Seminar 8

24.11.2022

G - grup, $S \subseteq G$ finită

$<S>$ = cel mai mic subgrup al lui G care conține S.

$$= \bigcap_{\substack{H \leq G \text{ (subgrup)} \\ S \subseteq H}} H = \left\{ x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \,\middle|\, n \in \mathbb{N}^*,\ x_i \in S \ \forall i \in \overline{1,n},\ \varepsilon_i \in \{\pm 1\} \right\}$$

$$<x> = \left\{ x^{\varepsilon_1} \cdots x^{\varepsilon_n} \,\middle|\, n \in \mathbb{N}^*,\ \varepsilon_i \in \{\pm 1\},\ \forall i = \overline{1,n} \right\}$$

$= \{ x^k \mid k \in \mathbb{Z} \}$ - subgrupul lui G generat de x.

<u>Exemplu</u> $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ (produsul direct)

$$= \{ (x,y) \mid x,y \in \mathbb{Z}_2 \} = \{ \underbrace{(\hat{0},\hat{0})}_{e}, \underbrace{(\hat{1},\hat{0})}_{u}, \underbrace{(\hat{0},\hat{1})}_{v}, \underbrace{(\hat{1},\hat{1})}_{u+v} \}$$

$<e> = e$

$$<(\hat{1},\hat{0})> = \{ k(\hat{1},\hat{0}) \mid k \in \mathbb{Z} \} = \{ (\hat{k},\hat{0}) \mid k \in \mathbb{Z} \} =$$

$$= \{ (\hat{0},\hat{0}), (\hat{1},\hat{0}) \}$$

$$<(\hat{0},\hat{1})> = \{(\hat{0},\hat{0}),(\hat{0},\hat{1})\}$$

$$<(\hat{1},\hat{1})> = \{(\hat{0},\hat{0}),(\hat{1},\hat{1})\}$$

<u>Obs</u> G - grup abelian și $x,y \in G$, $<x,y> = \{x^k y^l \mid k,l \in \mathbb{Z}\}$

$$<(\hat{0},\hat{1}),(\hat{1},\hat{0})> = \{k(\hat{0},\hat{1}) + l(\hat{1},\hat{0}) \mid k,l \in \mathbb{Z}\}$$
$$= \{(\hat{l},\hat{k}) \mid k,l \in \mathbb{Z}\}$$
$$= \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = <u,v \mid u^2 = v^2 = e,\ uv = vu>$$
$$= \{u^i v^j \mid 0 \le i,j \le 1\}$$

1) $f: \mathbb{Z}_2 \times \mathbb{Z}_2 \to G$ morfism de grupuri

$<u,v> = \mathbb{Z}_2 \times \mathbb{Z}_2$

$f(u) = x$, $f(v) = y \Rightarrow f(u+v) = f(u) \cdot f(v)$
$= xy$

$f((\hat{0},\hat{0})) = e$

$e = f((\hat{0},\hat{0})) = f(u+u) = f(u)\cdot f(u) = x \cdot x = x^2$

analog $e = y^2$

$f(u+v) = f(v+u) \Leftrightarrow f(u)\cdot f(v) = f(v)\cdot f(u)$
$\Leftrightarrow xy = yx$

A da un morfism de grupuri de la $\mathbb{Z}_2 \times \mathbb{Z}_2$ în G $\Leftrightarrow$

a da $x, y \in G$ a.î. $x^2 = y^2 = e$ și $xy = yx$

$f : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$

$f((\hat{0}, \hat{0})) = (\hat{0}, \hat{0})$

$f(u) = x \, ; \, 2x = (\hat{0}, \hat{0}) \Rightarrow x \in \mathbb{Z}_2 \times \mathbb{Z}_2$

$f(v) = y \Rightarrow \underbrace{2y = (\hat{0}, \hat{0})}_{\text{~~toate~~ conform}} \Rightarrow y \in \mathbb{Z}_2 \times \mathbb{Z}_2$

$\Rightarrow$ sunt 16 morfisme de grupuri

Câte sunt bijective?

$f((\hat{0}, \hat{0})) = (\hat{0}, \hat{0})$

$f(u) = x, \; x \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(\hat{0}, \hat{0})\}$

$f(v) = y, \; y \in \mathbb{Z}_2 \times \mathbb{Z}_2 \setminus \{(\hat{0}, \hat{0}), x\}$

Deci sunt 6 automorfisme

c) $S_3 = \{e, \underset{\tau}{(12)}, \underset{\sigma\tau}{(13)}, \underset{\sigma^2\tau}{(23)}, \underset{\sigma}{(123)}, \underset{\sigma^2}{(132)}\}$

$\tau\sigma \neq \sigma\tau$ deci S_3 nu este ciclic pentru că nu e abelian

$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\,3)$

$$\tau^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = e$$

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1\,3\,2)$$

$$\sigma^2\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2\,3)$$

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2\,3) = \sigma^2\tau \Leftrightarrow \sigma\tau = \tau\sigma^2$$

$$\sigma^3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = e$$

deci $ord(\sigma) = 3$

$$\langle \sigma, \tau \rangle = \{x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n} \mid x_i \in \{\sigma, \tau\}, n \in \mathbb{N}^*, \varepsilon_i \in \{\pm 1\}\}$$

$$= \{x_1 \dots x_n \mid n \in \mathbb{N}^*, x_i \in \{\sigma, \sigma^2, \tau\}\} =$$

$$= \{\sigma^k \tau^l \mid k, l \in \mathbb{N}\} = \{\sigma^i \tau^j \mid i \in \{0,1,2\}, j \in \{0,1\}\} = S_3$$

Dacă G grup, $Aut(G) = \{f : G \to G \mid f$ izom de gr$\}$

$Aut(G)$ este gr. în raport cu comp. funcțiilor

$Aut(\mathbb{Z}_2 \times \mathbb{Z}_2)$ izom cu S_3, izom de gr

$f_1 = id_{\mathbb{Z}_2 \times \mathbb{Z}_2}$

$f_2(e) = e$, $f_2(u) = v$, $f_2(v) = u+v$, $f_2(u+v) = u$

$f_3(e)$, $f_3(u) = v$, $f_3(v) = u$, $f_3(u+v) = u+v$

$f_4(e) = e$, $f_4(u) = u$, $f_4(v) = u+v$, $f_4(u+v) = v$

$f_5(e) = e$, $f_5(u) = u+v$, $f_5(v) = v$, $f_5(u+v) = u$

$f_6(e) = e$, $f_6(u) = u+v$, $f_6(v) = u$, $f_6(u+v) = v$

$\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{f_1, \ldots f_6\}$

$f_2^2(u) = f_2(f_2(u)) = f_2(v) = u+v \Rightarrow f_2^3(u) = f_2(u+v) =$
$= u$

$f_2^2(v) = f_2(f_2(v)) = f_2(u+v) = u \Rightarrow f_2^2 = f_6$, $f_2^3 = f_1$

$f_3^2 = f_1 \quad f_4^2 = f_1 \quad f_5^2 = f_1$

$$\left.\begin{aligned} f_2 \circ f_3(u) &= f_2(v) = u+v \\ f_2 \circ f_3(v) &= f_2(u) = v \end{aligned}\right\} \Rightarrow f_2 \circ f_3 = f_5$$

$$\left.\begin{aligned} f_2^2 \circ f_3(u) &= f_2^2(v) = \cancel{u+v}\ u \\ f_2^2 \circ f_3(v) &= f_2^2(u) = u+v \end{aligned}\right\} f_2^2 \circ f_3 = f_4$$

$f_3 \circ f_2(u) = f_3(v) = u \quad f_3 \circ f_2(v) = f_3(u+v) = u+v$

$\Rightarrow f_3 \circ f_2 = f_4$

$= f_2^2 \circ f_3$

$\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) = \langle f_2, f_3 \mid f_2^3 = f_1 = f_3^2, f_3 \circ f_2 = f_2^2 \circ f_3 \rangle$

$f_1 \to e$, $f_2 \to \sigma$, $f_3 \to \tau$, $f_4 \to \sigma\tau$, $f_5 \to \tau\sigma$, $f_6 \to \sigma^2$

$\Downarrow$

$S_3 = \langle \sigma, \tau \mid \sigma^3 = e = \tau^2, \tau\sigma = \sigma\tau = \sigma^2\tau \rangle$