Notes on quasi-invariant measures

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Abstract

We give a summary of quasi-invariant measures on (left) coset spaces of locally compact groups. We carefully show how to translate these results to right coset spaces.

1 Introduction

Let G be a locally compact (for me, always Hausdorff) group. We write elements of G as s, t etc. Recall that the left Haar measure is a Radon measure μ on G which is left invariant: $\lambda(sE) = \lambda(E)$ for Borel E and $s \in G$. We write the resulting integral as

$$\int_G f(s) \ d\lambda(s) = \int_G d(s) \ ds,$$

that is, suppress λ if context allows. The left Haar measure is unique up to a scalar.

For each $s \in G$ the measure $E \mapsto \lambda(Es)$ is left-invariant and so there is a scalar $\nabla(s) \in (0, \infty)$ with $\lambda(Es) = \nabla(s)\lambda(E)$. This function is continuous, and is the *modular function*.

We fix a right Haar measure on G by defining $\rho(E) = \lambda(E^{-1})$ where of course $E^{-1} = \{s^{-1} : s \in E\}$. Then $d\rho(s) = \nabla(s^{-1})d\lambda(s)$, that is,

$$\int_G f(s)\nabla(s^{-1}) \ d\lambda(s) = \int_G f(s) \ d\rho(s),$$

for, say, continuous compactly supported f. We also have that

$$\int_G f(s^{-1}) \ ds = \int_G f(s)\nabla(s^{-1}) \ ds, \qquad \int_G f(s) \ d\rho(s) = \int_G f(s)\nabla(s) \ d\rho(s).$$

Some additional properties of the Haar measure are:

- $\lambda(U) > 0$ for any non-empty open subset U;
- $\lambda(K) < \infty$ for any compact K.

Here we have followed [Fol]; see also [HR]. We remark that different sources can use slightly different conventions.

2 Quasi-invariant measures

Here I am following [Fol]; for further details and bibliographical comments see this book. In this section, fix a locally compact group G and a closed subgroup H. Then the left coset space G/H, with the quotient topology, is locally compact. Let $q : G \to G/H$ be the quotient map. Define $P : C_{00}(G) \to C_{00}(G/H)$ a map between the spaces of compactly supported continuous functions, by

$$(Pf)(sH) = \int_H f(su) \ du.$$

This is well-defined as du is left-invariant on H. Given $f \in C_{00}(G/H)$ the map $f \circ q$ is continuous and $P((f \circ q)g) = fP(g)$.

Proposition 2.1. *P maps $C_{00}(G)$ into $C_{00}(G/H)$. Indeed, for $g \in C_{00}(G/H)$ there is $f \in C_{00}(G)$ with $Pf = g$ and $q(\operatorname{supp} f) = \operatorname{supp} g$. If $g \geq 0$ we can choose $f \geq 0$.*

We are interested in when G/H carries a left invariant measure.

Theorem 2.2. *There is a left G-invariant Radon measure μ on G/H if and only if ∇_H is the restriction of ∇_G to H. If so, μ is unique up to a scalar, and we can choose μ so that*

$$\int_G f(s)\ ds = \int_{G/H} Pf\ d\mu = \int_{G/H} \int_H f(su)\ du\ d\mu(sH)$$

for $f \in C_{00}(G)$.

We remark that if H is normal, then this holds and μ is the Haar measure on the group G/H. The relation is known as Weil's formula.

In general, we say that a Radon measure μ on G/H is *quasi-invariant* if, defining $\mu_s(E) = \mu(sE)$ for $s \in G$, the measures μ_s are all equivalent (that is, mutually absolutely continuous). In fact, we work with the stronger condition that μ is *strongly quasi-invariant* if there is a continuous function $\lambda : G \times (G/H) \to (0, \infty)$ such that $d\mu_s(u) = \lambda(s, u) d\mu(u)$.

Definition 2.3. A *rho-function* for the pair (G, H) is a continuous function $\rho : G \to (0, \infty)$ such that

$$\rho(su) = \frac{\nabla_H(u)}{\nabla_G(u)} \rho(s) \qquad (s \in G, u \in H).$$

Proposition 2.4. *For any locally compact group G and closed subgroup H, there is a rho-function for (G, H).*

Theorem 2.5. *Given any rho-function ρ for the pair (G, H), there is a strongly quasi-invariant measure μ on G/H with*

$$\int_{G/H} Pf\ d\mu = \int_G f(s)\rho(s)\ ds \qquad (f \in C_{00}(G)),$$

and with

$$\frac{d\mu_s}{d\mu}(tH) = \frac{\rho(st)}{\rho(t)} \qquad (s, t \in G).$$

We remark that a quasi-invariant measure μ on G/H satisfies that $\mu(U) > 0$ for any non-empty open U.

Theorem 2.6. *Every strongly quasi-invariant measure on G/H arises from a rho-function as above, and all such measures are strongly equivalent.*

An immediate corollary is that all strongly quasi-invariant measures on G/H have the same null sets. In fact, this is true for all quasi-invariant measures. For the following, recall that a subset F is *locally Borel* if $E \cap F$ is Borel whenever F is Borel with $\mu(F) < \infty$, and a locally Borel set E is *locally null* if $\mu(E \cap F) = 0$ whenever F is Borel with $\mu(F) < \infty$.

Proposition 2.7. *All strongly quasi-invariant measures on G/H have the same locally null sets.*

Theorem 2.8. *A set $E \subseteq G/H$ is locally null (with respect to any quasi-invariant measure on G/H) if and only if $q^{-1}(E) \subseteq G$ is locally null (with respect to the Haar measure on G).*

There is also a useful summary in [BHV, Appendix B].

3 For right coset spaces

In this section, we provide some formulas for quasi-invariant measures on $H\backslash G$ the right coset space. Let G^{op} be the group G with the same inverse, but the reversed product. We write $s^{\mathrm{op}} = s^o$ for $s \in G$ considered as an element of G^{op}, so that $s^o t^o = (ts)^o$ and $(s^o)^{-1} = (s^{-1})^o$. The left Haar measure on G^{op}, say $\lambda_{G^{\mathrm{op}}}$, is just the right Haar measure on G, and so

$$\int_{G^{\mathrm{op}}} f^o(t)\ dt = \int_{G^{\mathrm{op}}} f^o(s^o)\ d\lambda_{G^{\mathrm{op}}}(s^o) = \int_G f(s)\nabla(s^{-1})\ d\lambda_G(s) = \int_G f(s^{-1})\ ds.$$

for $f \in C_{00}(G)$, say, where $f^o(s^o) = f(s)$ so $f^o \in C_{00}(G^{\mathrm{op}})$. It follows that $G \to G^{\mathrm{op}}, s \mapsto (s^{-1})^o$ is a group isomorphism which preserves the measure.

Lemma 3.1. *The modular function on G^{op} is $\nabla_{G^{\mathrm{op}}}(s^o) = \nabla_G(s^{-1})$.*

Proof. For a Borel $E \subseteq G$ let $E^o = \{s^o : s \in E\}$ a Borel subset of G^{op}. For $s^o \in G^{\mathrm{op}}$ we have that $\nabla_{G^{\mathrm{op}}}(s^o)\lambda_{G^{\mathrm{op}}}(E^o) = \lambda_{G^{\mathrm{op}}}(E^o s^o)$. Let f be the Borel function on G with $f^o = \chi_{E^o s^o}$ so $f(t) = 1$ if and only if $t^o \in E^o s^o$, equivalently $t \in sE$, so $f = \chi_{sE}$. Thus

$$\nabla_{G^{\mathrm{op}}}(s^o)\lambda_{G^{\mathrm{op}}}(E^o) = \int_{G^{\mathrm{op}}} \chi_{E^o s^o} = \int_G \chi_{sE}(t)\nabla_G(t^{-1})\ dt = \int_G \chi_E(t)\nabla_G((st)^{-1})\ dt,$$

by left-invariance. This in turn equals

$$\nabla_G(s^{-1})\int_G \chi_E(t)\nabla_G(t^{-1})\ dt = \nabla_G(s^{-1})\int_{G^{\mathrm{op}}} \chi_{E^o}.$$

Hence $\nabla_{G^{\mathrm{op}}}(s^o) = \nabla_G(s^{-1})$. $\square$

Given a closed subgroup H of G, notice that H^{op} is identified with $\{s^o : s \in H\}$, a closed subgroup of G^{op}.

Lemma 3.2. *A function $\rho' : G^{\mathrm{op}} \to (0,\infty)$ is a rho-function for $(G^{\mathrm{op}}, H^{\mathrm{op}})$ if and only if $\rho(s) = \rho'((s^{-1})^o)$ defines a rho-function for (G, H).*

Proof. Let $s \in G, u \in H$. We calculate that $\rho(su) = \rho'((s^{-1})^o(u^{-1})^o)$ while

$$\frac{\nabla_H(u)}{\nabla_G(s)}\rho(s) = \frac{\nabla_H(u)}{\nabla_G(s)}\rho'((s^{-1})^o).$$

So ρ is a rho-function on G if and only if

$$\rho'(s^o u^o) = \frac{\nabla_H(u^{-1})}{\nabla_G(s^{-1})}\rho'(s^o) = \frac{\nabla_{H^{\mathrm{op}}}(u^o)}{\nabla_{G^{\mathrm{op}}}(s^o)}\rho'(s^o) \qquad (s^o \in G^{\mathrm{op}}, u^o \in H^{\mathrm{op}}),$$

that is, ρ' is a rho-function for $(G^{\mathrm{op}}, H^{\mathrm{op}})$. $\square$

The map $G \to G^{\mathrm{op}}; s \mapsto s^o$ is an anti-isomorphism of groups which maps H to H^{op}. Furthermore, a right coset Hs is mapped to $\{(us)^o : u \in H\} = \{s^o v : v \in H^{\mathrm{op}}\} = s^o H^{\mathrm{op}}$ a left coset. Thus this map drops to a well-defined map $H\backslash G \to G^{\mathrm{op}}/H^{\mathrm{op}}; Hs \mapsto s^o H^{\mathrm{op}}$. This map is a homeomorphism by the definition of the quotient topology.

Let μ^{op} be a strongly quasi-invariant measure on $G^{\mathrm{op}}/H^{\mathrm{op}}$ with respect to a rho-function ρ' for $(G^{\mathrm{op}}, H^{\mathrm{op}})$. Let ρ be the associated rho-function for (G, H). Thus

$$\int_{G^{\mathrm{op}}/H^{\mathrm{op}}} P^{\mathrm{op}} f^o\ d\mu^{\mathrm{op}} = \int_{G^{\mathrm{op}}} f^o(s^o)\rho'(s^o)\ ds^o \qquad (f \in C_{00}(G)). \tag{1}$$

Now, $P^{\mathrm{op}} f^o \in C_{00}(G^{\mathrm{op}}/H^{\mathrm{op}})$ is

$$(P^{\mathrm{op}} f^o)(s^o H^{\mathrm{op}}) = \int_{H^{\mathrm{op}}} f^o(s^o t^o)\ dt^o = \int_{H^{\mathrm{op}}} f^o((ts)^o)\ dt^o = \int_H f(t^{-1}s)\ dt. \tag{2}$$

Define a measure μ on $H\backslash G$ by $\mu(E) = \mu^{\mathrm{op}}(E^o)$. Given a Borel $E \subseteq H\backslash G$ let $E^o \subseteq G^{\mathrm{op}}/H^{\mathrm{op}}$ be the image (so $Hs \in E$ if and only if $s^o H^{\mathrm{op}} \in E^o$). Extend this notation to $F \in C_{00}(H\backslash G)$, so that $F(Hs) = F^o(s^o H^{\mathrm{op}})$. Consider now $\mu^{\mathrm{op}}_{s^o}$ which satisfies $\mu^{\mathrm{op}}_{s^o}(E^o) = \mu^{\mathrm{op}}(s^o E^o) = \mu^{\mathrm{op}}((Es)^o) = \mu(Es)$. We have by Theorem 2.5,

$$\mu(Es) = \mu^{\mathrm{op}}_{s^o}(E^o) = \int_{G^{\mathrm{op}}/H^{\mathrm{op}}} \chi_{E^o} \, d\mu^{\mathrm{op}}_{s^o} = \int_{G^{\mathrm{op}}/H^{\mathrm{op}}} \chi_{E^o} \frac{d\mu^{\mathrm{op}}_{s^o}}{d\mu^{\mathrm{op}}} \, d\mu^{\mathrm{op}}$$
$$= \int_{G^{\mathrm{op}}/H^{\mathrm{op}}} \chi_{E^o}(t^o H^{\mathrm{op}}) \frac{\rho'(s^o t^o)}{\rho'(t^o)} \, d\mu^{\mathrm{op}}(t^o H^{\mathrm{op}}).$$

We remark that by the defining relation to be a rho-function, $\rho'(s^o t^o)/\rho'(t^o)$ depends only on the coset $t^o H^{\mathrm{op}}$, so defines $F^o(t^o H^{\mathrm{op}})$. Then

$$F(Ht) = \frac{\rho'(s^o t^o)}{\rho'(t^o)} = \frac{\rho(s^{-1}t^{-1})}{\rho(t^{-1})}$$

also depends only on Ht. In conclusion,

$$\mu(Es) = \int_{H\backslash G} \chi_E(Ht) \frac{\rho(s^{-1}t^{-1})}{\rho(t^{-1})} \, d\mu(Ht).$$

If we define ${}_s\mu(E) = \mu(Es)$ then

$$\frac{d_s\mu}{d\mu}(Ht) = \frac{\rho(s^{-1}t^{-1})}{\rho(t^{-1})} \tag{3}$$

Given $f \in C_{00}(G)$ define $P'f \in C_{00}(H\backslash G)$ by integrating over the right Haar measure of H, that is,

$$(P'f)(Hs) = \int_H f(ts) \, d\rho_H(t),$$

where ρ_H is the right Haar measure on H. Then, from (2),

$$(P'f)^o(s^o H^{\mathrm{op}}) = (P'f)(Hs) = \int_H f(ts) \, d\rho_H(t) = \int_H f(t^{-1}s) \, dt = (P^{\mathrm{op}} f^o)(s^o H^{\mathrm{op}}). \tag{4}$$

It now follows from (1) that

$$\int_{H\backslash G} P'f \, d\mu = \int_{G^{\mathrm{op}}} f^o(s^o)\rho'(s^o) \, ds^o = \int_G f(s^{-1})\rho(s) \, ds = \int_G f(s)\rho(s^{-1}) \, d\rho_G(s), \tag{5}$$

where ρ_G is the right Haar measure on G.

In summary, we have shown the following.

Theorem 3.3. *Given a rho-function ρ for the pair (G, H) there is strongly quasi-invariant measure μ on $H\backslash G$ with*

$$\int_{H\backslash G} \int_H f(ts) \, d\rho_H(t) \, d\mu(Hs) = \int_G f(s)\rho(s^{-1}) \, d\rho_G(s),$$

and with

$$\frac{d_s\mu}{d\mu}(Ht) = \frac{\rho(s^{-1}t^{-1})}{\rho(t^{-1})}.$$

It perhaps makes sense to make a new definition.

Definition 3.4. A *right rho-function* for the pair (G, H) is a continuous function $\rho : G \to (0, \infty)$ such that

$$\rho(us) = \frac{\nabla_G(u)}{\nabla_H(u)}\rho(s) \qquad (s \in G, u \in H).$$

This definition is designed so that ρ is a right rho-function if and only if $s \mapsto \rho(s^{-1})$ is a rho-function. We hence immediately have

Theorem 3.5. *Given a right rho-function ρ for the pair (G, H) there is strongly quasi-invariant measure μ on $H\backslash G$ with*

$$\int_{H\backslash G}\int_H f(ts)\ d\rho_H(t)\ d\mu(Hs) = \int_G f(s)\rho(s)\ d\rho_G(s).$$

Further, if ${}_s\mu(E) = \mu(Es)$ for $E \subseteq H\backslash G$, then

$$\frac{d_s\mu}{d\mu}(Ht) = \frac{\rho(ts)}{\rho(t)}.$$

Furthermore, we have that

Proposition 3.6. *$E \subseteq H\backslash G$ is locally null if and only if $q^{-1}(E)$ is locally null in G.*

References

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