13.10.2022

Seminar 2

Relații

- O relație este un triplet $\alpha = (A, B, \rho)$ unde

A, B = mulțimi $(\neq \emptyset)$

$\rho \subseteq A \times B = \{(a, b) \mid a \in A, b \in B\}$

- O funcție $f = (A, B, \Gamma_f)$ este o relație (între A și B) cu prop. că $\forall a \in A, \exists! b \in B$ a.î. $(a, b) \in \Gamma_f$

există unic

Γ_f = graficul lui f : $\Gamma_f = \{(a, f(a)) \mid a \in A\}$

Notația $f : A \to B$, $\forall a \mapsto \exists! f(a)$

Compunerea relațiilor

$\alpha = (A, B, \rho)$, $\beta(B, C, \rho')$

$\beta \circ \alpha = (A, C, \rho' \circ \rho = \{(a, c) \mid \exists b \in B$ a.î. $(a, b) \in \rho, (b, c) \in \rho'\})$

compunerea relațiilor α și β

Notație $(a,b) \in \rho \Leftrightarrow a \rho b$

$(a,b) \notin \rho \Leftrightarrow a \not\rho b$

1 Ex 1) $\forall \alpha = (A, B, \rho), \beta = (B, C, \delta), \gamma = (C, D, \varepsilon)$

$(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)$ (asociativitate)

2) Dacă $\Delta_A = \{(a,a) \mid a \in A\}$ și $1_A = (A, A, \Delta_A)$

$\alpha \circ 1_A = \alpha \ \forall \alpha = (A, B, \rho)$

$1_B \circ \alpha = \alpha \ \forall a = (A, B, \rho)$

R1) $A = \{2, 4, 6, 8\}, B = \{1, 3, 5, 7\}$

$\rho = \{(x,y) \mid x \geq 6 \lor y \geq 1\} \subseteq A \times B$

$\rho = \{(x,y) \mid x \geq 6 \lor y \geq 1\} = \{(x,y) \mid x \in \{6,8\} \lor y = 1\}$

$= \{(6,1), (6,3), (6,5), (6,7), (8,1), (8,3), (8,5), (8,7), (2,1), (4,1)\} \subseteq A \times B$

R2) $A = B = \mathbb{N}$

$\rho = \{(3,5), (5,3), (3,3), (5,5)\}$

$\sigma = \{(x,y) \mid x \leq y\} \subseteq \mathbb{N} \times \mathbb{N}$

$\gamma = \{(x,y) \mid y - x = 12\} \subseteq \mathbb{N} \times \mathbb{N}$

$\rho \circ \gamma = \{(a,c) \mid a \in \mathbb{N}, c \in \mathbb{N}, \exists b \in \mathbb{N}$ a.î. $(a,b) \in \gamma, (b,c) \in \rho\}$

$\rho \circ \nabla = \{(a,b) \mid a \leq b,\ (b,c) \in \{(3,5),(5,3),(3,3),(5,5)\}\} =$

$= \{(0,5),(1,5),(2,5),(3,5),(0,3),(1,3),(2,3),(3,3),(4,3),(5,3),(4,5),(5,5)\}$

$\nabla \circ \rho = \{(a,c) \mid a \in \mathbb{N},\ c \in \mathbb{N},\ (\exists) b \in \mathbb{N}$ a.î. $(a,b) \in \rho,\ (b,c) \in \nabla\} = \{(a,c) \mid (\exists) b$ a.î. $(a,b) \in \rho,\ b \leq c\}$

$\nabla \circ \rho = \{(3,c) \mid c \geq 5\} \cup \{(3,3),(3,4)\} \cup \{(5,c),\ c \geq 3\}$

$\rho \circ \sigma = \{(a,c) \mid a \in \mathbb{N},\ c \in \mathbb{N},\ (\exists) b \in \mathbb{N}$ a.î. $(a,b) \in \gamma,\ (b,c) \in \rho\} = \{(a,c) \mid (\exists) b$ a.î. $b-a=12,\ (b,c) \in \rho\} = \emptyset$

$\sigma \circ \rho = \{(a,c) \mid a \in \mathbb{N},\ c \in \mathbb{N},\ (\exists) b \in \mathbb{N}$ a.î. $(a,b) \in \rho,\ (b,c) \in \gamma\} = \{(a,c) \mid (\exists) b$ a.î. $c-b=12,\ (a,b) \in \rho\} =$

$= \{(3,17),(3,15),(5,15),(5,17)\}$ $\square$

Def

$\alpha = (A,B,\rho)$ relație

$\alpha^{-1} = (B,A,\rho^{-1})$ cu $\rho^{-1} = \{(b,a) \mid (a,b) \in \rho\}$

ex. $\gamma^{-1} = \{(x,y) \mid (y,x) \in \gamma\} = \{(x,y) \mid x-y=12\}$

$|\underline{\sigma \circ \rho, \rho \circ \sigma}| \to$ Temă

R3) $\rho = \{(2a, 3b) \mid (a, b) \in \mathbb{Z}\}$

$\rho^n = ?,\ n \in \mathbb{Z}$

$\rho^2 = \{(x, z) \mid \exists y \in \mathbb{Z} \text{ a.î. } (x, y) \in \rho, (y, z) \in \rho\}$

$\rho^2 = \{(x, z) \mid \exists a, b \in \mathbb{Z} \text{ a.î. } x = 2a, y = 3b \wedge (3b, z) \in \rho\}$

$= \{(2a, z) \mid a \in \mathbb{Z}, \exists b \in \mathbb{Z} \text{ a.î. } (3b, z) \in \rho\} =$

$= \{(2a, 3b') \mid a, b' \in \mathbb{Z}\} = \rho \Rightarrow \underbrace{\rho \circ \rho \circ \ldots \circ \rho}_{n \text{ ori}} = \rho$

$\forall n \in \mathbb{N}^*$

$\sigma = \rho^{-1} = \{(3a, 2b) \mid a, b \in \mathbb{Z}\}$

$\sigma^2 = \sigma \circ \sigma = \{(x, z) \mid x, z \in \mathbb{Z}, \exists y \in \mathbb{Z} \text{ a.î. } (x, y) \in \sigma,$
$(y, z) \in \sigma\} =$

$= \{(3a, z) \mid a \in \mathbb{Z} \text{ a.î. } \exists b \in \mathbb{Z} \text{ a.î. } (2b, z) \in \sigma\} =$

$= \{(3a, 2b') \mid a, b' \in \mathbb{Z}\} = \sigma$

Inductiv, $\sigma^n = \sigma, \forall n \in \mathbb{N}^*$

$$\Rightarrow \rho^n = \begin{cases} \sigma, & n \in \mathbb{Z} \setminus \mathbb{N} \\ \rho, & n \in \mathbb{N}^* \\ 1_{\mathbb{Z}}, & n = 0 \end{cases}$$

$\boxtimes$

$$(\alpha\circ\beta)^{-1} = (\beta^{-1}\circ\alpha^{-1})$$

$$(\underbrace{\alpha\circ\ldots\circ\alpha}_{n \text{ ori}})^{-1} = \underbrace{\alpha^{-1}\circ\ldots\circ\alpha^{-1}}_{n \text{ ori}}$$

$$(\alpha^{n})^{-1} = (\alpha^{-1})^{n}$$

R4) $\rho = \{(a, a+3) \mid a\in\mathbb{Z}\}$

$\rho\circ\rho = \rho^2 = \{(a,c) \mid a\in\mathbb{Z}, c\in\mathbb{Z}, \exists b\in\mathbb{Z}$ a.î. $(a,b)\in\rho,$

$(b,c)\in\rho\} = \{(a,c) \mid b=a+3, c=b+3$

$\rho^2 = \{\underbrace{(a,a+6)}_{a,c} \mid a\in\mathbb{Z} \mid c=a+6\} = \{(a,a+6), a\in\mathbb{Z}\}$

Pp că $\rho^n = \{(a, a+3n) \mid a\in\mathbb{Z}\}$

Atunci $\rho^n\circ\rho^{\cancel{n+1}} = \{(a,c) \mid a\in\mathbb{Z}, c\in\mathbb{Z} \mid \exists b\in\mathbb{Z}$ a.î.

$(a,b)\in\rho, (b,c)\in\rho^n\} = \{(a,c) \mid a=a, b=\cancel{a+3}, c=\underset{a+3}{\cancel{3a}}+3n\}$

Deci $\rho^n = \{(a, a+3+3n) \mid a\in\mathbb{Z}, n\in\mathbb{N}\}$

$= \{(a+3(n+1)) \mid a\in\mathbb{Z}\}$ (adevărat)

$\rho^{-1} = \{(a+3, a), a\in\mathbb{Z}\} = \{(a, a-3), a\in\mathbb{Z}\}$

$$\rho^{-n} = \{(a, a-3n), a \in \mathbb{Z}\} \ \forall n \in \mathbb{N}^*$$

$$\rho \circ \rho^{-1} = \{(a, b) \mid 3 \mid a-b\} \subseteq \mathbb{Z} \times \mathbb{Z}$$

$$(\rho \cup \rho^{-1})^{4} \Rightarrow \rho^{2} = ?$$ (temă)