Tensor products of weights

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1 Introduction

Let M, N be von Neumann algebras with normal semifinite faithful weights φ, ψ. We consider two different ways to form $\varphi\otimes\psi$, and show that they are equal. We use [4] as our main reference for weight theory. Having written most of this, we realised that [3, Section 8] gives a self-contained account, and is probably the best reference.

1.1 Operator valued weights

We can define $(\varphi\otimes\mathrm{id})$ as an operator valued weight $M\bar{\otimes}N \to N$, see [4, Chapter IX, Section 4]. Indeed, letting $\widehat{N}_+$ be the extended positive cone of N, we define

$$\varphi\otimes\mathrm{id}\colon (M\bar{\otimes}N)_+ \to \widehat{N}_+; \quad (\varphi\otimes\mathrm{id})(x)(\omega) = \varphi((\mathrm{id}\otimes\omega)(x)) \qquad (\omega\in N_*^+).$$

For a given $x \geqslant 0$ set $m = (\varphi\otimes\mathrm{id})(x)$. Then m is positive homogeneous and additive. We seek to show that $m\colon N_*^+ \to [0,\infty]$ is lower semi-continuous. Let $\omega_0 \in N_*^+$, and let $\omega_i \to \omega_0$ in N_*^+, so setting $x_i = (\mathrm{id}\otimes\omega_i)(x)$, we have that $x_i \to (\mathrm{id}\otimes\omega_0)(x)$ σ-weakly in M^+. As φ is normal, $\liminf_i \varphi(x_i) \geqslant \varphi((\mathrm{id}\otimes\omega_0)(x))$, because φ is σ-weakly lower semi-continuous, see [4, Theorem VII.1.11]. That is, $\liminf_i m(\omega_i) \geqslant m(\omega_0)$, as required to show that m is lower semi-continuous.

It is easy to see that $\varphi\otimes\mathrm{id}$ is itself additive and positive homogeneous, and an N-bimodule map, that is, an operator-valued weight. If $x_i \uparrow x$ then $(\mathrm{id}\otimes\omega)(x_i) \uparrow (\mathrm{id}\otimes\omega)(x)$ for each $\omega \in N_*^+$, and so $\varphi((\mathrm{id}\otimes\omega)(x_i)) \uparrow \varphi((\mathrm{id}\otimes\omega)(x))$ by definition of φ being normal. Thus $\varphi\otimes\mathrm{id}$ is a normal operator-valued weight. Notice that for $x \in \mathfrak{n}_\varphi$ and $y \in N$, we have that $x\otimes y \in \mathfrak{n}_{\varphi\otimes\mathrm{id}}$ as $(\varphi\otimes\mathrm{id})(x^*x\otimes y^*y)(\omega) = \varphi(x^*x)\omega(y^*y)$ for all $\omega\in N_*^+$, and so $(\varphi\otimes\mathrm{id})(x^*x\otimes y^*y) = \varphi(x^*x)y^*y < \infty$. Hence $\mathfrak{n}_{\varphi\otimes\mathrm{id}}$ is σ-weakly dense in $M\bar{\otimes}N$ because $\mathfrak{n}_\varphi \odot N$ is.

In particular, we may define

$$(\varphi\otimes\psi)(x) = \psi\big((\varphi\otimes\mathrm{id})(x)\big),$$

where ψ is extended to $\widehat{N}_+$ in the obvious way, [4, Corollary IX.4.9]. Defining

$$\Phi_\varphi = \{\omega\in M_*^+ : \omega(x) \leqslant \varphi(x) \ (x\in M_+)\},$$

we have that $\varphi(x) = \sup_{\omega\in\Phi_\varphi}\omega(x)$ for $x\in M_+$, see [4, Theorem VII.1.11]. Similarly form Φ_ψ. Then, just following the definitions,

$$\begin{aligned}(\varphi\otimes\psi)(x) = \psi\big((\varphi\otimes\mathrm{id})(x)\big) &= \sup_{\omega\in\Phi_\psi}\varphi((\mathrm{id}\otimes\omega)(x)) = \sup_{\omega\in\Phi_\psi}\sup_{\tau\in\Phi_\varphi}(\tau\otimes\omega)(x)\\ &= \sup_{\tau\in\Phi_\varphi}\psi((\tau\otimes\mathrm{id})(x)) = \varphi\big((\mathrm{id}\otimes\psi)(x)\big),\end{aligned}$$

and so the definition is symmetric.

If we wish to use a reference, then [3, Proposition 8.3] combined with this calculation immediately shows that $\varphi\otimes\psi = \varphi\bar{\otimes}\psi$, the latter defined in the next section using Hilbert algebra techniques.

1.2 Tensor products of Hilbert algebras

This is the usual definition of $\varphi \otimes \psi$, see [4, Definition VIII.4.2]. Let $\mathfrak{A}_\varphi \subseteq H_\varphi$ be the full left Hilbert algebra given by φ, where $(H_\varphi, \Lambda_\varphi, \pi_\varphi)$ is the GNS construction for φ. Thus $\mathfrak{A}_\varphi = \Lambda_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$. Similarly we define $\mathfrak{A}_\psi$ and so forth.

Lemma 1.1. *The algebraic tensor product* $\mathfrak{A}_\varphi \odot \mathfrak{A}_\psi$ *is a left Hilbert algebra in* $H_\varphi \otimes H_\psi$.

Proof. This is routine, with all operators defined using the tensor product. Showing that $a \otimes b \mapsto a^\sharp \otimes b^\sharp$ is perhaps the only non-trivial step, but this follows by using $D^\flat$, and the relations from [4, Lemma VI.1.5], for example. □

Tensor products of unbounded operators have the expected definitions and properties, see [2, Section 7.5] for example. When T_1, T_2 are densely defined and closed, then $T_1 \odot T_2$ is densely and pre-closed, and we define $T_1 \otimes T_2$ to be the closure. One non-trivial result is that then $(T_1 \otimes T_2)^* = T_1^* \otimes T_2^*$, see [2, Proposition 7.26] or [4, Lemma VIII.4.1].

We consider the left Hilbert algebra $\mathfrak{A} = \mathfrak{A}_\varphi \odot \mathfrak{A}_\psi$. We then see that $S_\mathfrak{A} = S_{\mathfrak{A}_\varphi} \otimes S_{\mathfrak{A}_\psi} = (J_\varphi \otimes J_\psi)(\Delta_\varphi^{1/2} \otimes \Delta_\psi^{1/2})$ and so forth. For $\xi \in \mathfrak{A}$ we denote by $\pi_l(\xi)$ the bounded operator formed by left multiplication by ξ. It is easy to see that

$$\pi_l\big(\Lambda_\varphi(a) \otimes \Lambda_\psi(b)\big) = a \otimes b \qquad (a \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*, b \in \mathfrak{n}_\psi \cap \mathfrak{n}_\psi^*).$$

By density, the (left) von Neumann algebra associated to $\mathfrak{A}$, namely $\pi_l(\mathfrak{A})''$, is hence $M \bar{\otimes} N$. We define $\varphi \bar{\otimes} \psi$ to be the weight associated to the left Hilbert algebra $\mathfrak{A}$.

Recall that $\mathfrak{B}' \subseteq H_\varphi \otimes H_\psi$ is the space of right bounded vectors, so $\eta \in \mathfrak{B}'$ exactly when there is a bounded operator $\pi_r(\eta)$ with $\pi_r(\eta)\xi = \pi_l(\xi)\eta$ for each $\xi \in \mathfrak{A}$. For $\eta \in \mathfrak{B}'$ we define $\omega_\eta^l \in (M \bar{\otimes} N)_*$ by $\omega_\eta^l(x) = (\eta | x\eta)$. Now define

$$\Phi_{l,0} = \big\{\omega_\eta^l : \eta \in \mathfrak{B}', \|\pi_r(\eta)\| \leqslant 1\big\}.$$

Then, [4, Lemma VII.2.4], we have that

$$(\varphi \bar{\otimes} \psi)(x) = \sup\{\omega(x) : \omega \in \Phi_{l,0}\} \qquad (x \in (M \bar{\otimes} N)_+).$$

Given $\xi \in \mathfrak{B}'_\varphi, \eta \in \mathfrak{B}'_\psi$ it is easy to see that $\xi \otimes \eta \in \mathfrak{B}'$ with $\pi_r(\xi \otimes \eta) = \pi_r(\xi) \otimes \pi_r(\eta)$. It follows that

$$\{\omega_\xi^l \otimes \omega_\eta^l : \xi \in \Phi_{l,0}^\varphi, \eta \in \Phi_{l,0}^\psi\} \subseteq \Phi_{l,0},$$

and so as $\varphi(x) = \sup\{\omega(x) : \omega \in \Phi_{l,0}^\varphi\}$, and simiarly for ψ, arguing as in the end of last section, we conclude that

$$(\varphi \bar{\otimes} \psi)(x) \geqslant \psi((\varphi \otimes \mathrm{id})(x)).$$

To show the other inclusion using these techniques seems hard. One can show (see [3, (7), Section 8] for example) that $\mathfrak{A}'_\varphi \odot \mathfrak{A}'_\psi$ generates $\mathfrak{A}'$ as a right Hilbert algebra (or use that J intertwines $\mathfrak{A}'$ and $\mathfrak{A}''$, [4, Theorem VI.1.19(ii)]). Presumably a similar result holds for the right bounded vectors $\mathfrak{B}'$. Using the right version of [4, Theorem VI.1.26(ii)], we can hence approximate elements of $\mathfrak{B}'$ by elements in $\mathfrak{B}'_\varphi \odot \mathfrak{B}'_\psi$, when forming $\Phi_{l,0}$. However, we would wish to approximate by rank-one tensors, and this seems out of reach.

Let $a \in \mathfrak{n}_\varphi$ have polar decomposition $a = u|a|$, so $u \in M$, and $|a| = u^*a \in \mathfrak{n}_\varphi$ as this is a left ideal. Hence also $|a| \in \mathfrak{n}_\varphi^*$ as $|a|$ is self-adjoint. Similarly let $b = v|b| \in \mathfrak{n}_\psi$. As above, then $|a| \otimes |b| = \pi_l(\Lambda_\varphi(|a|) \otimes \Lambda_\varphi(|b|))$ and hence by definition, [4, Section VII.2],

$$(\varphi \bar{\otimes} \psi)(a^*a \otimes b^*b) = \|\Lambda_\varphi(|a|) \otimes \Lambda_\varphi(|b|)\|^2 = \|(u^* \otimes v^*)(\Lambda_\varphi(a) \otimes \Lambda_\psi(b))\|^2 = \|\Lambda_\varphi(a) \otimes \Lambda_\psi(b)\|^2.$$

For the last equality, we clearly have "$\leqslant$", but also $\Lambda_\varphi(a) = u\Lambda_\varphi(|a|)$, and so $\|\Lambda_\varphi(a) \otimes \Lambda_\psi(b)\| \leqslant \|\Lambda_\varphi(|a|) \otimes \Lambda_\varphi(|b|)\|$. In particular, $a \odot b \in \mathfrak{n}_{\varphi \bar{\otimes} \psi}$. Furthermore, for $c \in \mathfrak{n}_\varphi, d \in \mathfrak{n}_\psi$, we have $c^*a \otimes d^*b \in \mathfrak{m}_{\varphi \bar{\otimes} \psi}$ with

$$(\varphi \bar{\otimes} \psi)(c^*a \otimes d^*b) = (\Lambda_\varphi(c) \otimes \Lambda_\psi(d) | \Lambda_\varphi(a) \otimes \Lambda_\psi(b)) = \varphi(c^*a)\psi(d^*b).$$

1.2.1 Proceeding via modular automorphism groups

We follow the strategy from [3]. Deciding when two normal semi-finite weights are equal is hard, but use can be made of the modular automorphism groups. Write $\varphi\bar{\otimes}\psi$ as above, and $\varphi\otimes\psi$ for the weight constructed from operator-valued weight theory in Section 1.1. We have that

$$\Delta_{\varphi\bar{\otimes}\psi} = \Delta_\varphi \otimes \Delta_\psi \qquad \Longrightarrow \qquad \sigma_t^{\varphi\bar{\otimes}\psi} = \sigma_t^\varphi \otimes \sigma_t^\psi,$$

compare [4, Proposition VIII.4.3].

However, determining $\sigma_t^{\varphi\otimes\psi}$ seems hard, and we proceed as suggested by [3]. I have not found suitable references in [4], but the machinery used is that of Spatial Derivative theory. Define

$$\Phi_\varphi^0 = \{t\omega \in M_*^+ : 0 < t < 1, \omega(x) \leqslant \varphi(x) \ (x \in M_+)\},$$

so that Φ_φ^0 is a directed set (for a proof, compare [1, Proposition 3.5], using the ideas of [4, page 55]). We consider (finite) weights of the form $\tau\otimes\omega$ as τ,ω vary over $\Phi_\varphi^0, \Phi_\psi^0$ respectively. This forms a directed set, increasing to $\varphi\otimes\psi$. By [3, Proposition 7.17], we have that

$$\sigma_t^{\tau\otimes\omega}(x) \to \sigma_t^{\varphi\otimes\psi} \qquad (x \in M\bar{\otimes}N),$$

for each t (infact, uniformly on bounded intervals). However, also $\sigma_t^\tau \to \sigma_t^\varphi$ and $\sigma_t^\omega \to \sigma_t^\psi$ pointwise, and so

$$\sigma_t^{\varphi\otimes\psi}(x) = (\sigma_t^\varphi \otimes \sigma_t^\psi)(x) \qquad (x \in M \odot N).$$

As each automorphism is σ-weakly continuous, we see that $\sigma_t^{\varphi\otimes\psi} = \sigma_t^\varphi \otimes \sigma_t^\psi$ on all of $M\bar{\otimes}N$.

We now use [4, Proposition VIII.3.16], which tells us that as the modular automorphism groups agree, to show that $\varphi\otimes\psi = \varphi\bar{\otimes}\psi$, it suffices to show equality on a σ-weakly dense $*$-subalgebra $\mathfrak{m}_0 \subseteq \mathfrak{m}_{\varphi\bar{\otimes}\psi}$. We will use $\mathfrak{m}_0 = \mathfrak{m}_\varphi \odot \mathfrak{m}_\psi$. Indeed, we established above that $\mathfrak{m}_\varphi \odot \mathfrak{m}_\psi \subseteq \mathfrak{m}_{\varphi\otimes\psi}$ with $\varphi\bar{\otimes}\psi = \varphi\otimes\psi$ on this $*$-subalgebra. It follows that $\varphi\bar{\otimes}\psi = \varphi\otimes\psi$.

References

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