Commentary on "K_0 of purely infinite simple regular rings"

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Abstract

We give the details for some of the results from the start of [1].

We assume throughout that R is a (unital) ring. We write $\mathsf{Mod}\text{-}R$ for the category of right R-modules, and for $M, N \in \mathsf{Mod}\text{-}R$ we write $\operatorname{Hom}_R(M, N)$ for the morphisms: the right R-module homomorphisms. We write $\operatorname{End}_R(M)$ for $\operatorname{Hom}_R(M, M)$.

For us, a ring is *simple* exactly when it has no proper two-sided ideals. We warn the reader that this is not completely standard terminology, compare [2, Chapter XVII, Section 5] for example.

1 Idempotents

Definition 1.1. *Let R be a ring, and let $e, f \in R$ be idempotents. Then e, f are (Murray-von Neumann)* equivalent *if there are $x \in eRf, y \in fRe$ with $e = xy, f = yx$. In this case, we write $e \sim f$.*

Suppose we merely have $x, y \in R$ with $e = xy, f = yx$. As $e^2 = e$ we have $xy = xyxy = xfy = xf^2y$ and so $xy = e = e^3 = exye = exffye$. Similarly we can show that $yx = fyeexf$, and so it is no loss of generality to suppose that $x \in eRf$ and $y \in fRe$ as in the definition.

When e is an idempotent, eR is a complemented (right) submodule of R.

Lemma 1.2. *For idempotents $e, f \in R$ we have that $e \sim f$ if and only if $eR \cong fR$ in $\mathsf{Mod}\text{-}R$.*

Proof. If $e \sim f$ with $e = xy, f = yx$ as in the definition, then define $\theta : eR \to fR$ by $\theta(a) = ya$. This is defined, as if $a \in eR$ then $ea = a$ so $ya = y(xy)a = fya \in fR$. Clearly $\theta \in \operatorname{Hom}_R(eR, fR)$. Similarly, there is $\phi \in \operatorname{Hom}_R(eR, fR)$ given by $\phi(b) = xb$. Then $\theta\phi(b) = yxb = fb = b$ for $b \in fR$, and $\phi\theta(a) = xya = ea = a$ for $a \in eR$. Hence $eR \cong fR$ in $\mathsf{Mod}\text{-}R$.

Conversely, let e, f be idempotents, and let $\theta \in \operatorname{Hom}_R(eR, fR)$ be an isomorphism. Let $y = \theta(e)$ and $x = \theta^{-1}(f)$. For $a \in eR$ as $a = ea$ we have $\theta(a) = \theta(ea) = \theta(e)a = ya$; similarly for $b \in fR$ we have $\theta^{-1}(b) = xb$. Thus, for $a \in eR$ we have $a = \theta^{-1}(\theta(a)) = xya$ and for $b \in fR$ we have $b = yxb$. In particular, $e = xye$ and $f = yxf$. As $y \in fR$ and $x \in eR$ we have $fy = y, ex = x$. Set $a = fye = ye$ and set $b = exf = xf$. Then $ab = yexf = yxf = f$ and $ba = xfye = xye = e$ and so $e \sim f$. $\square$

Definition 1.3. *Two idempotents e, f are* orthogonal *when $ef = fe = 0$, written $e \perp f$.*

If $e \perp f$ then $(e+f)^2 = e+f$ so $e+f$ is an idempotent. Conversely, if $2x = 0$ implies $x = 0$ in R, then if $(e+f)^2 = e+f$ then $ef + fe = 0$. Then $0 = e0 = ef + efe$ and $0 = 0e = efe + fe$ so $ef = fe$ and so $ef = fe = 0$. We need this condition, for with $R = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $e = (1, 1), f = (0, 1)$ we have that $ef = fe = f \neq 0$ but $e + f = (1, 0)$ which is an idempotent.

When $e \perp f$, it is easy to see that $(e + f)R \cong eR \oplus fR$.

Lemma 1.4 ([1, Lemma 1.1])**.** *Let $e \in R$ be an idempotent, and suppose there are right ideals $(A_i)_{i=1}^n$ in R with $eR = A_1 \oplus \cdots \oplus A_n$. Then there exist pairwise orthogonal idempotents $e_i \in R$ with $e = \sum_i e_i$ and $A_i = e_i R$ for each i.*

Proof. Consider $S = \mathrm{End}_R(eR)$ which acts on the left of eR. That $eR = A_1 \oplus \cdots \oplus A_n$ is a decomposition of right R-modules means that the projection onto A_i gives an idempotent f_i in S. Thus the f_i are pairwise orthogonal, sum to the identity, and $f_i(eR) = A_i$ for each i.

Define $\theta : S \to eRe$ by $\theta(f) = f(e)$. For $f \in \mathrm{End}_R(eR)$ we have $f(e) = f(e^2) = f(e)e$ and so $f(e) \in eRe$ and θ is defined. Furthermore, for $x \in R$ we have $f(ex) = f(e)x$ while for any $eye \in eRe$ we have that $x \mapsto eyex$ defines a member of $\mathrm{End}_R(eR)$. Thus θ is an isomorphism, and setting $e_i = \theta(f_i)$ gives the required conclusion. □

Definition 1.5. *An idempotent $e \in R$ is* infinite *if there are orthogonal idempotents $f, g \in R$ with $e = f + g$ and $e \sim f$ with $g \neq 0$.*

By the Lemma, e is infinite if and only if eR is isomorphic to a proper direct summand of itself. In this case, we say that eR is *directly infinite.*

Lemma 1.6. *Let R be a ring and let e be an infinite idempotent. If $e \sim f$ then also f is infinite.*

Proof. As e is infinite there are orthogonal idempotents a, b with $e = a+b$ and $e \sim a$. As a useful aside, we claim that $ea = ae = ea$ and $eb = be = b$. Indeed, $a+b = e = e^2 = e(a+b) = (a+b)e$. Multiply by a on either side to get $a = a(a+b)e = ae$ and $a = e(a+b)a = ea$. The same argument works for b.

There are $u \in eRa, v \in aRe$ with $uv = e, vu = a$. As $e \sim f$ there are $x \in eRf$ and $y \in fRe$ with $xy = e$ and $yx = f$. Set $c = yax$ and $d = ybx$. Then $c^2 = yaxyax = yaeax = yax = c$, and similarly $d^2 = d$, and $cd = yaxybx = yaebx = yabx = 0$, and similarly $dc = 0$. Then $c + d = y(a+b)x = yex = yx = f$ so f is the sum of the orthogonal idempotents c, d.

Observe now that $(yux)(yvx) = yuevx = yuaeavx = yuavx = yuvx = yex = yx = f$ and $(yvx)(yux) = yveux = yvux = yax = c$ so $f \sim c$, showing that f is infinite. □

Lemma 1.7. *Let $P \in \mathsf{Mod}\text{-}R$ and suppose that $P \cong Q \oplus A$ with Q being directly infinite. Then also P is directly infinite.*

Proof. That Q is directly infinite means that $Q \cong Q \oplus B$ for some $0 \neq B \in \mathsf{Mod}\text{-}R$. Then

$$P \cong Q \oplus A \cong Q \cong B \oplus A \cong P \oplus B,$$

and so P is directly infinite. □

Definition 1.8. *A ring R is* purely infinite *if every nonzero right ideal of R contains an infinite idempotent.*

2 Some ring theory

Given a set I write R^I for the direct sum of I-many copies of R. One construction of this is to consider all functions $f : I \to R$ which have $f(i) \neq 0$ for only finitely many i, and endow with point-wise operations. Similarly define M^I for $M \in \mathsf{Mod}\text{-}R$.

Recall that $F \in \mathsf{Mod}\text{-}R$ is *free* if P is isomorphic to R^I for some I. A module $P \in \mathsf{Mod}\text{-}R$ is *projective* if given any $N, M \in \mathsf{Mod}\text{-}R$, any $g \in \mathrm{Hom}_R(P, M)$ and any surjective $f \in \mathrm{Hom}_R(N, M)$, there is some $h \in \mathrm{Hom}_R(P, N)$ with $fh = g$:

$$\begin{tikzcd} & N \arrow[d, "f"] \\ P \arrow[ur, dotted, "\exists h"] \arrow[r, "g"'] & M \end{tikzcd}$$

The following is standard:

Lemma 2.1. *For $P \in$ Mod-R the following are equivalent:*

1. *P is projective;*
2. *P is the direct summand of a free module. That is, there is a free module F and $Q \in$ Mod-R with $F \cong P \oplus Q$;*
3. *The functor $\mathrm{Hom}_R(P, \_)$, from Mod-R to the category of Abelian groups, is exact.*

We shall say that $X \in$ Mod-R *generates* Mod-R, or is a *generator*, if, whenever we have $M, N \in$ Mod-R and $f, g \in \mathrm{Hom}_R(M, N)$ with $f\alpha = g\alpha$ for all $\alpha \in \mathrm{Hom}_R(X, M)$, then $f = g$. This is the usual Category Theory notion of a "generator" or *separator*, though we warn the reader that this is not completely settled, compare [3].

As Mod-R is Abelian, it suffices to take $g = 0$ in the definition. To be more prosaic, if given $h \in \mathrm{Hom}_R(M, N)$ there exists $\alpha \in \mathrm{Hom}_R(X, M)$ with $h\alpha \neq 0$ then for $f, g \in \mathrm{Hom}_R(M, N)$ set $h = f - g$, find a suitable α, and then note that $h\alpha \neq 0$ exactly when $f\alpha \neq g\alpha$.

For the following, compare the definition of "generator" given in [2, Chapter XVII, Section 7].

Lemma 2.2. *$X \in$ Mod-R is a generator if and only if for each $M \in$ Mod-R there is some index set I and a surjective $f \in \mathrm{Hom}_R(X^I, M)$. If M is finitely generated, we may choose I to be finite.*

Proof. If X is a generator, then let $N \subseteq M$ be sum of $\alpha(X)$ as α varies through $\mathrm{Hom}_R(X, M)$. Notice that N can be obtained as the image of some morphism $X^I \to M$ for some index set I. If $N \neq M$ then M/N is non-zero, and so the quotient map $h : M \to M/N$ is non-zero. Thus there is $\alpha \in \mathrm{Hom}_R(X, M)$ with $h\alpha \neq 0$. However, this means that $\alpha(X)$ is not a subset of N, contradiction.

Suppose now that M is finitely generated, with generators $m_1, \cdots, m_n$. We have already shown that for each i there is are $\alpha_1, \cdots, \alpha_j \in \mathrm{Hom}_R(X, M)$ and $x_1, \cdots, x_j \in X$ with $m_i = \sum_k \alpha_k(x_k)$. Thus for $a \in R$ we have that $m_i a = \sum_k \alpha_k(x_k a)$. It follows that we can find a finite I and $\alpha : X^I \to M$ surjective.

Conversely, if every module is a homomorphic image of X^I for some I, then let $h \in \mathrm{Hom}_R(M, N)$ be non-zero. Select I and $\alpha \in \mathrm{Hom}_R(X^I, M)$ a surjection. Such an α is given by a family $(\alpha_i)_{i \in I}$ in $\mathrm{Hom}_R(X, M)$ in the obvious way. If $h\alpha_i = 0$ for all i, then also $h\alpha = 0$ which contradicts α being surjective and $h \neq 0$. Thus for some i with have $h\alpha_i \neq 0$, as required to show that X is a generator. □

The following is inspired by [3].

Lemma 2.3. *Let $X \in$ Mod-R be projective. Then $X \in$ Mod-R is a generator if and only if for each non-zero $M \in$ Mod-R we have that $\mathrm{Hom}_R(X, M) \neq 0$.*

Proof. The "only if" case follows immediately from the lemma above (and holds for any X). With X assumed projective, suppose that $\mathrm{Hom}_R(X, M) \neq 0$ for all $M \neq 0$. We first give a category-theoretic proof. Let $h \in \mathrm{Hom}_R(M, N)$ be non-zero. The following sequence is exact

$$0 \to \ker(h) \to M \to M/\ker(h) \to 0,$$

and so as X is projective, the following sequence is also exact

$$0 \to \mathrm{Hom}_R(X, \ker(h)) \to \mathrm{Hom}_R(X, M) \to \mathrm{Hom}_R(X, M/\ker(h)) \to 0.$$

As $h \neq 0$ we have that $\ker(h) \neq M$ and so $\mathrm{Hom}_R(X, M/\ker(h)) \neq 0$ by hypothesis. By exactness, there is some $\alpha \in \mathrm{Hom}_R(X, M)$ which does not map to 0 in $\mathrm{Hom}_R(X, M/\ker(h))$, that is, with $h\alpha \neq 0$. This is exactly what we needed to prove to show that X is a generator.

Let us now give a more direct proof. Let $h \in \mathrm{Hom}_R(M, N)$ be non-zero, and again consider $M/\ker(h)$ which is non-zero as h is non-zero, so by assumption, there is $\beta : X \to M/\ker(h)$ non-zero. As X is projective, there is α making the diagram commute.

$$\begin{tikzcd} & M \arrow[d] \\ X \arrow[ur, dotted, "\exists\alpha"] \arrow[r, "\beta"'] & M/\ker(h) \end{tikzcd}$$

If $h\alpha = 0$ then $\alpha(X) \subseteq \ker(h)$, which implies that $\beta = 0$ which is not so. Thus $h\alpha \neq 0$, as required to show that X is a generator. $\square$

For $n \geq 1$ consider R^n as a right R-module. If $R^n = X \oplus Y$ for some submodules X, Y then there is an idempotent $e \in \mathrm{Hom}_R(R^n)$ with $X = e(R^n)$. Furthermore, $\mathrm{Hom}_R(R^n)$ may be identified with $M_n(R)$ *acting on the left* of R^n.

Given $M \in \mathsf{Mod}\text{-}R$ let $\mathrm{Ann}(M) = \{x \in R : mx = 0 \ (m \in M)\}$. For $x \in \mathrm{Ann}(M)$ and $y, z \in R$ we have that $myxz = (my)x(z) = 0z = 0$ for all $m \in M$, so $yxz \in \mathrm{Ann}(M)$. Thus $\mathrm{Ann}(M)$ is an ideal in R. If $I \trianglelefteq R$ is an ideal, then let $M = R/I$ as a right module. Then $x \in \mathrm{Ann}(M)$ exactly when $(y + I)x = 0$ for all $y \in R$, that is, $Rx \subseteq I$, that is, $x \in I$. So $\mathrm{Ann}(R/I) = I$ and hence all ideals of R arise in this way.

Finally, we need one small part of the Morita equivalence between R and $M_n(R)$, which is the claim that every $M_n(R)$ module is of the form M^n (thought of as row vectors, with $M_n(R)$ acting on the right by matrix multiplication) for some $M \in \mathsf{Mod}\text{-}R$. This claim can be proved directly if one wishes.

Proposition 2.4. *Let $e \in M_n(R)$ be an idempotent, and set $P = eR^n$ considered as a right R-module. P is a generator if and only if e is not contained in any proper two-sided ideal of $M_n(R)$.*

Proof. For $M \in \mathsf{Mod}\text{-}R$ non-zero, we wish to show that $\mathrm{Hom}_R(P, M) \neq 0$. Let R^n have basis $(e_i)_{i=1}^n$ so every $x \in R^n$ can be written as $x = \sum_{i=1}^n e_i x_i$ for some $x_i \in R$. We identify

$$\mathrm{Hom}_R(R^n, M) \cong \mathrm{Hom}_R(R, M) \oplus \cdots \oplus \mathrm{Hom}_R(R, M) \cong M^n,$$

so $f \in \mathrm{Hom}_R(R^n, M)$ is identified with $(\xi_i) \in M^n$ when $f(x) = \sum \xi_i x_i$ for $x \in R^n$.

Similarly, given $f \in \mathrm{Hom}_R(P, M)$ we can set $\xi_i = f(ee_i)$ as $P = eR^n$. We now identify which families (ξ_i) can occur; perhaps there is an easy way to see this, but I will resort to calculation. Let e have matrix (x_{ij}) so $ee_i = \sum_j e_{ji} e_j \in R^n$. Given some $(\xi_i) \in M^n$ the map $ex \mapsto \sum \xi_i x_i$ is well-defined exactly when $ex = 0 \implies \sum_i \xi_i x_i = 0$. If we consider $\xi = (\xi_i)$ as a row vector, and $x = (x_i)$ as a column vector, we may write $\xi x = 0$. That $ex = 0$ is equivalent to $(1 - e)x = x$. So our condition is that for all y, with $x = (1 - e)y$, we need that $0 = \xi x = \xi(1 - e)y = \xi y - \xi e y$, where now $e \in M_n(R)$ acts on the right of the row vector ξ. That is, $\xi y = \xi e y$ for all y, so $\xi = \xi e$.

Thus P is a generator if and only if for every M there is $\xi \in M^n$ with $\xi e = \xi$; equivalently, $M^n e \neq 0$. Notice that we have turned M^n into a right $M_n(R)$ module in the obvious way, and so our condition is equivalent to $e \notin \mathrm{Ann}(M^n) \trianglelefteq M_n(R)$. As any ideal in $M_n(R)$ arises as $ann(M^n)$ for some M, the result follows. $\square$

Let $e_{ij} \in M_n(R)$ be the matrix with 1 in the (i, j)th place. The diagonal embedding $R \to M_n(R)$ is a ring homomorphism which turns $M_n(R)$ into a left R-module, with the action of $x \in R$ just left multiplication on every matrix element. Thus a general $x \in M_n(R)$ can be written uniquely as

$$x = \sum_{i,j=1}^{n} x_{ij} e_{ij}$$

for some (x_{ij}) in R.

Let $I \trianglelefteq M_n(R)$ be some non-zero ideal. As $1 = \sum_i e_{ii}$ there is $x \in I$ and some i with $e_{ii}x \neq 0$. Similarly, we can then find j with $e_{ii}xe_{jj} \neq 0$. So there is $y \in R$ with $ye_{ij} \in I$. By multiplying by suitable e_{kl} on the left and right, we see that $ye_{ij} \in I$ for all i, j. Let $J \subseteq R$ be the collection of such y. Clearly J is an ideal, and so $M_n(J) \subseteq I$. However, our argument shows that if $x = (x_{ij}) \in I$ then each $x_{ij} \in J$. Thus $I = M_n(J)$. If $J \trianglelefteq R$ is an ideal, then $M_n(J)$ is an ideal, and so we have completely classified the ideals of $M_n(R)$.

We can finally prove the following; the argument leading here is inspired by [4].

Proposition 2.5. *Let R be a simple ring. Any finitely-generated projective module $P \in$ Mod-R is a generator for Mod-R.*

Proof. As P is finitely-generated there is a surjective morphism $R^n \to P$ for some n, and as P is projective, there is $P' \in$ Mod-R with $R^n \cong P \oplus P'$. Let $e \in M_n(R)$ be the resulting idempotent with $eR^n \cong P$. By the previous proposition, eR^n is a generator if and only e is not contained in a proper ideal of $M_n(R)$. As R is simple, the preceding discussion shows that $M_n(R)$ is simple, and so the result follows. □

3 Purely infinite rings

We continue to follow [1].

Proposition 3.1 ([1, Lemma 1.4])**.** *Let R be a simple ring, and let P and Q be finitely generated projective modules in Mod-R. If P is directly infinite, then there is a non-zero $A \in$ Mod-R with $P \cong Q \oplus A$.*

Proof. As P is directly infinite there is $0 \neq B \in$ Mod-R with $P \cong P \oplus B$. Thus also $P \cong P \oplus B \cong P \oplus B \oplus B$; by induction, $P \cong P \oplus B^n$ for any $n \geq 1$. As B is the direct summand of a projective, it is itself projective. There is some m and a surjective morphism $R^m \to P \cong P \oplus B$, and so also there is a surjective morphism $R^m \to B$, so B is finitely generated. By Proposition 2.5, B is a generator, and so Lemma 2.2 gives $n \geq 1$ and a surjection $B^n \to Q$. As Q is projective there is a splitting and we can find $C \in$ Mod-R with $B^n \cong Q \oplus C$. Thus

$$P \cong P \oplus B^n \cong P \oplus Q \oplus C \cong Q \oplus (P \oplus C),$$

and $P \oplus C \neq 0$ as $P \neq 0$. □

Proposition 3.2 ([1, Proposition 1.5])**.** *Let R be a purely infinite, simple ring. All non-zero finitely generated projective right R-modules are directly infinite; equivalently, all non-zero idempotents in $M_n(R)$ are infinite.*

If P, Q are non-zero finitely generated projective right R-modules, there is $A \in$ Mod-R with $P \cong Q \oplus A$.

Proof. As R is purely infinite, there is some infinite idempotent $e \in R$. Set $P = eR$ so P is a finitely generated, projective, directly infinite right R-module. Let Q be a non-zero finitely generated projective right R-module, so by Proposition 3.1, there is $A \in$ Mod-R with $P \cong Q \oplus A$. As $P = eR$ we may identify Q with a non-zero right ideal I in R. Thus I contains an infinite idempotent f, and so $fR \subseteq I$ is a directly infinite right R-module which is a direct summand of Q. The claim now follows from Lemma 1.7.

The equivalence with all non-zero idempotents in $M_n(R)$ being infinite follows as if $e \in M_n(R)$ is an idempotent, then eR^n is a finitely generated projective right R-module, and so directly infinite, which is equivalent to e being infinite. The converse is similar.

The final claim is immediate from Proposition 3.1. $\square$

We now come to our main theorem.

Theorem 3.3. *Let R be a simple ring. Then R is purely infinite if and only if:*

1. *R is not a division ring; and*
2. *for every $0 \neq a \in R$ there are $b, c \in R$ with $bac = 1$.*

Proof. Assume R is purely infinite. Then R contains an infinite idempotent, and so R is not a division ring. Given $0 \neq a \in R$, the non-zero right ideal aR contains an infinite idempotent e. By the final claim of Proposition 3.2 there is $A \in \mathsf{Mod}\text{-}R$ with $eR \cong R \oplus A$. By Lemma 1.4 we can find orthogonal idempotents f, g with $e = f + g$ and with $fR \cong R$, that is, $f \sim 1$, by Lemma 1.2. Thus there are $x \in fR, y \in Rf$ with $xy = f, yx = 1$. As $f \in eR$ (compare the proof of Lemma 1.6) and as $e \in aR$, we have $f \in aR$ so $f = ar$ for some $r \in R$. Hence

$$1 = yxyx = yfx = (y)a(rx),$$

as required.

Conversely, suppose the two conditions hold. Let I be a non-zero right ideal in R. Either $I \neq R$ in which case set $J = I$, or, as R is not a division ring, there is a proper right ideal J in R. [1] Let $a \in J$ be non-zero, so there are $b, c \in R$ with $bac = 1$. Then $e = acb$ is an idempotent in $aR \subseteq J$, so $e \neq 1$ and $e \in I$. As $(eac)(be) = e$ and $(be)(eac) = 1$ we have that $e \sim 1$. Thus 1 is infinite, but then as e is equivalent to 1, also e is infinite, by Lemma 1.6. Hence R is purely infinite. $\square$

References

[1] P. Ara, K. R. Goodearl and E. Pardo, K_0 of purely infinite simple regular rings, K-Theory **26** (2002), no. 1, 69–100. MR1918211

[2] S. Lang, *Algebra*, revised third edition, Graduate Texts in Mathematics, 211, Springer-Verlag, New York, 2002. MR1878556

[3] Q. Yuan, Generators, blog post available at `https://qchu.wordpress.com/2015/05/17/generators/`

[4] Q. Yuan, When do projective modules give generators?, URL (version: 2020-10-28): `https://math.stackexchange.com/q/3884117`

[1] This seems to be well-known. Such an ideal exists, for if not, we would have that $xR = R$ for all $x \in R$, and so for each x there is y with $xy = 1$. If $e \in R$ is an idempotent then there is z with $ez = 1$, so $1 = ez = eez = e1 = e$. As $xy = 1$ we have that yx is an idempotent, so $yx = 1$. Thus R is a division ring, contradiction.