Direct sums of Operator spaces

Matthew Daws

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Abstract

We give a short, self-contained account of the $\oplus_1$ and $\oplus_\infty$ norms of operator spaces, and the duality between them.

1 Introduction

An *operator space* is a Banach space E together with a norms on each matrix space $M_n(E)$, for $n = 1, 2, \cdots$, which satisfy Ruan's axioms. Alternatively, E comes with an embedding into $\mathcal{B}(H)$, the bounded operators on a Hilbert space, where $M_n(E)$ inherits the norm from $M_n(\mathcal{B}(H)) = \mathcal{B}(H^n)$. We refer the reader to [3, 4, 5] for further details.

Given a family of operators spaces $(E_i)_{i\in I}$ there is an obvious way to form an analogue of the Banach space ℓ^∞-direct-sum, namely $E = \oplus^\infty_{i\in I} E_i$ which consists of all families $(x_i)_{i\in I}$ where $x_i \in E_i$ for each i, and $\sup_i \|x_i\| < \infty$. The matrix norms are defined as follows. We identify $M_n(E)$ with $\oplus^\infty_{i\in I} M_n(E_i)$ as vector spaces in the obvious way, and then define

$$\|x\|_n = \sup_{i\in I} \|x_i\|_n \qquad (x = (x_i) \in \oplus^\infty_{i\in I} M_n(E_i)).$$

Verifying Ruan's axioms is routine, and so we have an operator space structure on $\oplus^\infty_{i\in I} E_i$.

We can similarly define the Banach space $\oplus^1_{i\in I} E_i$ consisting of all sequences $x = (x_i)$ with $\|x\| = \sum_i \|x_i\| < \infty$. It is standard (copy the proof that the dual of ℓ^1 is ℓ^∞) that the dual space of $\oplus^1_{i\in I} E_i$ is $\oplus^\infty_{i\in I} E_i^*$ with the dual-pairing element wise. Furthermore, if I is finite, then the dual space of $\oplus^\infty_{i\in I} E_i$ is $\oplus^1_{i\in I} E_i^*$. Finally, for arbitrary I, the natural map from $\oplus^1_{i\in I} E_i$ into the dual of $\oplus^\infty_{i\in I} E_i^*$ is isometric.

Motivated by this, we *define* the operator space structure on $\oplus^1_{i\in I} E_i$ to be that given by the embedding into the dual of $\oplus^\infty_{i\in I} E_i^*$. This note will be concerned with other characterisations of this operator space structure.

1.1 In the literature

The book by Effros and Ruan only considers the ∞-sum, see [3, page 38]. Pisier's book treats both sums in [4, Section 2.6], and defines the 1-sum via the "universal property" discussed below. That the duality results hold is stated without proof. To the best of our knowledge, [5] does not consider the direct sum (which is not surprising, as it is not, as such, a book about operator spaces). The ∞-sum is considered in [2, Section 1.2.17] and the 1-sum in [2, Section 1.4.13] using our "duality" definition.

To the best of our knowledge, the 1-sum was first introduced in [1, Section 3]. Here $\oplus^1_{i\in I} E_i$ is again defined by the injection into the dual of $\oplus^\infty_{i\in I} E_i^*$. Again, that the duality results hold are stated without proof.

2 The ∞-sum

As above, we identify $M_n(\oplus^\infty_{i\in I} E_i)$ with $\oplus^\infty_{i\in I} M_n(E_i)$, and so induce an operator space structure on $\oplus^\infty_{i\in I} E_i$. For each $i \in I$ define $\iota_i : E_i \to \oplus^\infty_{i\in I} E_i$ and $P_i : \oplus^\infty_{i\in I} E_i \to E_i$ to be the natural injection and quotient maps. It is easy to see that ι_i is a complete isometry, and P_i is a complete quotient map.

We can alternatively characterise $\oplus^\infty$ by the following universal property. Let (E_i) be a family of operator spaces, and let F be an operator space. Given a family (T_i) of complete contractions $T_i : F \to E_i$, there is a complete contraction $T : F \to \oplus^\infty_{i\in I} E_i$ given by $T(x) = (T_i(x))$. Indeed, $\oplus^\infty$ does satisfy this property, as for $x \in M_n(F)$ we have

$$\|(T)_n(x)\| = \sup_i \|(T_i)_n(x)\| \le \sup_i \|T_i\|_{cb}\|x\| \le \|x\|,$$

and so T is a complete contraction.

Let us be more precise about this: we mean that there exists an operator space E and complete contractions $\pi_i : E \to E_i$ such that if F is any operator space, and $T_i : F \to E_i$ a family of complete contractions, then there is a unique complete contraction $T : F \to E$ with $\pi_i \circ T = T_i$ for each i. That is, (E, π_i) is a *product* in the category of operator spaces, see Section 6 below. As shown below, E is necessarily unique if it exists, and furthermore, we have maps $\kappa_i : E_i \to E$ with $\pi_i \circ \kappa_j = \delta_{i,j}$.

Consider $F = \oplus^\infty_{i\in I} E_i$ and the complete contractions $T_i : F \to E_i$ being $T_i = P_i$. Then there is a complete contraction $T : F \to E$ with $\pi_i \circ T = P_i$. Given $x = (x_i) \in M_n(F) = \oplus^\infty_{i\in I} M_n(E_i)$ and $\epsilon > 0$ there is $j \in I$ with $\|x_j\|_{M_n(E_j)} > \|x\| - \epsilon$. Thus

$$x_j = \pi_j(T(x)) \implies \|x\| - \epsilon < \|x_j\| \le \|T(x)\| \le \|x\|.$$

As $\epsilon > 0$ was arbitrary, we conclude that T is a complete isometry. By uniqueness of E, we must have that T is a completely isometric isomorphism.

3 Equivalence of definitions

Consider the following two operator space structures on $\oplus^1_{i\in I} E_i$:

- The "dual structure" given by the embedding into $\oplus^\infty_{i\in I} E_i^*$;
- The "universal property structure" is that for any operator space F and any family (T_i) of complete contractions $T_i : E_i \to F$, then the operator $\oplus T_i : \oplus^1_{i\in I} E_i \to F; (x_i) \mapsto \sum_i T_i(x_i)$ exists, and is a complete contraction.

Notice that as a *Banach space*, it is immediately clear that the dual structure gives the usual Banach space structure on $\oplus^1_{i\in I} E_i$. Further, working just with Banach spaces, it is clear that the usual norm on $\oplus^1_{i\in I} E_i$ does satisfy the given universal property. Consider now applying the universal property to the family of maps (ι_i) where $\iota_i : E_i \to \oplus^1_{j\in I} E_j$. This shows that the universal property, for Banach spaces, determines the usual norm on $\oplus^1_{i\in I} E_i$.

Thus, if the universal property structure exists, then as a Banach space it is just the Banach space direct sum $\oplus^1_{i\in I} E_i$. For example, $M_n(\oplus^1_{i\in I} E_i)$ can be identified, as a vector space, with $\oplus^1_{i\in I} M_n(E_i)$.

Proposition 3.1. *There exists an operator space structure on $\oplus^1_{i\in I} E_i$ which satisfies the universal property. Indeed, given a sufficiently large Hilbert space H, the collection $\mathcal{C}$ of families of complete contractions $T_i : E_i \to \mathcal{B}(H)$ is a set, and given $u = (u_i) \in M_n(\oplus^1_{i\in I} E_i) \cong \oplus^i_{i\in I} M_n(E_i)$ we define $\|u\|_n$ to be the supremum over $\mathcal{C}$ of $\|\sum_i (T_i)_n(u_i)\|$, this norm computed in $M_n(\mathcal{B}(H))$.*

Proof. Let J be a set so that each E_i contains a dense subset of cardinality at most $|J|$, say $E_i' \subseteq E_i$. Let F be an operator space, and let (T_i) be a family of complete contractions $T_i : E_i \to F$. Let F_0 be the collection

$$\Big\{ \sum_{i\in I'} T_i(x_i) : x_i \in E_i', I' \subseteq I \text{ is finite} \Big\}.$$

Let F_1 be the rational-linear span of F_0, so F_1 is dense in the closed linear span of F_0 which equals the closure of $\{\sum_i T_i(x_i) : (x_i) \in \oplus^1_{i\in I} E_i\}$. Thus, we may replace F by the closure of F_1, and we lose no information. Thus, in the definition of the universal property structure, we can limit F to having a dense subset of cardinality of at most $|I \times J|$. If we set $H = \ell^2(I \times J)$ then we can completely isometrically embed such F into $\mathcal{B}(H)$. In conclusion, it suffices to consider families of complete contractions mapping into $\mathcal{B}(H)$ for this fixed H.

The family $\mathcal{C}$ is a set. Given $(T_i) \in \mathcal{C}$ and $u = (u_i) \in M_n(\oplus^1_{i\in I} E_i)$ we have that $\sum_i \|(T_i)_n(u_i)\| \leq \sum_i \|u_i\| < \infty$ and so $\|u\|_n$ is well-defined. Given the argument above, it is now clear that given any operator space F and any family of complete contractions (S_i) with $S_i : E_i \to F$, then for any $u = (u_i) \in M_n(\oplus^1_{i\in I} E_i)$ we have that $\|\sum_i S_i(u_i)\|$ computed in $M_n(F)$, is $\leq \|u\|_n$. Thus $\oplus S_i$ is a complete contraction, as required. $\square$

Having established that the universal property structure exists, we now want to show that this equals to dual structure. Given $\mu = (\mu_i) \in M_n(\oplus^\infty_{i\in I} E_i^*)$ each $\mu_i \in M_n(E_i^*) = \mathcal{CB}(E_i, M_n)$; let $T_i : E_i \to M_n$ be the induced map. If $\|\mu\| \leq 1$ then $\|\mu_i\|_n \leq 1$ and so T_i is a complete contraction, for each i. Given $x = (x_i) \in M_m(\oplus^\infty_{i\in I} E_i)$ the dual pairing is

$$\langle\langle \mu, x\rangle\rangle = \sum_i \langle\langle \mu_i, x_i\rangle\rangle \in M_{mn} = \sum_i (T_i)_n(x_i) = (\oplus_i T_i)_n(x).$$

From this calculation, it follows that the identity on $\oplus^i_{i\in I} E_i$ gives a complete contraction from the universal property structure to the dual structure.

Conversely, given an operator space F and $x \in M_n(F)$, we know that

$$\|x\|_n = \sup\big\{\|\langle\langle \mu, x\rangle\rangle\| : \mu \in M_m(F^*), \|\mu\|_m \leq 1, m \in \mathbb{N}\big\}.$$

(This is equivalent to $F \to F^{**}$ being a complete contraction. Further, by Smith's lemma, we can just fix $m = n$, see the discussion in [3, Section 3.2].) For $\mu \in M_m(F^*) = \mathcal{CB}(F, M_m)$ a complete contraction, and given $T_i : E_i \to F$ a complete contraction, also $S_i = \mu \circ T_i : E_i \to M_m$ is a complete contraction. Given $x = (x_i) \in M_n(\oplus^1_{i\in I} E_i)$ we hence see that

$$\begin{aligned}
&\sup\big\{\|\sum_i (T_i)_n(x)\| : (T_i) \text{ complete contractions to } F\big\} \\
&= \sup\big\{\|\langle\langle \mu, \sum_i (T_i)_n(x)\rangle\rangle\| : (T_i) \text{ complete contractions}, \|\mu\| \leq 1\big\} \\
&\leq \sup\big\{\|(S_i)_n(x)\| : (T_i) \text{ complete contractions to } M_m\big\}.
\end{aligned}$$

However, this final supremum is just the dual structure norm. In conclusion, we have hence shown that the universal property structure and the dual structure agree.

The universal property structure is useful for proving existence of certain maps, but as we have seen, the dual structure is easier to work with if one wishes to compute the norm of an element.

4 Duality results

We now turn our attention of the duals of the 1-sum and ∞-sum. The first proof is routine, but we find that the second claim is non-trivial to show.

Proposition 4.1. *The dual of $\oplus^1_{i\in I} E_i$ is $\oplus^\infty_{i\in I} E_i^*$ for the natural duality.*

Proof. For Banach spaces, this is true, and so we need only show that the operator space structure on $(\oplus^1_{i\in I} E_i)^*$ is $\oplus^\infty_{i\in I} E_i^*$. By the definition of the (dual) operator space structure on $\oplus^1_{i\in I} E_i$ we have that

$$\Big\|\sum_i \langle\langle \mu_i, x_i\rangle\rangle\Big\|_{M_{nm}} \leq \|(\mu_i)\|_n \|(x_i)\|_m \qquad \big((\mu_i) \in M_n(\oplus^\infty_{i\in I} E_i^*), (x_i) \in M_m(\oplus^1_{i\in I} E_i)\big).$$

Denote by $\|\cdot\|_n^*$ the norm on $M_n(\oplus^\infty_{i\in I} E_i^*)$ induced by identifying this with $(\oplus^1_{i\in I} E_i)^*$. Thus $\|\mu\|_n^* \leq \|\mu\|_n$.

Conversely, given $\mu = (\mu_i) \in M_n(\oplus^\infty_{i\in I} E_i^*)$ fix i and $\epsilon > 0$, so as $M_n(E_i^*) = \mathcal{CB}(E_i, M_n)$, there is $x \in M_m(E_i)$ with $\|x\|_m \leq 1$ and $\|\langle\langle \mu_i, x\rangle\rangle\| \geq \|\mu_i\| - \epsilon$. Then

$$\|\langle\langle \mu, (\iota_i)_m(x)\rangle\rangle\| = \|\langle\langle \mu_i, x\rangle\rangle\| \geq \|\mu_i\| - \epsilon,$$

so as $\|(\iota_i)_m(x)\| \leq 1$, it follows that $\|\mu\|_n^* \geq \|\mu_i\| - \epsilon$. As ϵ, i were arbitrary, it follows that $\|\mu\|_n^* \geq \sup_i \|\mu_i\| = \|\mu\|$.

Hence the norms on $M_n(\oplus^\infty_{i\in I} E_i^*)$ agree, as required. $\square$

Proposition 4.2. *The natural embedding of $\oplus^1_{i\in I} E_i^*$ into the dual of $\oplus^\infty_{i\in I} E_i$ is a complete isometry.*

Proof. By definition, the natural pairing gives rise to a complete isometry from $\oplus^1_{i\in I} E_i^*$ to the dual of $\oplus^\infty_{i\in I} E_i^{**}$. That is, given $\mu = (\mu_i) \in M_n(\oplus^1_{i\in I} E_i^*)$ and $\epsilon > 0$ there is $x = (x_i) \in M_m(\oplus^\infty_{i\in I} E_i^{**})$ with $\|\langle\langle x, \mu\rangle\rangle\| \geq \|\mu\| - \epsilon$ and $\|x\| \leq 1$. Now, $\langle\langle x, \mu\rangle\rangle = \sum_i \langle\langle x_i, \mu_i\rangle\rangle$.

For an operator space E we have that $M_m(E^{**}) = M_m(E)^{**}$ as a Banach space (see [3, Proposition 7.1.6], compare [3, Lemma 4.1.1]). Following [3, Section 4.1] let $T_m(E^*)$ be $M_m(E^*)$ together with the norm induced by the pairing between $M_m(E^*)$ and $M_m(E)$ given by

$$\langle \mu, x\rangle = \sum_{j,k=1}^n \langle \mu_{jk}, x_{jk}\rangle \in \mathbb{C} \qquad \big(x \in M_m(E), \mu \in T_m(E^*)\big).$$

Then $M_m(E)^* = T_m(E^*)$ and $T_m(E^*)^* = M_m(E^{**})$.

Given $\lambda \in E^*$ and j_0, k_0 define $\mu_{jk} = \lambda$ if $j = j_0, k = k_0$, and 0 otherwise. Then $\mu \in T_m(E^*)$ and $\langle \mu, x\rangle = \langle \lambda, x_{j_0,k_0}\rangle$. In this way, given $\phi \in M_n(E^*)$, there is a finite-dimensional subspace $X \subseteq T_m(E^*)$ depending on ϕ, such that if $\phi \in M_m(E^{**})$ and $x \in M_m(E)$ satisfy

$$\langle \Phi, \mu\rangle = \langle \mu, x\rangle \qquad (\mu \in X),$$

then $\langle\langle \Phi, \phi\rangle\rangle = \langle\langle \phi, x\rangle\rangle$.

For each $i \in I$ choose X_i using $\mu_i \in M_n(e_i)$. By Goldstein's Theorem, we can find $y_i \in M_m(E_i)$ with $\|y_i\| < \|x_i\| + \epsilon \leq 1 + \epsilon$ and with $\langle x_i, \mu\rangle = \langle \mu, y_i\rangle$ for $\mu \in X_i$. It follows that $\langle\langle x_i, \mu_i\rangle\rangle = \langle\langle \mu_i, y_i\rangle\rangle$. Then $y = (y_i) \in M_m(\oplus^\infty_{i\in I} E_i)$ with $\|y\| \leq 1 + \epsilon$ and

$$\langle\langle \mu, y\rangle\rangle = \sum_i \langle\langle \mu_i, y_i\rangle\rangle = \sum_i \langle\langle x_i, \mu_i\rangle\rangle = \langle\langle x, \mu\rangle\rangle.$$

As $\epsilon > 0$ was arbitrary, it follows that $\oplus^\infty_{i\in I} E_i$ norms $\oplus^1_{i\in I} E_i^*$, as required. ☐

Corollary 4.3. *If I is finite, then the dual of $\oplus^\infty_{i\in I} E_i$ is $\oplus^1_{i\in I} E_i^*$.*

Proof. The underlying Banach spaces agree, and so do the operator space structures by the above result. ☐

5 For operators

Proposition 5.1. *For any family (E_i) of operator spaces, and an operator space F, we have that:*

1. $\mathcal{CB}(\oplus^1_{i\in I} E_i, F) = \oplus^\infty_{i\in I} \mathcal{CB}(E_i, F)$;
2. $\mathcal{CB}(F, \oplus^\infty_{i\in I} E_i) = \oplus^\infty_{i\in I} \mathcal{CB}(F, E_i)$.

Proof. By the universal property, if (T_i) is a bounded family in $\oplus^\infty_{i\in I} \mathcal{CB}(E_i, F)$ then $T = \oplus_i T_i \in \mathcal{CB}(\oplus^1_{i\in I} E_i, F)$. Conversely, given $T \in \mathcal{CB}(\oplus^1_{i\in I} E_i, F)$ let $T_i = T \circ \iota_i \in \mathcal{CB}(E_i, F)$ so $\|T_i\|_{cb} \leq \|T\|_{cb}$ and $T = \oplus_i T_i$. Thus, at the level of Banach spaces, we have the required equality.

By definition, we have that

$$M_n\big(\mathcal{CB}(\oplus^1_{i\in I} E_i, F)\big) = \mathcal{CB}(\oplus^1_{i\in I} E_i, M_n(F)),$$
$$M_n\Big(\oplus^\infty_{i\in I} \mathcal{CB}(E_i, F)\Big) = \oplus^\infty_{i\in I} M_n\big(\mathcal{CB}(E_i, F)\big) = \oplus^\infty_{i\in I} \mathcal{CB}(E_i, M_n(F)).$$

Thus also the nth matrix levels agree, and we have shown (1).

For (2) given $(T_i) \in \oplus^\infty_{i\in I} \mathcal{CB}(F, E_i)$ we can define $T : F \to \oplus^\infty_{i\in I} E_i$ by $T(x) = (T_i(x))$. Then, for $x \in M_n(F)$, we have from $M_n(\oplus^\infty_{i\in I} E_i) = \oplus^\infty_{i\in I} M_n(E_i)$ that

$$\|(T)_n(x)\| = \sup_{i\in I} \|(T_i)_n(x)\| \leq \|x\| \sup_{i\in I} \|T_i\|_{cb}.$$

Thus $T \in \mathcal{CB}(F, \oplus^\infty_{i\in I} E_i)$. Conversely, given $T \in \mathcal{CB}(F, \oplus^\infty_{i\in I} E_i)$ define $T_i = P_i \circ T \in \mathcal{CB}(F, E_i)$ so that $\|T_i\|_{cb} \leq \|T\|_{cb}$ for each i. We have hence established that $\mathcal{CB}(F, \oplus^\infty_{i\in I} E_i) = \oplus^\infty_{i\in I} \mathcal{CB}(F, E_i)$ as Banach spaces. Again, that the operator spaces structures are also equal follows from

$$M_n\big(\mathcal{CB}(F, \oplus^\infty_{i\in I} E_i)\big) = \mathcal{CB}(F, \oplus^\infty_{i\in I} M_n(E_i)),$$
$$M_n\big(\oplus^\infty_{i\in I} \mathcal{CB}(F, E_i)\big) = \oplus^\infty_{i\in I} M_n(\mathcal{CB}(F, E_i)) = \oplus^\infty_{i\in I} \mathcal{CB}(F, M_n(E_i)).$$

□

By the usual duality between $\mathcal{CB}$ and the operator space projective tensor product $\widehat{\otimes}$, [3, Chapter 7], we have the following.

Corollary 5.2. *For any family (E_i) of operator spaces, and an operator space F, we have that:*

1. $(\oplus^1_{i\in I} E_i)\widehat{\otimes} F \cong \oplus^1_{i\in I}(E_i \widehat{\otimes} F)$;
2. $F\widehat{\otimes}(\oplus^1_{i\in I} E_i) \cong \oplus^1_{i\in I}(F\widehat{\otimes} E_i)$.

Proof. We have that $((\oplus^1_{i\in I} E_i)\widehat{\otimes} F)^* \cong \mathcal{CB}(\oplus^1_{i\in I} E_i, F^*)$ and $(\oplus^1_{i\in I} E_i \widehat{\otimes} F)^* \cong \oplus^\infty_{i\in I} \mathcal{CB}(E_i, F^*)$, the isomorphisms respecting our other identifications.

The second claim follows by commutativity of the projective tensor product, or from observing that $(F\widehat{\otimes}(\oplus^1_{i\in I} E_i))^* = \mathcal{CB}(F, \oplus^\infty_{i\in I} E_i^*)$ while $(\oplus^1_{i\in I}(F\widehat{\otimes} E_i))^* \cong \oplus^\infty_{i\in I} \mathcal{CB}(F, E_i^*)$. □

6 Category theory aspects

We shall consider the category of operator spaces and complete contractions. Given (E_i) a family of operator spaces, the category theoretic *product* is an operator space E and morphisms $\pi_i : E \to E_i$ such that, for any F and morphisms $T_i : F \to E_i$ there is a unique $T : F \to E$ with $\pi_i \circ T = T_i$ for each i. As a diagram, we have

$$\begin{tikzcd} F \arrow[r, "T_i"] \arrow[dr, dashed, "\exists!\, T"'] & E_i \\ & E \arrow[u, "\pi_i"'] \end{tikzcd}$$

If a product exists, it is unique. Let us remind ourselves why this is true. Let (E', π_i') be another product. Consider the universal property of E applied to the maps $\pi_i' : E' \to E_i$ which yields a map ϕ; and similarly consider the universal property of E' applied to the maps $\pi_i : E \to E_i$ which yields a map ϕ'. Finally, consider the universal property of E applied to the map $\pi_i : E \to E_i$, which of course has the diagram as shown on the right.

$$\begin{tikzcd} E' \arrow[r, "\pi_i'"] \arrow[dr, dashed, "\exists!\, \phi"'] & E_i \\ & E \arrow[u, "\pi_i"'] \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\pi_i"] \arrow[dr, dashed, "\exists!\, \phi'"'] & E_i \\ & E' \arrow[u, "\pi_i'"'] \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\pi_i"] \arrow[dr, dashed, "\mathrm{id}"'] & E_i \\ & E \arrow[u, "\pi_i"'] \end{tikzcd}$$

Consider now $\phi \circ \phi'$ which has the diagram

$$\begin{tikzcd} E \arrow[r, "\phi'"] \arrow[dr, "\pi_i"'] & E' \arrow[r, "\phi"] \arrow[d, "\pi_i'"] & E \arrow[dl, "\pi_i"] \\ & E_i & \end{tikzcd}$$

By uniqueness, we must hence have that $\phi \circ \phi' = \mathrm{id}$. Similarly $\phi' \circ \phi = \mathrm{id}$. Hence ϕ (and ϕ') is a completely isometric isomorphism.

We showed above that $\oplus^\infty_{i\in I} E_i$ is a concrete realisation of the product.

The *coproduct* is the same concept, with the "arrows reversed". It follows from discussion above that $\oplus^1_{i\in I} E_i$ is a concrete realisation of the coproduct.

With this in mind, Proposition 5.1 would follows from abstract category theoretic considerations.

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