Cohen–Hewitt Factorisation: A brief history

Matthew Daws

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Abstract

We review some of the original literature around the Cohen–Hewitt factorisation theorem.

We fix some notation and terminology. Let A be a Banach algebra. We will always assume that the product is contractive: $\|ab\| \leq \|a\|\|b\|$ for $a, b \in A$. A *bounded left approximate identity* (blai) for A is a bounded net (e_i) with $\lim_i \|e_i a - a\| = 0$ for each $a \in A$. The *bound* of (e_i) is $\limsup_i \|e_i\|$.

For us, a *left module* over A is a Banach space E which is algebraically a left module, such that the action is contractive, meaning $\|a \cdot x\| \leq \|a\|\|x\|$ for $a \in A, x \in E$. We denote by $\overline{A \cdot E}$ the closed linear span of the set $\{a \cdot x : a \in A, x \in E\}$. We say that E is essential if $E = \overline{A \cdot E}$.

1 Cohen's paper

This is [2][1] translated into slightly more modern language.

Theorem 1.1 ([2, Theorem 1])**.** *Let A be a Banach algebra with a bounded left approximate identity. For each $a \in A$ and $\epsilon > 0$ there are $b, c \in A$ with:*

1. *$a = bc$;*

2. *c belongs the the closed left ideal generated by a;*

3. *$\|a - c\| < \epsilon$.*

2 Hewitt's paper

The statement concerns, in modern language, left modules. This paper works with a Banach algebra A and a left module E such that A possesses a blai (e_i) which additionally satisfies $\lim_i \|e_i \cdot x - x\| = 0$ for each $x \in E$. We see that this is equivalent to A having some blai, and E being essential.

Theorem 2.1 ([6, Theorem 2.5])**.** *Let A be a Banach algebra with a blai of bound C, and let E be an essential left module over A. For each $x \in E$ and $\epsilon > 0$ there is $a \in A, y \in E$ with:*

1. *$x = a \cdot y$;*

2. *y belongs to the closure of $\{b \cdot x : b \in A\}$;*

3. *$\|x - y\| \leq \epsilon$;*

4. *$\|a\| \leq C$.*

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[1]A small rant: Why does Duke insist that one has a subscription to access a paper from ≥ 70 years ago?

3 Curtis–Figá-Talamanca paper

We explore now the main statement of [3]. This is stated in a rather different way to Theorem 2.1, but we shall show how they are similar. For a Banach space X we write $\mathcal{B}(X)$ for the algebra of bounded linear maps on X.

Theorem 3.1. *Let W, X be Banach spaces and $\sigma : W \to \mathcal{B}(X)$ a bounded linear map. Let $F \subseteq W$ be a bounded set with dense linear span in W, and set $E = \sigma(F)$.*

Suppose that for each $\{e_1, \cdots, e_n\} \subseteq E$ and $\epsilon > 0$ there is $e \in E$ with $\|ee_i - e_i\| < \epsilon$. Then $W \cdot X = \{\sigma(w)(x) : w \in W, x \in X\}$ is a closed subspace of X, and for each $y \in W \cdot X, \delta > 0$ there exists $z \in Y, w \in W$ with $\|y - z\| < \delta$ and $y = \sigma(w)(z)$.

This is stated in (from a "modern perspective") a somewhat strange way. However, a simple approximation shows that for each finite set $T_1, \cdots, T_k \in \operatorname{lin} \sigma(W)$ there is $e \in E$ with $\|eT_i - T_i\| < \epsilon$ for each i. Indeed, just approximate each T_i by a (finite) linear combination of elements of E. Now given a product $T = T_1 \cdots T_k$ in the algebra generated by $\sigma(W)$, set $S = T_2 \cdots T_k$ and find $e \in E$ with $\|eT_1 - T_1\| < \epsilon \|S\|^{-1}$ to show that $\|eT - T\| \leq \|eT_1 - T_1\|\|S\| < \epsilon$. Arguing in this way, we find that A, the Banach algebra generated by $\sigma(W)$, has a blai (bounded as F is assumed bounded, so E must be).

The conclusion is also different, as we look simply at $W \cdot X$ not $A \cdot X$. Set $Y = W \cdot X \subseteq X$. Clearly $\sigma(w)(X) \subseteq Y$ for all $w \in W$, and so $\sigma(w)(Y) \subseteq Y$ for each $w \in W$, so $\sigma(w_1)\sigma(w_2)(Y) \subseteq \sigma(w_1)(Y) \subseteq Y$, and so forth. Thus the algebra generated by $\sigma(W)$ maps Y into Y (of course, once we know that Y is a subspace!) Thus if we set Y_0 to be the closure of Y, then Y_0 is a left A-module, essential because $W \subseteq A$ and A has a blai. However, the conclusion of Theorem 3.1 seems formally stronger, as we find $w \in W$, not in A.

Conversely, with A a Banach algebra and E a left module, with simply set $X = E$ and let σ be the left action map $A \to \mathcal{B}(E)$. Thus Theorem 3.1, absence the choice of z in the closure of $A \cdot y$, gives exactly Theorem 2.1.

In conclusion, while Theorem 3.1 seems a little more general, it is very clearly related to the Cohen–Hewitt result (and in fact, as the authors state in [3], none of their applications seems to use this extra generality).

4 Doran–Wichmann book

This is [5] which undertakes a careful (and in our opinion extremely readable) survey of the state of the art, circa 1979. The factorisation result is considered in [5, Section 16] and the statement is identical to Theorem 2.1. No references are given in the text, but extensive notes can be found starting in page 248. From this, it does appear to be reasonable to call Theorem 2.1 the Cohen–Hewitt Factorisation Theorem.

5 Further refinements

In [4, Theorem 2.9.24] a refinement of Theorem 2.1 is presented:

Theorem 5.1 ([4, Theorem 2.9.24])**.** *Let A be a Banach algebra with a blai of bound C, and let E be an essential left module over A. Let (α_n) be an increasing, unbounded sequence in $(1, \infty)$. For each $x \in E$ and $\epsilon > 0$ there is $a \in A$ and a sequence $(y_n) \in E$ with:*

1. $x = a^n \cdot y_n$ for each n;

2. $\|x - y_n\| \leq \epsilon$;

3. $\|y_n\| \leq \alpha_n^n \|x\|$;

4. $\|a\| \leq C$.

Then [4, Corollary 2.9.26] gives a statement equivalent to Theorem 2.1. The above is attributed to Cohen and Hewitt, and also [1].

Palmer's book [7] considers factorisation results in Section 5.2, using the language of representations instead of modules. The results stated are equivalent to those discussed above.

6 Further reading

A detailed history of the theorem can be found in [8, pages 1033-34].

References

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