Some notes on weights

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1 Introduction

I collect some notes about weights on von Neumann algebras. These vary from the trivial to slightly complicated. I constantly get confused about the basics, so these are just a personal aide de memoir. I try to give some references to where I found ideas.

2 The basics

We follow the notation of [7, Chapter VII]. Much the same material can be found in [5, Section 10.14], with a brief summary in [4, Chapter I]. A very brief summary can be found in [3, Section 7.5], but this final source is, as usual, notable for its very careful proofs.

Given a weight $\varphi\colon M^+ \to [0,\infty]$ we set

$$\mathfrak{p}_\varphi = \{x \in M^+ : \varphi(x) < \infty\}, \quad \mathfrak{n}_\varphi = \{x \in M : \varphi(x^*x) < \infty\}, \quad \mathfrak{m}_\varphi = \operatorname{lin}\{x^*y : x, y \in \mathfrak{n}_\varphi\}.$$

We say that φ is *semi-finite* when $\mathfrak{m}_\varphi$ is σ-weakly dense in M. We shall mostly suppose that φ is semi-finite, normal and faithful.

As $\mathfrak{n}_\varphi$ is a left ideal, we see that $\mathfrak{m}_\varphi \subseteq \mathfrak{n}_\varphi$; similarly $\mathfrak{m}_\varphi \subseteq \mathfrak{n}_\varphi^*$ and so in fact $\mathfrak{m}_\varphi \subseteq \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$. So:

- φ semi-finite implies that each of $\mathfrak{n}_\varphi, \mathfrak{n}_\varphi^*$ and $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ are σ-weakly dense in M.

Let $x \in \mathfrak{n}_\varphi$ and $x = u|x|$ be the polar decomposition. Then $|x| = u^*x \in \mathfrak{n}_\varphi$.

Form the GNS construction $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$. I think [7] is too hasty here, and so we follow [3]. As φ is faithful, the map $\mathfrak{n}_\varphi \to H_\varphi; a \mapsto \Lambda_\varphi(a)$ is injective. Suppose that $\pi_\varphi(x) = 0$, which is equivalent to $\Lambda_\varphi(xa) = 0$ for each $a \in \mathfrak{n}_\varphi$, equivalently, $xa = 0$ for each $a \in \mathfrak{n}_\varphi$. As $\mathfrak{n}_\varphi$ is σ-weakly dense in M, by continuity of the multiplication, $xa = 0$ for each $a \in M$, so $x = 0$. A similarly careful treatment of why π_φ is normal can be found in [3].

By Kaplansky Density (the cleanest statement is [2, Theorem 5.3.5, Corollary 5.3.6]) the unit ball of $\mathfrak{m}_\varphi$ is σ-strongly dense in the unit ball of M. Further, for any $y \in M_+$ there is a net (y_i) in $\mathfrak{m}_\varphi^+$ with $y_i \to y$ σ-weakly, and with $\|y_i\| \leqslant \|y\|$ for each i. However, I do not think it is true in general that we can choose (y_i) increasing with $y_i \leqslant y$.

As $\mathfrak{n}_\varphi$ is a left ideal, it has an increasing approximate identity. Indeed, [6, Theorem I.7.4] shows that the set $\{x \in \mathfrak{n}_\varphi : \|x\| < 1\}$ is upward directed and forms a right approximate identity for $\mathfrak{n}$.

If we work a bit harder, we can ensure that our approximate identity has further properties, compare [5, Proposition 3.21], or indeed the proof of [6, Theorem I.7.4]. Here is one way to do this. Using functional calculus, for $a \geqslant 0$ we can form the element $(a+\epsilon)^{-1}a$, and then for $0 \leqslant a \leqslant 1$ we have the estimate (which holds for real-values, so by functional calculus, for operators) $\|a - a^2(a+\epsilon)^{-1}\| < \epsilon$. Thus $\|xa^2(a+\epsilon)^{-1} - x\| \leqslant \|x\|\epsilon + \|xa - x\|$. Thus if (a_i) is our original approximate identity, then $b_i = a_i^2(a_i+\epsilon)^{-1}$ will also be an approximate identity, still have $b_i \in \mathfrak{n}_\varphi \cap M_+$, but we now also have $b_i \in \mathfrak{m}_\varphi^+$.

- There is a (right) approximate identity (b_i) in $\mathfrak{m}_\varphi^+ \subseteq \mathfrak{n}_\varphi$ for $\mathfrak{n}_\varphi$, that is, $\|ab_i - a\| \to 0$ for $a \in \mathfrak{n}_\varphi$.

2.1 Semi-finiteness

We explore some equivalent conditions for what it means for a normal weight to be semi-finite.

Proposition 2.1. *Let $\mathfrak{n} \subseteq M$ be a left ideal, and let $\mathfrak{m} = \operatorname{lin}\{x^*y : x, y \in \mathfrak{n}\}$, which is a $*$-subalgebra of M contained in $\mathfrak{n}$. There is a projection $e \in M$ with $Me = \overline{\mathfrak{n}}^\sigma$ and $eMe = \overline{\mathfrak{m}}^\sigma$. As such, $\mathfrak{n}$ is σ-weakly dense in M if and only if $\mathfrak{m}$ is.*

Proof. We follow the sketch in [1, III.1.1.15]. As $\mathfrak{n}$ is a left ideal, it follows that $\mathfrak{m} \subseteq \mathfrak{n}$, and that $\mathfrak{m}$ is a $*$-algebra. From [6, Theorem I.7.4] the set $\{x \in \mathfrak{n}_\varphi : x \geqslant 0, \|x\| < 1\}$ is upward directed and forms a right approximate identity for $\mathfrak{n}$, say (a_i).

We show that a_i converges strongly to a projection e with $\overline{\mathfrak{n}}^\sigma = Me$. There are many ways to show this, but here is a simple-minded approach. Let M act on H, and again, we know that $a_i \to e$ σ-strongly,[1] where $e \in M^+$ is the upper bound. For $x \in \mathfrak{n}$ we have $xe = \lim_i xa_i = x$, the first limit holding strongly, say (the second limit is in norm, of course). Set $H_0 = \overline{\operatorname{lin}}\{x^*\xi : x \in \mathfrak{n}, \xi \in H\}$. As $a_i = a_i^* \to e$ strongly, it follows that $e(H) \subseteq H_0$. For $x \in \mathfrak{n}, \xi \in H$ we have $ex^*\xi = (xe)^*\xi = x^*\xi$ and so $e\eta = \eta$ for each $\eta \in H_0$. Hence $e = e^2$ is an idempotent with image H_0. Now let $\eta \in H_0^\perp, \xi \in H$, so $(e\eta|\xi) = \lim_i (a_i\eta|\xi) = \lim_i (\eta|a_i^*\xi) = 0$ as $a_i^*\xi \in H_0$ for each i. So $e(H_0^\perp) = \{0\}$ and e is the orthogonal projection onto H_0. As $xe = x$ for each $x \in \mathfrak{n}$, certainly $\mathfrak{n} \subseteq Me$, and hence $\overline{\mathfrak{n}}^\sigma \subseteq Me$. For $x \in M$, we have $xe = \lim_i xa_i \in \overline{\mathfrak{n}}^\sigma$, as the limit certainly holds in the σ-weak topology. So $Me = \overline{\mathfrak{n}}^\sigma$ as claimed.

Now let $x, y \in \mathfrak{n}$, so $x^*y \in \mathfrak{m}$ and hence $x^*y = ex^*ye \in eMe$. Hence $\overline{\mathfrak{m}}^\sigma \subseteq eMe$. Given $x \in eMe$, we have $x = exe = \lim_i a_i^*xa_i$ as $a_i = a_i^*$, the limit holding in the strong topology say, which follows from the estimate

$$\|(a_ixa_i - exe)\xi\| \leqslant \|a_ixa_i\xi - a_ixe\xi\| + \|a_ixe\xi - exe\xi\| \leqslant \|x\|\|(a_i - e)\xi\| + \|(a_i - e)xe\xi\|,$$

remembering that $\|a_i\| \leqslant 1$. As $a_i \in \mathfrak{n}$ also $xa_i \in \mathfrak{n}$ and so $a_i^*(xa_i) \in \mathfrak{m}$. We conclude that $x \in \overline{\mathfrak{m}}^\sigma$, showing the other inclusion, so $\overline{\mathfrak{m}}^\sigma = eMe$.

To finish, as $\mathfrak{m} \subseteq \mathfrak{n}$, if $\mathfrak{m}$ is σ-weakly dense in M, then obviously $\mathfrak{n}$ is as well. For the converse, we observe that $\overline{\mathfrak{n}}^\sigma = M$ if and only if $Me = M$, if and only if $e = 1$, which implies $\overline{\mathfrak{m}}^\sigma = eMe = M$. ☐

Corollary 2.2. *φ is semi-finite if and only if $\mathfrak{n}_\varphi$ is σ-weakly dense in M.*

In particular, this shows that the meaning of "semi-finite" for operator-valued weights, [7, Definition IX.4.14], is in accordance with the meaning for weights.

An alternative characterisation of *semifinite* comes from [1, Proposition III.2.2.20]. As $\mathfrak{m}_\varphi$ is a $*$-algebra, its norm closure is a C^*-algebra, which has an approximate unit, so by approximation, so does $\mathfrak{m}_\varphi$.[2]

Proposition 2.3. *Let φ be a normal weight on M. The following are equivalent:*

1. *φ is semifinite: $\mathfrak{m}_\varphi$ is σ-weakly dense in M;*
2. *for each (positive) approximate unit (a_i) for $\mathfrak{m}_\varphi$, for $x \in M_+$ we have $\varphi(x) \leqslant \liminf_i \varphi(a_ixa_i)$;*
3. *for each (positive) approximate unit (a_i) for $\mathfrak{m}_\varphi$, when $x \in M_+$ has $\varphi(x) = \infty$, we have $\lim_i \varphi(a_ixa_i) = \infty$;*
4. *for each (positive) approximate unit (a_i) for $\mathfrak{m}_\varphi$, when $x \in M_+$ has $\varphi(x) = \infty$, we have $\sup_i \varphi(a_ixa_i) = \infty$;*

[1] We have strong convergence, see [2, Lemma 5.1.4] for example. By replacing M with $M \otimes 1$ acting on $H \otimes \ell^2$, we convert strong convergence to σ-strong convergence.

[2] To maintain positivity, let (e_i) be an (increasing) positive approximate identity for $\overline{\mathfrak{m}_\varphi}$. For each i, let $b_i \in \mathfrak{m}_\varphi$ norm approximate $e_i^{1/2}$, and set $a_i = b_i^*b_i$, so $a_i \geqslant 0$ is norm close to e_i.

Proof. When $\mathfrak{m}_\varphi$ is σ-weakly dense in M, we have that $a_i \to 1$ σ-strongly, and so for each $x \in M_+$ we have that $a_i x a_i \to x$ σ-weakly. As φ is σ-weakly lower semicontinuous, $\varphi(x) \leqslant \liminf_i \varphi(a_i x a_i)$, as required to show (1) $\implies$ (2). Conversely, as above, $\overline{\mathfrak{m}_\varphi}^\sigma = pMp$ for some projection p, and $a_i \to p$ σ-strongly. Let $x \in M_+$ with $px = 0$. As $a_i p = a_i$ because $a_i \in \mathfrak{m}_\varphi$, we have $\varphi(x) \leqslant \liminf_i \varphi(a_i x a_i) = \liminf_i \varphi(a_i p x a_i) = 0$ so $\varphi(x) = 0$ so $x \in \mathfrak{m}_\varphi^+$, so $x = px = 0$. Set $x = 1 - p$ to conclude that $p = 1$ as required.

As the lim inf is infinite implies the limit is, clearly (2) $\implies$ (3). When $x \in \mathfrak{m}_\varphi^+$, the condition in (2) holds by lower semicontinuity, and so (3) $\implies$ (2). Clearly (3) $\implies$ (4). When (4) holds, towards a contradiction, suppose that (3) doesn't, so there is $K > 0$ with $J = \{i : \varphi(a_i x a_i) \leqslant K\}$ is cofinal (that is, for each i_0 there is $i \geqslant i_0$ with $\varphi(a_i x a_i) \leqslant K$, this being the opposite of the limit being infinite). Then $(a_j)_{j \in J}$ is a subnet of (a_i), and so is still an approximate identity, but as $\sup_{j \in J} \varphi(a_j x a_j) \leqslant K$, we contradict (4). Hence (4) $\implies$ (3) as we want. □

3 Hilbert algebras to von Neumann algebras

I find the links between the Hilbert algebra and weights to be a bit obscure in [7]: it would perhaps benefit from a nice summary somewhere. Similarly in [5].

Let $\mathfrak{A}$ be a full left Hilbert algebra associated to the weight φ. The passage between these objects is detailed in [7, Section VII.2]. We have

$$\mathfrak{n}_l = \pi_l(\mathfrak{B}) = \mathfrak{n}_\varphi, \quad \mathfrak{n}_r = \pi_r(\mathfrak{B}') = \mathfrak{n}_{\varphi'}, \quad \pi_l(\mathfrak{A}) = \mathfrak{n}_l \cap \mathfrak{n}_l^*, \quad \pi_r(\mathfrak{A}') = \mathfrak{n}_r \cap \mathfrak{n}_r^*,$$
$$\varphi(\pi_l(\eta)^*\pi_l(\xi)) = (\eta|\xi) \qquad (\xi, \eta \in \mathfrak{B}), \qquad \varphi'(\pi_r(\eta)^*\pi_r(\xi)) = (\eta|\xi) \qquad (\xi, \eta \in \mathfrak{B}').$$

For $\eta \in \mathfrak{B}'$ define $\omega_\eta^l \in M_*^+; x \mapsto (\eta|x\eta)$, and set

$$\Phi_{l,0} = \{\omega_\eta^l : \eta \in \mathfrak{B}', \|\pi_r(\eta)\| < 1\}.$$

Then

$$\varphi(x) = \sup\{\omega(x) : \omega \in \Phi_{l,0}\} \qquad (x \in M_+).$$

We state the following for the "left" algebra, but analogous results hold on the right. By [7, Theorem VI.1.19, Theorem VIII.1.2] we have the modular automorphism group (σ_t) on M given by

$$\pi_l(\sigma_t(x)) = \nabla^{it} x \nabla^{-it} \qquad (x \in M, t \in \mathbb{R}).$$

The map J is a bijection between $\mathfrak{A}$ and $\mathfrak{A}'$ and an anti-homomorphism, [7, Theorem VI.1.19]. I find it buried in [7] (but see [5, Theorem 10.12]) that

$$\pi_l(\nabla^{it}\xi) = \nabla^{it}\pi_l(\xi)\nabla^{-it}, \qquad J\pi_l(\xi)J = \pi_r(J\xi), \qquad (\xi \in \mathfrak{A}).$$

We extend this to left bounded vectors.

Lemma 3.1. *For $\xi \in \mathfrak{B}$ we have that $\pi_l(\nabla^{it}\xi) = \nabla^{it}\pi_l(\xi)\nabla^{-it}$.*

Proof. Let $\xi \in \mathfrak{B}$ and $\eta \in \mathfrak{A}'$ so also $\nabla^{-it}\eta \in \mathfrak{A}'$, and

$$\pi_r(\eta)\nabla^{it}\xi = \nabla^{it}\nabla^{-it}\pi_r(\eta)\nabla^{it}\xi = \nabla^{it}\pi_r(\nabla^{-it}\eta)\xi = \nabla^{it}\pi_l(\xi)\nabla^{-it}\eta.$$

As η was arbitrary, we conclude that $\nabla^{it}\xi \in \mathfrak{B}$ with $\pi_l(\nabla^{it}\xi) = \nabla^{it}\pi_l(\xi)\nabla^{-it}$ as required. □

This lemma is equivalently stated as $\pi_l(\nabla^{it}\Lambda(a)) = \sigma_t(a)$ for $a \in \mathfrak{n}_\varphi$, or further, equivalently as $\nabla^{it}\Lambda(a) = \Lambda(\sigma_t(a))$, as $\Lambda\pi_l = \mathrm{id}$ on $\mathfrak{n}_\varphi$.

The relation between left and right bounded vectors allows us to show other results. For example, given $b \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ we have $\Lambda(b) \in \mathfrak{A}$ so $J\Lambda(b) \in \mathfrak{A}'$ with $\pi_r(J\Lambda(b)) = JbJ$, and hence

$$JbJ\Lambda(a) = \pi_r(J\Lambda(b))\Lambda(a) = \pi_l(\Lambda(a))J\Lambda(b) = aJ\Lambda(b) \qquad (a \in \mathfrak{n}_\varphi, b \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*).$$

By [7, Proposition VI.1.24], the map $\Lambda\colon \mathfrak{n}_\varphi \to \mathfrak{B} \subseteq H$ is closed for σ-strong topology on M and the norm topology on H. Simiarly, when restricted to $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^* \to \mathfrak{A}$, the map is closed for the σ-strong* topology and the $\mathfrak{D}^\sharp$ norm, that is, $\|\xi\|_\sharp = (\|\xi\|^2 + \|\xi^\sharp\|^2)^{1/2}$.

We can approximate vectors, [7, Theorem VI.1.26]. Given $\xi \in \mathfrak{B}$ there is a sequence $(\xi_n) \subseteq \mathfrak{A}$ converging to ξ with $\|\pi_l(\xi_n)\| \leqslant \|\pi_l(\xi)\|$ for each n, whence $\pi_l(\xi_n) \to \pi_l(\xi)$ strongly, because $\pi_l(\xi_n)\eta \to \pi_l(xi)\eta$ for each $\eta \in \mathfrak{A}'$.

Similarly, if $\mathfrak{A}_0 \subseteq \mathfrak{A}$ is any left Hilbert algebra with $\mathfrak{A}_0'' = \mathfrak{A}$, then for $\xi \in \mathfrak{A}$ there is a sequence (ξ_n) in $\mathfrak{A}_0$ with $\|\xi_n - \xi\|_\sharp \to 0$ and $\|\pi_l(\xi_n)\| \leqslant \|\pi_l(\xi)\|$ for each n, whence $\pi_l(\xi_n) \to \pi_l(\xi)$ σ-strong*, using now that $\pi_l(\xi_n)^*\eta = \pi_l(\xi_n^\sharp)\eta = \pi_r(\eta)\xi_n^\sharp$ for $\eta \in \mathfrak{A}'$.

3.1 Tomita algebras and smearing

Given $\xi \in H$ we define

$$\xi_n = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \nabla^{it}\xi \, dt \qquad (n > 0).$$

(Notice we use a slightly different convention to [7, 5].) As $\mathbb{R} \to H, t \mapsto \nabla^{it}\xi$ is continuous, dominated convergence shows that $\xi_n \to \xi$ in norm, as $n \to \infty$. We have that $\xi_n \in D(\nabla^{iz})$ for any $z \in \mathbb{C}$, with

$$\nabla^{iz}(\xi_n) = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2(t-z)^2} \nabla^{it}\xi \, dt,$$

a fact proved by contour deformation.

In both, for example, [7, Proof of Theorem VI.2.2] or [5, Page 303], it is claimed that if $\xi \in \mathfrak{A}$ then $\pi_l(\xi_n) \to \pi_l(x)$ strongly. I do not see why this is immediate. Firstly, why is $\xi_n \in \mathfrak{B}$?

- For $\eta \in \mathfrak{A}'$ we certainly have that

$$\pi_r(\eta)\xi_n = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \pi_r(\eta)\nabla^{it}\xi \, dt = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2t^2} \nabla^{it}\pi_l(\xi)\nabla^{-it}\eta \, dt,$$

  compare the proof of Lemma 3.1. For any $x \in \mathcal{B}(H)$ the map $t \mapsto \nabla^{it}x\nabla^{-it}$ is strongly continuous, and so we can form the integral in the strong operator topology, to give meaning to

$$\pi_r(\eta)\xi_n = \frac{n}{\sqrt{\pi}} \Big(\int_{\mathbb{R}} e^{-n^2t^2} \nabla^{it}\pi_l(\xi)\nabla^{-it} \, dt \Big)\eta = \frac{n}{\sqrt{\pi}} \Big(\int_{\mathbb{R}} e^{-n^2t^2} \sigma_t(\pi_l(\xi)) \, dt \Big)\eta.$$

  Call this $\pi_l(\xi)_n\eta$. Thus $\xi_n \in \mathfrak{B}$ with $\pi_l(\xi_n) = \pi_l(\xi)_n$.

- An alternative argument is the following. We can approximate the intergal by a suitable Riemann sum, say given a partition $I_m = \{t_0 < t_1 < \cdots < t_{N(m)}\}$ we have

$$\xi_n \approx \frac{n}{\sqrt{\pi}} \sum_{k=1}^{N(m)} \frac{1}{t_k - t_{k-1}} e^{-n^2t_k^2} \nabla^{it_k}\xi = \xi_{n,m},$$

  say. Similarly $\pi_l(\xi)_n$ has the same approximation, but with $\sigma_{t_k}(\pi_l(\xi))$ replacing $\nabla^{it_k}\xi$. As the sum is finite, we clearly have $\pi_l(\xi_{n,m}) = \pi_l(\xi)_{n,m}$. As $\mathfrak{B} \to \mathcal{B}(H), \xi \mapsto \pi_l(\xi)$ is closed for the strong topology, as $\lim_m \xi_{n,m} = \xi_n$ and $\lim_m \pi_l(\xi)_{n,m} = \pi_l(\xi)_n$ we have that $\xi_n \in \mathfrak{B}$ with $\pi_l(\xi_n) = \pi_l(\xi)_n$.

We now wish to show that for any $\eta \in H$ we have

$$\pi_l(\xi_n)\eta = \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-n^2(t-z)^2} \sigma_t(\pi_l(\xi))\eta \, dt \to \pi_l(\xi)\eta.$$

This is clear, as $t \mapsto \sigma_t(\pi_l(\xi))\eta$ is continuous, so in the limit the integral converges to $\sigma_0(\pi_l(\xi))\eta$. The ingredients of this argument are in [7, Proof of Theorem VI.2.2], but I think it's useful to be explicit.

4 Weights on opposite algebras

Let M be a von Neumann algebra and M^{op} its opposite algebra, say $M^{op} = \{x^{op} : x \in M\}$ with product $x^{op}y^{op} = (yx)^{op}$. For a subset $X \subseteq M$ write $X^{op} = \{x^{op} : x \in X\} \subseteq M^{op}$.

Given an nsf weight φ on M we define φ^{op} on M^{op} by $\varphi(x^{op}) = \varphi(x)$ for $x \in M^{op+} = (M^+)^{op}$. Then $x^{op} \in \mathfrak{n}_{\varphi^{op}}$ if and only if $\varphi^{op}(x^{op*}x^{op}) = \varphi(xx^*) < \infty$ if and only if $x^* \in \mathfrak{n}_\varphi$. Thus $\mathfrak{p}_{\varphi^{op}} = \mathfrak{p}_\varphi^{op}$ and

$$\mathfrak{m}_{\varphi^{op}} = \operatorname{lin}\{x^{op*}y^{op} : x^{op}, y^{op} \in \mathfrak{n}_{\varphi^{op}}\} = \operatorname{lin}\{(yx^*)^{op} : x^*, y^* \in \mathfrak{n}_\varphi\} = \mathfrak{m}_\varphi^{op}.$$

Alternatively, $\mathfrak{m}_{\varphi^{op}} = \operatorname{lin} \mathfrak{p}_{\varphi^{op}} = \operatorname{lin} \mathfrak{p}_\varphi^{op} = \mathfrak{m}_\varphi^{op}$. By linearity, $\varphi^{op}(x^{op}) = \varphi(x)$ for each $x \in \mathfrak{m}_\varphi$.

We have the GNS construction $(L^2(\varphi^{op}), \pi^{op}, \Lambda^{op})$. For $x, y \in \mathfrak{n}_\varphi^*$ we have

$$(\Lambda^{op}(x^{op})|\Lambda^{op}(y^{op})) = \varphi^{op}(x^{op*}y^{op}) = \varphi(yx^*) = (\Lambda(y^*)|\Lambda(x^*)) = (J\Lambda(x^*)|J\Lambda(y^*)).$$

Hence by density there is a unitary $U \colon L^2(\varphi^{op}) \to L^2(\varphi)$ with $U\Lambda^{op}(x^{op}) = J\Lambda(x^*)$ for each $x \in \mathfrak{n}_\varphi^*$. For $x \in \mathfrak{n}_\varphi^*$ and $a \in M$ we have

$$U\pi^{op}(a^{op})U^*J\Lambda(x^*) = U\Lambda^{op}(a^{op}x^{op}) = J\Lambda((xa)^*) = J\pi(a^*)JJ\Lambda(x^*),$$

and so $U\pi^{op}(a^{op})U^* = J\pi(a^*)J$.

We next investigate how U intertwines the maps S, F, ∇, J; see [7, Lemma VI.1.5] for example. Recall that S is the closure of the map $\Lambda(x) \mapsto \Lambda(x^*)$ for $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$, and that

$$D(F) = \{\eta \in L^2(\varphi) : \Lambda(x) \mapsto (S\Lambda(x)|\eta) \text{ is bounded}\}; \quad (\eta|S\xi) = (\xi|F\eta) \qquad (\xi \in D(S), \eta \in D(F)).$$

As $F = F^{-1}$ we have that $\eta \in D(F) \Leftrightarrow F(\eta) \in D(F)$ and $F^2(\eta) = \eta$ for $\eta \in D(F)$. Similarly for S. In particular, for $\xi \in D(S), \eta \in D(F)$ also $\eta' = F(\eta) \in D(F)$ with $F\eta' = \eta$, and so $(S\xi|F\eta) = (S\xi|\eta') = (F\eta'|\xi) = (\eta|\xi)$.

Lemma 4.1. *We have that* $US^{op} = FU$ *(so* U *restricts to bijection* $D(S^{op}) \to D(F)$*) and that* $UF^{op} = SU$*.*

Proof. That $US^{op} = FU$ means that $D(US^{op}) = D(FU)$, equivalently, $D(S^{op}) = U^*D(F)$, equivalently, $UD(S^{op}) = D(F)$, using that U is unitary, with $US^{op}(\xi^{op}) = FU(\xi^{op})$ for $\xi^{op} \in D(S^{op})$.

Let $\xi^{op} \in D(S^{op})$ so there is a sequence $(x_n^{op}) \subseteq \mathfrak{n}_{\varphi^{op}}$ with $\Lambda(x_n^{op}) \to \xi^{op}, \Lambda(x_n^{op*}) \to S^{op}(\xi^{op})$. For $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ we have

$$\begin{aligned}(S\Lambda(x)|U(\xi^{op})) &= \lim_n (S\Lambda(x)|J\Lambda(x_n^*)) = \lim_n (\Lambda(x_n^*)|JS\Lambda(x)) = \lim_n (S\Lambda(x_n)|FJ\Lambda(x)) \\ &= \lim_n (J\Lambda(x)|\Lambda(x_n)) = \lim_n (J\Lambda(x_n)|\Lambda(x)) = \lim_n (U\Lambda^{op}(x_n^{op*})|\Lambda(x)) \\ &= (US^{op}(\xi^{op})|\Lambda(x)).\end{aligned}$$

So $U(\xi^{op}) \in D(F)$ with $FU(\xi^{op}) = US^{op}(\xi^{op})$.

Conversely, and for variety using a different proof strategy, let $\xi \in D(F) = D(\nabla^{-1/2})$ so $J\xi \in D(\nabla^{1/2}) = D(S)$ and hence there is a sequence (x_n) in $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ with $\Lambda(x_n) \to J\xi, \Lambda(x_n^*) \to SJ(\xi) = JF(\xi)$. So $U\Lambda(x_n^{op}) = J\Lambda(x_n^*) \to F(\xi)$ and $U\Lambda(x_n^{op*}) = J\Lambda(x_n) \to \xi$ showing that $U^*F(\xi) \in D(S^{op})$ with $S^{op}U^*F(\xi) = U^*(\xi)$. Set $\xi^{op} = U^*F(\xi)$ so $\xi^{op} \in S^{op}$ with $US^{op}(\xi^{op}) = \xi$ hence showing that U maps $D(S^{op})$ onto $D(F)$.

That $UF^{op} = SU$ follows similarly. ∎

So $\nabla^{op} = F^{op}S^{op} = U^*SUU^*FU = U^*SFU = U^*\nabla^{-1}U$. Hence $S^{op} = U^*FU = U^*J\nabla^{-1/2}U = U^*JU\nabla^{op\,1/2}$ and $S^{op} = U^*FU = U^*\nabla^{1/2}JU = \nabla^{op\,-1/2}U^*JU$, so by uniqueness, $J^{op} = U^*JU$.

Consider now the modular automorphism group (σ_t^{op}); we use [7, Section VIII.1] for a reference. We claim that $\sigma_t^{op}(x^{op}) = \sigma_{-t}(x)^{op}$ for each $x \in M$. This would leave φ^{op} invariant, and given $x^{op}, y^{op} \in \mathfrak{n}_{\varphi^{op}} \cap \mathfrak{n}_{\varphi^{op}}^* = (\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)^{op}$ there is a bounded continuous F on $\overline{\mathbb{D}}$, holomorphic on $\mathbb{D}$, with

$$F(t) = \varphi(\sigma_t(x)y), \qquad F(t+i) = \varphi(y\sigma_t(x)) \qquad (t \in \mathbb{R}).$$

Define $F'(z) = F(i-z)$, so if $z = x+iy$ for $y \in [0,1]$ then $i - z = -x + i(1-y)$ and so F' is also bounded and continuous on $\overline{\mathbb{D}}$ and holomorphic on $\mathbb{D}$. We have

$$\begin{aligned} F'(t) &= F(i-t) = \varphi(y\sigma_{-t}(x)) = \varphi^{op}(\sigma_{-t}(x)^{op}y^{op}), \\ F'(t+i) &= F(i-(t+i)) = F(-t) = \varphi(\sigma_{-t}(x)y) = \varphi^{op}(y^{op}\sigma_{-t}(x)^{op}). \end{aligned}$$

So F' satisfies the requirements for the pair x^{op}, y^{op}. By uniqueness, our claim follows. We show this is consistent with ∇^{op}. We have that $J\nabla = \nabla^{-1}J$, but J is also anti-linear, and so $J\nabla^{it} = \nabla^{it}J$ for all t. Hence

$$\begin{aligned} \pi^{op}\big(\sigma_t^{op}(x^{op})\big) &= \nabla^{op\,it}\pi^{op}(x^{op})\nabla^{op\,-it} = U^*\nabla^{-it}UU^*J\pi(x^*)JUU^*\nabla^{it}U = U^*J\nabla^{-it}\pi(x^*)\nabla^{it}JU \\ &= U^*J\pi\big(\sigma_{-t}(x^*)\big)JU = \pi^{op}(\sigma_{-t}(x)^{op}) \end{aligned}$$

showing again that $\sigma_t^{op}(x^{op}) = \sigma_{-t}(x)^{op}$.

4.1 Commutant acting on standard form

We suppress $\pi\colon M \to \mathcal{B}(L^2(\varphi))$ and regard M as a subalgebra of $\mathcal{B}(L^2(\varphi))$. Then $M' = JMJ$ is isomorphic to M^{op} via the map $\theta\colon M^{op} \to M'; x^{op} \mapsto Jx^*J$. Let φ^{op} induce φ'. Then $\mathfrak{n}_{\varphi'} = \theta(\mathfrak{n}_{\varphi^{op}}) = \{Jx^*J : x^* \in \mathfrak{n}_\varphi\} = J\mathfrak{n}_\varphi J$, and similarly $\mathfrak{p}_{\varphi'} = J\mathfrak{p}_\varphi J$ and $\mathfrak{m}_{\varphi'} = J\mathfrak{m}_\varphi J$. So $\varphi'(JxJ) = \varphi(x^*)$ for $x \in \mathfrak{m}_\varphi$. Then θ induces a unitary $u\colon L^2(\varphi') \to L^2(\varphi); \Lambda(Jx^*J) \mapsto \Lambda^{op}(x^{op})$.

Then $Uu\colon L^2(\varphi') \to L^2(\varphi)$ is $\Lambda'(JxJ) \mapsto J\Lambda(x)$ and $Uu\pi'(Ja^*J)u^*U^* = U\pi^{op}(a^{op})U^* = J\pi(a^*)J = Ja^*J$. By definition, $u\nabla'u^* = \nabla^{op} = U^*\nabla^{-1}U$ and so $Uu\nabla'u^*U^* = \nabla^{-1}$, and similarly $UuJ'u^*U^* = J$. Also $UuS'u^*U^* = US^{op}U^* = F$ and $UuF'u^*U^* = S$.

In summary, if we identify $L^2(\varphi')$ with $L^2(\varphi)$ via $\Lambda'(JxJ) = J\Lambda(x)$ for $x \in \mathfrak{n}_\varphi$, then π' is the identity representation, and the modular operators are $\nabla' = \nabla^{-1}, J' = J, S' = F, F' = S$.

We can check various things for consistency. For example, with $x' = Jx^*J = \theta(x^{op}) \in M'$, then we would expect $\sigma'_t(x') = \theta(\sigma_t^{op}(x^{op})) = \theta(\sigma_{-t}(x)^{op}) = J\sigma_{-t}(x)^*J$. We also have

$$\sigma'_t(x') = \nabla'^{it}x'\nabla'^{-it} = \nabla^{-it}Jx^*J\nabla^{it} = J\nabla^{-it}x^*\nabla^{it}J = J\sigma_{-t}(x^*)J = J\sigma_{-t}(x)^*J,$$

as expected.

4.2 Alternative construction

An alternative construction of φ' is given in [7, Theorem VII.1.17]. We show here that these definitions do agree. We recall the definition from [7]. Let $\Phi_\varphi = \{\omega \in M_*^+ : \omega \leqslant \varphi\}$ and $E_\varphi = \bigcup_{\lambda \geqslant 0} \lambda\Phi_\varphi$. For each $\omega \in E_\varphi$ there is $h_\omega \in M'_+$ with $(\Lambda(x)|h_\omega\Lambda(y)) = \omega(x^*y)$ for $x, y \in \mathfrak{n}_\varphi$. Then set $\mathfrak{p}_{\varphi'_0} = \{h_\omega : \omega \in E_\varphi\}$ and define

$$\varphi'_0(x) = \begin{cases} \|\omega\| & : x = h_\omega \in \mathfrak{p}_{\varphi'_0}, \\ \infty & : \text{otherwise.} \end{cases} \qquad (x \in M'_+).$$

Then φ'_0 is a nsf weight on M'.

We use the bijection between left Hilbert algebras and nsf weights, in particular Theorems 2.5 and 2.6 from [7, Section VII]; see also Section 3 above. Starting with φ on M we form the full left Hilbert algebra $\mathfrak{A}_\varphi = \Lambda(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$. This gives rise to a weight φ_l on $\mathcal{R}_l(\mathfrak{A}_\varphi) \cong M$ and $\varphi_l = \varphi$. We also obtain φ_r on $\mathcal{R}_r(\mathfrak{A}_\varphi) = \mathcal{R}_l(\mathfrak{A}_\varphi)'$ which agrees with φ'_0.

Let $\mathfrak{B}$ be the algebra of left bounded vectors, and similarly $\mathfrak{B}'$. As in Section 3 above, $J\pi_l(\xi)J = \pi_r(J\xi)$ for each $\xi \in \mathfrak{A} \Leftrightarrow J\xi \in \mathfrak{A}'$. Then $\xi \in \mathfrak{B}$ exactly when there is a bounded operator $\pi_l(\xi)$ satisfying $\pi_l(\xi)\eta = \pi_r(\eta)\xi$ for each $\eta \in \mathfrak{A}'$, equivalently,

$$\pi_l(\xi)J\eta = J\pi_l(\eta)J\xi \qquad (\eta \in \mathfrak{A}).$$

Then $\pi_l(\eta)J\xi = J\pi_l(\xi)J\eta$ for each $\eta \in \mathfrak{A}$ shows that $J\xi \in \mathfrak{B}'$ with $\pi_r(J\xi) = J\pi_l(\xi)J$. We can reverse this argument, and so we conclude that $J\mathfrak{B} = \mathfrak{B}'$.

We recall the construction of φ_r: we have

$$\varphi_r(x) = \sup\{\omega(x) : \omega \in \Phi_{r,0}\} \quad \text{where} \quad \Phi_{r,0} = \{\omega_\xi|_{M'} : \xi \in \mathfrak{B}, \|\pi_l(\xi)\| \leqslant 1\}.$$

However, we now see that

$$\Phi_{r,0} = \{\omega_{J\xi}|_{M'} : \xi \in \mathfrak{B}', \|\pi_r(\xi)\| \leqslant 1\},$$

which should be compared with $\Phi_{l,0}$ is Section 3. Given $x \in M'$ we have $JxJ \in M$ and $\omega_{J\xi}(x) = (J\xi|xJ\xi) = (\xi|JxJ\xi) = \omega_\xi(JxJ)$, so

$$\varphi_l(JxJ) = \sup\left\{\omega_\xi(JxJ) : \xi \in \mathfrak{B}', \|\pi_r(\xi)\| < 1\right\} = \varphi_r(x).$$

Hence φ_l and φ_r are related in the way we expect, and as $\varphi_r = \varphi_0'$, we conclude that $\varphi' = \varphi_0'$ as claimed.

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