10.11.2022

Închideri de relații

Seminar 6

$\rho \subseteq A \times A$

$R(\rho) = \rho \cup \Delta_A \quad (\rho \subseteq R(\rho) \to$ "cea mai mică" în raport cu "$\subseteq$")

reflexivă

diagonala

$S(\rho) = \rho \cup \rho^{-1} \quad (\rho \subseteq S(\rho)$ —"—"— $)$

$RS(\rho) = SR(\rho) = \rho \cup \rho^{-1} \cup \Delta_A \quad (\rho \subseteq RS(\rho)$ -"-"- $)$

$T(\rho) = \rho \cup \rho^2 \cup \ldots \cup = \bigcup_{n \geq 1} \rho^n;$

$\rho^n = \{(a,c) \in A \times A \mid \exists b_1 \ldots b_n \in A: a\rho b_1, b_1 \rho b_2, \ldots b_{n-1} \rho c\}$

$$E(\rho) = \underbrace{\Delta_A}_{\text{reflexivă}} \cup \underbrace{T}_{\text{tranzitivă}}(\underbrace{\rho \cup \rho^{-1}}_{\text{simetrică}}) = \bigcup_{n \geq 0} (\rho \cup \rho^{-1})^n$$

Exemplu $\rho = \{(1,2),(2,1),(3,5),(4,4)\} \subseteq A \times A$

$A = \{1,2,3,4,5\}$

$R(\rho), S(\rho), RS(\rho), E(\rho), T(\rho)$

$R(\rho) = \rho \cup \Delta_A = \rho \cup \{(1,1),(2,2),(3,3),(5,5)\}$

$S(\rho) = \rho \cup \rho^{-1} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$

$RS(\rho) = R(\rho) \cup \{(5,3)\} = \{\dots\}$

$T(\rho) = \bigcup_{n \geq 1} \rho^n$

$\rho^2 = \{(a,c),\ \exists b \text{ a.î. } (a,b) \in \rho,\ (b,c) \in \rho\} = \{(1,1),(2,2),(4,4)\}$

$\rho^3 = \{(a,c),\ \exists b \text{ a.î. } (a,b) \in \rho^2,\ (b,c) \in \rho\} = \rho \setminus \{(3,5)\}$

$\rho^4 = \rho^2 \circ \rho^2 = \{(a,c) \mid \exists b \text{ a.î. } (a,b) \in \rho^2 \text{ și } (b,c) \in \rho^2\} = \rho^2$

Deci $T(\rho) = \rho \cup \rho^2 \cup \rho^3 = \rho \cup \rho^2 = \{\Delta_A \setminus \{(3,3),(5,5)\}\} \cup$

$\cup \{(1,2),(2,1),(3,5)\}$

$E(\rho) = \bigcup_{n \geq 0} (\rho \cup \rho^{-1})^n$; Notăm $\rho \cup \rho^{-1} = Z$

$\tau^2 = \{(1,1),(2,2),(3,3),(4,4),(5,5)\} = \Delta_A$

$\tau^3 = \tau^2 \circ \tau = \Delta_A \circ \tau = \tau$

Deci $E(\rho) = \Delta_A \cup \tau = RS(\rho)$

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2) $\rho = \{(x,2x) \mid x \in \mathbb{N}\} \subseteq \mathbb{N}\times\mathbb{N}$

$E(\rho) = ?$

$E(\rho) = \bigcup_{n\geq 0} (\rho\cup\rho^{-1})^n$

$\tau = \rho\cup\rho^{-1} = \{(x,2x),(2x,x) \mid x\in\mathbb{N}^*\} \cup \{(0,0)\}$

$\tau^2 = \{(a,c) \mid \exists b\in\mathbb{N}$ a.î. $(a,b)\in\tau, (b,c)\in\tau\} =$

$= \Delta_{\mathbb{N}} \cup \{(x,4x) \mid x\in\mathbb{N}\} \cup \{(2x, \frac{x}{2}) \mid x \in 2\mathbb{N}\} =$

$= \Delta_{\mathbb{N}} \cup \{(x,4x),(4x,x) \mid x\in\mathbb{N}^*\}$

$\tau^3 = \{(a,c) \mid \exists b\in\mathbb{N}$ a.î. $(\overset{\substack{x\ \ 4x\\ x\ \ x}}{a,b})\in\tau^2, (\overset{\substack{4x\ \ 8x\\ x\ \ 2x}}{b,c})\in\tau\} =$

$= \tau \cup \{(x,8x),(x,2x) \mid x\in\mathbb{N}^*\} \cup \{(4x,2x) \mid x\in\mathbb{N}^*\} \cup$

$\{(4x,\frac{x}{2}) \mid x\in 2\mathbb{N}\} = \tau \cup \{(x,8x),(8x,x) \mid x\in\mathbb{N}^*\}$

$E(\rho) = \{(x,y) \mid \exists k\in\mathbb{Z}$ a.î. $y = 2^k x\} \overset{\text{not}}{=} \theta \supseteq \rho$

(θ = rel de echiv (vezi sem 5))

$\Rightarrow E(\rho) \subseteq \theta$

Fie $(x,y) \in \theta \Rightarrow \exists k \in \mathbb{Z}$ a.î. $y = 2^k x$

Căutăm $n \in \mathbb{N}^*$ a.î. $(x,y) \in \rho^n \Leftrightarrow \exists s_1 \dots s_{n-1} \in \mathbb{N}$

a.î. $x \rho s_1,\ s_1 \rho s_2,\ s_{n-1} \rho y$

Dacă $k \in \mathbb{N},\ x \rho 2x,\ 2x \rho 4x,\ 2^{k-1}x \rho 2^k x = y$

$\Rightarrow (x,y) \in \rho^k \ (n = k)$

Dacă $k \in \mathbb{Z}_-,\ x = 2^{-k} y,\ x \rho 2^{-1}x,\ 2^{-1}x \rho 2^{-2}x \dots$

$\dots 2^{k-1} x \rho 2^{-k} x = y \Rightarrow (y,x) \in \rho^{-k} (n=k)$ $\Big\}\ (x,y) \in \rho^{-k}$

ρ^{-k} simetrică $\Big\}\ (n = -k)$

$y \rho 2y,\ 2y \rho 4y, \dots 2^{-k-1} y \rho 2^{-k} y = x$

<u>Temă</u>

Pentru ρ de mai jos calculați $E(\rho)$

1) $\rho = \{(-1,0),(2,3),(1,1)\} \subset A \times A,\ A = \{-1,0,1,2,3\}$

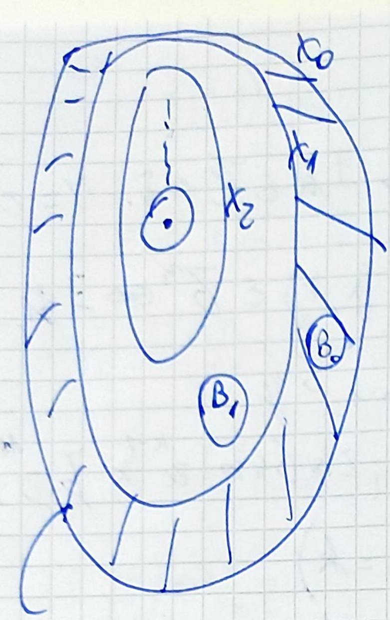
2) $\rho = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y = 3\} \subset \mathbb{Z} \times \mathbb{Z}$

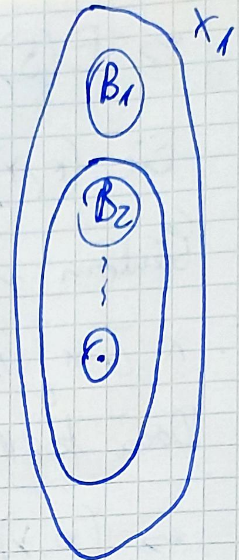
<u>Cantor - Bernstein</u>

$X_0 \supseteq X_1 \supseteq X_2,\ \exists f : X_0 \to X_2$ bij $\Rightarrow \exists g : X_0 \to X_1$ bij

$f(X_0) = X_2$

$X_3 := f(X_1) \subseteq f(X_0) = X_2$





$$\left.\begin{aligned} B_0 &= X_0 \setminus X_1 \\ B_1 &= X_1 - X_2 \\ &\vdots \\ B_n &= X_n - X_{n+1} \end{aligned}\right\} \text{sectoare}$$

$f(X_0) = X_2$

$X_3 := f(X_1) \subseteq f(X_0) = X_2$

$\vdots$

$X_n = f(X_{n-2}) \subseteq f(X_{n-3}) = X_{n-1}$

$$f(B_n) = f(X_n \setminus X_{n+1}) \underset{\boxed{\text{temă}}}{=} f(X_n) \setminus f(X_{n+1}) = X_{n+2} \setminus X_{n+3} =$$

$$= B_{n+2}$$

Dacă $A = \bigcap_{n \geq 0} X_n \Rightarrow X_0 = A \cup \bigcup_{n \geq 0} B_n$ $({}_4\supseteq^4$ doi din d

$X_1 = A \cup \bigcup_{n \geq 1} B_n \left({}_4\supseteq^4 - \cancel{A}\right)$

$X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n \supseteq \ldots$

$x \in X_0$ — $x \in A$, gata!

— $x \notin A$, iau $n \geq 0$ minim a.î. $x \notin X_n$

$\Downarrow$

$n \geq 1$ și $x \in X_{n-1}$

$\Rightarrow x \in X_{n-1} \setminus X_n = B_{n-1}$

$$\tilde{f} : X_0 = A \cup \bigcup_{n \geq 1} B_n \to X_1 = A \cup \bigcup_{n \geq 1} B_n$$

$$\tilde{f}(x) = \begin{cases} x, & x \in A \cup \bigcup_{n \geq 0} B_{2n+1} \\ f(x), & x \in \bigcup_{n \geq 0} B_{2n} \end{cases}$$

<u>Consecință</u> Dacă $\exists f : A \to B$ inj, $\exists g : B \to A$ inj $\Rightarrow \exists h : A \to B$ bij

$$\underbrace{A \supseteq g(B) \supseteq g(f(A))}_{\text{în bij}} \Rightarrow A \sim g(B)$$

$gf : A \to A$ inj $\Rightarrow A \sim gf(A)$ $\quad g : B \to A$ inj $\Rightarrow B \sim g(B)$

$\Rightarrow A \sim B$