Modeling Heat Flow in a House

Matthew Daw, William Terry, Nate Carlson

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Abstract

We use systems of ordinary differential equations and partial differential equations to model heat flow throughout a house. We first explore several models derived from Newtons Law of Cooling. These models describe the fluctuations in temperature in the rooms of a house over a fixed period of time $[0, T]$. Then we explore models that describe diffusion of heat in 2-d and 3-d space. Code for this project is publicly available at https://github.com/terrywillr/HeatProject

1 Background/Movitation

A model that explains the heat flow throughout a house can be highly valuable in today's world since heating a house is expensive, and excess energy expenditure can result in atmospheric pollution. To help minimize heating costs though, one must first have a model to study heat flow. As such, we explored using systems of differential equations to model house heating accounting for house size, insulation, and the placement of heat sources. While our models may not reach the complexity of a modern home heating environment, they do provide valuable insights on how the effects of various controllable factors can impact the flow and conservation of heat in a confined space. Our goal is to be able to demonstrate principles that hold true in a physical environment and can be applied to real life problems.

We explored two types of models in our project. The first models we used are derived from Newtons Law of Cooling which states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings [1]. We provide a derivation of a simple model describing the temperature of a one room one story house using Newtons law and then expand upon that model to account for more rooms and levels.

Next we model heat diffusion using the PDE form of the heat equation. In its simplest case, the one dimensional heat equation describes the heat in a straight rod over some time period $[t_0, T]$. We extend it the two dimensional case to describe heat diffusion over a square plate, as well as the three dimensional case where we are then able to describe and visualize the temperature changes of a 3d space over time.

2 Newtons Law of Cooling

2.1 The Simplest Model

Our base model describes the temperature in a single room, single story house and is derived from Newton's Law of Cooling which is given by

$$\frac{dT}{dt} = \frac{\alpha A}{C}(T_e - T)$$

Where T is the temperature of the room, and T_e is the temperature of the environment surrounding the room. The constant $\frac{\alpha A}{C}$ describes the transfer of heat between the enclosed room and outside environment. More specifically α is the heat transfer coefficient, A is the surface area of the room, and C is the capacity of the object through which the heat transfers. It is common to replace $\frac{\alpha A}{C}$ with a single constant k, referred to as the thermal conductivity, which gives the relation

$$\frac{dT}{dt} = k(T_e - T)$$

For our most simple case we have a one floor single room house with a single heat source. Let $x(t)$ be the function that gives the temperature of our room at time t. The room is surrounded on four sides by walls, above by a root, and below by the floor. All of these surfaces can influence the heat transfer in the room. Therefore let us define the following constants

- k_1: thermal conductivity of the floor
- k_2: thermal conductivity of the ceiling
- k_3: thermal conductivity of the walls

Additionally let the function $f(t)$ describe the heat generated from our heat source at time t. Putting this all together gives the following differential equation

$$\frac{dx}{dt} = k_1(T_f - x) + k_2(T_c - x) + k_3(T_w - x) + f(t) \tag{1}$$

Where T_f, T_c, and T_w are the tempuratures of the floor, ceiling, and walls of the room. A solution to this simple base model can be seen in figure 1. We made the simplifying assumption that the heat generating function produced constant output $f(t) = f_0$ meaning that there was no fluctuation in the heat generation overtime.

2.2 Adding More Variables

We can expand upon the model introduced in the previous section to account for larger house sizes and more heat sources. We now consider a model of temperature of a 2 story house. Let $x(t)$ and $y(t)$ denote the temperatures on the ground floor and 2nd floors respectively. Now we define the following constants for our system

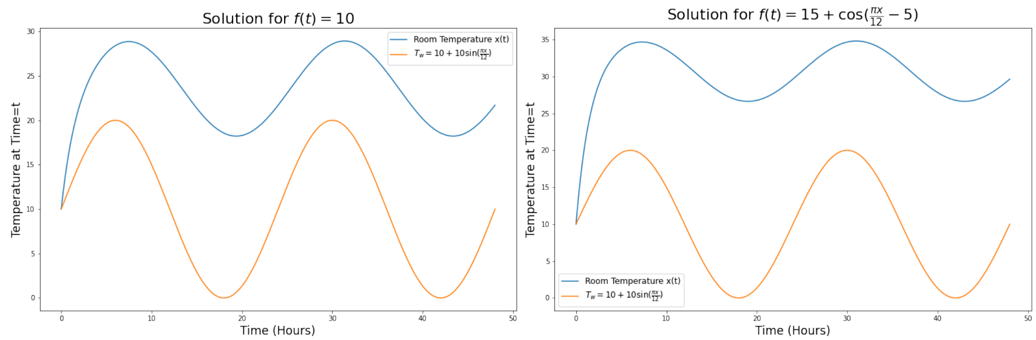

Figure 1: Solution to equation (1) for $t \in [0, 48]$. In each plot the orange line represents the outside tempurature and the blue line represents the temperature inside of the houseThe figure on the left is for $f(t) = 10$ a constant heat source, while the figure on the right is for a fluctuating heat source $f(t) = 15 + \cos(\frac{\pi x}{12} - 5)$. Note that we translated the heat generating function over so that it maximized heat production when the outside temperature was coldest. This results in a more steady curve for the house temperature.

- k_1: thermal conductivity/temperature of the ground floor
- k_2: thermal conductivity of the ceiling between the ground floor and second floor
- k_3: thermal conductivity of the walls on the ground floor
- k_4: thermal conductivity of the walls on the second floor

As in our first model define T_e to give the temperature of environment e. With these definitions we can now represent the heat flow in our two story house with the following ODE system.

$$\begin{aligned} \frac{dx}{dt} &= k_1(T_g - x) + k_2(y - x) + k_3(T_e - x) + f(t) \\ \frac{dy}{dt} &= k_2(x - y) + k_4(T_e - y) \end{aligned} \tag{2}$$

Where $f(t)$ is a constant heat source as in our last model. We can rewrite this system into vector matrix form

$$X'(t) = KX(t) + F(t) \tag{3}$$

where

$$X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, K = \begin{bmatrix} -k_1 - k_2 - k3 & k2 \\ k_2 & -k_2 - k4 \end{bmatrix}, F(t) = \begin{bmatrix} k_1 T_g(t) + k_3 T_e(t) + f(t) \\ k_4 T_e(t) \end{bmatrix}$$

To solve this system of equations, we will begin by solving the more simple homogeneous version of it first.

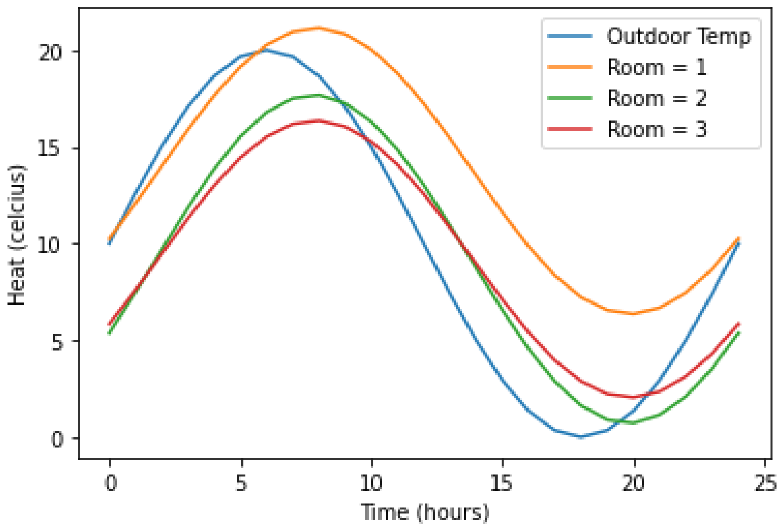

Figure 2: Heat versus time for three room system of equations

$$X'(t) = KX(t) \tag{4}$$

To do this, we will find the eigenvalues and eigen-vectors of K. Since in this situation K is a $2X2$ full rank matrix will yield two distinct pairs of eigen-vectors and eigen-values. Specifically,

$$\lambda_1 = -\frac{1}{2}, V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_2 = -\frac{9}{10}, V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{5}$$

It follows then that a general solution to our homogenous equation is

$$X_0(t) = C_1 e^{-\frac{9}{10}t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{-\frac{1}{2}t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6}$$

Next, we wish to solve the equation for the inhomogeous terms. We will begin by separating it into two distinct terms. Let $F = F_1 + F_2$ be defined as follows

$$F_1(t) = \begin{bmatrix} k_1 T_g(t) + k_3 T_e(t) \\ k_4 T_e(t) \end{bmatrix} \tag{7}$$

$$F_2(t) = \begin{bmatrix} f(t) \\ 0 \end{bmatrix} \tag{8}$$

To solve for F_1 we will be using the following ansantz

$$X_1 = \begin{bmatrix} D_1 + A_1 \sin(\frac{\pi}{12}t) + B_1 \cos(\frac{\pi}{12}t) \\ D_2 + A_2 \sin(\frac{\pi}{12}t) + B_2 \cos(\frac{\pi}{12}t) \end{bmatrix} \tag{9}$$

We are now ready to solve our original equation.

$$X_1'(t) = KX_1(t) + F_1(t) \tag{10}$$

If we it out a little we get the following

$$\begin{bmatrix} x_1'(t) \\ y_1'(t) \end{bmatrix} = \begin{bmatrix} -k_1 - k_2 - k_3 & k_2 \\ k_2 & -k_2 - k_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} \begin{bmatrix} 5 + 4\sin(\frac{\pi}{12}t) \\ 5 + 5\sin(\frac{\pi}{12}t) \end{bmatrix}$$

To simplify the equation a little, we will be using the following substitutions. $K_1 = -k_1 - k_2 - k_3$, $K_2 = k_2$, $K_3 = -k_2 - k_4$ so that

$$K = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} \tag{11}$$

We also will let $T_e = T_{e1} + T_{e2}\sin(\frac{\pi}{12})$

Some algebra then yields the following:

$$\begin{bmatrix} -K_1 & -K_2 & -\frac{\pi}{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -K_3 & K_4 \\ 0 & -\frac{\pi}{12} & -K_3 & -K_4 & 0 & 0 \\ -K_2 & -K_4 & 0 & -\frac{\pi}{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & -K_3 & -K_4 \end{bmatrix} \begin{bmatrix} A1 \\ A2 \\ B1 \\ B2 \\ D1 \\ D2 \end{bmatrix} = \begin{bmatrix} 0 \\ k_3 * T_{e2} \\ k_3 * T_{e2} + T_g * k_1 \\ 0 \\ k_4 * T_{e1} \end{bmatrix} \tag{12}$$

Using this, we can solve for our coefficients thereby allowing us to solve X_1 explicitly.

With this, we are now ready to solve for an in-homogeneous equation for

$$F_2 = \begin{bmatrix} f(t) \\ 0 \end{bmatrix} \tag{13}$$

If we assume that $f(t)$ is just a constant, then we solve this by simply looking for a vector such that

$$X_2 = \begin{bmatrix} G \\ H \end{bmatrix} \tag{14}$$

$$X_2' = KX_2 + F_2 \to 0 = K \begin{bmatrix} G \\ H \end{bmatrix} + \begin{bmatrix} f(t) \\ 0 \end{bmatrix} \tag{15}$$

A little algebra can easily explicitly solve for G and H.

According to the principle of superposition, a particular solution of the linear system can be expressed as

$$X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = X_0(t) + X_1(t) + X_2(t) \tag{16}$$

Letting $k_1 = \frac{1}{10}$, $k_2 = \frac{1}{5}$, $k_3 = \frac{2}{5}$, $k_1 = \frac{1}{2}$, $T_e(t) = 10 + 10\sin(t)$, $T_g = 10$ we end up with the following solution:

$$X(t) = C_1 e^{-\frac{9}{10}t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{-\frac{1}{2}t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 10 + 6.55\sin(\frac{\pi t}{12} - 3.55\cos(\frac{\pi t}{12}) \\ 10 + 7.58\sin(\frac{\pi t}{12} - 3.85\cos(\frac{\pi t}{12}) \end{bmatrix} + \begin{bmatrix} \frac{14}{9} f_0 \\ \frac{4}{9} f_0 \end{bmatrix} \tag{17}$$

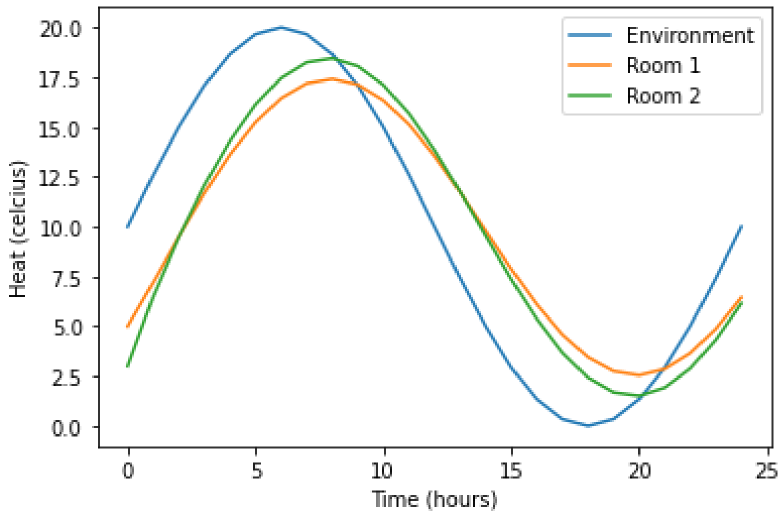

Figure 3: Heat versus time for two room system of equations

To solving for our final coefficients C_1 and C_2 is then a simple matter of setting initial values and solving for the initial values.

If we set $f_0 = 5$ then we get the figure 2

Making this system of equations work for an arbitrary number of rooms follows the exact same logic flow as described for two rooms. For an exact method of how to do this, we recommend reading the code in code provided in the github link.

3 Heat Equation

3.1 2-d Heat Equation: Modeling Heat Diffusion in a Room

Another interesting approach to exploring the heat flow through a house is to model the diffusion of heat through a solid medium. Inside of a house the temperatures of adjacent rooms may differ, so in order to understand how heat behaves in our environment it is critical to be able to model how the temperature in one room will affect the surrounding rooms. To model this behavior we started with the heat equation in two dimensions in the region $[0, a]X[0, b] \subset \mathbb{R}^2$.

$$\begin{aligned} u_t &= k(u_{xx} + u_{yy}) \\ u(0, x, 0) &= u(0, x, a) = f(x) \\ u(0, 0, y) &= u(0, b, y) = g(y) \end{aligned} \tag{18}$$

We used a centered finite difference scheme to approximate the solution the heat equation.

$$u_{i,j}^{k+1} - k\left(\frac{u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k}{\Delta x^2} + \frac{u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k}{\Delta y^2}\right) = 0$$

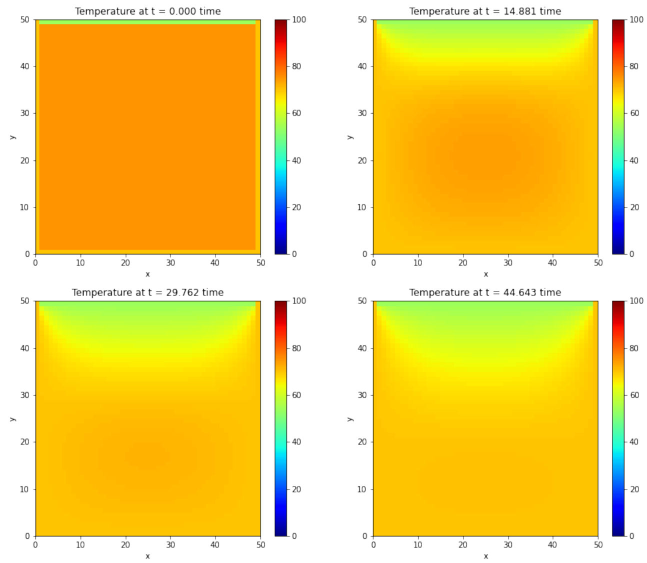

Figure 4: Heat map of our approximation to the 2-d Heat Equation (5) with constant boundary conditions

If we assume $\Delta x = \Delta y$ then we can rewrite the expression above as

$$u_{i,j}^{k+1} = \gamma(u_{i+1,j}^k + u_{i-1,j}^k + u_{i,j+1}^k + u_{i,j-1}^k - 4u_{i,j}^k) + u_{i,j}^k$$

Where $\gamma = k\frac{\Delta t}{\Delta x^2}$. This approximation is stable under the CFL conditions $\Delta t \leq \frac{\Delta x^2}{4k}$. In figure 2 we include a visualization of the heat map produced by this finite difference scheme.

3.2 3-d Heat Equation

We also extend the results of the previous section to 3 dimensions. Consider an arbitrary subregion $V \subseteq \mathbb{R}^3$ with temperature $u(\vec{x}, t)$ defined for all $\vec{x} = (x, y, z) \in V$. The heat equation in 3 dimensions is defined as

$$u_t = k(u_{xx} + u_{yy} + u_{zz}) \tag{19}$$

As before we can approximate solutions to (6) with a finite difference method

$$u_{i,j,k}^{l+1} - k\left(\frac{u_{i+1,j,k}^l - 2u_{i,j,k}^k + u_{i-1,j,k}^k}{\Delta x^2} + \frac{u_{i,j+1,k}^l - 2u_{i,j,k}^l + u_{i,j-1,k}^k}{\Delta y^2} + \frac{u_{i,j,k+1}^l - 2u_{i,j,k}^l + u_{i,j,k-1}^l}{\Delta z^2}\right) = 0$$

If we assume $\Delta x = \Delta y = \Delta z$ then we can rewrite our numerical scheme above as

$$u_{i,j,k}^{l+1} = \gamma(u_{i+1,j,k}^l + u_{i-1,j,k}^l + u_{i,j+1,k}^l + u_{i,j-1,k}^l + u_{i,j,k+1}^l + u_{i,j,k-1}^l - 6u_{i,j,k}^l) + u_{i,j,k}^l$$

where $\gamma = k\frac{\Delta t}{\Delta x^2}$. The above numerical approximation is stable under the CLF conditions $\Delta t \leq \frac{\Delta x^2}{6k}$.

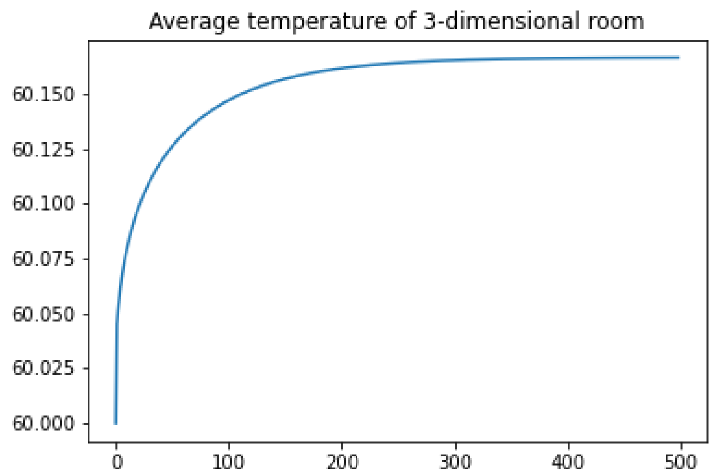

Figure 5: Using our approximate solution to (6) U, for each $\ell \in [0, T]$ we take the average temperature of the room U^ℓ at that time and plot the curve. We can see that the overall temperature of the room increases over the course of time.

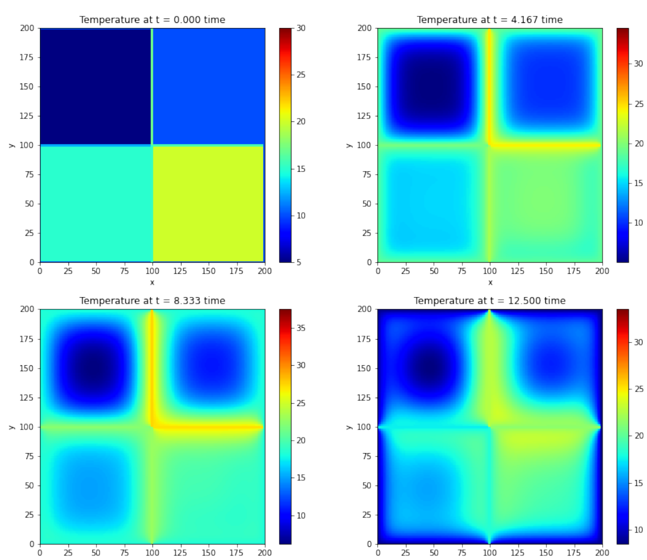

Figure 6: A visualization of the diffusion of heat between 4 adjacent rooms in a house.

3.3 Diffusion Between Rooms

We again consider the 2-dimensional case, but this time with a 4-room house instead of a simple one room house. Modeling temperature diffusion through multiple rooms is a much harder problem, since in this case we would have to consider how heat interacts between the air and walls and how temperature diffuses in the walls themselves. To avoid solving this complex and dynamic problem, we instead calculate the boundary temperatures of the walls separately, using the model described in section 2.

First we use a simple model to calculate the temperature of four rooms over time along with environmental temperature. These sinusoidal functions describing one-dimensional temperature change over time can then be implemented as boundary conditions of the form found in (5). The four temperature curves describing interior temperature are then used as boundary conditions for the inner walls of our 4-room house, with the environmental temperatures used as our exterior boundary conditions. Though these boundary conditions are not completely indicative of realistic heat-flow, we get a good approximation of how heat would diffuse through multiple rooms. Since we now have proper boundary conditions for each of the four rooms, we can use (5) on each room separately, the results of which are visualized in figure 4.

4 Analysis/Conclusions

In our models based off of Newtons Law of Cooling we were able to model the periodic fluctuations of temperature that naturally occur in a day. The changes in temperature in our models were consistent with what we expected to see. A drop in the temperature of the environments surrounding a room resulted in a temperature drop inside of the room. We also were able to see that having a variable heat source can counteract the dips in outside temperature thus stabilizing the internal temperature of the house.

Now that we have a simulation for an arbitrary house we have everything we need to optimize controlling heat flow to minimize heat flow while still maximizing comfort.

References

[1] Differential equations. Home Heating. (n.d.). Retrieved December 9, 2021, from https://math24.net/home-heating.html.