Seminar 5

03.11.2022

Relații de echivalență

$\rho \subseteq A \times A$ relație ($A \neq \phi$)

ρ s.n. relație de echivalență dacă este – reflexivă

$(a \rho a, \forall a \in A \iff \Delta_A \subseteq \rho)$

diagonala

– simetrică: $\forall a, b \in A$, $a \rho b \Rightarrow b \rho a$ $(\rho \subseteq \rho^{-1})$

$\iff \underline{\rho = \rho^{-1}}$

– tranzitivă $\forall a, b, c \in A$ $a \rho b$, $b \rho c$, $a \rho c$

$a \rho b \wedge b \rho c \Rightarrow a \rho c \iff \rho^2 \subseteq \rho$

$A/\rho = \{\hat{a} \mid a \in A\}$ $\quad \hat{a} = \{b \in A \mid b \rho a\}$

~~$A/\rho = \{\hat{a} \mid a \in A\}$~~

clasa de echiv. a lui a

$\forall \hat{a}, \hat{b} \in A/\rho : \hat{a} = \hat{b}$ sau $\hat{a} \cap \hat{b} \neq \phi$

$A = \biguplus_{\hat{a} \in A/\rho} \hat{a}$ reuniune disjunctă

$A \xrightarrow{\pi} A/\rho$, $\pi(a) = \hat{a}$ este funcție surj. canonică

Obs $\hat{a} = \hat{b} \Leftrightarrow a \rho b$

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[dr, "f"'] & A/\rho \arrow[d, dashed, "\bar{f}"] \\ & B \end{tikzcd}$$

(proprietatea de universalitate a mulțimii factor)

$(\rho = \rho_f)$

$a \rho_f b \Leftrightarrow f(a) = f(b)$

Exemplu $A = \mathbb{Z}$; $a \rho b \Leftrightarrow |a| = |b|$

ρ este relație de echivalență

$$\hat{a} = \{ b \in \mathbb{Z} \mid b \rho a \} = \{ b \in \mathbb{Z} \mid |b| = |a| \} =$$

$$= \begin{cases} \{0\}, & a = 0 \\ \{\pm a\}, & a \neq 0 \end{cases}$$

bijecție cu $\mathbb{N}$ (numărabil)

$$\mathbb{Z}/\rho = \{ \{0\}, \{\pm 1\}, \{\pm 2\} \dots \} \simeq \mathbb{N}$$

$$\hat{a} \longrightarrow |a| \in \mathbb{N}$$

$$\begin{tikzcd} \mathbb{Z}\ A \arrow[r] \arrow[dr, "f"'] & A/\rho\ \ \mathbb{Z}/\rho \arrow[d, dashed] \\ & B\ \ \mathbb{N} \end{tikzcd}$$

(Bine def: $a \rho b \Rightarrow f(a) = f(b)$

$$f(a) = |a|$$

$$\overline{f}(\hat{a}) = f(a)$$

1) Pe $\mathbb{R}$: $x \rho y \Leftrightarrow x = y \vee x + y = 3$

i) ρ = rel. de echiv.

ii) $\mathbb{R}/\rho = ?$

i) ~~reflexiva~~ $\Rightarrow a = a \vee a + a = 3$ (A)

$a \rho b \Rightarrow a = b \vee a + b = 3 \Leftrightarrow b = a \vee b + a = 3$

$\Leftrightarrow b \rho a$

$a \rho b \wedge b \rho c \Leftrightarrow (a = b \vee b + a = 3) \wedge (b = c \vee b + c = 3)$

$\Downarrow$

$b \rho c$

$\Leftrightarrow (a = b \wedge b = c) \vee (a = b \wedge b + c = 3) \vee (a + b = 3 \wedge$

$b = c) \vee (a + b = 3 \wedge b + c = 3) \Leftrightarrow a = b = c \vee$

$(a = b \wedge a + c = 3) \vee (a + c = 3 \wedge b = c) \vee (a = c \wedge$

$b + c = 3) \Rightarrow a = c \vee a + c = 3 \Leftrightarrow a \rho c$

$\hat{a} = \{b \in \mathbb{R} \mid b \rho a\} = \{b \in \mathbb{R} \mid b = a \vee b + a = 3\}$

$= \{a, 3 - a\}$ cu $a \neq \frac{3}{2}$ $= \begin{cases} \{a, 3-a\}, & a \neq \frac{3}{2} \\ \{\frac{3}{2}\}, & a = \frac{3}{2} \end{cases}$

$\bar{f} : \mathbb{R}/\rho \to [0, \infty)$ $\bar{f}(\hat{a}) = \left|a - \frac{3}{2}\right|$, $f(a) = \left|a - \frac{3}{2}\right|$

$\hat{a} = \hat{b} \overset{?}{\Rightarrow} f(a) = f(b)$

$\hat{a} = \hat{b} \Leftrightarrow a \rho b \Leftrightarrow a = b \vee a + b = 3$

I $a = b \Rightarrow f(a) = f(b)$

II $a + b = 3 \Rightarrow b = 3 - a$

$f(b) = \left|3 - a - \frac{3}{2}\right| = \left|\frac{3}{2} - a\right| = \left|a - \frac{3}{2}\right| = f(a)$

Inj $\bar{f}(\hat{a}) = \bar{f}(\hat{b}) \Leftrightarrow \left|a - \frac{3}{2}\right| = \left|b - \frac{3}{2}\right| \Leftrightarrow a - \frac{3}{2} = \pm\left(b - \frac{3}{2}\right)$

$\Leftrightarrow a = b \vee a + b = 3 \Leftrightarrow a \rho b \Leftrightarrow \hat{a} = \hat{b}$

Obs $\bar{g} : \mathbb{R}/\rho \to \left(-\infty, \frac{9}{4}\right]$

$\bar{g}(\hat{a}) = a(3 - a)$, $g(a) = a(3 - a)$

c) $f : A \to B$

$\rho_f : a \rho_f a' \Leftrightarrow f(a) = f(a')$

ρ_f relație de echivalență = ? (temă)

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x^3 - 3x + 5, & x \geq 1 \\ 2x^2 - 3x + 1, & x < 1 \end{cases}$

Det $\hat{1}$, $\hat{3}$, $\hat{0}$ – temă

$\exists a \in \mathbb{R}$ a.î. $\hat{a}$ să aibă 5 elemente?

$a \in \rho_f \Leftrightarrow f(a) = f(1) \Leftrightarrow f(a) = 3$

i) $x^3 - 3x + 5 = 3, \; x \geq 1$

$x^3 - 3x + 2 = 0$

1 este soluție

$x^3 - x^2 + x^2 - x - 2x + 2 = 0$

$(x-1)(x^2 + x - 2) = 0$

$\left.\begin{array}{l} x_1 = 1 \\ x_{2,3} = 1, -2 \end{array}\right\}$ de unde $\hat{1} = 1$

ii) $2x^2 - 3x + 1 = 3, \; x < 1$

$2x^2 - 3x - 2 = 0$

$x_{1,2} = \dfrac{3 \pm \sqrt{9+16}}{4} = \dfrac{3 \pm 5}{4} = \begin{cases} 2 \\ \boxed{-\frac{1}{2}} \end{cases}$

Deci $\hat{1} = \left\{1, -\frac{1}{2}\right\}$

3) $\mathbb{N}$ f $a \rho b \Leftrightarrow k \in \mathbb{N}$ a.î. $b = 2^k \cdot a$

ρ relație de echiv?

$a \rho a \Leftrightarrow \exists k \in \mathbb{N}$ a.î. $a = 2^k \cdot a$

$a = 2^0 \cdot a$

$a \rho b \Rightarrow b \rho a$

$a \rho b \Rightarrow b = 2^k \cdot a \Leftrightarrow a = 2^{-k} \cdot b$

$k \in \mathbb{N} \Rightarrow -k \notin \mathbb{N}$

$\left.\begin{array}{l} a = 3 \\ b = 6 = 2^1 \cdot 3 \end{array}\right\}$ $a \rho b$, dar $b \not\rho a$

$a \rho b \wedge b \rho c \Rightarrow a \rho c$

$$\left.\begin{array}{l} b = 2^k \cdot a \\ c = 2^p \cdot b \end{array}\right\} \Rightarrow c = 2^{k+p} \cdot a \Rightarrow a \rho c$$

<u>Obs</u> $a \rho b,\ b \rho a \Rightarrow a = b$

$$\left.\begin{array}{l} b = 2^k \cdot a \\ a = 2^p \cdot b \end{array}\right\} \Rightarrow b = 2^k \cdot 2^p \cdot a = 2^{k+p} \cdot b \Rightarrow 2^{k+p} = 1 \vee b = 0$$

$k = p = 0 \vee b = 0$

$$\left.\begin{array}{l} b = 2^k \cdot a \\ 0 = 2^k \cdot a \Rightarrow a = 0 \end{array}\right\} \Rightarrow a = b$$

<u>Obs</u> $a \rho b \Leftrightarrow \exists k \in \mathbb{Z}$ a.î. $b = 2^k \cdot a$ atunci ρ rel. de echivalență

$$\hat{a} = \{ b \in \mathbb{N} \mid \exists k \in \mathbb{Z} \text{ a.î. } b = 2^k \cdot a \} =$$

$$= \{ 2^k \cdot a \mid k \in \mathbb{Z} \} \cap \mathbb{N}$$

$\hat{0} = \{0\}$

$\hat{1} = \{1, 2, 4 \ldots\}$

$\hat{2} = \hat{1}$

$\hat{3} = \{3, 6, 12 \ldots 2^n \cdot 3\}$

Pe $\mathbb{Z}$: $a \rho b \Leftrightarrow a \mid b \vee b \mid a$

$a \rho a \Leftrightarrow a \mid a \vee a \mid a$ (A)

$a \rho b \Leftrightarrow a \mid b \vee b \mid a \Leftrightarrow b \mid a \vee a \mid b \Rightarrow b \rho a$

$a \varrho b \wedge b \varrho c \Rightarrow a \varrho c$

$\cancel{a \varrho b \vee b \mid a}$

$a \varrho b \Leftrightarrow a \mid b \vee b \mid a$

$b \varrho c \Leftrightarrow b \mid c \vee c \mid b$

$2 \varrho 6$

$6 \varrho 3$ $\Rightarrow 2 \mid 3 \vee 3 \mid 2$ (F)