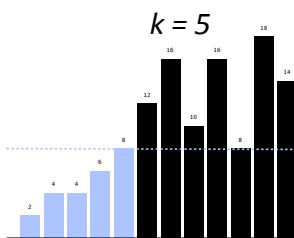
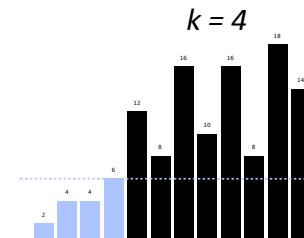
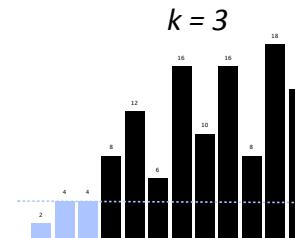
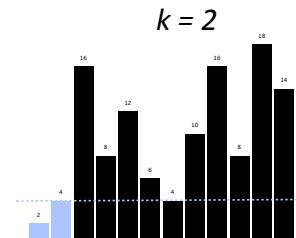
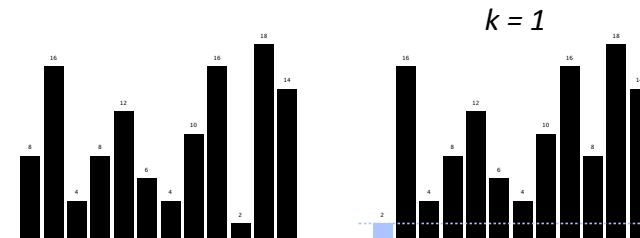
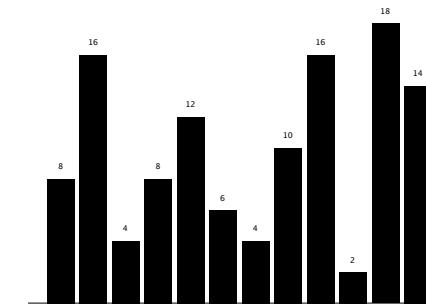
| (as presented) | Selection | Insertion | Mergesort | Quicksort |
|----------------|-----------|-----------|-----------|-----------|
| In-place? | Yes | Yes | No | Yes |
| Stable? | No | Yes | Yes | No |
| Adaptive? | No | Yes | No | No |

Selection sort

Walk from left to right through the array.

On each step, **select** the element that goes in the current location in sorted order and put it there.

After k steps, the first k elements are in sorted order and are in their final positions.



Selection sort

```
public static void selectionSort(Comparable[] a)
{
    int N = a.length;

    for (int i = 0; i < N-1; i++)
    {
        int min = i;
        for (int j = i+1; j < N; j++)
        {
            if (less(a[j], a[min]))
                min = j;
        }
        swap(a, i, min);
    }
}
```

For each index from the first to the next-to-last ...

Find the minimum element from this index to the right...

and swap it into its correct sorted position.

Selection sort

```
public void selectionSort(Comparable[] a {  
    int N = a.length;  
    for (int i = 0; i < N-1; i++) {  
        int min = i;  
        for (int j = i+1; j < N; j++) {  
            if (less(a[j], a[min]))  
                min = j;  
        }  
        swap(a, i, min);  
    }  
}
```

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 8 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 2 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 2 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 8 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

| | | | | | | | | | | | |
|---|---|----|---|----|---|---|----|----|---|----|----|
| 2 | 4 | 16 | 8 | 12 | 6 | 4 | 10 | 16 | 8 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Invariants

On the k^{th} iteration of the i loop:

The first k elements are in their correct sorted position.

Elements $k+1$ through the last element are all greater than or equal to the first k elements.

| | | | | | | | | | | | |
|---|---|---|---|----|---|----|----|----|---|----|----|
| 2 | 4 | 4 | 8 | 12 | 6 | 16 | 10 | 16 | 8 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

| | | | | | | | | | | | |
|---|---|---|---|----|---|----|----|----|---|----|----|
| 2 | 4 | 4 | 6 | 12 | 8 | 16 | 10 | 16 | 8 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Selection sort

```
public void selectionSort(Comparable[] a) {  
    int N = a.length;  
    for (int i = 0; i < N-1; i++) {  
        int min = i;  
        for (int j = i+1; j < N; j++) {  
            if (less(a[j], a[min]))  
                min = j;  
        }  
        swap(a, i, min);  
    }  
}
```

Time complexity

Selection sort is $O(N^2)$.

$(N^2-N)/2$ comparisons (calls to less)

$N-1$ exchanges (calls to swap)

Selection sort is **not adaptive** to its input,
so all arrangements of data in the array
will require a quadratic amount of work.

| $O(N^2)$ <i>already sorted</i> | | | | | |
|-----------------------------------|---|---|---|----|----|
| 2 | 4 | 6 | 8 | 10 | 12 |
| 0 | 1 | 2 | 3 | 4 | 5 |

$\sim N^2$ comparisons
 $\sim N$ exchanges

| $O(N^2)$ <i>"almost" sorted</i> | | | | | |
|------------------------------------|---|----|----|---|---|
| 6 | 8 | 10 | 12 | 2 | 4 |
| 0 | 1 | 2 | 3 | 4 | 5 |

$\sim N^2$ comparisons
 $\sim N$ exchanges

| $O(N^2)$ <i>in reverse order</i> | | | | | |
|-------------------------------------|----|---|---|---|---|
| 12 | 10 | 8 | 6 | 4 | 2 |
| 0 | 1 | 2 | 3 | 4 | 5 |

$\sim N^2$ comparisons
 $\sim N$ exchanges

| $O(N^2)$ <i>in random order</i> | | | | | |
|------------------------------------|----|---|---|---|----|
| 4 | 12 | 8 | 2 | 6 | 10 |
| 0 | 1 | 2 | 3 | 4 | 5 |

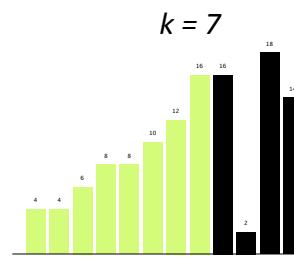
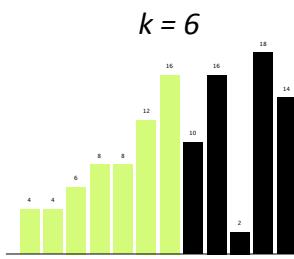
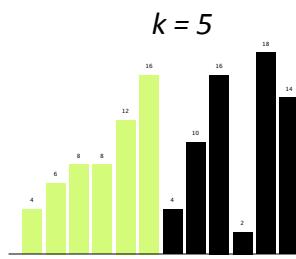
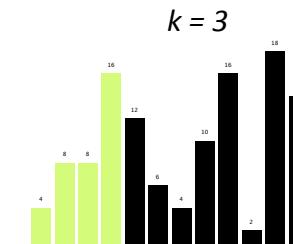
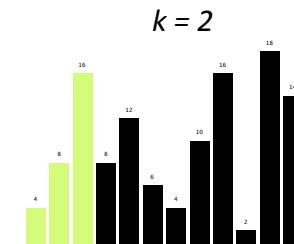
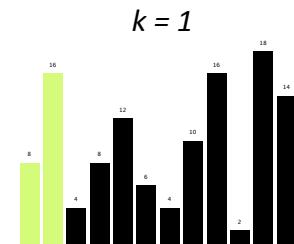
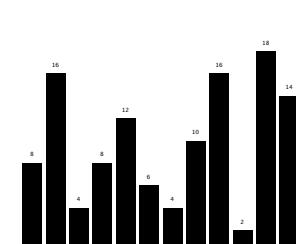
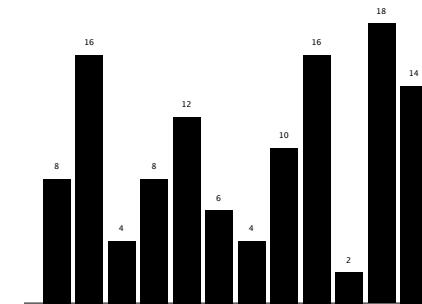
$\sim N^2$ comparisons
 $\sim N$ exchanges

Insertion sort

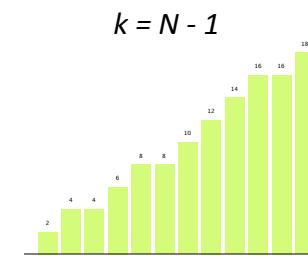
Walk from left to right through the array.

On each step, **insert** the element in the current location in sorted order to its left.

After k steps, the first $k + 1$ elements are in sorted order relative to each other.



• • •



Insertion sort

```
public static void insertionSort(Comparable[] a)
{
    int N = a.length;
    for (int i = 1; i < N; i++)
    {
        int j = i;
        while ((j > 0) && (less(a[j], a[j-1])))
        {
            swap(a, j, j-1);
            j--;
        }
    }
}
```

For each element from the second to the last ...

While the current element is less than the element to its left...

swap this element with the element immediately to the left.

Insertion sort

```
public void insertionSort(Comparable[] a) {  
    int N = a.length;  
    for (int i = 1; i < N; i++) {  
        int j = i;  
        while ((j > 0) &&  
               (less(a[j], a[j-1]))) {  
            swap(a, j, j-1);  
            j--;  
        }  
    }  
}
```

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 8 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 2 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 8 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 2 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

| | | | | | | | | | | | |
|---|---|----|---|----|---|---|----|----|---|----|----|
| 4 | 8 | 16 | 8 | 12 | 6 | 4 | 10 | 16 | 2 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Invariants:

On the k^{th} iteration of the i loop:

The first $k+1$ elements are in sorted order relative to each other.

| | | | | | | | | | | | |
|---|---|---|----|----|---|---|----|----|---|----|----|
| 4 | 8 | 8 | 16 | 12 | 6 | 4 | 10 | 16 | 2 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Elements $k+2$ through the last element are in their original positions and have not been accessed.

| | | | | | | | | | | | |
|---|---|---|----|----|---|---|----|----|---|----|----|
| 4 | 8 | 8 | 12 | 16 | 6 | 4 | 10 | 16 | 2 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Insertion sort

```
public void insertionSort(Comparable[] a) {  
    int N = a.length;  
    for (int i = 1; i < N; i++) {  
        int j = i;  
        while ((j > 0) && (less(a[j], a[j-1]))) {  
            swap(a, j, j-1);  
            j--;  
        }  
    }  
}
```

Time complexity

Insertion sort is $O(N^2)$.

$\leq (N^2-N)/2$ comparisons (calls to less)

$\leq (N^2-N)/2$ exchanges (calls to swap)

Insertion sort is **adaptive** to its input, so some arrangements of data in the array will require less work than others.

| $O(N)$ <i>already sorted</i> | | | | | |
|---------------------------------|---|---|---|----|----|
| 2 | 4 | 6 | 8 | 10 | 12 |
| 0 | 1 | 2 | 3 | 4 | 5 |

$\sim N$ comparisons
0 exchanges

| $O(N)$ <i>"almost" sorted</i> | | | | | |
|----------------------------------|---|----|----|---|---|
| 6 | 8 | 10 | 12 | 2 | 4 |
| 0 | 1 | 2 | 3 | 4 | 5 |

$\sim N$ comparisons
 $\sim N$ exchanges

| $O(N^2)$ <i>in reverse order</i> | | | | | |
|-------------------------------------|----|---|---|---|---|
| 12 | 10 | 8 | 6 | 4 | 2 |
| 0 | 1 | 2 | 3 | 4 | 5 |

$\sim N^2$ comparisons
 $\sim N^2$ exchanges

| $O(N^2)$ <i>in random order</i> | | | | | |
|------------------------------------|----|---|---|---|----|
| 4 | 12 | 8 | 2 | 6 | 10 |
| 0 | 1 | 2 | 3 | 4 | 5 |

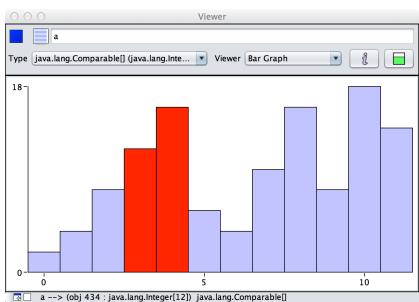
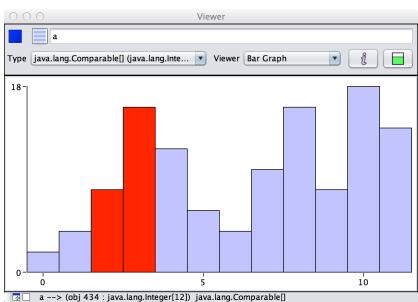
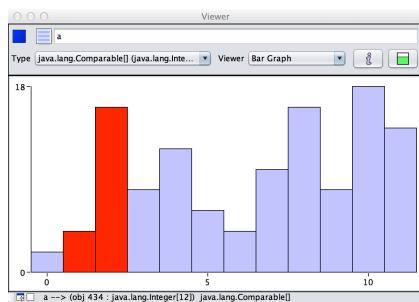
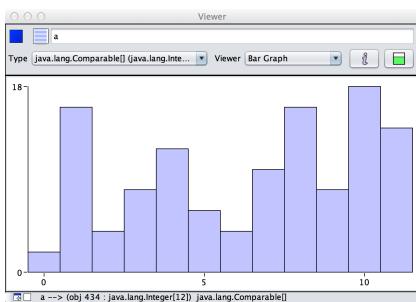
$\sim N^2$ comparisons
 $\sim N^2$ exchanges

Participation



Q: Given the following sequence of images from a jGRASP viewer showing the successive changes in the elements of an array, what sorting algorithm is being used by the underlying program?

- A. Insertion sort ←
- B. Selection sort



Sorting more efficiently

Insertion sort and selection sort both have $O(N^2)$ performance, so they don't scale very well.

| | 1,000 | 10,000 | 100,000 | 1,000,000 | 10,000,000 |
|---------------|-------------|----------|-----------|-----------|-------------|
| $O(\log N)$ | < 1 ns | 10 ns | 132 ns | 166 ns | 199 ns |
| $O(N)$ | 1 ns | 10 ns | 100 ns | 1 ms | 10 ms |
| $O(N \log N)$ | 10 ns | 100 ns | 2 ms | 20 ms | 0.2 sec |
| $O(N^2)$ | 1 ms | 0.1 sec | 10 sec | 17 min | 28 hours |
| $O(N^3)$ | 1 sec | 17 min | 12 days | 32 yrs | 32,000 yrs |
| $O(N^4)$ | 17 min | 4 months | 3,200 yrs | 3.2 M yrs | 3.17E15 yrs |
| $O(2^N)$ | 3.4E284 yrs | ?? | ?? | ?? | ?? |
| $O(N!)$ | ?? | ?? | ?? | ?? | ?? |

Sorting more efficiently

How could we sort more efficiently?

| | 1,000 | 10,000 | 100,000 | 1,000,000 | 10,000,000 |
|--------------------|-------------|----------------|---------------|---------------|-----------------|
| O(log N) | < 1 ns | 10 ns | 132 ns | 166 ns | 199 ns |
| O(N) | 1 ns | 10 ns | 100 ns | 1 ms | 10 ms |
| O(NlogN) | 10 ns | 100 ns | 2 ms | 20 ms | 0.2 sec |
| O(N ²) | 1 ms | 0.1 sec | 10 sec | 17 min | 28 hours |
| O(N ³) | 1 sec | 17 min | 12 days | 32 yrs | 32,000 yrs |
| O(N ⁴) | 17 min | 4 months | 3,200 yrs | 3.2 M yrs | 3.17E15 yrs |
| O(2 ^N) | 3.4E284 yrs | ?? | ?? | ?? | ?? |
| O(N!) | ?? | ?? | ?? | ?? | ?? |

Counting sort, radix sort, etc.
O(N)

Impose additional constraints on the problem.

Example: The values being sorted must be integers in a given range. => Counting sort O(N)

Sorting more efficiently

How could we sort more efficiently?

| | 1,000 | 10,000 | 100,000 | 1,000,000 | 10,000,000 |
|--------------------|--------------|----------------|---------------|---------------|-----------------|
| O(log N) | < 1 ns | 10 ns | 132 ns | 166 ns | 199 ns |
| O(N) | 1 ns | 10 ns | 100 ns | 1 ms | 10 ms |
| O(NlogN) | 10 ns | 100 ns | 2 ms | 20 ms | 0.2 sec |
| O(N ²) | 1 ms | 0.1 sec | 10 sec | 17 min | 28 hours |
| O(N ³) | 1 sec | 17 min | 12 days | 32 yrs | 32,000 yrs |
| O(N ⁴) | 17 min | 4 months | 3,200 yrs | 3.2 M yrs | 3.17E15 yrs |
| O(2 ^N) | 3.4E284 yrs | ?? | ?? | ?? | ?? |
| O(N!) | ?? | ?? | ?? | ?? | ?? |

Merge sort, quicksort,
O(NlogN)

Use a divide-and-conquer algorithm

Example: Divide the array in half, sort each half, then combine the sorted halves. => Merge sort O(NlogN)

Divide and conquer

Divide and conquer is an algorithm design technique where we **divide** the problem into two or more smaller parts, solve (**conquer**) each part, and then **combine** the solutions for the parts into a solution for the whole problem.

Example: Find the maximum element in an array.

Typical strategy: iteration

| | | | | | | | |
|----|----|---|---|----|---|----|---|
| 14 | 10 | 4 | 6 | 12 | 8 | 16 | 2 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

16

```
public int max(int[] a) {  
    int m = a[0];  
    for (int i : a) {  
        if (i > m)  
            m = i;  
    }  
    return m;  
}
```

Divide and conquer strategy

Just for illustration!

| | | | | | | | |
|----|----|---|---|----|---|----|---|
| 14 | 10 | 4 | 6 | 12 | 8 | 16 | 2 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

Divide: Partition the array into two halves.

| | | | |
|----|----|---|---|
| 14 | 10 | 4 | 6 |
| 0 | 1 | 2 | 3 |

| | | | |
|----|---|----|---|
| 12 | 8 | 16 | 2 |
| 4 | 5 | 6 | 7 |

Conquer: Find the largest value in each half.

largest in left: 14

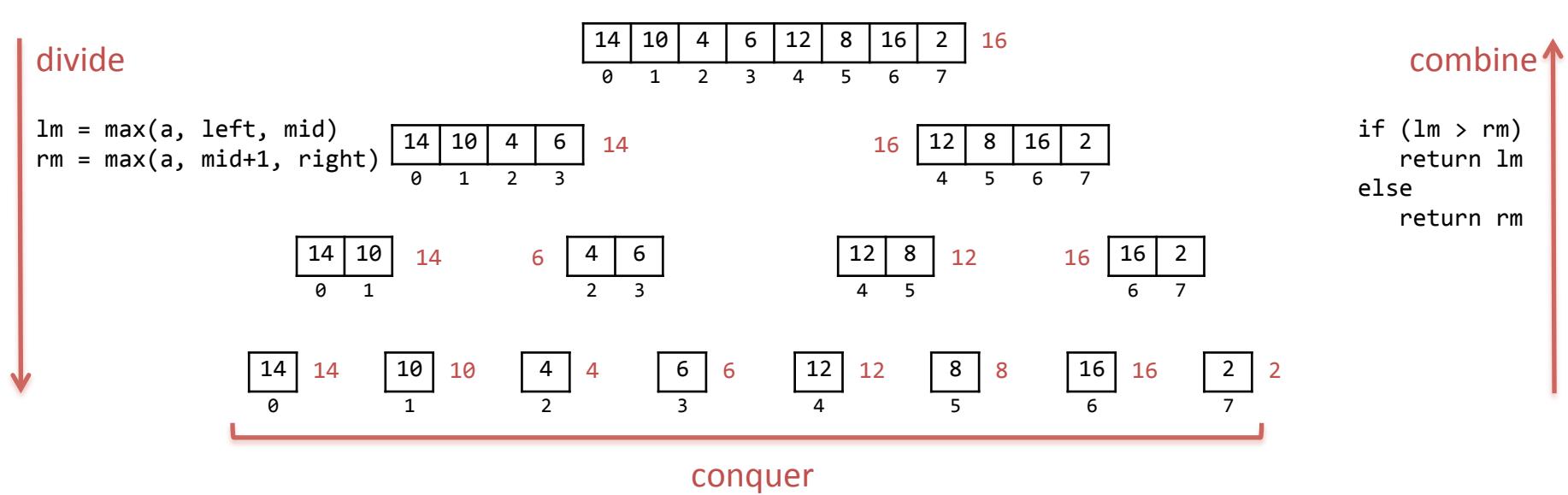
largest in right: 16

Combine: Pick the larger of these two values. **16**

Divide and conquer

Divide and conquer algorithms are usually expressed **recursively**, and the division is repeated until each part is small enough to be solved directly or trivially. (*more to come . . .*)

```
public int max(int[] a, int left, int right) { . . . }
```



Divide and conquer

A divide-and-conquer solution for the maximum element in an array:

Just for illustration!

```
public static int max(int[] a, int l, int r) {  
    if (l == r) {  
        return a[l]; } conquer  
    }  
  
    int mid = (l + r) / 2;  
    int lm = max(a, l, mid); } divide  
    int rm = max(a, mid + 1, r);  
  
    if (lm > rm )  
        return lm;  
    else  
        return rm; } combine  
}
```

D&C sorting?

divide

| | | | | | | | |
|----|----|---|---|----|---|----|---|
| 14 | 10 | 4 | 6 | 12 | 8 | 16 | 2 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

| | | | |
|----|----|---|---|
| 14 | 10 | 4 | 6 |
| 0 | 1 | 2 | 3 |

| | | | |
|----|---|----|---|
| 12 | 8 | 16 | 2 |
| 4 | 5 | 6 | 7 |

| | |
|----|----|
| 14 | 10 |
| 0 | 1 |

| | |
|---|---|
| 4 | 6 |
| 2 | 3 |

| | |
|----|---|
| 12 | 8 |
| 4 | 5 |

| | |
|----|---|
| 16 | 2 |
| 6 | 7 |

| |
|----|
| 14 |
| 0 |

| |
|----|
| 10 |
| 1 |

| |
|---|
| 4 |
| 2 |

| |
|---|
| 6 |
| 3 |

| |
|----|
| 12 |
| 4 |

| |
|---|
| 8 |
| 5 |

| |
|----|
| 16 |
| 6 |

| |
|---|
| 2 |
| 7 |

conquer

combine

| | |
|----|----|
| 10 | 14 |
| 0 | 1 |

| | |
|---|---|
| 4 | 6 |
| 2 | 3 |

| | |
|---|----|
| 8 | 12 |
| 4 | 5 |

| | |
|---|----|
| 2 | 16 |
| 6 | 7 |

| | | | |
|---|---|----|----|
| 4 | 6 | 10 | 14 |
| 0 | 1 | 2 | 3 |

| | | | |
|---|---|----|----|
| 2 | 8 | 12 | 16 |
| 4 | 5 | 6 | 7 |

| | | | | | | | |
|---|---|---|---|----|----|----|----|
| 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

Merge sort

Merge sort is a comparison sort based on the **divide-and-conquer** strategy.

First described by John von Neumann in 1945 as part of his work on EDVAC.

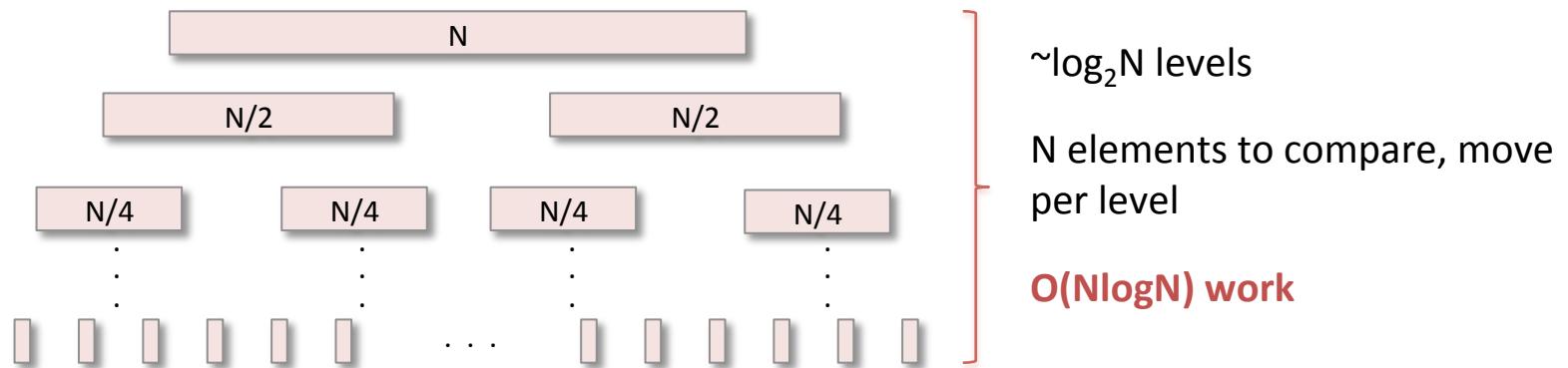


Asymptotically optimal for comparison sorting – $O(N \log N)$, but requires $\sim N$ extra memory (it's **not in-place**).

Divide: Divide the array in half.

Conquer: Sort each half (recursively).

Combine: Merge the sorted halves into a single array.

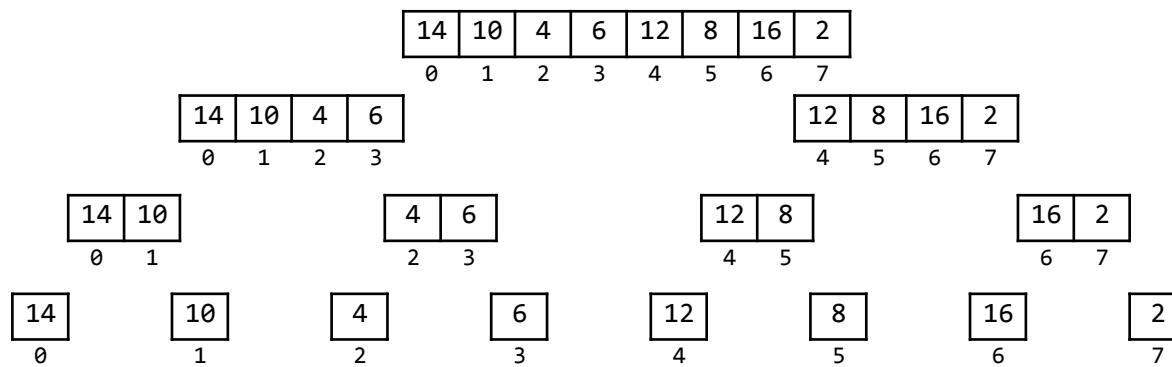


Merge sort

```
public void mergeSort(Comparable[] a, int left, int right) {  
    if (right <= left) return; }conquer  
    int mid = left + (right - left) / 2;  
    mergeSort(a, left, mid); }divide  
    mergeSort(a, mid + 1, right); }  
    merge(a, left, mid, right); }combine  
}
```

Merge sort

```
public static void mergeSort(Comparable[] a, int left, int right) {  
    if (right <= left) return;  
  
    int mid = left + (right - left) / 2; }  
    mergeSort(a, left, mid); }  
    mergeSort(a, mid + 1, right); } }  
    merge(a, left, mid, right); }
```



Merge sort

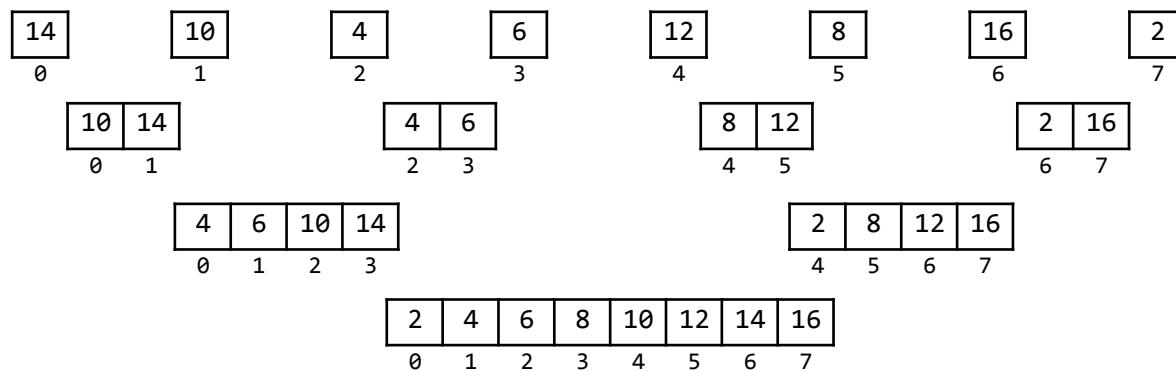
```
public static void mergeSort(Comparable[] a, int left, int right) {  
    if (right <= left) return; }conquer  
  
    int mid = left + (right - left) / 2;  
    mergeSort(a, left, mid);  
    mergeSort(a, mid + 1, right);  
  
    merge(a, left, mid, right);  
}
```



Sorting a one-element array is trivial – it's already sorted so there's nothing to do.

Merge sort

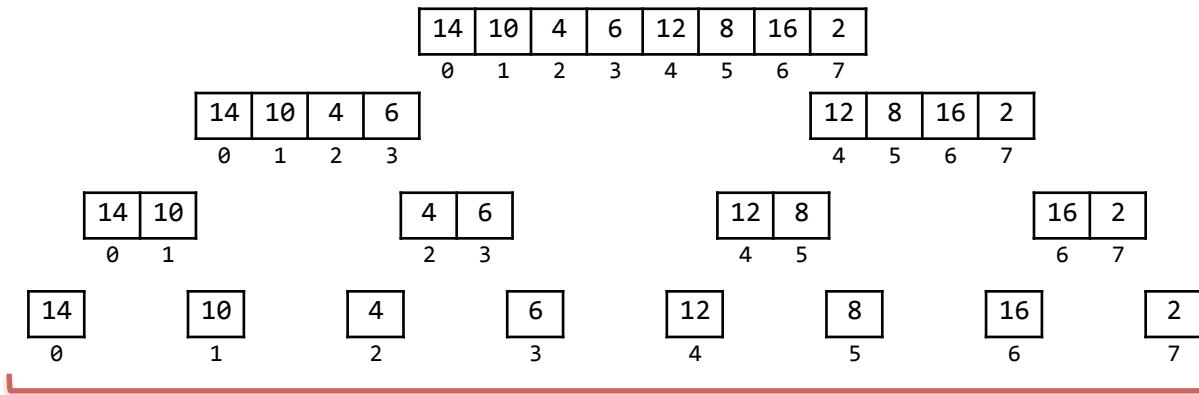
```
public static void mergeSort(Comparable[] a, int left, int right) {  
    if (right <= left) return;  
  
    int mid = left + (right - left) / 2;  
    mergeSort(a, left, mid);  
    mergeSort(a, mid + 1, right);  
  
    merge(a, left, mid, right); } combine
```



Merge sort

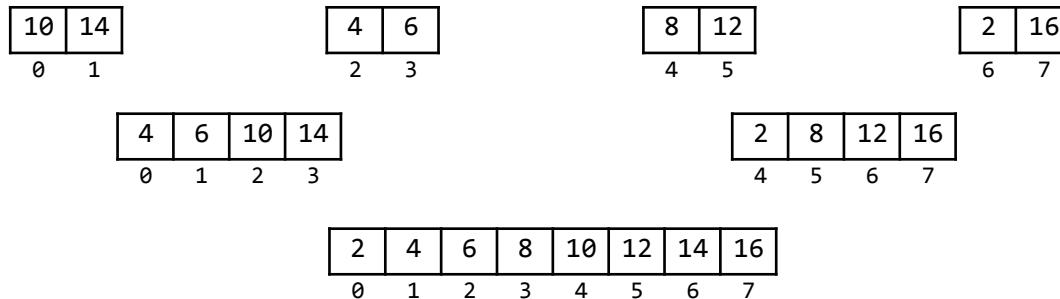
Truth in advertising: order of execution

divide



conquer

combine



Merge sort

The real work in merge sort happens during the combine phase: merging two sorted arrays into one sorted array.

```
public static void mergeSort(Comparable[] a, int left, int right) {  
    if (right <= left) return;  
  
    int mid = left + (right - left) / 2;  
    mergeSort(a, left, mid);  
    mergeSort(a, mid + 1, right);  
  
    merge(a, left, mid, right);  
}
```



Merge sort

The real work in merge sort happens during the combine phase: merging two sorted arrays into one sorted array.

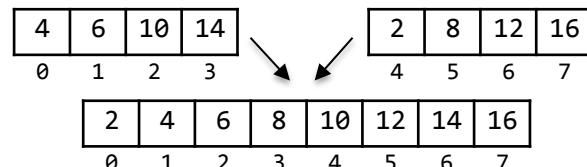
```
public void merge(Comparable[] a, int left, int mid, int right) {  
  
    for (int k = left; k <= right; k++)    aux[k] = a[k];  
  
    int i = left; j = mid + 1;  
    for (int k = left; k <= right; k++) {  
        if          (i > mid)                  a[k] = aux[j++];  
        else if     (j > right)                a[k] = aux[i++];  
        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else  
    }  
}
```

Time complexity:

$O(N)$

A field in the enclosing class.
 $O(N)$ extra space

$O(N)$



Merge sort

```
public void merge(Comparable[] a, int left, int mid, int right) {  
  
    for (int k = left; k <= right; k++)    aux[k] = a[k];  
  
    int i = left; j = mid + 1;  
    for (int k = left; k <= right; k++) {  
        if          (i > mid)                  a[k] = aux[j++];  
        else if     (j > right)                a[k] = aux[i++];  
        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```

| | left | | | | right | | | |
|---|------|---|----|----|-------|---|----|----|
| a | 4 | 6 | 10 | 14 | 2 | 8 | 12 | 16 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

| aux | 4 | 6 | 10 | 14 | 2 | 8 | 12 | 16 |
|-----|---|---|----|----|---|---|----|----|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

Merge sort

```
public void merge(Comparable[] a, int left, int mid, int right) {  
  
    for (int k = left; k <= right; k++)    aux[k] = a[k];  
  
    int i = left; j = mid + 1;  
    for (int k = left; k <= right; k++) {  
        if          (i > mid)                  a[k] = aux[j++];  
        else if     (j > right)                a[k] = aux[i++];  
        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```

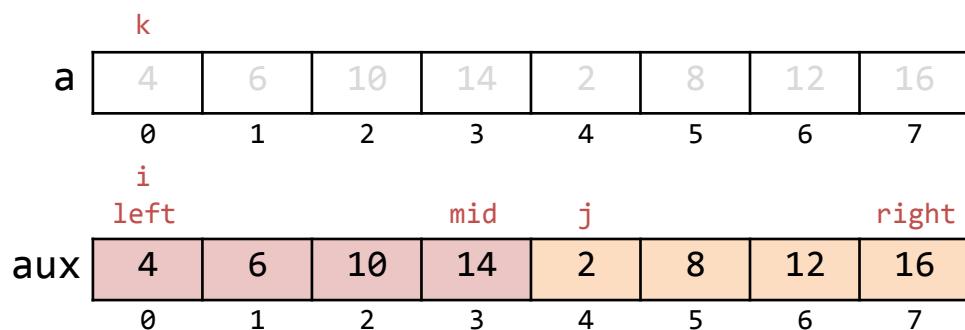
| | | | | | | | | |
|---|---|---|----|----|---|---|----|----|
| a | 4 | 6 | 10 | 14 | 2 | 8 | 12 | 16 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

| | | | | | | | | |
|-----|---|------|-----|----|-------|---|----|----|
| | i | left | mid | j | right | | | |
| aux | 4 | 6 | 10 | 14 | 2 | 8 | 12 | 16 |

| | | | | | | | | |
|--|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

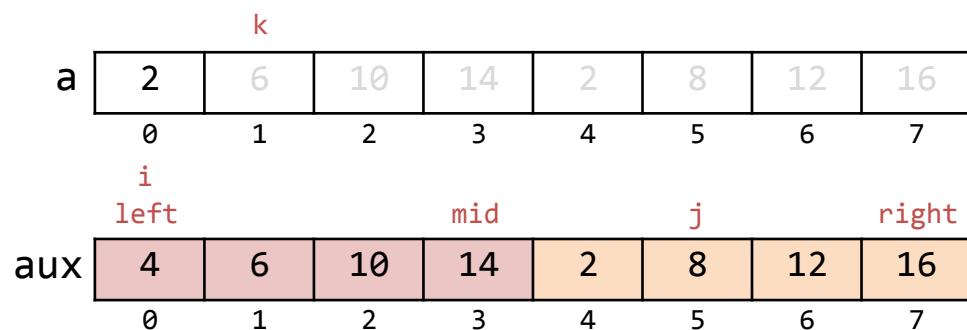
Merge sort

```
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    for (int k = left; k <= right; k++)    aux[k] = a[k];  
  
    int i = left; j = mid + 1;  
    for (int k = left; k <= right; k++) {  
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        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



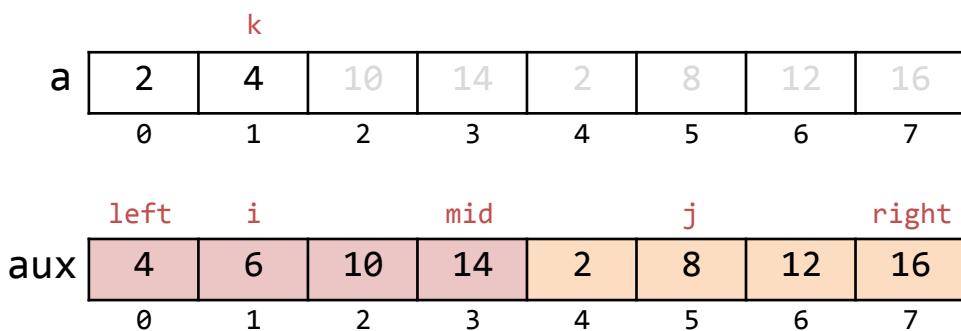
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        else if     (j > right)                a[k] = aux[i++];  
        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



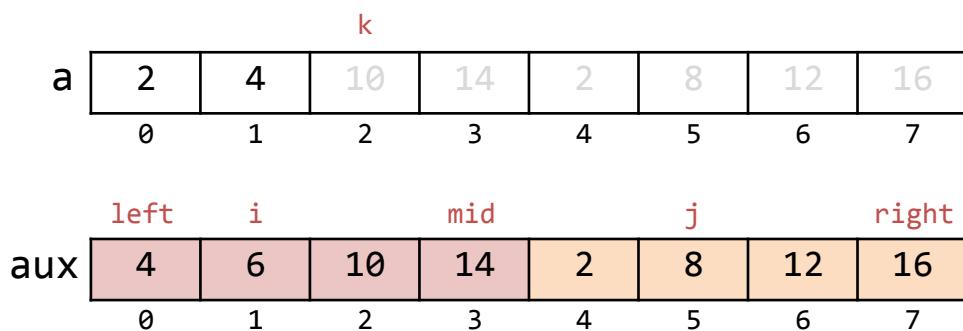
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        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



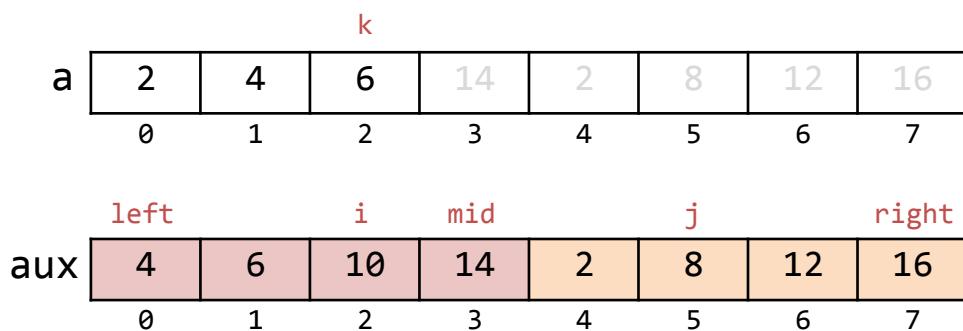
Merge sort

```
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    for (int k = left; k <= right; k++)    aux[k] = a[k];  
  
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        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



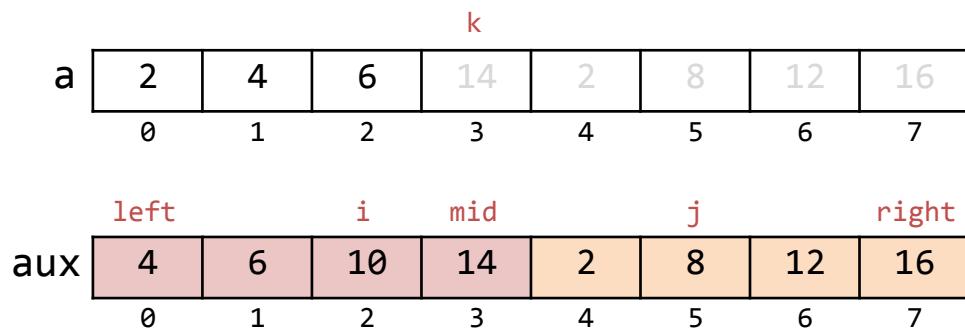
Merge sort

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        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



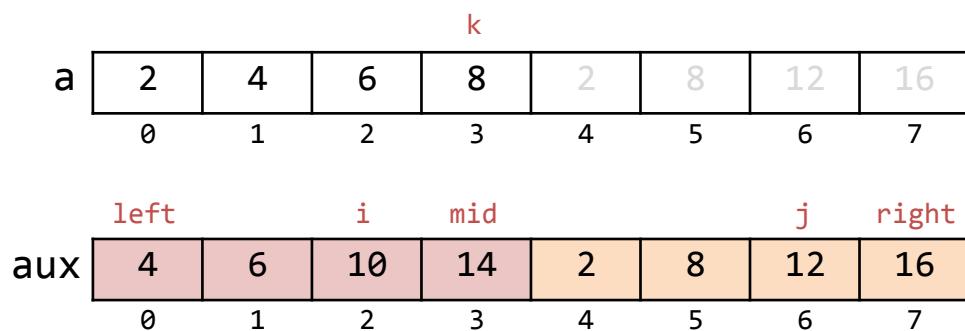
Merge sort

```
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    for (int k = left; k <= right; k++)    aux[k] = a[k];  
  
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    for (int k = left; k <= right; k++) {  
        if          (i > mid)                  a[k] = aux[j++];  
        else if     (j > right)                a[k] = aux[i++];  
        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



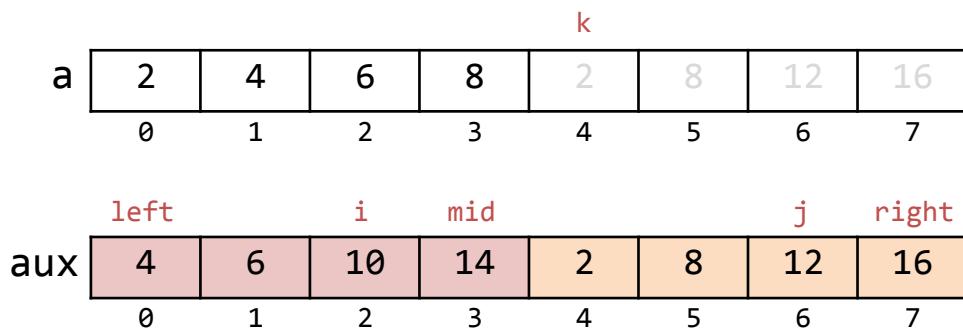
Merge sort

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    int i = left; j = mid + 1;  
    for (int k = left; k <= right; k++) {  
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        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



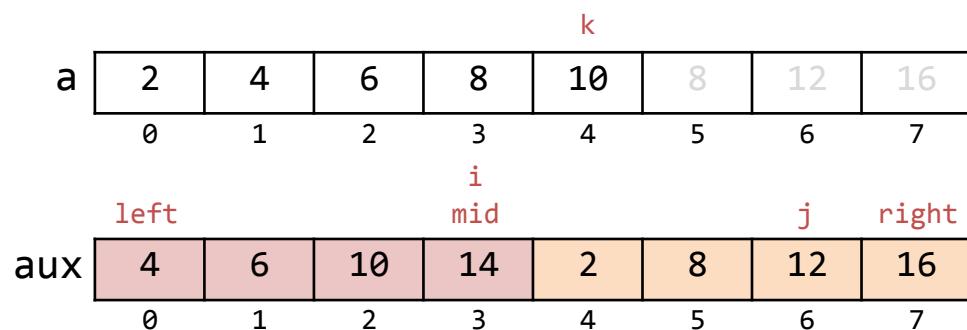
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        else                      a[k] = aux[i++];  
    }  
}
```



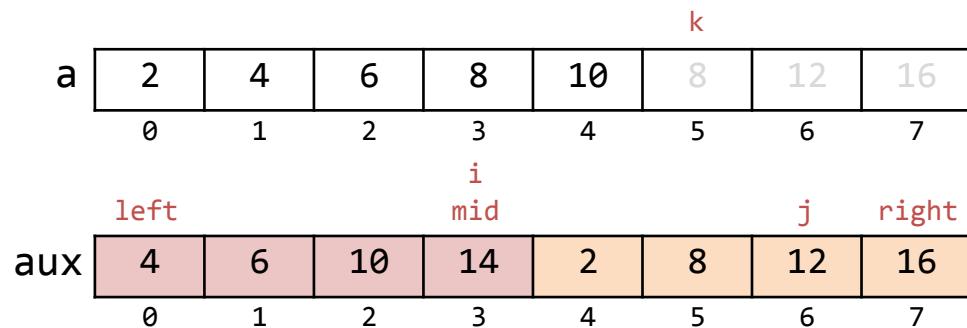
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        else                      a[k] = aux[i++];  
    }  
}
```



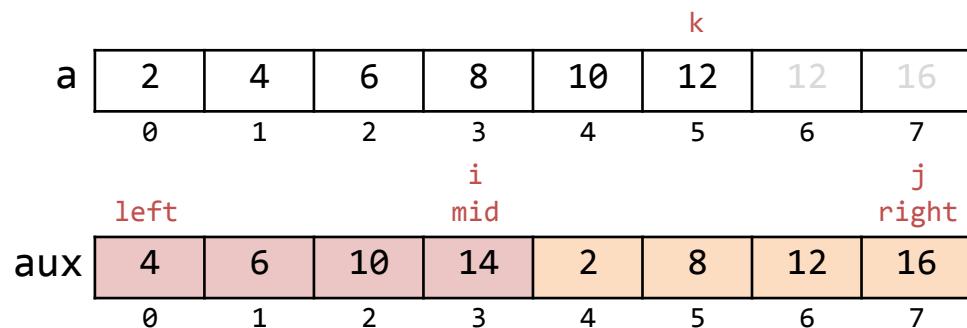
Merge sort

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        else if     (j > right)                a[k] = aux[i++];  
        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



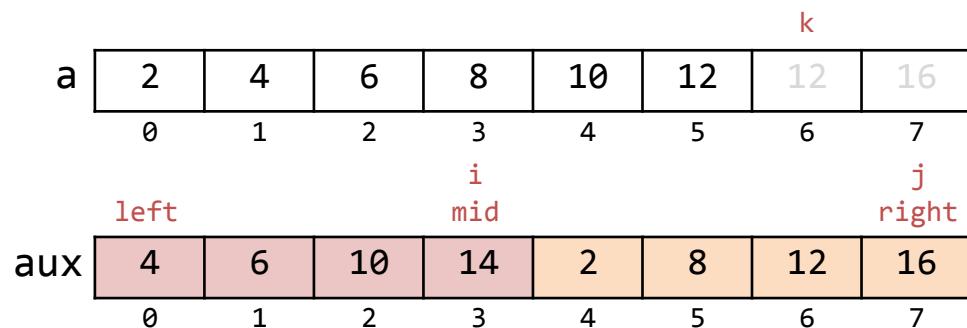
Merge sort

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        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



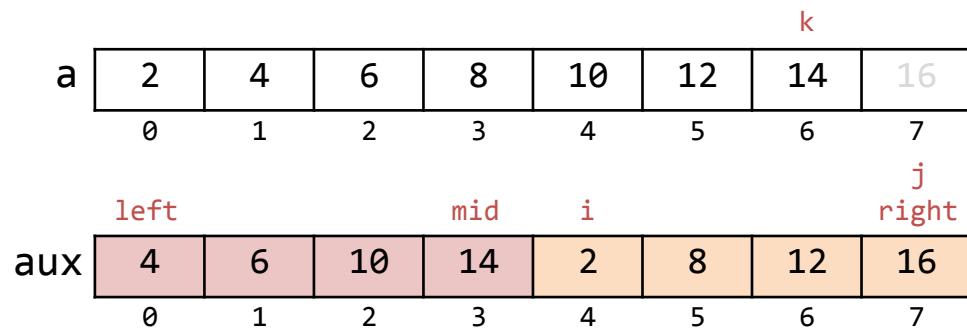
Merge sort

```
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        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



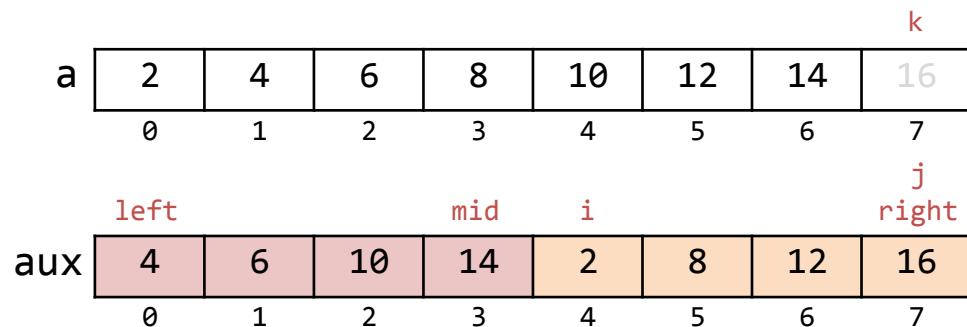
Merge sort

```
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    for (int k = left; k <= right; k++) {  
        if          (i > mid)                  a[k] = aux[j++];  
        else if     (j > right)                a[k] = aux[i++];  
        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



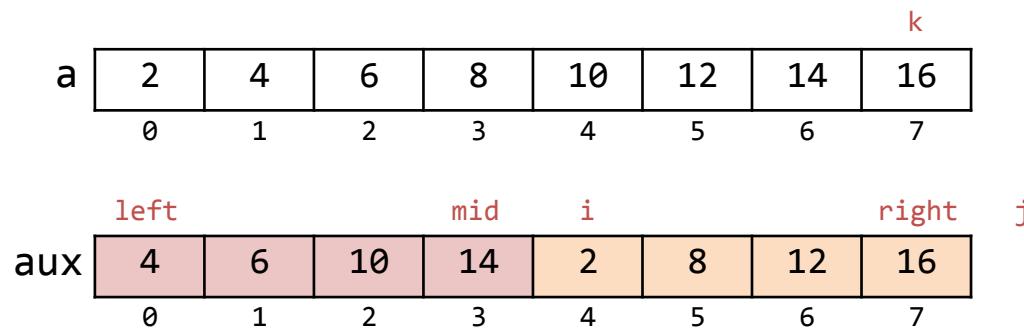
Merge sort

```
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    for (int k = left; k <= right; k++)    aux[k] = a[k];  
  
    int i = left; j = mid + 1;  
    for (int k = left; k <= right; k++) {  
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        else if     (j > right)                a[k] = aux[i++];  
        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



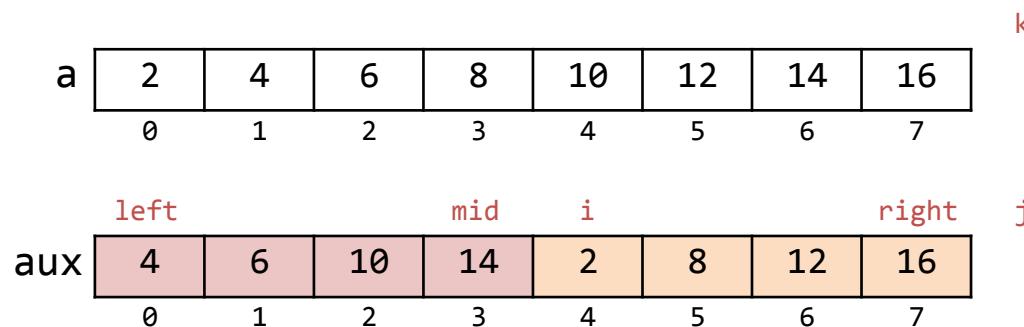
Merge sort

```
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    for (int k = left; k <= right; k++)    aux[k] = a[k];  
  
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        if          (i > mid)                  a[k] = aux[j++];  
        else if     (j > right)                a[k] = aux[i++];  
        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



Merge sort

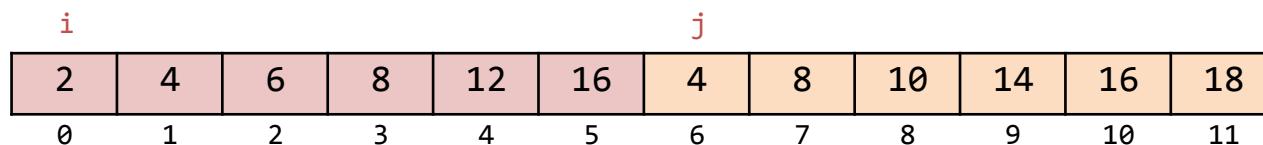
```
public void merge(Comparable[] a, int left, int mid, int right) {  
  
    for (int k = left; k <= right; k++)    aux[k] = a[k];  
  
    int i = left; j = mid + 1;  
    for (int k = left; k <= right; k++) {  
        if          (i > mid)                  a[k] = aux[j++];  
        else if     (j > right)                a[k] = aux[i++];  
        else if     (less(aux[j], aux[i]))    a[k] = aux[j++];  
        else                      a[k] = aux[i++];  
    }  
}
```



Merge sort

```
public void merge(Comparable[] a, int left, int mid, int right) {  
  
    for (int k = left; k <= right; k++)    aux[k] = a[k];  
  
    int i = left; j = mid + 1;  
    for (int k = left; k <= right; k++) {  
        if          (i > mid)                  a[k] = aux[j++];  
        else if     (j > right)                a[k] = aux[i++];  
        else if     (less(aux[j], aux[i]))    a[k] = aux[j++]; ←  
        else  
            a[k] = aux[i++];  
    }  
}
```

Note that this makes merge sort *stable*.



Quicksort

Quicksort is a comparison sort based on the **divide-and-conquer** strategy.

First described by Tony Hoare in 1960 as part of his work on machine translation.



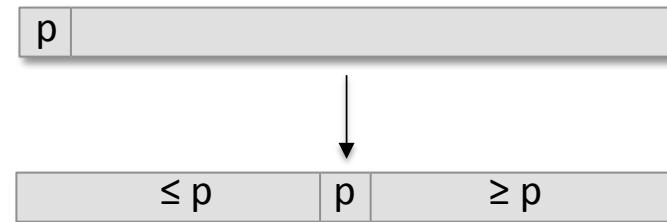
Asymptotically optimal for comparison sorting – $O(N \log N)$, but only in the average case. Has worst-case time that is $O(N^2)$.

Divide: Select a **pivot** then **partition** the array so that:

- pivot is in its correct sorted position
- no larger element is to the left of pivot
- no smaller element is to the right of pivot

Conquer: Sort each partition (recursively)

Combine: Nothing to do



The partitioning can be done in linear time.

The choice of pivot determines the size of the partitions.

Quicksort

```
public void qsort(Comparable[] a, int left, int right)
{
    if (right <= left) } conquer
        return;

    int j = partition(a, left, right);
    qsort(a, left, j-1); } divide
    qsort(a, j+1, right);
}
```

Partitioning does the real work of the sort, so there's nothing to do in the combine phase.

Quicksort

```
public void qsort(Comparable[] a, int left, int right)
{
    if (right <= left)
        return;

    int j = partition(a, left, right);
    qsort(a, left, j-1);
    qsort(a, j+1, right);
}
```

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 2 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 8 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

pivot = 10 ≤ 10 10 ≥ 10

| | | | | | | | | | | | |
|---|---|---|---|---|---|----|----|----|----|----|----|
| 2 | 4 | 8 | 6 | 4 | 8 | 10 | 14 | 16 | 16 | 18 | 12 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Quicksort

```
public void qsort(Comparable[] a, int left, int right)
{
    if (right <= left)
        return;

    int j = partition(a, left, right);
    qsort(a, left, j-1);
    qsort(a, j+1, right);
}
```

The partition method takes the array and (at least) the left and right indexes of the current portion of the array, and returns the resulting index of the pivot value after partitioning.

Notice that the partition method above chooses the pivot value internally. We could move that decision out to the qsort method, as the following example assumes.

Quicksort

There are several ways to approach partitioning. Here's one that's easy to follow and has the pivot location provided as a parameter:

```
private int partition(T[] a, int left, int right, int pivotIndex) {
    T pivot = a[pivotIndex];
    swap(a, pivotIndex, right); // move pivot to the end
    int p = left; // p will become the final index of pivot
    for (int i = left; i < right; i++) {
        if (less(a[i], pivot)) {
            swap(a, i, p);
            p++;
        }
    }
    swap(a, p, right); // move pivot to its correct location
    return p;
}
```

Quicksort

```
private int partition(T[] a, int left, int right, int pivotIndex) {  
    T pivot = a[pivotIndex];  
    swap(a, pivotIndex, right); // move pivot to the end  
    int p = left; // p will become the final index of pivot  
    for (int i = left; i < right; i++) {  
        if (less(a[i], pivot)) {  
            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```

| left | pivotIndex | | | | | | | | | | | right |
|------|------------|---|---|----|---|---|----|----|---|----|----|-------|
| 2 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 8 | 18 | 14 | |

0 1 2 3 4 5 6 7 8 9 10 11

Quicksort

```
private int partition(T[] a, int left, int right, int pivotIndex) {  
    T pivot = a[pivotIndex];  
    swap(a, pivotIndex, right); // move pivot to the end  
    int p = left; // p will become the final index of pivot  
    for (int i = left; i < right; i++) {  
        if (less(a[i], pivot)) {  
            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```

| left | pivotIndex | | | | | | | | | | | right |
|------|------------|---|---|----|---|---|----|----|---|----|----|-------|
| 2 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 8 | 18 | 14 | |

Quicksort

```
private int partition(T[] a, int left, int right, int pivotIndex) {  
    T pivot = a[pivotIndex];  
    swap(a, pivotIndex, right); // move pivot to the end  
    int p = left; // p will become the final index of pivot  
    for (int i = left; i < right; i++) {  
        if (less(a[i], pivot)) {  
            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```

| left | | | | | | | | | | | | | pivotIndex | right | |
|------|----|---|---|----|---|---|----|----|---|----|----|--|------------|-------|--|
| 2 | 16 | 4 | 8 | 12 | 6 | 4 | 14 | 16 | 8 | 18 | 10 | | | | |

Quicksort

```
private int partition(T[] a, int left, int right, int pivotIndex) {  
    T pivot = a[pivotIndex];  
    swap(a, pivotIndex, right); // move pivot to the end  
    int p = left; // p will become the final index of pivot  
    for (int i = left; i < right; i++) {  
        if (less(a[i], pivot)) {  
            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```

| i | p | left | right |
|---|---|------|-------|
| 0 | 1 | 2 | 11 |

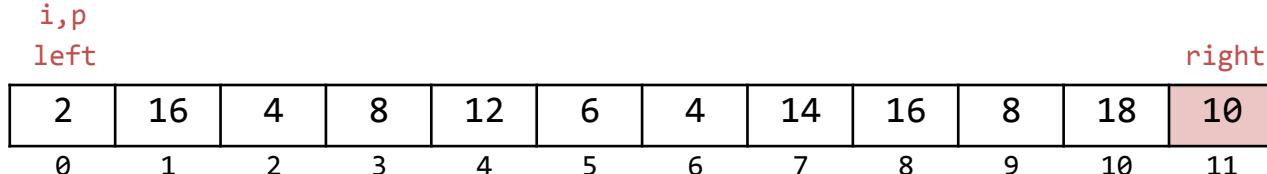
The array elements are: 2, 16, 4, 8, 12, 6, 4, 14, 16, 8, 18, 10. The pivot element is 10, highlighted in pink.

Quicksort

```
private int partition(T[] a, int left, int right, int pivotIndex) {  
    T pivot = a[pivotIndex];  
    swap(a, pivotIndex, right); // move pivot to the end  
    int p = left; // p will become the final index of pivot  
    for (int i = left; i < right; i++) {  
        if (less(a[i], pivot)) {  
            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```

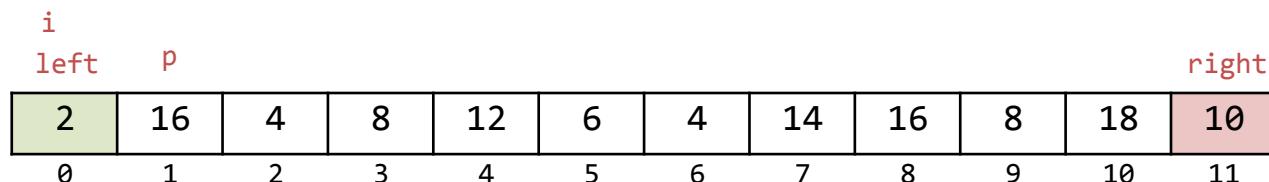
| i | p | left | right |
|---|---|------|-------|
| 0 | 1 | 2 | 11 |

The array elements are: 2, 16, 4, 8, 12, 6, 4, 14, 16, 8, 18, 10. The pivot element is 10, highlighted in pink.



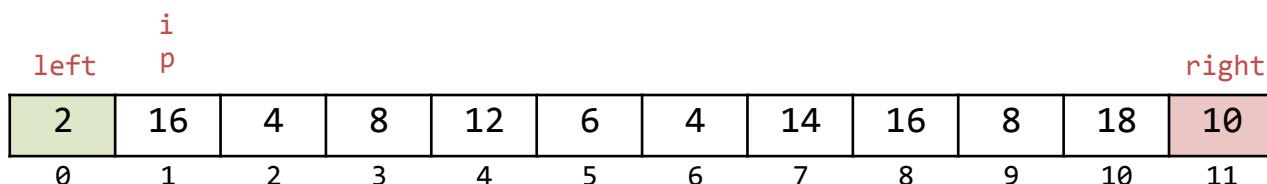
Quicksort

```
private int partition(T[] a, int left, int right, int pivotIndex) {  
    T pivot = a[pivotIndex];  
    swap(a, pivotIndex, right); // move pivot to the end  
    int p = left; // p will become the final index of pivot  
    for (int i = left; i < right; i++) {  
        if (less(a[i], pivot)) {  
            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



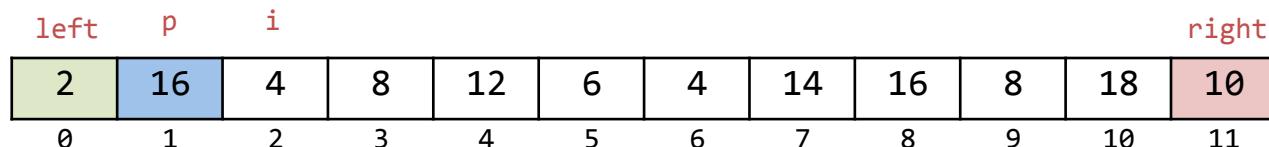
Quicksort

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    T pivot = a[pivotIndex];  
    swap(a, pivotIndex, right); // move pivot to the end  
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    for (int i = left; i < right; i++) {  
        if (less(a[i], pivot)) {  
            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



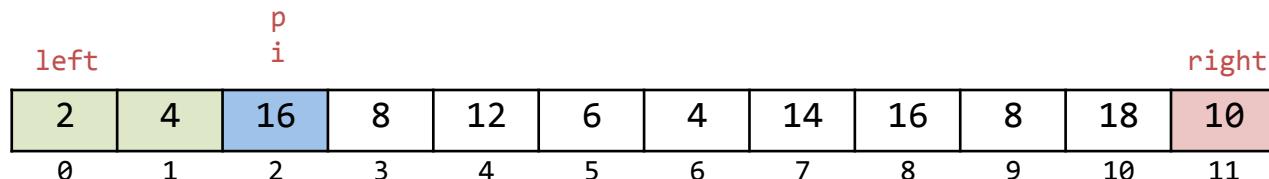
Quicksort

```
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    T pivot = a[pivotIndex];  
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            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



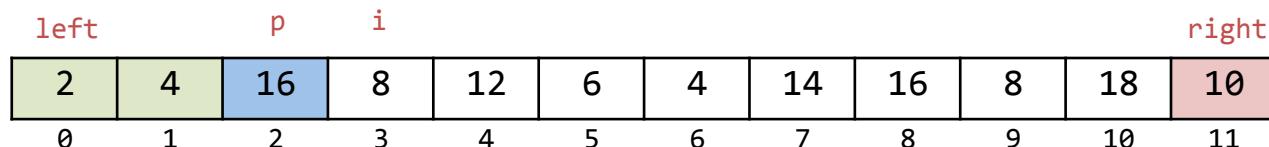
Quicksort

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    T pivot = a[pivotIndex];  
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        if (less(a[i], pivot)) {  
            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



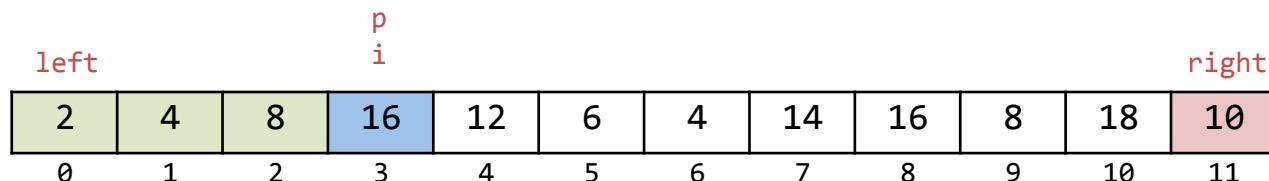
Quicksort

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            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



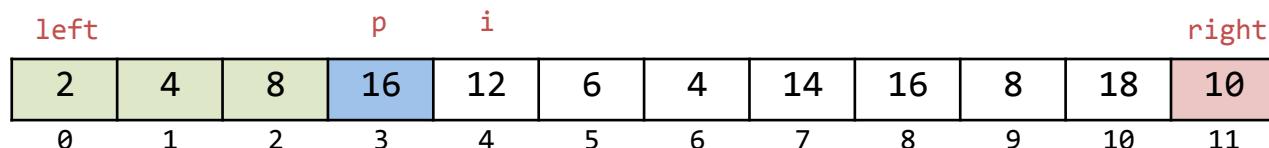
Quicksort

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            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



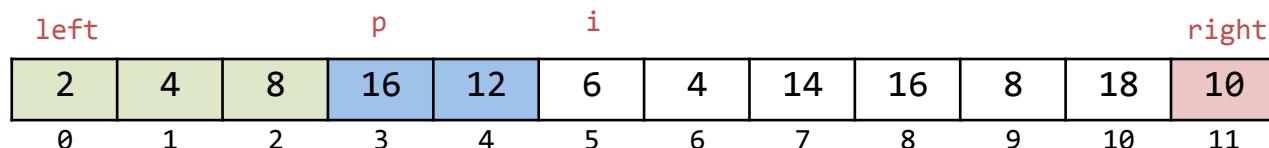
Quicksort

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            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



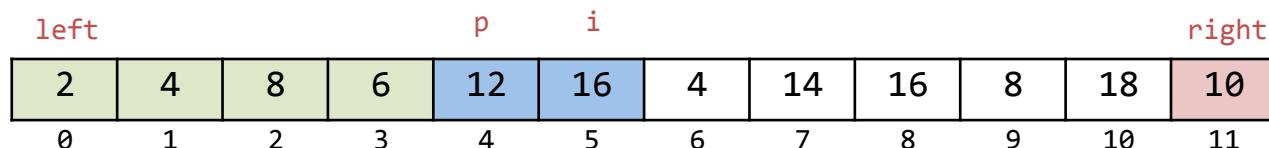
Quicksort

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            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



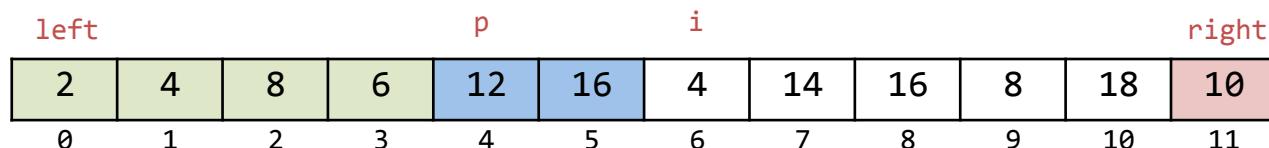
Quicksort

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    T pivot = a[pivotIndex];  
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            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
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```



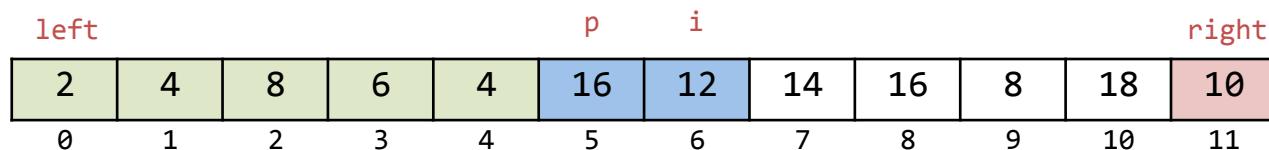
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            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
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}
```



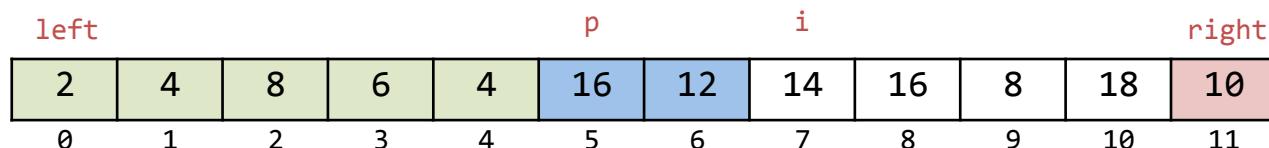
Quicksort

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    return p;  
}
```



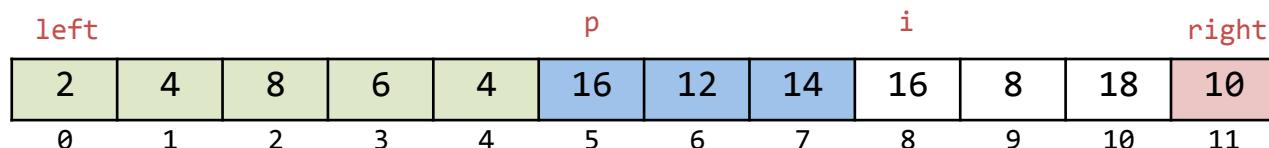
Quicksort

```
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    T pivot = a[pivotIndex];  
    swap(a, pivotIndex, right); // move pivot to the end  
    int p = left; // p will become the final index of pivot  
    for (int i = left; i < right; i++) {  
        if (less(a[i], pivot)) {  
            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



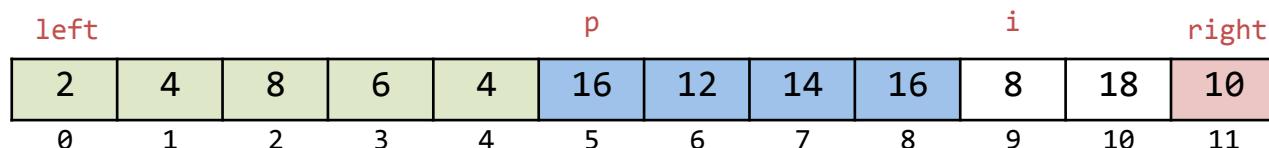
Quicksort

```
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        if (less(a[i], pivot)) {  
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            p++;  
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    swap(a, p, right); // move pivot to its correct location  
    return p;  
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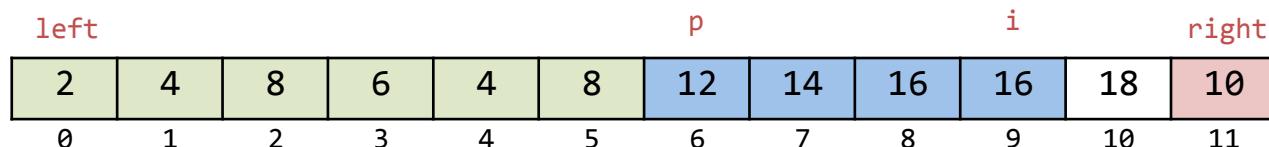
Quicksort

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    T pivot = a[pivotIndex];  
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            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



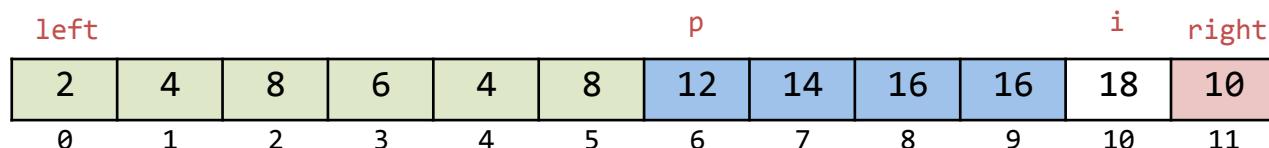
Quicksort

```
private int partition(T[] a, int left, int right, int pivotIndex) {  
    T pivot = a[pivotIndex];  
    swap(a, pivotIndex, right); // move pivot to the end  
    int p = left; // p will become the final index of pivot  
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        if (less(a[i], pivot)) {  
            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



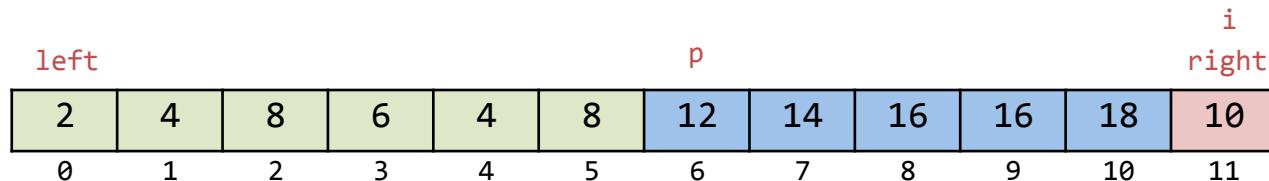
Quicksort

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private int partition(T[] a, int left, int right, int pivotIndex) {  
    T pivot = a[pivotIndex];  
    swap(a, pivotIndex, right); // move pivot to the end  
    int p = left; // p will become the final index of pivot  
    for (int i = left; i < right; i++) {  
        if (less(a[i], pivot)) {  
            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



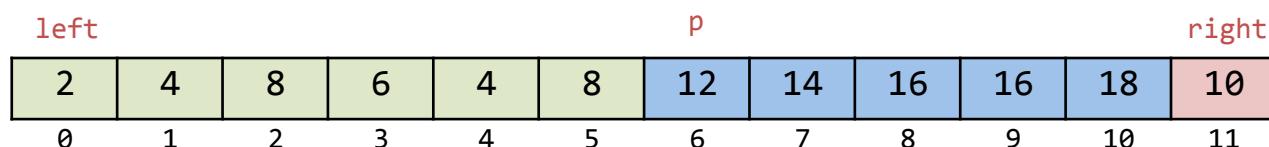
Quicksort

```
private int partition(T[] a, int left, int right, int pivotIndex) {  
    T pivot = a[pivotIndex];  
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            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



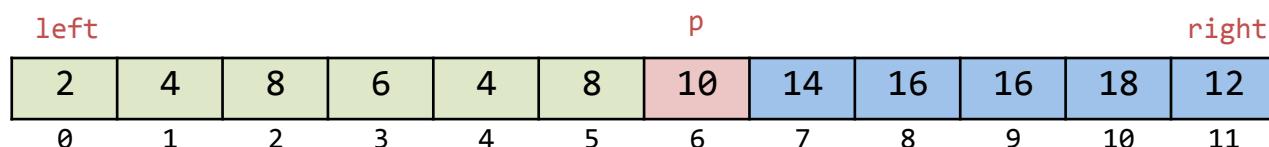
Quicksort

```
private int partition(T[] a, int left, int right, int pivotIndex) {  
    T pivot = a[pivotIndex];  
    swap(a, pivotIndex, right); // move pivot to the end  
    int p = left; // p will become the final index of pivot  
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        if (less(a[i], pivot)) {  
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            p++;  
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    swap(a, p, right); // move pivot to its correct location  
    return p;  
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```



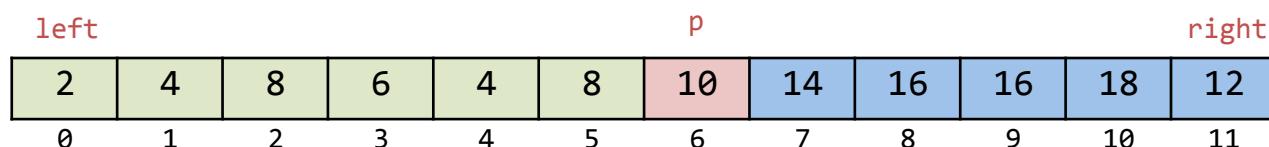
Quicksort

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    T pivot = a[pivotIndex];  
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Quicksort

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        if (less(a[i], pivot)) {  
            swap(a, i, p);  
            p++;  
        }  
    }  
    swap(a, p, right); // move pivot to its correct location  
    return p;  
}
```



Quicksort

```
public void qsort(Comparable[] a, int left, int right)
{
    if (right <= left)
        return;

    int j = partition(a, left, right);
    qsort(a, left, j-1);
    qsort(a, j+1, right);
}
```

The choice of pivot value determines the size of each partition, and therefore determines the number of divide steps that will be necessary.

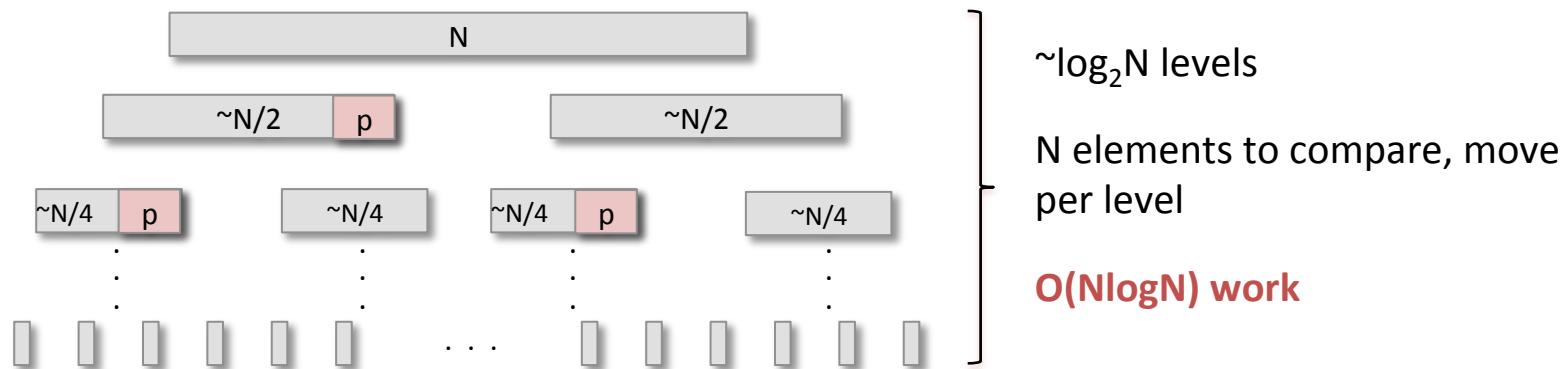


“It becomes evident that sorting on the basis of Quicksort is somewhat like a gamble in which one should be aware of how much one may afford to lose if bad luck were to strike.” – Niklaus Wirth

Quicksort

The choice of pivot value determines the size of each partition, and therefore determines the number of divide steps that will be necessary.

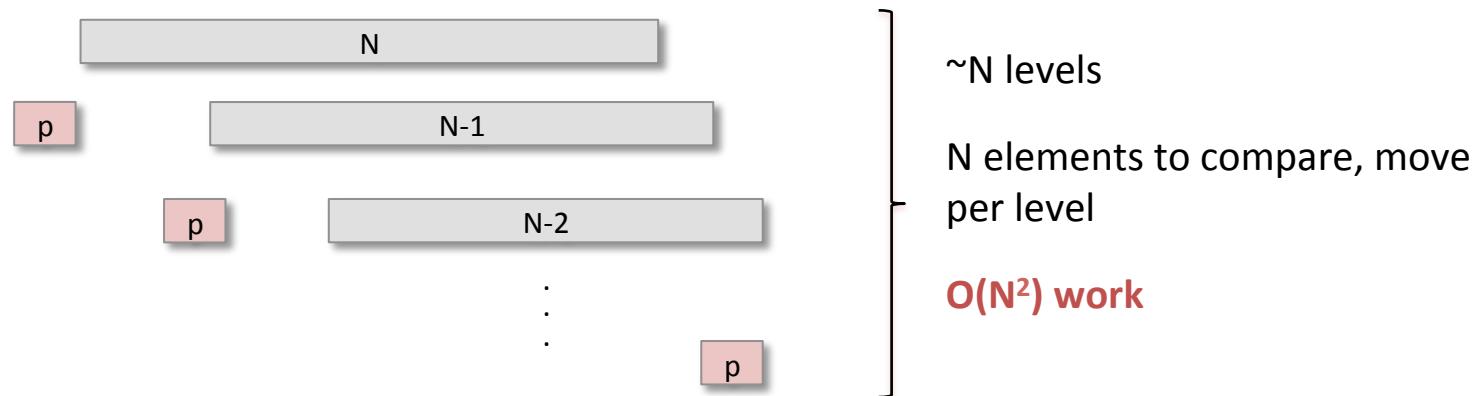
Best case pivot choices at each step lead to partitions that are approximately the same size.



Quicksort

The choice of pivot value determines the size of each partition, and therefore determines the number of divide steps that will be necessary.

Worst case pivot choices at each step lead to one partition that is empty.



Quicksort

Example partition based on a given pivot value for the following array:

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 2 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 8 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |



pivot = 6

| | | | | | | | | | | | |
|---|---|---|---|----|----|----|----|----|---|----|----|
| 2 | 4 | 4 | 6 | 12 | 14 | 16 | 10 | 16 | 8 | 18 | 8 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Quicksort

Example partition based on a given pivot value for the following array:

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 2 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 8 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |



pivot = 10

| | | | | | | | | | | | |
|---|---|---|---|---|---|----|----|----|----|----|----|
| 2 | 4 | 8 | 6 | 4 | 8 | 10 | 14 | 16 | 16 | 18 | 12 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Quicksort

Example partition based on a given pivot value for the following array:

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 2 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 8 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |



pivot = 2

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 2 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 8 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Quicksort

Example partition based on a given pivot value for the following array:

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 2 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 8 | 18 | 14 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |



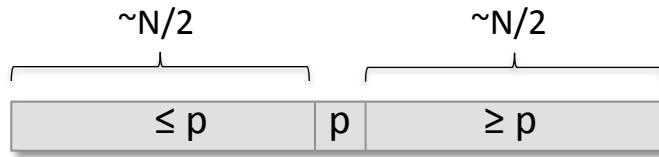
pivot = 18

| | | | | | | | | | | | |
|---|----|---|---|----|---|---|----|----|---|----|----|
| 2 | 16 | 4 | 8 | 12 | 6 | 4 | 10 | 16 | 8 | 14 | 18 |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Quicksort

Choosing a pivot value

Median value



The median of a list of N items can be found in $O(N)$ time using a *selection* algorithm.

But, this linear time overhead would be added to each divide step in quicksort.

Median of three

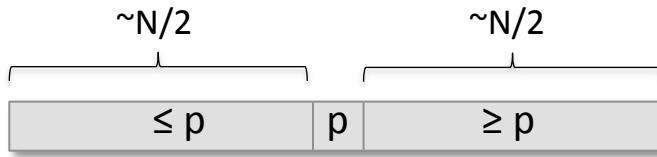
Choose three elements of the array (usually first, middle, last) and then use the median of those three values as the pivot.

This is a good compromise and a popular choice in quicksort implementations. The 3270 analysis of quicksort will show that even though the partitions may not be $\approx N/2$ at any given divide step, the overall performance will be $O(N\log N)$ on average.

Quicksort

Choosing a pivot value

Randomly pick



Use a random number generator (PRNG) to pick an index and use the element at that index as the pivot. Again, 3270 will show that this will lead to $O(N\log N)$ average case complexity.

But, some PRNGs can be slow and this overhead would be added to each divide step.

Shuffle once, pick first element

Randomize the order of elements in the array once up front (before the D&C part of the sort begins) and then just pick the first element as the pivot each time.

This is really just a variation of the first approach, but it pulls all calls to the PRNG out of the sort and make the pivot choice trivial (and fast).

Quicksort

Randomize the order of elements in an array in $O(N)$ time.

```
private void shuffle(Comparable[] a) {
    int N = a.length;
    java.util.Random rng = new java.util.Random();
    for (int i = N - 1; i > 0; i--) {
        int j = rng.nextInt(i + 1);
        swap(a, i, j);
    }
}
```



[The Knuth Shuffle](#)

Quicksort

A “randomized” quicksort:

```
public void quicksort(Comparable[] a) {  
    shuffle(a);  
    qsort(a, 0, a.length - 1);  
}
```

Shuffling happens once, before the sort begins. So, the overhead associated with randomization is minimized.

```
public void qsort(Comparable[] a, int left, int right) {  
    if (right <= left)  
        return;  
  
    int j = partition(a, left, right);  
    qsort(a, left, j-1);  
    qsort(a, j+1, right);  
}
```

The partition method can use $a[left]$ as the pivot – trivial and fast pivot selection.

Quicksort

There are many variations on the implementation of quicksort, but when it's done well, quicksort is typically the sort-of-choice in many situations.

Quicksort has a worst case complexity of $O(N^2)$. However, a good pivot choice strategy makes the worst case highly unlikely.

Because of this, we categorize quicksort by its average case complexity – **$O(N \log N)$** .

Note that quicksort is an **in-place sort**, but is usually **not stable**.

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Engineering a Sort Function

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SUMMARY
We recount the history of a new `qsort` function for C library. Our function is clearer, faster and more reliable than existing `qsort`s. It sorts using elements by a new sampling scheme, partitions by a novel solution to Dijkstra's Dutch National Flag problem, and it swaps efficiently. Its behavior was assessed with timing and debugging testbeds, and with a program to certify performance. The design techniques apply in domains beyond sorting.

KEY WORDS Quicksort Sorting algorithm Performance tuning Algorithm design and implementation Testing

INTRODUCTION
C libraries have long included a `qsort` function to sort an array, usually implemented by Hoare's Quicksort.¹ Because existing `qsort`s are flawed, we built a new one. This paper summarizes its evolution.

Compared to existing library sorts, our new `qsort` is faster—typically about twice as fast—cleaner, and more reliable under nonrandom inputs. It uses some standard Quicksort tricks, blends others, and introduces some new tricks of its own. Our approach to building a `qsort` is relevant to engineering other algorithms.

The `qsort` on our home system, based on Scowen's "Quicksort",² had served faithfully since Lee McMahon wrote it almost two decades ago. Shipped with the landmark Seventh Edition Unix System,³ it became a model for other `qsort`s. Yet in the summer of 1991 our colleagues Allan Wilks and Rick Becker found that a `qsort` run that should have taken a few minutes was chewing up hours of CPU time. Had they not interrupted it, it would have gone on for weeks.⁴ They found that it took n^2 comparisons to sort an "organ-pipe" array of $2ni$ integers: 123, 333, 321.

Shopping around for a better `qsort`, we found that a `qsort` written at Berkeley in 1983 would consistently quadratic in n , and that it was the only `qsort` to do so among many timer tests, particularly on runs of random zeros and ones.⁵ In fact, among a dozen different Unix libraries we found no `qsort` that could not easily be driven to quadratic behavior; all were derived from the Seventh Edition or from the 1983 Berkeley function. The Seventh



[Jon Bentley](#)

[A great interview](#)

[A great video](#)

Participation



Q: Suppose an array with its elements in reverse (descending) order is passed to a sorting method and you observe that the method takes time proportional to N^2 . Which sorting algorithm is definitely NOT implemented in this method?

- A. Insertion sort
- B. Selection sort
- C. Quicksort
- D. Merge sort



Sorts used in the JCF

The JCF offers sorting methods in both the Arrays class and the Collections class.

Quicksort is used for primitives.

"The sorting algorithm is a Dual-Pivot Quicksort by Vladimir Yaroslavskiy, Jon Bentley, and Joshua Bloch. This algorithm offers $O(n \log(n))$ performance on many data sets that cause other quicksorts to degrade to quadratic performance, and is typically faster than traditional (one-pivot) Quicksort implementations."

Merge sort is used for references.

Implementation note: This implementation is a stable, adaptive, iterative mergesort that requires far fewer than $n \lg(n)$ comparisons when the input array is partially sorted, while offering the performance of a traditional mergesort when the input array is randomly ordered. If the input array is nearly sorted, the implementation requires approximately n comparisons. Temporary storage requirements vary from a small constant for nearly sorted input arrays to $n/2$ object references for randomly ordered input arrays. The implementation takes equal advantage of ascending and descending order in its input array, and can take advantage of ascending and descending order in different parts of the same input array. It is well-suited to merging two or more sorted arrays: simply concatenate the arrays and sort the resulting array. The implementation was adapted from Tim Peters's list sort for Python (TimSort). It uses techniques from Peter McIlroy's "Optimistic Sorting and Information Theoretic Complexity", in Proceedings of the Fourth Annual ACM-SIAM Symposium on Discrete Algorithms, pp 467-474, January 1993.

Stability

```
private static class Data implements Comparable<Data> {  
    private Integer field1;  
    private Integer field2;  
  
    public Data(Integer f1, Integer f2) { . . . }  
  
    public int compareTo(Data that) {  
        if (this.field1 < that.field1)  
            return -1;  
        else if (this.field1 > that.field1)  
            return 1;  
        else  
            return 0;  
    }  
    . . .  
}
```

```
Data d1 = new Data(4, 1);  
Data d2 = new Data(3, 9);
```

d1.compareTo(d2) ==

Compares
only on
field1

Stability

```
Data[] a = {new Data(5, 1), new Data(4, 2), new Data(3, 3), new Data(5, 4),
            new Data(2, 5), new Data(1, 6), new Data(5, 7)};
```

| | | | | | | |
|-------|-------|-------|-------|-------|-------|-------|
| (5,1) | (4,2) | (3,3) | (5,4) | (2,5) | (1,6) | (5,7) |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |



| | | | | | | |
|-------|-------|-------|-------|-------|-------|-------|
| (1,6) | (2,5) | (3,3) | (4,2) | (5,1) | (5,4) | (5,7) |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |

Three duplicates according to natural order.

The duplicates are in the same relative order as before the sort.

Merge sort is stable.

Stability

```
Data[] a = {new Data(5, 1), new Data(4, 2), new Data(3, 3), new Data(5, 4),
            new Data(2, 5), new Data(1, 6), new Data(5, 7)};
```

| | | | | | | |
|-------|-------|-------|-------|-------|-------|-------|
| (5,1) | (4,2) | (3,3) | (5,4) | (2,5) | (1,6) | (5,7) |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |

↓
quicksort(a)

| | | | | | | |
|-------|-------|-------|-------|-------|-------|-------|
| (1,6) | (2,5) | (3,3) | (4,2) | (5,1) | (5,7) | (5,4) |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 |

Three duplicates according to natural order.

The duplicates are not in the same relative order as before the sort.

Quicksort is not stable.

Quicksort could be stable, but typical implementations aren't in the interest of efficiency.

Stability

```
class Student implements Comparable<Student> {
    private String fname;
    private String lname;
    private int    section;

    . . .

    public String toString() {
        return lname + ", " + fname + ", " + section;
    }

    public int compareTo(Student s) {
        return this.toString().compareTo(s.toString());
    }
}
```

Stability

```
Student[] roll = new Student[7];
// new Student("last name", "first name", section)
roll[0] = new Student("Wei", "Patricia", 1);
roll[1] = new Student("Dipaolo", "Jordan", 1);
roll[2] = new Student("Cullins", "Janna", 2);
roll[3] = new Student("Center", "Wallace", 2);
roll[3] = new Student("Trojacek", "Evelyn", 3);
roll[4] = new Student("Buffum", "Tawana", 3);
roll[5] = new Student("Laber", "Sharyl", 3);
roll[6] = new Student("Hageman", "Rachel", 2);

shuffle(roll);

Comparator<Student> orderByName = new CompareStudentsByName();
Comparator<Student> orderBySection = new CompareStudentsBySection();
```

Stability

```
print(a);

quicksort(roll, orderByName);
print(a);

quicksort(roll, orderBySection);
print(a);
```

```
% java SortStabilityExample
Laber,      Sharyl,      3
Hageman,    Rachel,     2
Cullins,    Janna,      2
Dipaolo,    Jordan,     1
Buffum,     Tawana,     3
Trojacek,   Evelyn,    3
Wei,        Patricia,   1
```

Stability

```
print(a);

quicksort(roll, orderByName);
print(a);

quicksort(roll, orderBySection);
print(a);
```

```
% java SortStabilityExample
```

```
. . .
```

| | | |
|-----------|-----------|---|
| Buffum, | Tawana, | 3 |
| Cullins, | Janna, | 2 |
| Dipaolo, | Jordan, | 1 |
| Hageman, | Rachel, | 2 |
| Laber, | Sharyl, | 3 |
| Trojacek, | Evelyn, | 3 |
| Wei, | Patricia, | 1 |

Stability

```
print(a);

quicksort(roll, orderByName);
print(a);

quicksort(roll, orderBySection);
print(a);
```

```
% java SortStabilityExample
. . .
. . .
Wei,      Patricia, 1
Dipaolo,  Jordan,   1
Hageman,  Rachel,   2
Cullins,  Janna,    2
Laber,    Sharyl,   3
Buffum,   Tawana,   3
Trojacek, Evelyn,  3
```

No longer sorted by name.

Stability

```
print(a);

mergesort(roll, orderByName);
print(a);

mergesort(roll, orderBySection);
print(a);
```

```
% java SortStabilityExample
Laber,      Sharyl,      3
Hageman,    Rachel,     2
Cullins,    Janna,      2
Dipaolo,    Jordan,     1
Buffum,     Tawana,     3
Trojacek,   Evelyn,    3
Wei,        Patricia,   1
```

Stability

```
print(a);

mergesort(roll, orderByName);
print(a);

mergesort(roll, orderBySection);
print(a);
```

```
% java SortStabilityExample
```

```
. . .
```

| | | |
|-----------|-----------|---|
| Buffum, | Tawana, | 3 |
| Cullins, | Janna, | 2 |
| Dipaolo, | Jordan, | 1 |
| Hageman, | Rachel, | 2 |
| Laber, | Sharyl, | 3 |
| Trojacek, | Evelyn, | 3 |
| Wei, | Patricia, | 1 |

Stability

```
print(a);

mergesort(roll, orderByName);
print(a);

mergesort(roll, orderBySection);
print(a);
```

```
% java SortStabilityExample
```

```
. . .
```

| | | |
|-----------|-----------|---|
| Dipaolo, | Jordan, | 1 |
| Wei, | Patricia, | 1 |
| Cullins, | Janna, | 2 |
| Hageman, | Rachel, | 2 |
| Buffum, | Tawana, | 3 |
| Laber, | Sharyl, | 3 |
| Trojacek, | Evelyn, | 3 |

Sorted by name within section.

Summary

Here's a summary of the four sorting algorithms we've talked about – as illustrated by the given implementations – with respect to time complexity and basic properties of sorting.

| | Selection | Insertion | Mergesort | Quicksort |
|--------------|-----------|-----------|---------------|---------------|
| Worst case | $O(N^2)$ | $O(N^2)$ | $O(N \log N)$ | $O(N^2)$ |
| Average case | $O(N^2)$ | $O(N^2)$ | $O(N \log N)$ | $O(N \log N)$ |
| Best case | $O(N^2)$ | $O(N)$ | $O(N \log N)$ | $O(N \log N)$ |
| In-place? | Yes | Yes | No | Yes |
| Stable? | No | Yes | Yes | No |
| Adaptive? | No | Yes | No | No |