Seminar 10

08. 11. 2022

1) Fie G un grup a.î. $x^2 = e$, $\forall x \in G$ atunci G este abelian

$x^2 = e$, pt $\forall x \in G \iff x = x^{-1}$, pt $\forall x \in G$.

Fie $x, y \in G$, atunci $xy = (xy)^{-1} \iff xy = y^{-1}x^{-1}$

$\iff xy = yx$

2) G grup cu p elemente (p prim) $\Rightarrow G = \langle x \rangle$, $\forall x \in G \setminus \{e\}$

$\Rightarrow G \simeq (\mathbb{Z}_p, +)$

$G = \{e, x, \ldots, x^{p-1}\} \longrightarrow \mathbb{Z}_p : x^j \to \hat{j}$

G grup cu 4 elemente

$\Rightarrow G \simeq (\mathbb{Z}_4, +)$ sau $G \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2, +)$

Fie $x \in G$, $x \neq e \Rightarrow \text{ord}(x) \mid 4 \Rightarrow \text{ord}(x) \in \{2, 4\}$

(I) $\exists x \in G$ a.î. $\text{ord}(x) = 4$

$|\langle x \rangle| = 4 = |G| \Rightarrow G = \langle x \rangle \Rightarrow G$ grup ciclic cu 4 elemente $\Rightarrow G \simeq (\mathbb{Z}_4, +)$

$\text{ord}(x) = 4 \Rightarrow \text{ord}(x) = 2 \ \forall x \in G \setminus \{e\}$

(ii) $\nexists x \in G$ a.î. $\text{ord}(x) = 4 \Rightarrow G$ abelian

$\Rightarrow x^2 = e \ \forall x \in G \Rightarrow G$ abelian

Fie $x \in G \setminus \{e\} \Rightarrow \langle x \rangle \subsetneq G$

$\Rightarrow \exists y \in G \setminus \langle x \rangle \Rightarrow \underbrace{|\{e, x, y, xy\}|}_{\text{cardinal}} = 4 \Rightarrow$

$\Rightarrow G = \{e, x, y, xy\}$ cu $\text{ord}(x) = \text{ord}(y) = 2$

$xy = yx$

$G \to \mathbb{Z}_2 \times \mathbb{Z}_2 : x \to (1,0),\ y \to (0,1),\ xy \to (1,1)$

(3) Fie $G = \{A \in M_2(\mathbb{C}) \mid |A| \neq 0\}$ gr. cu $\cdot$. Considerăm

$Q = \langle A, B \rangle < G$

$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$A^2 = A \cdot A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2 \Rightarrow$

$\Rightarrow A^3 = -A \Rightarrow A^4 = I_2$, deci $\text{ord}(A) = 4$

$B^2 = B \cdot B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2 \Rightarrow$

$\Rightarrow B^3 = -B \Rightarrow B^4 = I_2$

$\text{ord}(B) = 4$

$AB = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$$BA = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -AB$$

$$Q = \langle A, B \mid ord(A) = ord(B) = 4,\ A^2 = B^2 = -I_2,\ AB = -BA \rangle$$

$$Q = \{\pm A^u B^v \mid u, v \in \mathbb{N}\} = \{\pm A^u B^v \mid u, v \in \{0,1\}\}$$

$$= \{\pm I_2, \pm A, \pm B, \pm AB\} \Rightarrow |Q| = 8,\ Q \text{ s.n.}$$

grupul cuaternionilor.

<u>Subgrupurile lui Q</u>

Fie $H \leq Q \Rightarrow ord(H) \in D_8 = \{1, 2, 4, 8\}$

$|H| = 1 \Rightarrow H = \{I_2\}$

$|H| = 2 \Rightarrow H = \langle X \rangle,\ ord(X) = 2$

$Q = \{\pm I_2, \pm A, \pm B, \pm AB\} \Rightarrow X \in \{-I_2\}$

$$(\pm AB)^2 = (\pm AB)(\pm AB) = A\underbrace{BA}_{-AB}B = -A^2B^2 = -I_2$$

$\Rightarrow H = \langle -I_2 \rangle = \{I_2, -I_2\}$

$|H| = 4 \Rightarrow H = \langle X \mid ord(X) = 4 \rangle$ sau $H = \langle X, Y \mid ord(X) = ord(Y) = 2, XY = YX \rangle$, I_2 e singurul element de ordin 2 din Q

$\Rightarrow H = \langle X \mid ord(X) = 4 \rangle$

$\Rightarrow H \langle A \rangle = \{\pm I_2, \pm A\} = \langle -A \rangle$ sau $H = \langle B \rangle = \{\pm I_2, \pm B\} = \langle -B \rangle$ sau $H = \langle AB \rangle = \{\pm I_2, \pm AB\} = \langle -AB \rangle$

$|H| = 8 \Rightarrow H = Q$

Def H subgrup în G s.n. ~~subgrupul~~ subgrup normal dacă $\forall x \in G$ și $h \in H$, avem $xhx^{-1} \in H$ $(H \trianglelefteq G)$

$H \trianglelefteq G \Rightarrow (G/H)_s = (G/H)_d = G/H$ grup în raport cu înmulțirea indusă de pe $G: \hat{x} \cdot \hat{y} = \widehat{xy}$.

Obs $\{e\} \trianglelefteq G$, $G \trianglelefteq G$

$[G:H] = 2 \Rightarrow H \trianglelefteq G$

Subgrupurile normale ale lui Q

$H = \{I_2\} \Rightarrow H \trianglelefteq Q$

$H = \{\pm I_2\}$

$\forall x \in Q$, $x(\pm I_2)x^{-1} = \pm x x^{-1} = \pm I_2 \in H \Rightarrow$

$\Rightarrow H \trianglelefteq Q$

$H \leq Q$, $|H| = 4 \Rightarrow [G:H] = \frac{|G|}{|H|} \Rightarrow [Q:H] =$

$= \frac{|Q|}{|H|} = \frac{8}{4} = 2 \Rightarrow H \trianglelefteq Q$

$|H| = 8 \Rightarrow H = Q \Rightarrow H \trianglelefteq Q$

Obs Toate subgrupurile lui Q sunt normale, iar Q nu e abelian.

Grupurile factor

Obs $G/\{e\} \simeq G$

$[x]_s = \{xh \mid h \in H\} = xH$

$[x]_d = \{hx \mid h \in H\} = Hx$

$H \trianglelefteq G$, $[H]_s = [H]_d = \hat{x}$

Obs: $G/\{e\} \simeq G$ 2. $G/G \simeq \{e\}$

$\|$

$\{\hat{x} = \{x\} \mid x \in G\}$

$H = \{\pm i_2\} \Rightarrow |Q/H| = 4$

$Q/H = \{\underbrace{\hat{I}_2 = \{\pm I_2\}}_{\text{ordin } 1}, \hat{A} = \{\pm A\}, \hat{B} = \{\pm B\}, \widehat{AB} = \{\pm AB\}\}$

$\hat{A}^2 = \hat{A}\hat{A} = \widehat{-I} = \hat{I}_2 \Rightarrow ord(\hat{A}) = 2$

$\hat{B}^2 = \hat{B^2} = \widehat{-i_2} = \hat{i}_2 = ord(B) = 2$

$\widehat{AB}^2 = \widehat{(AB)^2} = \widehat{-i_2} = \hat{i}_2 \quad ord(\widehat{AB}) = 2$

$\Rightarrow Q/H \simeq (\mathbb{Z}_2 \times \mathbb{Z}_2)$

Dacă $ord(H) = 4 \Rightarrow |Q/H| = 2 \Rightarrow Q/H \simeq \mathbb{Z}_2$

$H = \langle A \rangle = \{\pm i_2, \pm A\}$

$Q/H = \{\hat{i}_2 = \{\pm i_2, \pm A\}, \hat{B} = \{\pm B, \pm AB\}\}$

$ord(H) = 8 \Rightarrow H = Q \Rightarrow Q/H \simeq \{e\}$

$$G = S_3 = \langle \sigma, \tau \mid \sigma^3 = \tau^2 = e, \tau\sigma = \sigma^2\tau \rangle$$

$$H = \langle \tau \rangle = \{e, \tau\}$$

stânga

$$(G/H)_s = \{[e]_s = \{e, \tau\}, [\sigma]_s = \sigma H = \{\sigma, \sigma\tau = \tau\sigma^2\},$$

$$[\sigma^2]_s = \sigma^2 H = \{\sigma^2, \sigma^2\tau = \tau\sigma\}\}$$

$$(G/H)_d = \{[e]_d = \{e, \tau\}, [\sigma]_d = H\sigma = \{\sigma, \tau\sigma\},$$

$$[\sigma^2]_d = H\sigma^2 = \{\sigma^2, \tau\sigma^2\}\}$$

Deci $H \ntrianglelefteq S_3$

<u>Temă</u> Det. subgr. lui S_3 și subgr. normale și grupurile factor.