# Smaller notes

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February 13, 2026

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## 1 Unions of cosets

This is a careful write-up of my question [4] at math*overflow* together with the answer [11]. The question is:

> Let $G$ be a locally compact group, and let $K, L$ be cosets of $G$ (not assumed open or closed) which each have empty interior. Does also $K \cup L$ have empty interior?

The answer is "no". The counter-example comes from considering $G = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ the infinite product of the cyclic group of order 2. We shall write $0, 1$ for the elements of $\mathbb{Z}/2\mathbb{Z}$, so $1+1 = 0$. We give $G$ the product topology, so $G$ is compact and Hausdorff. We shall write elements of $G$ as infinite sequences $x = (x_n)$ in $\mathbb{Z}/2\mathbb{Z}$. Notice that $G$ is compact, abelian, and every $x \in G$ satisfies that $x + x = 0$.

The topology has a base of "cylinder" sets, given as follows. Let $n \in \mathbb{N}$ and $y = (y_1, y_2, \cdots, y_n)$ a finite sequence in $\mathbb{Z}/2\mathbb{Z}$. Define

$$\mathcal{O}_{n,y} = \big\{ x = (x_k) \in G : x_i = y_i \ (i \leq n) \big\}.$$

These sets form a base for the topology on $G$. Notice that the intersection of two such sets is of the same form (or is empty).

Furthermore, notice that for $s, t \in \mathbb{Z}/2\mathbb{Z}$ either $s = t$ or $s = t + 1$. Then

$$G \setminus \mathcal{O}_{n,y} = \{x : \exists\, 1 \le i \le n, x_i \ne y_i\} = \bigcup_{i=1}^{n} \{x : x_i \ne y_i\}$$
$$= \bigcup_{i=1}^{n} \{x : x_i = y_i + 1\}$$

which is the union of (many) basic open sets. Thus $\mathcal{O}_{n,y}$ is also closed.

Finally, notice that $\mathcal{O}_{n,y}$ is a subgroup exactly when $y_i = 0$ for $i \le n$, and so every $\mathcal{O}_{n,y}$ is a coset in $G$.

**Lemma 1.1.** *$G$ has countably many open subgroups.*

*Proof.* Consider a basic open set $\mathcal{O}_{n,y}$. Given $x, z \in \mathcal{O}_{n,y}$, as $x$ and $z$ agree in the first $n$ coordinates, we see that $(x + z)_k = 0$ for $k \le n$. It follows that $\mathcal{O}_{n,y} + \mathcal{O}_{n,y} = \mathcal{O}_{n,0}$ an open subgroup.

Now let $H$ be an arbitrary open subgroup, so we can write $H$ as some union of basic open sets. Let $n$ be minimal with $\mathcal{O}_{n,y} \subseteq H$ for some $y$. Thus $\mathcal{O}_{n,0} \subseteq H$ as $H$ is a subgroup. Any open basic open set $\mathcal{O}_{m,z} \subseteq H$ must have $n \le m$, and so we see that $\mathcal{O}_{m,z} + \mathcal{O}_{n,0} = \mathcal{O}_{n,z} \subseteq H$.

As $H$ is the union of basic open sets, we conclude that there are finitely many $x_1, \cdots, x_k$ with $H = \bigcup_{i=1}^{k} \mathcal{O}_{n,x_i}$. Let $y_i \in (\mathbb{Z}/2\mathbb{Z})^n$ be the projection of $x_i$ onto the first $n$ coordinates. As $H$ is a subgroup, it follows that $\{y_i : 1 \le i \le k\}$ is a subgroup of $(\mathbb{Z}/2\mathbb{Z})^n$, say $K$. Furthermore, $H$ is exactly the collection of all $x \in G$ such that the projection of $x$ onto the first $n$ coordinates is in $K$.

It follows that open subgroups of $H$ can be described by $n \in \mathbb{N}$ and a subgroup $K$ of $(\mathbb{Z}/2\mathbb{Z})^n$. There are only countably many such choices. $\square$

**Corollary 1.2.** *There are countably many closed subgroups of $G$ of index 2.*

*Proof.* Let $H \le G$ be a closed subgroup of index 2. Then $G \setminus H$ is a coset of $H$ and so is closed, and so $H$ is open. The result follows. $\square$

We now consider arbitrary subgroups of $G$. It is instructive to consider the bijection between $G$ and $\mathcal{P}(\mathbb{N})$ the power set of $\mathbb{N}$, given by $x = (x_n)$ mapping to the set $A \subseteq \mathbb{N}$ where $n \in A$ if and only if $x_n = 0$. If $x, y \in G$ biject with $A, B$, respectively, then $x + y$ bijects with $C$ where $n \in C$ if and only if $x_n + y_n = 0$, that is, $x_n = y_n = 0$ or $x_n = y_n = 1$, that is, $n \in A \cap B$ or $n \in \mathbb{N} \setminus (A \cup B)$. Thus $C = (A \cap B) \cup (\mathbb{N} \setminus (A \cup B)) = \mathbb{N} \setminus (A \triangle B)$.

We recall the notion of a *filter* on $\mathbb{N}$. This is a subset $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$ with $\emptyset \notin \mathcal{F}$, with, if $A \in \mathcal{F}$ and $A \subseteq B$ then also $B \in \mathcal{F}$, and with $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$. An *ultrafilter* $\mathcal{U}$ is a maximal filter; alternatively, $\mathcal{U}$ is a filter with the property that if $A \in \mathcal{P}(\mathbb{N})$ then either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$.

**Lemma 1.3.** *There are $2^{\mathfrak{c}}$ subgroups of index 2 in $G$.*

*Proof.* Let $\mathcal{U}$ be an ultrafilter, and let $H \subseteq G$ be the associated subset. Given $A, B \in \mathcal{U}$ consider $C = \mathbb{N} \setminus (A \triangle B)$. Then $A \cap B \subseteq C$ and so $C \in \mathcal{U}$. It follows that $H$ is a subgroup of $G$. Furthermore, given $A \notin \mathcal{U}$ we know that $\mathbb{N} \setminus A \in \mathcal{U}$. Thus if $x \notin H$ then $1 + x \in H$, and as $\emptyset \notin \mathcal{U}$ also $1 \notin H$. Thus $H$ is proper, and $G$ is the union of $H$ and $1 + H$, so $H$ has index 2.

It is well-known (see for example [9]) that there are $2^c$ ultrafilters on $\mathbb{N}$, and so there are (at least) $2^{\mathfrak{c}}$ subgroups of index 2 in $G$. As $G$ bijects with $\mathcal{P}(\mathbb{N})$ we have $|G| = 2^{\aleph_0} = \mathfrak{c}$ and so $|\mathcal{P}(G)| = 2^{\mathfrak{c}}$. Thus there are at most $2^{\mathfrak{c}}$ subgroups of any index. $\square$

There hence exists a subgroup $H$ of index 2 which is not closed. (In fact this follows more directly from the existence of non-principle ultrafilters, and the proof of Lemma 1.1.) Thus $H$ is not open, and so cannot contain any non-empty open set (if $\emptyset \neq U \subseteq H$ is open then using the group operations we can cover $H$ by translates of $U$ which shows that $H$ is open, contradiction). As $G \setminus H$ is a coset of $H$ it follows that $G \setminus H$ cannot contain any non-empty open set. Thus $H$ is dense in $G$. We have also now answered our original question, as both $H$ and its coset have empty interior, and yet their union is all of $G$.

# 2 Semi-direct products

This is standard material. Let $G$ be a group with a subgroup $H$ and a normal subgroup $N$. The following statements are equivalent:

1. $G = NH = \{nh : n \in N, h \in H\}$ and $N \cap H = \{e\}$;
2. for each $g \in G$ there are unique $n \in N, h \in H$ with $g = nh$;
3. for each $g \in G$ there are unique $n \in N, h \in H$ with $g = hn$;
4. for the inclusion $i : H \to G$ and the quotient $\pi : G \to G/N$, the composition $\pi \circ i : H \to G/N$ is an isomorphism;
5. there is a homomorphism $G \to H$ that is the identity on $H$ and has kernel $N$;
6. there is a split short-exact sequence $1 \to N \to G \to H \to 1$.

Let us show these equivalences. If (1) holds then $G = NH$ but if there was a possibly non-unique way to write $g = n_1h_1 = n_2h_2$ then $n_2^{-1}n_1 = h_2h_1^{-1} \in N \cap H = \{e\}$ so $n_1 = n_2$ and $h_1 = h_2$, so (2) holds. Conversely, the uniqueness clause in (2) shows that if $g \in N \cap H$ then we must have $g = e$. So (1)⇔(2).

That (1)⇔(3) is similar, using the group inverse to show that $G = NH$ if and only if $G = G^{-1} = H^{-1}N^{-1} = HN$.

Considering (4), $\ker(\pi \circ i) = N \cap H$ and the image of $\pi \circ i$ is $\{Nh : h \in H\} = \{Nnh : n \in N, h \in H\}$, so immediately (1) implies (4), while conversely, if $\pi \circ i$ is onto, then given $g \in G$ there is $h \in H$ with $Nh = Ng$ so $g \in Nh \subseteq NH$ and hence (1) holds. So (1)⇔(4).

For (5) consider $p : G \to H$ a homomorphism with $p(h) = h$ for $h \in H$ and $\ker(p) = N$. Then $N \cap H = \{e\}$ while for $g \in G$ let $h = p(g) \in H$ so $p(hg^{-1}) = hp(g)^{-1} = e$ and hence $hg^{-1} = n$ for some $n \in N$, so $g = n^{-1}h \in NH$, so (1) holds. Conversely, given (1), also (4), let $p : G \to H$ be the composition of $(\pi \circ i)^{-1} \circ \pi : G \to H$. Then $p$ is a homomorphism, $\ker(p) = \ker(\pi) = N$, and for $h \in H$ we have $p(h) = h_1$ where $h_1 \in H$ is the necessarily unique element with $h_1N = \pi(h) = hN$, so $h_1 = h$ by uniqueness. Hence (5) holds.

For (6) we have

$$1 \longrightarrow N \xhookrightarrow{\iota} G \underset{\theta}{\overset{\pi}{\rightleftarrows}} H \longrightarrow 1 \tag{1}$$

where $\iota$ is an inclusion, $\pi$ a surjection with $\ker(\pi) = \iota(H)$, and we have $\pi \circ \theta = \mathrm{id}_H$. We identify $N$ with its image in $G$ which is certainly normal, and then $\pi$ gives an isomorphism between $G/N$ and $H$. That $\pi \circ \theta = \mathrm{id}_H$ means $\theta$ is injective, so we identify $H$ with its image in $G$. Then $p = \theta \circ \pi$ is an idempotent homomorphism, which is the identity on $H$, and has $\ker(p) = \ker(\pi) = N$, so (5) holds. Conversely, given (5), we have inclusions $N \to G$ and $H \to G$ giving $\iota$ and $\theta$. Consider the map $p : G \to H$, and set $\pi = p$. Then $\ker(\pi) = \ker(p) = N$ while $\pi$ is onto, and $\pi \circ \theta = \mathrm{id}_H$, so (6) holds.

Any of these equivalent conditions define what it means for $G$ to be the *(inner) semidirect product* denoted $G = N \rtimes H$ (sometimes $G = H \ltimes N$).

**Definition 2.1.** The *(outer) semidirect product* of groups $N, H$ is defined by specifying a homomorphism $\varphi : H \to \mathrm{Aut}(N)$ and setting $G = N \times H$ as a set, with product

$$(n_1, h_1)(n_2, h_2) = (n_1\varphi(h_1)(n_2), h_1h_2).$$

The identity of $G$ is $(e_N, e_H)$ and the inverse is $(n, h)^{-1} = (\varphi(h^{-1})(n^{-1}), h^{-1})$.

Given such a construction, we identify $N$ with $\{(n, e_H) : n \in N\}$ and $H$ with $\{(e_N, h) : h \in H\}$. These are seen to be subgroups of $G$. Then $NH = \{(n, e)(e, h) : n \in N, h \in H\} = G$ and $N \cap H = \{e\}$. Finally, the product satisfies

$$hnh^{-1} = (e, h)(n, e)(e, h)^{-1} = (\varphi(h)(n), h)(e, h^{-1}) = (\varphi(h)(n)\varphi(h)(e), e) = \varphi(h)(n) \in N,$$

for each $n \in N, h \in H$. Thus $N$ is a normal subgroup and we have verified condition (1).

Conversely, if we have an inner semidirect product $G = NH$ with $N \trianglelefteq G$, then define $\varphi(h)(n) = hnh^{-1} \in N$, so that $\varphi(h)$ is an automorphism of $N$ for each $h$, and $\varphi : H \to \mathrm{Aut}(N)$ is an homomorphism. We define $\theta : G \to N \times H$ by $\theta(nh) = (n, h)$. As a slight aside, notice that this map is a homomorphism if and only if $\theta(h)\theta(n) = \theta(hn) = \theta((hnh^{-1})h) = \theta(\varphi(h)(n)h)$, which is true by definition. This "commutation relation", showing how elements of $N$ and $H$ commute pass each other, is often a useful way to think of a semi-direct product.

Thus inner and outer semidirect products are canonically isomorphic.

Finally, we claim that $G$ is isomorphic to $N \rtimes H$ if and only if there is a split short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r, hook, "\beta"] & G \arrow[r, two heads, "\alpha"] & H \arrow[l, dotted, bend left, "\gamma"] \arrow[r] & 1 \end{tikzcd}$$

This is just condition (6). Notice

$$\varphi(h)(n) = hnh^{-1} = \beta^{-1}\big(\gamma(h)\beta(n)\gamma(h^{-1})\big)$$

gives the action of $H$ on $N$.

## 2.1 For topological groups

When $G$ is a topological group, it seems natural to consider the continuous automorphisms of $N$, and to ask for $\varphi : H \to N$ to be continuous. Forming the (outer) semidirect product, we form the topological product $N \times H$ and the product as above. We require that $H \times N \to N; (h, n) \mapsto \varphi(h)(n)$ is (jointly) continuous, and then we see that the product on $N \rtimes H$ is (jointly) continuous. The inverse is continuous, as it is the composition of $(h, n) \mapsto (h^{-1}, n^{-1})$ with the continuous action map.

Then $N, H$ are closed subgroups of $N \rtimes H$. Conversely, start with $G = NH$ for some closed $N, H$ with $N \trianglelefteq G$. Then in (4) while $i, \pi$ are continuous, so $\pi \circ i$ is continuous, we require that $(\pi \circ i)^{-1}$ be continuous as well. This is equivalent to $p$ in (5) being continuous. For (6), in the split short exact sequence (1), we have that $\iota, \pi$ are continuous, and we also require that $\theta$ is continuous. Then $\theta$ is injective, and furthermore is a homeomorphism onto its range, because $\pi \circ \theta = \mathrm{id}_H$.

It does not seem obvious what conditions we would like to add to (1), (2) or (3) to ensure the correct continuity conditions. However, notice that given $G = NH$ with $N \trianglelefteq H$, with $N, H$ closed, we can still define $\varphi(h)(n) = hnh^{-1}$, and continuity of the product in $G$ will ensure that $\varphi(h)$ is continuous for $h$, and that $H \times N \to N; (h, n) \mapsto \varphi(h)(n)$ is continuous. Thus we can form the outer semidirect product. The canonical isomorphism is $G \to N \times H, nh \mapsto (n, h)$. The inverse of this is always continuous, but this map itself might not be. It is continuous exactly when $(n_i h_i)$ is a net converging to $e$, we must have $n_i \to e$ and $h_i \to e$. If $G$ is locally compact, this fails to happen exactly when we can find $(n_i) \subseteq N, (h_i) \subseteq H$ with $n_i \to \infty, h_i \to \infty$ and yet $n_i h_i \to e$.

**Remark 2.2.** It is implicitly claimed on page 10 of [7] that in this setting, the map from the outer semidirect product $N \rtimes H$ to $G$ is an isomorphism of locally compact groups. The following counter-example, from [12], shows that this need not be the case. Let $K$ be an infinite compact group and let $K_d$ be $K$ with the discrete topology, so $K \neq K_d$ as topological groups.

Let $G = K_d \times K$ the direct product of groups, so $G$ is locally compact. Let $N = K_d \times \{e\}$ a closed subgroup, and let $H = \{(g,g) : g \in K\} \subseteq G$ the diagonal, which is also a closed, normal subgroup. Then it's easy to see that $N \cap H = \{e\}$ and $NH = G$. However, both $N$ and $H$ have the discrete topology (as the subspace topologies from $G$) and so the outer semidirect product $N \rtimes H$ will also be discrete. Hence $N \rtimes H \to G$ is continuous, but the inverse is not.

In this example, the map $p$ from (5) is given by $p(g,h) = (h,h)$ which is obviously a projection onto the diagonal $H$ with kernel $N$. This is not continuous, as there is a net $(h_i)$ in $K$ which converges to $e$ without eventually being equal to $e$. Then for any fixed $g$, we have $(g, h_i) \to (g, e)$ in $G$ but $(h_i, h_i) \not\to (e, e)$. A similar remark applies to (6), while in (4) we see that $\pi \circ i$ does not have continuous inverse.

In [7], see page 9, in the locally compact case the Haar measure on $N \rtimes H$ is computed. For each $h \in H$ the measure on $N$ given by $\lambda_N^h(E) = \lambda_N(\varphi(h)(E))$, for each Borel $E$, is left-invariant, and so there is $\delta(h) > 0$ with $\lambda_N^h(E) = \delta(h)\lambda_N(E)$ for all $E$. One can check that $\delta : H \to (\mathbb{R}^+, \times)$ is a continuous homomorphism, and then a left Haar measure on $N \rtimes H$ is given by

$$\int_{N \rtimes H} f(n,h)\ d(n,h) = \int_H \int_N f(n,h)\delta(h)^{-1}\ dn\ dh. \qquad (f \in C_{00}(N \rtimes H)).$$

The modular function is

$$\Delta_{N \rtimes H}(n,h) = \Delta_N(n)\Delta_H(h)\delta(h)^{-1}.$$

# 3 Modules over algebras of operators

The following is surely folklore: people will know the construction, but I'm not aware of a reference.

We consider homomorphisms $\pi\colon \mathcal{B}(E) \to \mathcal{B}(X)$ for Banach spaces $E$ and $X$. Any such $\pi$ restricts to a homomorphism $\pi\colon \mathcal{F}(E) \to \mathcal{B}(X)$, where $\mathcal{F}(E)$ is the finite-rank operators. For $e \in E, e^* \in E^*$ we write $e \otimes e^*$ for the rank-one operator $E \ni f \mapsto \langle e^*, f\rangle e$.

Choose $e_0 \in E, e_0^* \in E^*$ with $\langle e_0^*, e_0\rangle = 1$, so that $e_0 \otimes e_0^* \in \mathcal{F}(E)$ is an idempotent, and hence so too is $\pi(e_0 \otimes e_0^*)$. Set

$$Y_0 = \{x \in X : \pi(e_0 \otimes e_0^*)x = x\},$$

a closed subspace (as the range of an idempotent) in $X$. Define $\phi\colon E \otimes Y_0 \to X$ by $e \otimes y \mapsto \pi(e \otimes e_0^*)y$.

**Lemma 3.1.** *$\phi$ is an injective map, the image of which is a $\mathcal{B}(E)$-submodule of $X$. Infact, the image of $\phi$ is* $\operatorname{lin}\{\pi(T)x : x \in X, T \in \mathcal{F}(E)\}$.

*Proof.* Given $u \in E \otimes Y_0$ we can find linearly independent $e_1, \cdots, e_n$ in $E$ with $u = \sum_{i=1}^n e_i \otimes y_i$ for some $y_i$ in $Y$. Then there are $e_j^* \in E^*$ with $\langle , e_j^*\rangle e_i = \delta_{i,j}$. Suppose $\phi(u) = 0$ we see that

$$0 = \pi(e_0 \otimes e_j^*) \sum_{i=1}^n \phi(e_i \otimes y_i) = \sum_{i=1}^n \pi(e_0 \otimes e_j^*)\pi(e_i \otimes e_0^*)y_i = \pi(e_0 \otimes e_0^*)y_j = y_j,$$

as $y_j \in Y_0$, and using that $\pi$ is a homomorphism. Hence $y_j = 0$ for all $j$ and so $u = 0$, showing that $\phi$ is injective.

Given $T \in \mathcal{B}(E)$ and $\phi(e \otimes y)$ we see that $\pi(T)\phi(e \otimes y) = \pi(T)\pi(e \otimes e_0^*)y = \pi(T(e) \otimes e_0^*)y = \phi(T(e) \otimes y)$, and so the image of $\phi$ is a $\mathcal{B}(E)$-submodule.

The space which we claim equals the image of $\phi$ is $\operatorname{lin}\{\pi(e \otimes e^*)x : x \in X, e \in E, e^* \in E^*\}$. Clearly the image of $\phi$ is contained in this. Conversely, given $x' = \pi(e \otimes e^*)x$, set $y = \pi(e_0 \otimes f^*)x'$ where $f^* \in E^*$ is chosen so that $\langle f^*, e\rangle = 1$ (such a functional exists via Hahn–Banach). Then $y = \pi(e_0 \otimes e^*)x$ and so $\pi(e_0 \otimes e_0^*)y = y$, hence $y \in Y_0$. Then $\phi(e \otimes y) = \pi(e \otimes e_0^*)y = \pi(e \otimes e_0^*)\pi(e_0 \otimes e^*)x = \pi(e \otimes e^*)x = x'$ and so $x'$ is in the image of $\phi$, as required. $\square$

In particular, the image of $\phi$ does not depend upon the choice of $e_0, e_0^*$. Let's explore this a little more: suppose we also have $\langle e_1^*, e_1\rangle = 1$ and analogously define $Y_1$ and $\phi_1$. Define $\alpha_{1,0}\colon Y_0 \to Y_1$ by $y \mapsto \pi(e_1 \otimes e_0^*)y$. It's easy to see that $\alpha_{1,0}$ does map into $Y_1$. Analogously define $\alpha_{0,1}$. For $y \in Y_0$ we see that

$$\alpha_{0,1}\alpha_{1,0}y = \pi(e_0 \otimes e_1^*)\pi(e_1 \otimes e_0^*)y = \pi(e_0 \otimes e_0^*)y = y,$$

and similarly, $\alpha_{1,0}\alpha_{0,1}$ is the identity on $Y_1$. Furthermore, for $y \in Y_0$,

$$\phi_1(e \otimes \alpha_{1,0}y) = \pi(e \otimes e_1^*)\alpha_{1,0}y = \pi(e \otimes e_1^*)\pi(e_1 \otimes e_0^*)y = \pi(e \otimes e_0^*)y = \phi(e \otimes y).$$

So we have explicitly found intertwiners between the different maps.

Of course, this construction only tells us about homomorphisms $\mathcal{F}(E) \to \mathcal{B}(X)$. If we have a homomorphism of the generalised Calkin algebra $\mathcal{B}(E)/\mathcal{K}(E)$ then we learn nothing.

When $\mathcal{B}(X)$ is an essential $\mathcal{A}(E)$-module, the map $\phi$ realises $X$ as the completion of $E \otimes Y_0$. The action of $\mathcal{A}(E)$ (or $\mathcal{B}(E)$) is just the standard action on the $E$ tensor factor.

If $S \in \mathcal{B}(X)$ commutes with each $\pi(T)$, for $T \in \mathcal{A}(E)$, then $y \in Y_0$ implies $Sy \in Y_0$ and then $\phi(e \otimes Sy) = S\phi(e \otimes y)$. Letting $S_0$ be the restriction of $S$ to $Y_0$, if we view $X$ as the completion of $E \otimes Y_0$, then $1 \otimes S_0$ is the operator $S$. However, in general there seems no reason to suspect that every member of $\mathcal{B}(Y_0)$ occurs as some $S_0$.

# 4 Polar decompositions

This material can be found in a variety of sources, but I wanted something in my own presentation to reference. Fix Hilbert spaces $H, K$.

Let $T \in \mathcal{B}(H,K)$ and use the continuous functional calculus to define $|T| = (T^*T)^{1/2} \in \mathcal{B}(H)$. This is the unique positive operator $S$ with $S^2 = T^*T$. We start with some elementary facts about operators on Hilbert spaces.

**Lemma 4.1.** *Let $S \in \mathcal{B}(H,K)$. Then:*

*(1)* $\operatorname{Im}(S)^\perp = \ker S^*$ *and so* $\overline{\operatorname{Im}}(S) = (\ker S^*)^\perp$.

*(2)* $\ker S = \ker S^*S = \ker |S|$.

*(3) $S$ is an isometry if and only if $S^*S = 1$.*

*Proof.* We see that $\xi \in \operatorname{Im}(S)^\perp$ if and only if $(\xi|S\eta) = 0$ for all $\eta$, if and only if $S^*\xi = 0$, so (1) follows. For (2) we note that

$$S\xi = 0 \implies S^*S\xi = 0 \implies (\xi|S^*S\xi) = 0 \implies \|S\xi\|^2 = 0 \implies S\xi = 0,$$

and so we have equivalence throughout. As $|S|^*|S| = S^*S$ also $\ker |S| = \ker S^*S$.

For (3), if $S^*S = 1$ then $\|S\xi\|^2 = (\xi|S^*S\xi) = \|\xi\|^2$ for each $\xi \in H$ and so $S$ is an isometry. For the converse, we use the polarisation identity, so for $\xi, \eta \in H$,

$$(S\xi|S\eta) = \frac{1}{4}\sum_{k=0}^{3} i^k \|S\xi + (-i)^k S\eta\|^2 = \frac{1}{4}\sum_{k=0}^{3} i^k \|\xi + (-i)^k \eta\|^2 = (\xi|\eta),$$

in the middle step using that $S$ is an isometry. It follows that $S^*S = 1$. □

Next we look at partial isomeries in the abstract.

**Lemma 4.2.** *Let $A$ be a $C^*$-algebra and let $u \in A$. The following are equivalent:*

*(1) $u^*u$ is a projection;*

*(2) $uu^*u = u$;*

*(3) $u^*uu^* = u^*$;*

*(4) $uu^*$ is a projection.*

*Proof.* If (1) holds, then $u^*uu^*u = u^*u$ and so $(uu^*u - u)^*(uu^*u - u) = u^*uu^*uu^*u - u^*uu^*u - u^*uu^*u + u^*u = 0$, so (2) holds, by the $C^*$-condition (as $a^*a = 0 \implies a = 0$ for $a \in A$). If (2) holds then multiply on the left by $u^*$ to see that $u^*u$ is idempotent; clearly $u^*u$ is self-adjoint, and so (1) holds. Replacing $u$ by $u^*$ shows that (3) and (4) are equivalent. Finally, (2) and (3) are equivalent by taking adjoints. □

**Definition 4.3.** A *partial isometry* is $u \in A$ satisfying any of the above equivalent conditions. The projection $u^*u$ is the *initial projection* and $uu^*$ is the *final projection*.

**Lemma 4.4.** *For $U \in \mathcal{B}(H,K)$ we have that $U$ is a partial isometry if and only if there is a closed subspace $H_0 \subseteq H$ such that $U$ restricted to $H_0$ is an isometry, and $U$ restricted to $H_0^\perp$ is $0$. In this case, $H_0$ is the image of the initial projection, and $U(H_0)$ is the image of the final projection.*

*Proof.* Let $U$ be a partial isometry, and let $H_0$ be the image of the initial projection $U^*U$. For $\xi = U^*U\xi \in H_0$, we have $\|U\xi\|^2 = (\xi|U^*U\xi) = (\xi|\xi) = \|\xi\|^2$, so conclude that $U$ is an isometry on $H_0$. By Lemma 4.1(2), we have $\ker U = \ker U^*U = H_0^\perp$.

Conversely, the restriction of $U$ to $H_0$ is an isometry, and so $(U\xi|U\eta) = (\xi|\eta)$ for all $\xi, \eta \in H_0$, see Lemma 4.1(3). Let $\xi \in H$ and write $\xi = \xi_0 + \xi_1 \in H_0 \oplus H_0^\perp$, and similarly for $\eta$, so that $(\eta|U^*U\xi) = (U\eta|U\xi) = (U\eta_0|U\xi_0) = (\eta_0|\xi_0) = (\eta|\xi_0)$. Thus $U^*U\xi = \xi_0$, and so $U^*U$ is the projection onto $H_0$. Hence $U$ is a partial isometry with initial space $H_0$. Obviously $\mathrm{Im}(UU^*) \subseteq \mathrm{Im}(U)$, but $UU^*U = U$ so $UU^*\xi = \xi$ for any $\xi \in U(H_0)$, and we conclude that $\mathrm{Im}(UU^*) = U(H_0)$. □

For a partial isometry $U$ on a Hilbert space, the *initial space* is the range of the projection $U^*U$ and the *final space* is the range of $UU^*$.

We can now construct the polar decomposition. Given $T \in \mathcal{B}(H, K)$ form $|T| = (T^*T)^{1/2}$, and define

$$U\colon |T|(H) = \{|T|\xi : \xi \in H\} \to K; \quad U|T|\xi = T\xi.$$

For $\xi \in H$ we have that $\|T\xi\|^2 = (\xi|T^*T\xi) = \||T|\xi\|^2$ and so $U$ is an isometry, and hence extends by continuity to the closure of $|T|(H)$. Set $U$ to be 0 on the orthogonal complement of $|T|(H)$, so $U$ is now defined on all of $H$. By construction, $U|T| = T$, and by Lemma 4.4, $U$ is a partial isometry with initial space $\overline{\mathrm{Im}}|T|$ and final space $\overline{\mathrm{Im}}T$. By Lemma 4.1, $\overline{\mathrm{Im}}|T| = (\ker |T|)^\perp = (\ker T)^\perp = \overline{\mathrm{Im}}(T^*)$ and $\overline{\mathrm{Im}}T = (\ker T^*)^\perp$. Also notice that $\ker U = (\mathrm{Im}\,|T|)^\perp = \ker T$.

**Proposition 4.5.** *The polar decomposition is unique in the sense that if $T = VS$ with $S$ positive and $V$ a partial isometry with initial space $\overline{\mathrm{Im}}S$ then $U = V$ and $S = |T|$. We have that $|T|$ belongs to the $C^*$-algebra generated by $T^*T$, and when $H = K$, $U$ belongs to the von Neumann algebra generated by $T$.*

*Proof.* We have $T^*T = SV^*VS = S^2$ as $V^*V$ is the projection onto $\overline{\mathrm{Im}}S$, and so by uniqueness of positive square-roots, $|T| = S$. Then $U|T| = V|T|$ so $U$ and $V$ agree on $\overline{\mathrm{Im}}|T|$ which is the initial space of both partial isomeries, and hence $U = V$.

By the continuous functional calculus, $|T| \in C^*(T^*T)$. When $H = K$, the von Neumann algebra generated by $T$ is the bicommutant $\{T, T^*\}''$. Let $S$ commute with $T$ and $T^*$; we need to show that $SU = US$. Let $\eta \in \ker U = \ker T$ so $TS\eta = ST\eta = 0$ so $S\eta \in \ker T$ so $US\eta = 0$. As $S$ and $|T|$ commute, for $\xi \in H, \eta \in \ker U$, we have

$$SU(|T|\xi + \eta) = ST\xi = TS\xi = U|T|S\xi = US(|T|\xi + \eta).$$

As $|T|(H) + \ker U$ is dense in $H$, we conclude that $SU = US$, as required. □

Suppose now that $T \in \mathcal{B}(H)$ is self-adjoint, so we can write $T = T_+ - T_-$ for some $T_+, T_-$ positive with $T_+T_- = 0 = T_-T_+$. Let $H_\pm = \overline{\mathrm{Im}}T_\pm$ so $H_+$ and $H_-$ are mutually orthogonal as $(T_+\xi|T_-\eta) = (\xi|T_+T_-\eta) = 0$ for all $\xi, \eta \in H$. Let $U$ be the operator which is 1 on $H_+$, $-1$ on $H_-$ and 0 on $(H_+ \oplus H_-)^\perp$. Then $U = U^*$ and $U^*U$ is the projection onto $H_+ \oplus H_-$, so $U$ is a partial isometry. As $\overline{\mathrm{Im}}T = (\ker T)^\perp = (H_+ \oplus H_-)^{\perp\perp}$ we see that $U^*U$ is the projection onto $\overline{\mathrm{Im}}T$. As $|T| = T_+ + T_-$ (by uniqueness, or functional calculus) we see that $U|T| = UT_+ + UT_- = T_+ - T_- = T$ and so by uniqueness, we have constructed the polar decomposition.

# 5 von Neumann regular elements

Again, we present some surely well-known results, with proofs, for applications to follow in the next section.

**Definition 5.1.** Let $A$ be an algebra (or even just a ring). An element $x \in A$ is *von Neumann regular* if there is $y \in A$ with $xyx = x$.

**Proposition 5.2.** *Let $E$ be a Banach space and let $x \in \mathcal{B}(E)$. The following are equivalent:*

1. *$x$ is von Neumann regular in $\mathcal{B}(E)$;*
2. *$x$ has closed image and both $\ker(x)$ and $\operatorname{Im}(x)$ are complemented subspaces of $E$.*

*In this case, when $xyx = x$, we have that $xy = e$ is a projection onto $\operatorname{Im}(x)$ and $1 - yx = f$ is a projection onto $\ker(x)$. Conversely, given $e$ and $f$ projections onto $\operatorname{Im}(x)$, which is closed, and $\ker(x)$, respectively, we can choose $y$ with $xyx = x, xy = e$ and with $yx = 1 - f$.*

*Proof.* Suppose there is $y \in \mathcal{B}(E)$ with $xyx = x$. Then $(xy)^2 = xyxy = xy$ and $(yx)^2 = yxyx = yx$ so $e = xy$ and $f = 1 - yx$ are projections. We have $e(E) \subseteq x(E)$, but as $ex = xyx = x$ also $e(H) \supseteq x(E)$, so $x(E) = e(E)$ and in particular, $x$ has closed, complemented image. As $xf = x - xyx = 0$ we have $f(E) \subseteq \ker(x)$, but if $x\xi = 0$ then $f\xi = \xi - yx\xi = \xi$ so we conclude that $f$ is a projection onto $\ker(x)$.

Conversely, let $x$ have closed image, with $e$ a projection onto $x(E)$ and $f$ a projection onto $\ker(x)$. Let $x_0$ be the restriction of $x$ to a map $\operatorname{Im}(1 - f) \to \operatorname{Im}(e) = \operatorname{Im}(x)$. If $x_0\xi = 0$ then $(1 - f)\xi = \xi$ so $f\xi = 0$, but also $x\xi = 0$, so $f\xi = \xi$, so $\xi = 0$ and we conclude that $x_0$ is injective. Given any $\xi \in E$ we have that $xf\xi = 0$ so $x\xi = x(1 - f)\xi$ and hence $x_0(1 - f)\xi = x\xi$ and we conclude that $x_0$ is surjective. By the Open Mapping Theorem, $x_0$ is invertible. Set $y = x_0^{-1}e \colon E \to \operatorname{Im}(1 - f) \subseteq E$. Then $xy = xx_0^{-1}e = x_0x_0^{-1}e = e$. Given $\xi \in E$ set $\eta = (1 - f)\xi$ so as before, $x\xi = x\eta = x_0\eta$ as $\eta \in \operatorname{Im}(1 - f)$. Hence $yx\xi = x_0^{-1}ex\xi = x_0^{-1}x\xi = x_0^{-1}x_0\eta = \eta = (1 - f)\xi$. So $yx = 1 - f$. Finally, we have $xyx = ex = x$, as desired. □

For the following, compare also Proposition 6.3 below.

**Proposition 5.3.** *Let $H$ be a Hilbert space, and let $x \in \mathcal{B}(H)$. Then $x$ is von Neumann regular if and only if $x$ has closed image. When $x$ is von Neumann regular and self-adjoint, we can choose a self-adjoint $y$ with $xyx = x$ and with $xy = yx$ a (self-adjoint) projection. Further, when $x$ is positive, we can choose $y$ positive.*

*Proof.* In a Hilbert space, all closed subspaces are complemented, so $x$ is von Neumann regular if and only if $x$ has closed image. Now let $x$ be self-adjoint with closed image. As $\operatorname{Im}(x) = (\ker x)^\perp$, Lemma 4.1, we can choose $e$ to be the orthogonal projection onto $\operatorname{Im}(x)$, and then $f = 1 - e$ will be the orthogonal projection onto $\ker(x)$. Following the previous proof, let $x_0$ be the restriction of $x$ to $\operatorname{Im}(1 - f) = \operatorname{Im}(e) = \operatorname{Im}(x)$, so $x_0 \colon \operatorname{Im}(x) \to \operatorname{Im}(x)$ is invertible, and set $y = x_0^{-1}e$. Then $xy = e, yx = 1 - f = e$. As $x$ is self-adjoint, also $x_0$ is self-adjoint, and so $x_0^{-1}$ is self-adjoint. As $y = ey = ex_0^{-1}e$, we conclude that $y$ is self-adjoint. The same argument shows that when $x$ is positive, also $y$ is positive. □

# 6 Well-supported elements

This definition is given in [1, Definition II.3.2.8]. An element $x$ in a $C^*$-algebra $A$ is *well-supported* if $\sigma(x^*x) \setminus \{0\}$ is closed, that is, $\sigma(x^*x) \subseteq \{0\} \cup [\epsilon, \infty)$ for some $\epsilon > 0$. I haven't found an reference for elementary properties of such elements, so here give some elementary proofs, following Blackadar for the results.

As $\sigma(x^*x) \cup \{0\} = \sigma(xx^*) \cup \{0\}$, we see that $x$ is well-supported if and only if $x^*$ is. Let $x$ be well-supported. Define $f\colon [0,\infty) \to [0,1]$ by $f(0) = 0$ and $f(t) = 1$ for $t > 0$. Then $f$ is continuous on $\sigma(x^*x)$ and so by the continuous functional calculus, $p = f(x^*x)$ exists, and $p$ is a projection with $px^*x = x^*x = x^*xp$.

**Lemma 6.1.** *Let $x$ be well-supported, and $f$ be as above, and set $p = f(x^*x)$ and $q = f(xx^*)$. Then $p$ is a projection, $xp = x$ and $xy = 0 \implies py = 0$. Also $q$ is a projection, $qx = x$ and $yx = 0 \implies yq = 0$.*

*Proof.* As observed above, $px^*x = x^*x = x^*xp$ and so $(px^* - x^*)(xp - x) = px^*xp - x^*xp - px^*x + x^*x = 0$ hence $xp = x$. If $xy = 0$ then $x^*xyy^* = 0$ and $yy^*x^*x = y(xy)^*x = 0$ so working in the commutative $C^*$-algebra generated by $x^*x, yy^*$ and 1, we see that $pyy^* = 0 = yy^*p$. Thus $pyy^*p = 0$ so $py = 0$.

As $x^*$ is also well-supported, $q$ is defined. We argue similarly: $(qx - x)(qx - x)^* = qxx^*q - xx^*q - qxx^* + xx^* = 0$ so $qx = x$. If $yx = 0$ then $y^*y$ commutes with $xx^*$ so $y^*yq = 0$ so $yq = 0$. □

The next lemma says that for a well-supported element, its polar decomposition exists in the $C^*$-algebra.

**Lemma 6.2.** *Let $x \in A$ be well-supported, let $\pi\colon A \to \mathcal{B}(H)$ be a $*$-homomorphism, and let $\pi(x) = u\pi(|x|)$ be the polar decomposition. Then $u \in \pi(A)$.*

*Proof.* Define $g\colon [0,\infty) \to [0,\infty)$ by $g(0) = 0$ and $g(t) = t^{-1/2}$, so again $g$ is continuous on $\sigma(x^*x)$, and so we can set $v = xg(x^*x) \in A$. Then $g(x^*x)|x| = p$ by the functional calculus, where $p$ is as above. So $v|x| = xp = x$. Also $v^*v = g(x^*x)x^*xg(x^*x) = p$ again by the functional calculus.

Let $q$ be the projection onto $\overline{\mathrm{Im}}(\pi(|x|)) = \ker(\pi(|x|))^\perp = \ker(\pi(x))^\perp = \overline{\mathrm{Im}}(\pi(x^*))$, and as $xp = x$ also $px^* = x^*$ so $p \geq q$. As $(1-q)x^* = 0$, so $x(1-q) = 0$, by Lemma 6.1, we have $p(1-q) = 0$. So $p = pq = q$, and we conclude that $v^*v$ is the projection onto $\overline{\mathrm{Im}}(\pi(|x|))$. By uniqueness, Proposition 4.5, $\pi(x) = \pi(v)\pi(|x|)$ is the polar decomposition. □

We now make links between well-supported elements and von Neumann regular elements, as considered in the previous section.

**Proposition 6.3.** *An element $x \in A$ is well-supported if and only if $x$ is* von Neumann regular *meaning that there is $y \in A$ with $xyx = x$.*

*Proof.* Let $x$ be well-supported. Again define $g(0) = 0, g(t) = t^{-1/2}$ for $t > 0$, so as in the proof of Lemma 6.2, with $u = xg(x^*x)$ we have that $x = u|x|$ is the polar decomposition. In particular, $uu^*u = u$. Set $y = g(x^*x)u^*$ so that $xyx = xg(x^*x)u^*u|x| = uu^*u|x| = u|x| = x$.

Conversely, let $x$ be von Neumann regular. Represent $A$ faithfully on some Hilbert space $H$, and let $x = u|x|$ be the polar decomposition, in $\mathcal{B}(H)$. Then $u^*x = |x|$ and so if we choose $y$ with $xyx = x$ then $|x|yu|x| = |x|$, showing that $|x|$ is also von Neumann regular, in $\mathcal{B}(H)$. We now apply Proposition 5.3, which tells us that $|x|$ has closed image, and we can choose $y \geq 0$ in $\mathcal{B}(H)$ with $|x|y|x| = |x|$. Furthermore, $e = y|x| = |x|y$ is a projection onto $\mathrm{Im}(|x|) = \ker(|x|)^\perp$.

Let $0 < \lambda \leq \|y\|^{-1}$. We show that $|x| - \lambda 1$ is bounded below. Let $\xi \in H$ so $|x|\xi = |x|e\xi = e|x|e\xi$. Hence, as $e$ and $1-e$ has orthogonal images,

$$\||x|\xi - \lambda\xi\|^2 = \||x|e\xi - \lambda e\xi - \lambda(1-e)\xi\|^2 = \||x|e\xi - \lambda e\xi\|^2 + \lambda^2\|(1-e)\xi\|^2.$$

We have $\|e\xi\| = \|y|x|\xi\| \leq \|y\|\||x|\xi\| = \|y\|\||x|y|x|\xi\| = \|y\|\||x|e\xi\|$ and so with $\eta = e\xi$ we see that $\||x|\eta\| \geq \|y\|^{-1}\|\eta\|$. So $\||x|\eta - \lambda\eta\| \geq \||x|\eta\| - \lambda\|\eta\| \geq (\|y\|^{-1} - \lambda)\|\eta\|$, and thus

$$\||x|\xi - \lambda\xi\|^2 \geq (\|y\|^{-1} - \lambda)^2\|e\xi\|^2 + \lambda^2\|(1-e)\xi\|^2 \geq (\|y\|^{-1} - \lambda)^2\|\xi\|^2.$$

So $|x| - \lambda$ is bounded below and hence is an isomorphism onto its range. As we are working on a Hilbert space, there is $z \in \mathcal{B}(H)$ with $z(|x| - \lambda) = 1$, hence also $(|x| - \lambda)z^* = 1$. So $z = z^* = (|x| - \lambda)^{-1}$ and we conclude $\lambda \notin \sigma(|x|)$.

So $\sigma(|x|) \subseteq \{0\} \cup [c, \infty)$ for some $c > 0$ (we can take $c = \|y\|^{-1}$). Hence $\sigma(x^*x) = \sigma(|x|^2) \subseteq \{0\} \cup [c^2, \infty)$, and $x$ is well-supported.[1] $\square$

**Remark 6.4.** The definition of being von Neumann regular might depend upon the algebra. However, notice that the proof of Proposition 6.3 shows that if $A \subseteq \mathcal{B}(H)$ and $x \in A$ is von Neumann regular in $\mathcal{B}(H)$, then there is $y \in A$ with $xyx = x$. Hence, for $C^*$-algebras, there is no dependence on the algebra.

According to Blackadar, sometimes well-supported elements are called *elements with closed range*, which is supported by the following, and the arguments in the previous proposition. First a lemma.

**Lemma 6.5.** *Let $A$ be a $C^*$-algebra and let $x \in A$ be positive. Denote by $C^*(x) \subseteq A$ the $C^*$-algebra generated by $x$. For any $r > 0$ we have that $x^{1/2}$ is in the closure of $x^rC^*(x) \subseteq x^rA$.*

*Proof.* We use a functional calculus argument. Let $\epsilon > 0$ and define $f\colon [0,\infty) \to [0,\infty)$ by $f(t) = t\epsilon^{-1/2-r}$ for $t < \epsilon$, and $f(t) = t^{1/2-r}$ for $t \geq \epsilon$. Then $f$ is continuous, and for $t \geq \epsilon$ we have $t^r f(t) = t^{1/2}$, while for $t < \epsilon$ we have $t^{1/2} - t^r f(t) = t^{1/2} - t^{1+r}\epsilon^{-1/2-r} = t^{1/2}(1 - (t/\epsilon)^{1/2+r}) \leq t^{1/2} \leq \epsilon^{1/2}$. Hence $\|x^r f(x) - x^{1/2}\| \leq \epsilon^{1/2}$. It follows that $x^{1/2}$ is in the closure of $x^rC^*(x)$. $\square$

**Proposition 6.6.** *Let $x \in A$. The following are equivalent:*

*(1) $Ax$ is closed;*

*(2) $xA$ is closed;*

*(3) $x^*Ax$ is closed;*

*(4) $x$ is well-supported.*

*Proof.* Let $x$ be well-supported, so von Neumann regular, so there is $y$ with $xyx = x$. Let $(a_n)$ be a sequence in $A$ with $a_nx \to a \in A$. Then $ayx = \lim_n a_nxyx = \lim_n a_nx = a$ and so $a = ayx \in Ax$. So (4) implies (1), and similarly (2). Similarly, if $x^*a_nx \to a \in A$ then $x^*y^*ayx = \lim_n x^*y^*x^*a_nxyx = \lim_n x^*a_nx = a$ so $a \in x^*Ax$, and so (4) implies (3).

Represent $A$ faithfully on a Hilbert space $H$. Then $Ax$ is closed in $A$ if and only if $Ax$ is closed in $\mathcal{B}(H)$. Let $x^* = v|x^*|$ be the polar decomposition, so $v^*x^* = |x^*|$. If $Ax$ is closed in $\mathcal{B}(H)$ then for a sequence $(a_n)$ in $A$ with $a_n|x^*| \to a \in \mathcal{B}(H)$, we have $a_nx = a_n|x^*|v^* \to av^*$ so $av^* \in Ax$, say $av^* = bx$. Then $b|x^*| = bxv = av^*v = \lim_n a_n|x^*|v^*v = \lim_n a_n|x^*| = a$ and so $a \in A|x^*|$ and we conclude that $A|x^*|$ is closed. Conversely, suppose $A|x^*|$ is closed,

[1] This argument could also be made, slightly more directly, by using the notion of the "approximate point spectrum".

let $a_n x \to a$, so $a_n|x^*| = a_n x v \to av \in A|x^*|$, say $av = b|x^*|$, so $bx = b|x^*|v^* = avv^* = \lim_n a_n x v v^* = \lim_n a_n|x^*|v^* v v^* = \lim_n a_n|x^*|v^* = \lim_n a_n x = a$ so $a \in Ax$ and we conclude $Ax$ is closed. So we've shown that $Ax$ is closed if and only if $A|x^*|$ is closed. Using the polar decomposition of $x$, one similarly shows that $xA$ is closed if and only if $|x|A$ is closed. Finally, the same argument shows that $x^*Ax$ is closed if and only if $|x^*|A|x^*|$ is closed.

Suppose that $x$ is positive and $xA$ is closed. Apply the lemma with $r = 1$ to conclude that $x^{1/2} \in \overline{xA} = xA$. Hence there is $a \in A$ with $xa = x^{1/2}$, and so $xaa^*x = x$ showing that $x$ is von Neumann regular, and so well-supported. The same argument shows that if $Ax$ is closed then $x$ is well-supported. Thus if (1) holds, then $A|x^*|$ is closed, so $|x^*|$ is well-supported, but by definition, this means $x^*$ is well-supported, so $x$ is well-supported. So (4) holds. Similarly (2) implies (4).

Finally, suppose (3) holds, so $|x^*|A|x^*|$ is closed. Apply the lemma with $r = 2$ to conclude that $|x^*|^{1/2} \in \overline{|x^*|^2 C^*(|x^*|)} = \overline{|x^*|C^*(|x^*|)|x^*|} \subseteq \overline{|x^*|A|x^*|} = |x^*|A|x^*|$. So there is $a \in A$ with $|x^*|^{1/2} = |x^*|a|x^*|$ and so $|x^*|(a|x^*|^2 a^*)|x^*| = |x^*|$ showing that $|x^*|$ is von Neumann regular, hence well-supported. Again, this implies (4). □

The above argument was inspired by [6, Theorem 8]. This paper [6] contains a little more about von Neumann regular elements, and following the citations will find similar papers.

**Remark 6.7.** We have tacitly worked with unital $C^*$-algebras, so we can smoothly apply the functional calculus. The continuous functional calculus is constructed by using Gelfand theory applied to a normal element $x \in A$. Then $C^*(1, x)$ is commutative and we find that it is isomorphic to $C(\sigma(x))$. Then the map sending $x$ to the "coordinate function" $t \in C(\sigma(x))$ extends to polynomials, and by density, then to $C^*(1, x) \cong C(\sigma(x))$. The inverse map is the continuous functional calculus.

When $A$ is non-unital, we embed $A$ into $A^+$ the unitisation. By construction, if $f \in C(\sigma(x))$ can be approximated by polynomials with zero constant term, then $f(x) \in A$ not $A^+$. By the locally compact space version of the Stone–Weierstrass theorem, [2, Corollary V.8.3], the collection of such $f$ is exactly $C_0(\sigma(x) \setminus \{0\})$.

Hence $|x| = (x^*x)^{1/2} \in A$. All of the continuous functions we apply to well-supported elements vanish at 0 by construction, and so work in the non-unital case. The same applies to the function we used in the proof of Lemma 6.5.

## 6.1 Application to positive maps

Let $A$ be a C*-algebra (often a von Neumann algebra) and let $\varphi\colon A \to A$ be a (completely) positive map. We say that $\varphi$ is *irreducible* if there is not a non-trivial projection $p$ (so $p \neq 0, p \neq 1$) with $\varphi(p) \leq Mp$ for some $M > 0$. See [5, Proposition 1] for example, and references therein, for further details and motivation.

**Lemma 6.8.** *Let $v \in A$ be positive and invertible, and let $e \in A$ be a non-trivial projection. Then $x = vev$ is a well-supported element of $A$. The projection $p$ associated to $x$, from Lemma 6.1, is non-trivial.*

*Proof.* There are a number of ways to show this: for example, it follows almost immediately from Proposition 6.6, as $Avev = Aev$ is closed if and only if $Ae$ is closed, because $v$ is invertible, and $Ae$ is always closed as $e$ is idempotent.

Let $p$ be given by Lemma 6.1 applied to $x = vev$. Then $xp = x$ and $xy = 0 \implies py = 0$. As $x \neq 0$ clearly $p \neq 0$. If $p = 1$ then $xy = 0 \implies y = 0$, but let $y = v^{-1}(1-e) \neq 0$, as $e \neq 1$, so see that $xy = vevv^{-1}(1-e) = ve(1-e) = 0$, contradiction. So $p$ is non-trivial. □

**Proposition 6.9.** *Let $v \in A$ be positive and invertible, let $\varphi\colon A \to A$ be positive, and set $\tilde{A}\colon A \to A; x \mapsto v^{-1}A(vxv)v^{-1}$, which is positive. Then $A$ is irreducible if and only if $\tilde{A}$ is irreducible.*

*Proof.* Let $\tilde{A}$ be reducible, say $\tilde{A}(e) \leq Me$ for some non-trivial projection $e$ and $M > 0$. By the lemma, $x = vev$ is positive and well-supported. Let $p \in A$ be the projection associated to $x$ so $p$ is non-trivial. As $x$ is well-supported, $\sigma(x^2) \subseteq \{0\} \cup [c^2, C^2]$ for some $0 < c < C$, and so $\sigma(x) \subseteq \{0\} \cup [c, C]$. Let $f(0) = 0, f(t) = 1$ for $t > 0$, so by construction, $p = f(x)$. As $cf(t) \leq t \leq Cf(t)$ for $t \in \sigma(x)$, we see that $cp \leq x \leq Cp$. Then $\tilde{A}(e) \leq Me$ means $A(vev) \leq Mvev$, so $A(p) \leq c^{-1}A(x) \leq c^{-1}Mx \leq c^{-1}CMp$, in the first step using that $A$ is positive. Thus $A$ is reducible. Swapping the roles of $v$ and $v^{-1}$ shows that $A$ reducible implies $\tilde{A}$ reducible. $\square$

This gives justification to the claim at the start of the proof of [3, Proposition 4.3]. There we work with a finite-dimensional $C^*$-algebra, for which it is obvious that all elements are well-supported. I thank Mateusz Wasilewski for correspondance which suggested the approach to proving Proposition 6.9 in the finite-dimensional case.

# 7 Inductive limits of Hilbert spaces

We use [10, Chapter XIV] as a reference, though this is standard material (and I suspect the main result of this section is in the literature, if I knew where to look). Let $(H_n)$ be a seqeuence of Hilbert spaces with $\iota_n\colon H_n \to H_{n+1}$ isometries. Then $(H_n, \iota_n)$ is an *inductive sequence of Hilbert spaces*.

For $m \geq n$ define $\iota_{m,n} = \iota_{m-1} \circ \iota_{m-2} \circ \cdots \circ \iota_n$ an isometry $H_n \to H_m$, and set $\iota_{n,n} = 1_{H_n}$. As it is easy to forget the convention, we note this:

$$\iota_{m,n}\colon H_n \hookrightarrow H_m \qquad (n \leq m).$$

Let $H_\infty$ be the family of sequences $\xi = (\xi_n)$ where $\xi_n \in H_n$ for each $n$, and such that there is some $m$ with $\xi_{n+1} = \iota_n(\xi_n)$ for $n \geq m$. That is, $\xi_n = \iota_{n,m}(\xi_m)$ for $n \geq m$. Notice that $H_\infty$ is a vector space for the pointwise operations, and we may define a pre-inner-product by

$$(\xi|\eta) = \lim_n (\xi_n|\eta_n).$$

This is well-defined, as given $\xi, \eta$ there is some $m$ so that for $n \geq m$ we have $\xi_n = \iota_{n,m}(\xi_m)$ and $\eta_n = \iota_{n,m}(\eta_m)$, and so $(\xi_n|\eta_n) = (\iota_{n,m}(\xi_m)|\iota_{n,m}(\eta_m)) = (\xi_m|\eta_m)$ as $\iota_{n,m}$ is an isometry. Hence the sequence $(\xi_n|\eta_n)$ is eventually constant, and so the limit certainly exists. Notice that $(\xi|\xi) = 0$ if and only if $(\xi_n|\xi_n) = 0$ for sufficiently large $n$, and so we do not have an inner-product, and must quotient by the null space $\{\xi : (\xi|\xi) = 0\}$. The completion of the resulting space is $H = \varinjlim(H_n, \iota_n)$ a Hilbert space.

For each $n$ we define

$$\iota_{\infty,n}\colon H_n \to H_\infty; \quad \xi \mapsto (0, 0, \cdots, 0, \xi, \iota_{n+1,n}(\xi), \iota_{n+2,n}(\xi), \cdots),$$

where the $\xi$ occurs in the $n$th position. This is an isometry, and so we can regard $\iota_{\infty,n}$ as a map to $H$. We have that $\iota_{\infty,n+1} \circ \iota_n(\xi) = (0, \cdots, 0, 0, \iota_{n+1,n}(\xi), \iota_{n+2,n}(\xi), \cdots)$ which equals $\iota_{\infty,n}(\xi)$ in $H$ (once we have quotiented $H_\infty$ by the null space). Thus we have the commutative diagram:

$$\begin{tikzcd}
H_1 \arrow[r, hook, "\iota_1"] \arrow[drr, hook, "\iota_{\infty,1}"'] & H_2 \arrow[r, hook, "\iota_2"] \arrow[dr, hook] & H_3 \arrow[r, hook, "\iota_3"] \arrow[d, hook, "\iota_{\infty,3}"] & \cdots \\
 & & H & \cdots
\end{tikzcd}$$

The space $H$ has the following universal property. Suppose $K$ is a Hilbert space and for each $n$ we have an isometry $u_n\colon H_n \to K$ with $u_{n+1} \circ \iota_n = u_n$, and such that the images of the $u_n$ are dense in $K$. As $u_{n+1}(H_{n+1}) \supseteq u_n(H_n)$, we see that $K$ is the closure of this increasing family of subspaces. For $\xi \in H_n$ define $U\colon u_n(\xi) \mapsto \iota_{\infty,n}(\xi) \in H$. As $u_n(\xi) = u_{n+1}\iota_n(\xi)$ and $\iota_{\infty,n+1}\iota_n(\xi) = \iota_{\infty,n}(\xi)$, we see that $U$ is well-defined on the union of the subspaces $u_n(H_n)$. As all the maps are isometries, also $U$ is an isometry, and so $U$ extends to an isometry $U\colon K \to H$. The image of $U$ contains each subspace $\iota_{\infty,n}(H_n)$, and as $H$ is the union of all these subspaces, $U$ is unitary.

## 7.1 Relation to the algebraic inductive limit

We recall the usual construction of the algebraic inductive limit. We consider the disjoint union of each space $H_n$ and quotient by the equivalence relation that $\xi_i \in H_i$ is related to $\xi_j \in H_j$ if $\iota_{m,i}(\xi_i) = \iota_{m,j}(\xi_j)$ for some $m \geq i, j$. The relation is transitive as $\iota_{n,m} \circ \iota_{m,i} = \iota_{n,i}$ whenever $n \geq m \geq i$.

The vector space operations on each $H_n$ drop to the equivalence classes. Given two classes $[\xi_i]$ and $[\xi_j]$ given $m \geq i, j$ we define $([\xi_i]|[\xi_j]) = (\iota_{m,i}(\xi_i)|\iota_{m,j}(\xi_j))_{H_m}$. As $\iota_{n,m}$ is an isometry for

$n \geq m$, we again see that this definition does not depend upon the choice of $m$. Furthermore, it's seen to be independent of the choice of vector giving the equivalence class.

The maps $\iota_{\infty,n} \colon H_n \to H_\infty$ respect the equivalence relation and give a bijection between the algebraic inductive limit and $H_\infty$, which is unitary for the inner-products we have defined. Thus we obtain the same construction.

## 7.2 As an inverse limit

The sequence $(H_n, \iota_n^*)$ is an inverse sequence, where a general inverse sequence is $(H_n, q_n)$ where $q_n \colon H_{n+1} \to H_n$ is a coisometry for each $n$. Of course, every inverse sequence arises from an inductive sequence, and conversely, due to our requirement that each $q_n$ be a coisometry.

We may define

$$q_{m,n} = q_n \circ q_{n+1} \circ \cdots \circ q_{m-1} \colon H_m \to H_n \qquad (n \leq m).$$

The *algebraic inverse limit* is defined as all sequences $\xi = (\xi_n)$ where $\xi_n \in H_n$ for each $n$, and $q_{m,n}(\xi_m) = \xi_n$ for each $n \leq m$; equivalently, $\xi_n = q_n(\xi_{n+1})$ for each $n$. This space is a vector space for the pointwise operations. We define $\varprojlim(H_n, q_n)$ to be the subspace of the algebraic inverse limit consisting of all bounded sequences. For each $n$ we define $q_{\infty,n} \colon \varprojlim(H_n, q_n) \to H_n$ to be the map $\xi \mapsto \xi_n$, that is, the projection onto the $n$th coordinate, which is a linear map.

Let $\xi = (\xi_n)$ with $\|\xi_n\| \leq K$ for each $n$. As each $q_n$ is, in particular, a contraction, $\|\xi_n\| = \|q_n(\xi_{n+1})\| \leq \|\xi_{n+1}\|$ and so the sequence of norms $(\|\xi_n\|)$ is increasing, and bounded above, and so converges. For $n \leq m$, as $\iota_{m,n} = q_{m,n}^* \colon H_n \to H_m$ is an isometry, $q_{m,n}^* \circ q_{m,n}$ is an orthogonal projection of $H_m$ onto the image of $q_{m,n}^*$, which is isometric with $H_n$. For a bounded sequence $(\xi_n)$ we hence have

$$\|\xi_m\|^2 = \|q_{m,n}^* \circ q_{m,n}(\xi_m)\|^2 + \|\xi_m - q_{m,n}^* \circ q_{m,n}(\xi_m)\|^2 = \|\xi_n\|^2 + \|\xi_m - \iota_{m,n}(\xi_n)\|^2.$$

Thus $\|\xi_m - \iota_{m,n}(\xi_n)\|$ is small when $n \leq m$ and $n$ is large. Given another bounded sequence $\eta$ in the inverse limit, we have

$$\begin{aligned}
(\xi_m|\eta_m) &= (\iota_{m,n}(\xi_n) + \xi_m - \iota_{m,n}(\xi_n)|\iota_{m,n}(\eta_n) + \eta_m - \iota_{m,n}(\eta_n)) \\
&= (\iota_{m,n}(\xi_n)|\iota_{m,n}(\eta_n)) + (\xi_m - \iota_{m,n}(\xi_n)|\iota_{m,n}(\eta_n)) + (\iota_{m,n}(\xi_n)|\eta_m - \iota_{m,n}(\eta_n)) \\
&\qquad + (\xi_m - \iota_{m,n}(\xi_n)|\eta_m - \iota_{m,n}(\eta_n)) \\
&= (\xi_n|\eta_n) + (q_{m,n}(\xi_m) - \xi_n|\eta_n) + (\xi_n|q_{m,n}(\eta_m) - \eta_n) + (\xi_m - \iota_{m,n}(\xi_n)|\eta_m - \iota_{m,n}(\eta_n)) \\
&= (\xi_n|\eta_n) + (\xi_m - \iota_{m,n}(\xi_n)|\eta_m - \iota_{m,n}(\eta_n)),
\end{aligned}$$

using that $\iota_{m,n}$ is an isometry, and that $q_{m,n}(\xi_m) = \xi_n$, and similarly for $\eta$. As the 2nd term is small, we see that the sequence $((\xi_n|\eta_n))$ is Cauchy and so converges. We may hence define an inner-product on the subspace of bounded sequences by $(\xi|\eta) = \lim_n (\xi_n|\eta_n)$. This is an inner-product, as $(\xi|\xi) = \lim_n \|\xi_n\|$ which in an increasing limit, and so equals 0 only when $\xi_n = 0$ for all $n$. So $\varprojlim(H_n, q_n)$ is an inner-product space.

Given $\xi = (\xi_n)$ in $\varprojlim(H_n, q_n)$, for large $N$, define $\eta = (\eta_n)$ by setting $\eta_n = \xi_n$ for $n \leq N$ and $\eta_n = \iota_{n,N}(\xi_N)$ for $n > N$. Then $q_n(\eta_{n+1}) = \eta_n$ for $n < N$, while for $n \geq N$ we have $q_n(\eta_{n+1}) = q_n q_{n+1,N}^*(\xi_N) = q_n q_n^* q_{n-1}^* \cdots q_N^*(\xi_N) = q_{n,N}^*(\xi_N) = \eta_n$. Thus $\eta$ is in the inverse limit, is bounded, and for $n \geq N$ we have $\|\xi_n - \eta_n\| = \|\xi_n - \iota_{n,N}(\xi_N)\|$ is small. We conclude that the collection of such $\eta$ is dense in $\varprojlim(H_n, q_n)$; notice also that $\eta \in H_\infty$.

Let $\xi = (\xi_n) \in H_\infty$, say $\xi_{n+1} = \iota_n(\xi_n)$ for $n \geq N$. Adjust $\xi$ by setting $\xi_n = q_{N,n}(\xi_N)$ for $n \leq N$, and notice that this does not change the equivalence class that $\xi$ defines in $\varinjlim(H_n, \iota_n)$. For $n < N$ we have $q_n(\xi_{n+1}) = q_n q_{N,n+1}(\xi_N) = q_{N,n}(\xi_N) = \xi_n$, while for $n \geq N$ we have $q_n(\xi_{n+1}) = \iota_n^* \iota_n(\xi_n) = \xi_n$. Hence $\xi \in \varprojlim(H_n, q_n)$ as of course $(\xi_n)$ is bounded, and notice that the norms of $\xi$ in both spaces is the same. If $\xi' = (\xi'_n) \in H_\infty$ agrees with $\xi$ in $\varinjlim(H_n, \iota_n)$ then $\xi_n = \xi'_n$ for sufficiently large $n$, and so we obtain the same sequence in $\varprojlim(H_n, q_n)$. We hence have an isometry $U\colon \varinjlim(H_n, q_n) \to \varprojlim(H_n, q_n)$. The previous paragraph shows that $U$ has dense range, and so $U$ is a unitary.

Thus the inverse and inductive limits agree; in particular, $\varprojlim(H_n, q_n)$ is a Hilbert space. Consider $q^*_{\infty,n}\colon H_N \to \varprojlim(H_n, q_n)$ which satisfies, for $\xi \in \varprojlim(H_n, q_n), \eta_n \in H_n$,

$$\begin{aligned}(\xi | q^*_{\infty,n}(\eta_n)) &= (\xi_n|\eta_n) = \lim_n (q_{m,n}(\xi_m)|\eta_n) = \lim_m (\xi_m|\iota_{m,n}(\eta_n)) \\ &= (\xi|(q_{n,1}(\eta_n), \cdots, q_{n-1}(\eta_n), \eta_n, \iota_n(\eta_n), \iota_{n+1,n}(\eta_n), \cdots))\end{aligned}$$

This shows that $q^*_{\infty,n}(\eta_n) = (q_{n,1}(\eta_n), \cdots, q_{n-1}(\eta_n), \eta_n, \iota_n(\eta_n), \iota_{n+1,n}(\eta_n), \cdots)$, where we can again check that this sequence is in $\varprojlim(H_n, q_n)$. As adjusting a finite number of elements doesn't change a sequence in $\varinjlim(H_n, q_n)$, we have shown that $U\iota_{\infty,n} = q_{\infty,n}U$. The inverse limit has the advantage that we obtain a concrete description of the entire Hilbert space, while the inductive limit only constructs a dense subspace. In particular, we have the following.

**Proposition 7.1.** *Let $(H_n, \iota_n)$ be an inductive sequence of Hilbert spaces. For each $\xi \in \varinjlim(H_n, \iota_n)$ there is a unique bounded sequence $(\xi_n)$ with $\xi_n \in H_n$ for each $n$, with $\iota^*_n(\xi_{n+1}) = \xi_n$ for each $n$, and such that $\iota_{\infty,n}(\xi_n) \to \xi$.*

*Conversely, if $(\xi_n)$ is a sequence with $\xi_n \in H_n$ for each $n$, and "Cauchy" in the sense that for $\epsilon > 0$ there is $N$ such that $\|\iota_{m,n}(\xi_n) - \xi_m\| < \epsilon$ for $m \geq n \geq N$, then there is $\xi \in \varinjlim(H_n, \iota_n)$ with $\iota_{\infty,n}(\xi_n) \to \xi$.*

*Proof.* Given $\xi \in \varinjlim(H_n, \iota_n)$, let $(\xi_n) = U(\xi) \in \varprojlim(H_n, \iota_n)$. Then

$$\begin{aligned}U\iota_{\infty,n}(\xi_n) &= q^*_{\infty,n}U(\xi) = U(q_{n,1}(\xi_n), \cdots, q_{n,n-1}(\xi_n), \xi_n, \cdots, \iota_{m,n}(\xi_n), \cdots) \\ &= (q_{n,1}(\xi_n), \cdots, q_{n,n-1}(\xi_n), \xi_n, \cdots, \iota_{m,n}(\xi_n), \cdots),\end{aligned}$$

and so

$$\|U\iota_{\infty,n}(\xi_n) - U\xi\| = \lim_m \|\iota_{m,n}(\xi_n) - \xi_m\|,$$

which as observed before is small, if $n$ is large. So $U\iota_{\infty,n}(\xi_n) \to U\xi$ and hence $\iota_{\infty,n}(\xi_n) \to \xi$.

For the second claim, we again work in $\varprojlim(H_n, \iota_n)$. For each $n$ define $\eta^{(n)} = q^*_{\infty,n}(\xi_n) \in \varprojlim(H_n, \iota_n)$, so by definition, $\eta^{(n)}_k = \iota_{k,n}(\xi_n)$ for $k \geq n$. Given $\epsilon > 0$ select $N$ as in the statement, so for $m \geq n \geq N$ and $k \geq m$ we have that

$$\|\eta^{(m)}_k - \eta^{(n)}_k\| = \|\iota_{k,m}(\xi_m) - \iota_{k,n}(\xi_n)\| \leq \|\iota_{k,m}(\xi_m) - \xi_k\| + \|\xi_k - \iota_{k,n}(\xi_n)\| < 2\epsilon.$$

Hence $\|\eta^{(m)} - \eta^{(n)}\| \leq 2\epsilon$ and so $(\eta^{(n)})$ is Cauchy, so convergent, in $\varprojlim(H_n, \iota_n)$. Thus $\iota_{\infty,n}(\xi_n)$ converges in $\varinjlim(H_n, \iota_n)$. $\square$

### 7.3 Application to infinite tensor products

Let $H$ be a Hilbert space with unit vector $\xi_0 \in H$. Define $H_n = H^{\otimes n}$ and define connecting maps $\iota_n\colon H_n \to H_{n+1}; u \mapsto u \otimes \xi_0$. By definition, the infinite tensor product is $(H, \xi_0)^{\otimes\infty} = \varinjlim(H_n, \iota_n)$. The above proposition shows that we can regard this space as the limit points of sequences $u_n \in H^{\otimes n}$ which are Cauchy in the sense that $u_n \otimes \xi_0^{\otimes(m-n)} - u_m$ is small for $m \geq n$ large.

Suppose that $u_n = \xi_1 \otimes \cdots \otimes \xi_n$ for each $n$. If $\lim_n \prod_{i\leq n} \|\xi_i\| = 0$ then $\|u_n\| \to 0$ and so $\|u_n \otimes \xi_0^{\otimes(m-n)} - u_m\| \leq \|u_n\| + \|u_m\|$ is small when $n, m$ are large. Then of course the limit in $(H,\xi_0)^{\otimes\infty}$ is 0. Similarly, if $\lim_n \prod_{i\leq n}\|\xi_i\| = \infty$ then $\iota_{\infty,n}(u_n)$ is unbounded, and so cannot converge.

So we assume that $0 < \lim_n \prod_{i\leq n}\|\xi_i\| < \infty$. For $m \geq n$ we have

$$\begin{aligned}
&\|\xi_1 \otimes \cdots \otimes \xi_n \otimes \xi_0 \otimes \cdots \otimes \xi_0 - \xi_1 \otimes \cdots \otimes \xi_m\|^2 \\
&= \|\xi_1 \otimes \cdots \otimes \xi_n \otimes (\xi_0 \otimes \cdots \otimes \xi_0 - \xi_{n+1} \otimes \cdots \otimes \xi_m)\|^2 \\
&= \prod_{i=1}^{n} \|\xi_i\|^2 \Big(1 + \prod_{i=n+1}^{m} \|\xi_i\|^2 - 2\Re \prod_{i=n+1}^{m} (\xi_0|\xi_i)\Big).
\end{aligned}$$

The first term is bounded away from 0 and $\infty$, so we look at the term in brackets, which must be small for large $n$.

We now make some remarks about [10, Lemma XIV.1.7], in particular giving a counter-example to one direction of this claim. We start by looking at some elementary analysis results about infinite products, for which we follow the lecture notes [8], although we allow an infinite product to converge to 0.

**Lemma 7.2.** *Let $(z_n)$ be a sequence in $\mathbb{C}$. The following are equivalent:*

1. $\lim_n \prod_{i\leq n} z_i$ *exists and is non-zero;*
2. *no $z_n$ is zero, and for each $\epsilon > 0$ there is $N$ such that for each $m \geq n \geq N$ we have $|1 - \prod_{i=n}^{m} z_i| < \epsilon$.*

*Proof.* Suppose $\lim_n \prod_{i\leq n} z_i \neq 0$, so no $z_i$ is 0, and setting $p_n = \prod_{i\leq n} z_i$, there is $\delta > 0$ so that $|p_n| > \delta$ for all $n$. There is $N$ so that $|p_n - p_m| < \epsilon\delta$ for each $m \geq n \geq N$. Then $|1 - \prod_{i=n+1}^{m} z_i| = |1 - p_m/p_n| = |p_n|^{-1}|p_n - p_m| < \epsilon$.

For the converse, given $\epsilon > 0$ select $N$. Then $|1-z_n| < \epsilon$ for each $n \geq N$ so $1-\epsilon < |z_n| < 1+\epsilon$. Set $q_m = z_N z_{N+1}\cdots z_m$ for $m \geq N$, so we also have $|1-q_m| < \epsilon$. In particular, we may suppose that $\epsilon < 1/2$ so that $1/2 < |q_m| < 3/2$ for all $m \geq N$. Increasing $N$ if necessary, we may suppose that for $m > n \geq N$,

$$\Big|\frac{q_m}{q_n} - 1\Big| = \Big|\prod_{i=n+1}^{m} z_i - 1\Big| < \tfrac{2}{3}\epsilon.$$

Then $|q_m - q_n| = |q_n||q_m/q_n - 1| < \frac{2}{3}\epsilon|q_n| < \frac{3}{2}\frac{2}{3}\epsilon = \epsilon$. Hence $(q_n)_{n\geq N}$ is Cauchy and so converges. As $\prod_{i=1}^{n} z_i = q_n \prod_{i=1}^{N-1} z_i$, also the infinite product converges. As $z_i \neq 0$ for each $i$, and as $|q_n| > 1/2$, we also see that the limit is non-zero. □

**Proposition 7.3.** *Let $(a_n)$ be a sequence of positive reals. Then $\lim_n \prod_{i\leq n}(1+a_i)$ converges if and only if $\sum_n a_n$ converges.*

*Proof.* By hypothesis, both series are increasing, and so converge if and only if they are bounded above. As $x > 0$ implies $1 + x < e^x$ we see that $\sum_{i\leq n} a_i \leq \prod_{i\leq n}(1 + a_i) \leq \exp(\sum_{i\leq n} a_i)$ and the result follows. □

Note of course that if $\lim_n \prod_{i\leq n}(1 + a_i)$ converges then its limit is $> 1$. We next come to a notion of “absolute convergence”.

**Corollary 7.4.** *Let $(a_n)$ be a sequence in $\mathbb{C}$ such that $\sum_n |a_n|$ converges. Then $\lim_n \prod_{i\leq n}(1+a_i)$ converges (and is non-zero if $a_i \neq -1$ for all $i$).*

*Proof.* Note that

$$\big|(1+a_n)(1+a_{n+1})\cdots(1+a_m)-1\big| = \big|a_n+\cdots+a_m+\sum_{n\leq i<j\leq m} a_ia_j+\cdots\big|$$
$$\leq |a_n|+\cdots+|a_m|+\sum_{n\leq i<j\leq m} |a_ia_j|+\cdots = (1+|a_n|)(1+|a_{n+1}|)(1+|a_m|)-1,$$

and so Lemma 7.2 shows that $\lim_n \prod_{i\leq n}(1+|a_n|)$ converges implies that $\lim_n \prod_{i\leq n}(1+a_n)$ converges (and is non-zero if $a_i\neq -1$ for all $i$). If $\sum_n |a_n|$ converges, then by Proposition 7.3, $\lim_n \prod_{i\leq n}(1+|a_i|)$ converges, and the result follows. $\square$

We now come to one direction of [10, Lemma XIV.1.7].

**Proposition 7.5.** *Let $(\xi_n)$ be a sequence in $H$, and set $u_n=\xi_1\otimes\cdots\otimes\xi_n$ for each $n$. If $\lim_n \|u_n\| = \lim_n \prod_{i\leq n}\|\xi_i\|$ converges and is non-zero, and $\sum_n |1-(\xi_n|\xi_0)|<\infty$, then $(u_n)$ converges in the infinite tensor product.*

*Proof.* From the discussion above, given $\epsilon>0$ we seek $N$ so that if $m\geq n\geq N$ we have

$$1+\prod_{i=n+1}^m \|\xi_i\|^2 - 2\Re\prod_{i=n+1}^m (\xi_0|\xi_i) < \epsilon.$$

Set $a_n = 1-(\xi_0|\xi_n)$ so by hypothesis, $\sum_n |a_n|<\infty$ and so Corollary 7.4 tells us that $\lim_n \prod_{i\leq n}(1+a_n) = \lim_n \prod_{i\leq n}(\xi_0|\xi_i)$ converges and is non-zero. By Lemma 7.2 there is $N$ so that for $m\geq n\geq N$ we have both

$$\Big|1-\sum_{i=n+1}^m (\xi_0|\xi_i)\Big| < \epsilon, \qquad \Big|1-\sum_{i=n+1}^m \|\xi_i\|^2\Big| < \epsilon.$$

For $z\in\mathbb{C}$, as $|2-2\Re z| = |2-z-\overline{z}| \leq |1-z|+|1-\overline{z}|$, we see that $|2-2\Re\sum_{i=n+1}^m(\xi_0|\xi_i)|<2\epsilon$, and so

$$\Big|1+\prod_{i=n+1}^m \|\xi_i\|^2 - 2\Re\prod_{i=n+1}^m (\xi_0|\xi_i)\Big| = \Big|\sum_{i=n+1}^m \|\xi_i\|^2 - 1 + 2 - 2\Re\prod_{i=n+1}^m (\xi_0|\xi_i)\Big| < 3\epsilon,$$

as we want. $\square$

The converse does not seem to hold, as we now show. We continue to follow [8].

**Proposition 7.6.** *Let $a_n\geq 0$ for each $n$. If $\sum_n a_n$ converges then $\lim_n \prod_{i\leq n}(1-a_i)$ converges. If $\lim_n \prod_{i\leq n}(1-a_i)$ converges then either the limit is $0$ or $\sum_n a_n$ converges.*

*Proof.* If $\sum_n a_n$ converges then Corollary 7.4 tells us that $\lim_n \prod_{i\leq n}(1-a_i)$ converges.

Conversely, suppose $\lim_n \prod_{i\leq n}(1-a_i)$ converges and is non-zero, so by Lemma 7.2, in particular, $1-(1-a_i)=a_i\to 0$, so there is $N$ such that $a_n<1$ for $n\geq N$. Towards a contraction, suppose that $\sum_n a_n$ diverges, so also $\lim_n \prod_{i\leq n}(1+a_i)$ diverges. As $(1-a_i)(1+a_i) = 1-a_i^2\leq 1$ for $i\geq N$, for $m\geq n\geq N$ we have

$$\prod_{i=n}^m(1-a_i)\prod_{i=n}^m(1+a_i) = \prod_{i=n}^m(1-a_i^2)\leq 1 \qquad\Longrightarrow\qquad \prod_{i=n}^m(1-a_i)\leq\Big(\prod_{i=n}^m(1+a_i)\Big)^{-1}.$$

Thus $\prod_{i=n}^m(1-a_i)$ is arbitrarily small, but by Lemma 7.2, is also arbitrarily close to 1, giving the claimed contraction. $\square$

We now come to our counter-example. Let

$$(a_n) = \Big(1 - \tfrac{1}{\sqrt{2}}, 1 + \tfrac{1}{2} + \tfrac{1}{\sqrt{2}}, 1 - \tfrac{1}{\sqrt{3}}, 1 + \tfrac{1}{3} + \tfrac{1}{\sqrt{3}}, \cdots\Big).$$

Then $\sum_n |1 - a_n| \geq \sum_n n^{-1/2} = \infty$. However, we claim that $\lim_n \prod_{i\leq n} a_i$ converges, and is non-zero. We have

$$\big(1 - \tfrac{1}{\sqrt{k}}\big)\big(1 + \tfrac{1}{k} + \tfrac{1}{\sqrt{k}}\big) = \tfrac{1}{k}\big(1 - \tfrac{1}{\sqrt{k}}\big) + 1 - \tfrac{1}{k} = 1 - \tfrac{1}{k\sqrt{k}}.$$

By Proposition 7.6, $\lim_n \prod_{i\leq n}(1 - \frac{1}{i\sqrt{i}})$ converges and is non-zero, and so by Lemma 7.2, for $\epsilon > 0$ there is $N$ so that if $m \geq n \geq N$ we have

$$\Big|1 - \prod_{i=2n-1}^{2m} a_i\Big| < \epsilon.$$

Then

$$\sum_{i=2n}^{2m} a_i = \big(1 - \tfrac{1}{\sqrt{n+1}}\big)^{-1} \prod_{i=2n-1}^{2m} a_i, \qquad \sum_{i=2n-1}^{2m+1} = \big(1 + \tfrac{1}{\sqrt{m}} + \tfrac{1}{\sqrt{m+1}}\big) \prod_{i=2n-1}^{2m} a_i,$$

which are then both within $2\epsilon$ of 1, if $N$ is large enough, and similarly for the final case of $\sum_{i=2n}^{2m+1} a_i$. By Lemma 7.2 applied in the other direction, we conclude that $\lim_n \prod_{i\leq n} a_i$ converges and is non-zero.

With our Hilbert space example from the previous section, set $\xi_n = a_n\xi_0$ for each $n$. Then

$$\xi_1 \otimes \cdots \otimes \xi_n = a_1 a_2 \cdots a_n \xi_0 \otimes \cdots \otimes \xi_0,$$

and so for $m \geq n$,

$$\begin{aligned} &\|\xi_1 \otimes \cdots \otimes \xi_n \otimes \xi_0 \otimes \cdots \otimes \xi_0 - \xi_1 \otimes \cdots \otimes \xi_m\| \\ &= \Big|\prod_{i=1}^{n} a_i - \prod_{i=1}^{m} a_i\Big| = \Big|\prod_{i=1}^{n} a_i\Big|\Big|1 - \prod_{i=n+1}^{m} a_i\Big|. \end{aligned}$$

As $\lim_n \prod_{i\leq n} a_i$ converges, this is arbitrarily small if $n$ is large enough. So the sequence $(\xi_1 \otimes \cdots \otimes \xi_n)$ converges in the infinite tensor product. However, $\sum_n |1 - (\xi_n|\xi_0)| = \sum_n |1 - a_n|$ diverges, contrary to [10, Lemma XIV.1.7].

Finally, we remark that we can use the relationsation of $(H, \xi_0)^{\otimes\infty}$ as the inverse limit to write down infinite tensors directly. We have that

$$\iota_n^*(\xi_1 \otimes \cdots \otimes \xi_{n+1}) = (\xi_0|\xi_{n+1})\xi_1 \otimes \cdots \otimes \xi_n.$$

Hence $(u_n) = (\xi_1 \otimes \cdots \otimes \xi_n)$ is in the inverse limit, we might write $\xi_1 \otimes \xi_2 \otimes \cdots \in (H, \xi_0)^{\otimes\infty}$, if and only if $(\xi_0|\xi_i) = 1$ for all $i > 1$, and $\lim_n \prod_{i\leq n} \|\xi_i\| < \infty$.

The boundedness condition implies that $\|\xi_i\| \to 1$, and so $\xi_i = \xi_0 + \eta_i$ say, where $\eta_i \in \{\xi_0\}^\perp$ and $\|\eta_i\| \to 0$. Then $\|\xi_i\|^2 = 1 + \|\eta_i\|^2$, and so $\lim_n \prod_{i\leq n} \|\xi_i\| < \infty$ if and only if $\lim_n \prod_{i\leq n}(1 + \|\eta_i\|^2) < \infty$, which by Proposition 7.3 is equivalent to $\sum_i \|\eta_i\|^2 < \infty$.

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