# Hilbert $C^*$-bimodules and conjugation theory

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## 1 Introduction

We summarise some of the ideas from [4], and clarify some of the proofs. We very closely study the [4, Section 2], giving almost complete proofs, and clarifying some points. Much of the rest of this document currently is then just a summary.[1]

After a little time, I found that actually, for my purposes, the paper [5] is a somewhat more gentle introduction to similar ideas.

## 2 bi-Hilbertian modules

We follow [4, Section 2]. We assume the basic theory of Hilbert $C^*$-modules, see Lance's book [6], for example.

For $x, y \in X$ we write $\theta_{x,y}$ for the adjointable operator $z \mapsto x(y|z)$. Write $\mathcal{K}_0(X)$ for the linear span of such operators, write $\mathcal{K}(X)$ for the (operator) norm closure, and write $\mathcal{L}(X)$ for the algebra of all adjointable operators.

An $A$-$B$-bimodule is a Hilbert $C^*$-module $X$ over $B$, with a $*$-homomorphism $\phi\colon A \to \mathcal{L}(X)$ turning $X$ into a left $A$-module. We caution that terminology is confused here: something this notion is termed a "correspondence": sometimes a "bimodule" means something somewhat stronger, see [8, Section 1.5] for example. We write ${}_AX_B$ if we wish to stress which algebra is acting on the left/right.

We also have the notion of a *left* Hilbert $C^*$-module. An $A$-$B$-bimodule ${}_AX_B$ is *bi-Hilbertian* if it is both a (right) Hilbert $C^*$-module over $B$, a left Hilbert $C^*$-module over $A$, such that both structures lead to bimodules in the above sense, and such that the induced norms are equivalent: there are constants $\lambda, \lambda' > 0$ such that

$$\lambda' \|(x|x)_B\| \leqslant \|{}_A(x|x)\| \leqslant \lambda \|(x|x)_B\| \qquad (x \in X).$$

That we get bimodules means, for example, that $(x|ay)_B = (a^*x|y)_B$ and ${}_A(x|yb) = {}_A(xb^*|y)$. We stress that no further compatibility is assumed between $(\cdot|\cdot)_B$ and ${}_A(\cdot|\cdot)$ (but see Section 2.2 below).

**Proposition 2.1** (See [4, Proposition 2.4])**.** *Suppose that* $\lambda' \|(x|x)_B\| \leqslant \|{}_A(x|x)\|$ *for each* $x \in X$*. Then for any* $x_1, \cdots, x_n \in X$ *we have*

$$\lambda' \Big\| \sum_{i=1}^n \theta_{x_i,x_i} \Big\| \leqslant \Big\| \sum_{i=1}^n {}_A(x_i|x_i) \Big\|.$$

*Proof.* I don't follow the (end of the) proof given in [4], so we give the details.

Let $T \in M_n(B)$ be the matrix with $i,j$ entry $(x_i|x_j)_B$. As shown in [3, Lemma 2.1], we have that $\|T\| = \|\sum_{i=1}^n \theta_{x_i,x_i}\|$. Further, we have that

$$\|T\| = \sup \Big\{ \Big\| \sum_{i,j=1}^n b_i^*(x_i|x_j)b_j \Big\| : \Big\| \sum_{i=1}^n b_i^* b_i \Big\| \leqslant 1 \Big\}.$$

[1] Due to lack of time and/or motivation.

For any choice of $(b_i)$ with $\sum_i b_i^* b_i \leqslant 1$, set $y = \sum_i x_i b_i$, so that

$$\lambda' \|(y|y)_B\| \leqslant \|{}_A(y|y)\| = \Big\| \sum_{j=1}^{n} {}_A(y b_j^* | x_j) \Big\| \leqslant \big\|(y b_j^*)_j\big\| \big\|(x_j)_j\big\|. \tag{1}$$

The final inequality comes from Cauchy–Schwarz applied to the Hilbert $C^*$-module $X \otimes \mathbb{C}^n$ with $A$-valued inner product $((w_i)|(z_i)) = \sum_i {}_A(w_i|z_i)$. Thus, by definition,

$$\big\|(y b_j^*)_j\big\| = \Big\| \sum_j {}_A(y b_j^* | y b_j^*) \Big\|^{1/2} = \Big\| \sum_j {}_A(y b_j^* b_j | y) \Big\|^{1/2} \leqslant \|y\|_A \Big\| \sum_j b_j^* b_j \Big\|^{1/2} \leqslant \|y\|_A. \tag{2}$$

As $\|{}_A(y|y)\| = \|y\|_A^2$, combining (1) and (2) shows that

$$\|y\|_A^2 \leqslant \|y\|_A \Big\| \sum_j {}_A(x_j|x_j) \Big\|^{1/2} \quad \implies \quad \|y\|_A^2 \leqslant \Big\| \sum_{j=1}^{n} {}_A(x_j|x_j) \Big\|.$$

Taking the supremum over such $(y_i)$ yields the result. □

As shown in [4, Proposition 1.4], there is a directed set $\Lambda$ such that for each $\lambda \in \Lambda$, there is a finite set $u_\lambda \subseteq X$ so that, when we set $T_\lambda = \sum_{x \in u_\lambda} \theta_{x,x}$, we have that $(T_\lambda)_{\lambda \in \Lambda}$ is an increasing bounded approximate identity for $\mathcal{K}(X)$, of norm $\leqslant 1$. We term $(u_\lambda)$ a *generalised basis for* $X$.

Let ${}_A X_B$ be bi-Hilbertian. We wish to show that there is a linear map

$$F \colon \mathcal{K}_0(X) \to A; \quad \theta_{x,y} \mapsto {}_A(x|y).$$

The only non-trivial thing to check is that if $\sum_{i=1}^n \theta_{x_i,y_i} = 0$ then $\sum_{i=1}^n {}_A(x_i|y_i) = 0$. This seems not to be obvious; here is one way to show this. Let $(u_\lambda)$ be a generalised basis, and for each $\lambda \in \Lambda$, set

$$F_\lambda \colon \mathcal{L}(X_B) \to A; \quad T \mapsto \sum_{y \in u_\lambda} {}_A(Ty|y).$$

As $\lim_\lambda \sum_{y \in u_\lambda} \theta_{y,y}(z) = \lim_\lambda \sum_{y \in u_\lambda} y(y|z)_B = y$ for $\|\cdot\|_B$, and hence also for $\|\cdot\|_A$ as these norms are equivalent, we see that

$$F_\lambda(\theta_{x,z}) = \sum_{y \in u_\lambda} {}_A(x(z|y)_B|y) = \sum_{y \in u_\lambda} {}_A(x|y(y|z)_B) \to {}_A(x|z).$$

In fact, this only uses that $\|{}_A(x|x)\| \leqslant \lambda \|(x|x)_B\|$ for all $x \in X$. We can now conclude that

$$\sum_{i=1}^{n} \theta_{x_i,y_i} = 0 \quad \implies \quad \sum_{i=1}^{n} {}_A(x_i|y_i) = \sum_{i=1}^{n} \lim_\lambda F_\lambda(\theta_{x_i,y_i}) = \lim_\lambda F_\lambda\Big( \sum_{i=1}^{n} \theta_{x_i,y_i} \Big) = 0.$$

So $F$ exists.

We can now give some properties of $F$; see Lemma A.1 below for context, especially for (d).

**Proposition 2.2** ([4, Lemma 2.6])**.** *The map $F \colon \mathcal{K}(X_B) \to A; \theta_{x,y} \mapsto {}_A(x|y)$ satisfies:*

*(a)* $F(T^*T) \geqslant 0$ *for* $T \in \mathcal{K}_0(X_B)$*;*

*(b)* $F(T)^* = F(T^*)$ *for* $T \in \mathcal{K}_0(X_B)$*;*

*(c) letting $\phi \colon A \to \mathcal{L}(X_B)$ be the left-module action, we have $F(\phi(a)T) = aF(T)$ and $F(T\phi(a)) = F(T)a$ for $a \in A, T \in \mathcal{K}_0(X_B)$;*

*(d) supposing also the lower inequality $\lambda' \|(x|x)_B\| \leqslant \|{}_A(x|x)\|$, we have $\|F(T)\| \geqslant \lambda' \|T\|$ for any $T$ of the form $\sum_{i=1}^n \theta_{x_i,x_i}$.*

*Proof.* Part (a) follows from Lemma A.1.Part (b) is immediate. Part (c) follows from the identities $\phi(a)\theta_{x,y} = \theta_{a\cdot x,y}$ and $\theta_{x,y}\phi(a) = \theta_{x,a^*\cdot y}$. Proposition 2.1 gives (d). □

When do we have the analogue of Proposition 2.1 for the right-hand inequality? This doesn't always hold, but the following gives some equivalent conditions.

**Proposition 2.3** ([4, Proposition 2.7]). *The following are equivalent:*

1. *there is $\lambda > 0$ so that for all $x_1, \cdots, x_n \in X$, we have*
$$\Big\|\sum_{j=1}^{n} {}_A(x_i|x_i)\Big\| \leqslant \lambda\Big\|\sum_{j=1}^{n} \theta_{x_i,x_i}\Big\|;$$
2. *there is $\lambda > 0$ so that for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in X$, we have*
$$\Big\|\sum_{j=1}^{n} {}_A(x_i|y_i)\Big\| \leqslant \lambda\Big\|\sum_{j=1}^{n} \theta_{x_i,y_i}\Big\|;$$
3. $F(T) \geqslant 0$ *for each* $T \in \mathcal{K}_0(X_B) \cap \mathcal{K}(X_B)^+$, *and* $\lambda = \sup_\lambda \big\|F(\sum_{y\in u_\lambda} \theta_{y,y})\big\| < \infty$ *for some generalised basis* $(u_\lambda)$.

*When these conditions hold, the smallest constants $\lambda > 0$ for which 1 and 2 hold agree, and these agree with the $\lambda$ in 3. In particular, condition 3 does not depend on the choice of generalised basis.*

The smallest such $\lambda > 0$ is the *right numerical index* of $X$, denoted $r\text{-}I[X]$. We define the contragradient module $\overline{X}$, Appendix A.1, which is a $B$-$A$-bimodule, and then the *left numerical index* is $l\text{-}I[X] = r\text{-}I[\overline{X}]$.

Notice that condition 2 is simply the statement that $\|F\| \leqslant \lambda$.

**Proposition 2.4** ([4, Corollary 2.11]). *Let $X$ have finite right numerical index. Let $\lambda'$ be the constant in our assumed inequality $\lambda'\|(x|x)_B\| \leqslant \|{}_A(x|x)\|$ for each $x \in X$, and let $\phi\colon A \to \mathcal{L}(X)$ be the left action. We have that $\phi(F(T)) \geqslant \lambda' T$ for $T \in \mathcal{K}(X)^+$.*

*Proof.* By Proposition 2.2(d) combined with Lemma A.1, we have that $\|F(T)\| \geqslant \lambda'\|T\|$ for any $T = S^*S$ with $S \in \mathcal{K}_0(X)$. By continuity, the same holds for $T = S^*S$ with $S \in \mathcal{K}(X)$, that is, for any $T \in \mathcal{K}(X)^+$.

We now adapt an argument from [2, Pages 90–91]. For $\epsilon > 0$ we have that $\epsilon + F(T)$ is invertible (as $F(T) \geqslant 0$) and positive, and so we can set $S = T^{1/2}\phi((\epsilon + F(T))^{-1/2}) = T^{1/2}(\epsilon + \phi(F(T)))^{-1/2} \in \mathcal{K}(X_B)$. Then

$$F(S^*S) = F\big((\epsilon + \phi(F(T)))^{-1/2}T(\epsilon + \phi(F(T)))^{-1/2}\big) = (\epsilon + F(T))^{-1/2}F(T)(\epsilon + F(T))^{-1/2},$$

in the last step using Proposition 2.2(c). So $F(S^*S) \leqslant 1$, and hence $\lambda'\|S^*S\| \leqslant \|F(S^*S)\| \leqslant 1$. Hence $\lambda' S^*S \leqslant 1$ and so $\lambda'(\epsilon + \phi(F(T)))^{-1/2}T(\epsilon + \phi(F(T)))^{-1/2} \leqslant 1$. Multiplying both sides by $(\epsilon + \phi(F(T)))^{1/2}$ we obtain $\lambda' T \leqslant \epsilon + \phi(F(T))$ and letting $\epsilon \to 0$, we conclude that $\lambda' T \leqslant \phi(F(T))$ as claimed. □

## 2.1 Examples

### 2.1.1 Conditional expectations

Let $B \subseteq A$ be an inclusion of $C^*$-algebras, and let $E\colon A \to B$ be a conditional expectation. This means that $E$ is contractive and $E(b) = b$ for $b \in B$; then automatically $E$ is (completely) positive and a bimodule map over $A$, see [10, Theorem III.3.4]. Assume there is a constant $\lambda > 0$ so that $\|E(a)\| \geqslant \lambda\|a\|$ for each $a \in A^+$.

We turn ${}_AX_B = A$ into a $A$-$B$-bimodule for the obvious actions, $B$-valued inner-product $(x|y)_B = E(x^*y)$ and $A$-valued inner-product ${}_A(x|y) = xy^*$. As $E$ is contractive, and by our assumption, we have for each $x \in X$,

$$\|(x|x)_A\| = \|E(x^*x)\| \leqslant \|x^*x\| = \|x\|^2 = \|x^*\|^2 = \|xx^*\| = \|{}_B(x|x)\| \leqslant \lambda^{-1}\|E(x^*x)\| = \lambda^{-1}\|(x|x)_A\|,$$

and so the two inner-products give equivalent norms.

Consider the contragradient module ${}_B\overline{X}_A$, see Appendix A.1, so the right inner-product is $(\overline{x}|\overline{y})_A = {}_A(x|y) = xy^*$. The map $u\colon \overline{X} \to A; \overline{x} \mapsto x^*$ is hence linear, bijective, and an isometry for the right-norm. Using (6) we see that

$$u\theta_{\overline{x},\overline{y}}u^{-1}(z) = u\theta_{\overline{x},\overline{y}}(\overline{z^*}) = u(\overline{{}_A(z^*|y)\cdot x}) = u(\overline{z^*y^*x}) = x^*yz, \tag{3}$$

and so $\theta_{\overline{x},\overline{y}}$ is identified with $x^*y$. Hence [3, Lemma 2.1] (also used in the proof of Proposition 2.1) shows that

$$\Big\|\sum_{i=1}^n x_i^*x_i\Big\|_A = \Big\|\sum_{i=1}^n \theta_{\overline{x_i},\overline{x_i}}\Big\| = \big\|\big((\overline{x_i}|\overline{x_j})_A\big)_{i,j}\big\|_{M_n(A)} = \big\|\big({}_A(x_i|x_j)\big)_{i,j}\big\|_{M_n(A)} = \big\|\big(x_ix_j^*\big)_{i,j}\big\|_{M_n(A)}.$$

Analogously, [3, Lemma 2.1] applied to $X$ shows that

$$\big\|\big(E(x_i^*x_j)\big)_{i,j}\big\|_{M_n(A)} = \big\|\big((x_i|x_j)_B\big)_{i,j}\big\|_{M_n(A)} = \Big\|\sum_{i=1}^n \theta_{x_i,x_i}\Big\|. \tag{4}$$

As the two inner-products give equivalent norms, as ${}_AX = A$ is complete, also $X_B$ is complete, and so [2, Theorem 1] implies that the map $(E - \lambda\,\mathrm{id}_A)$ is positive (this is essentially the argument we used in the proof of Proposition 2.4 above). In fact, [2, Theorem 1] gives us more: there is $\lambda' > 0$ so that $(E - \lambda'\,\mathrm{id}_A)$ is completely positive. Together with (3) and (4) we obtain

$$\Big\|\sum_{i=1}^n \theta_{x_i,x_i}\Big\| = \big\|\big(E(x_i^*x_j)\big)_{i,j}\big\|_{M_n(A)} \geqslant \lambda'\big\|\big(x_i^*x_j\big)_{i,j}\big\|_{M_n(A)} = \Big\|\sum_{i=1}^n x_ix_i^*\Big\|_A = \Big\|\sum_{i=1}^n {}_A(x_i|x_i)\Big\|_A.$$

Hence we've verified the first condition in Proposition 2.3, and so $X$ has finite right numerical index, with constant at most $1/\lambda'$. [4, Proposition 2.12] shows we have equality, $r\text{-}I[X] = 1/\lambda'$.

Furthermore, to verify [4, Proposition 2.12] for $\overline{X}$, for constant $\lambda'' > 0$, we wish to show that

$$\Big\|\sum_{i=1}^n {}_B(\overline{x_i}|\overline{x_i})\Big\| \leqslant \lambda''\Big\|\sum_{i=1}^n \theta_{\overline{x_i},\overline{x_i}}\Big\|.$$

Using the map $u$ again, this is equivalent to

$$\Big\|E\Big(\sum_{i=1}^n x_i^*x_i\Big)\Big\| = \Big\|\sum_{i=1}^n (x_i|x_i)_B\Big\| \leqslant \lambda''\Big\|\sum_{i=1}^n x_i^*x_i\Big\|.$$

Clearly this holds with $\lambda'' = 1$ and this is the best constant. So $l\text{-}I[X] = 1$.

Following Watatani, [11, Definition 1.2.2], a *quasi-basis* for $E$ is a finite family $(u_1, v_1), \cdots, (u_n, v_n)$ in $A \times A$ with

$$\sum_{i=1}^n u_iE(v_ix) = \sum_{i=1}^n E(xu_i)v_i = x \qquad (x \in A). \tag{5}$$

The *index* of $E$ is $\mathrm{Index}(E) = \sum_i u_iv_i$. Then [11, Proposition 2.6.2] shows that

$$E(a) \geqslant \|\mathrm{Index}(E)\|^{-1}a \qquad (a \in A^+).$$

Hence such a conditional expectation fits into this framework (note that in [11] we always assume that $E$ is faithful).

### 2.1.2 Finite basis

Remember that $X$ has a finite basis if there are $(u_i)_{i=1}^n$ in $X$ with $\sum_i \theta_{u_i,u_i} = \mathrm{id}_X$. By comparison, notice that (5) can be re-written as

$$\sum_i \theta_{u_i,v_i^*}(x) = \sum_i u_i\cdot(v_i^*|x) = x, \quad \sum_i \theta_{v_i^*,u_i}(x^*) = \sum_i v_i^*E(u_i^*x^*) = x^* \qquad (x \in X),$$

and so a quasi-basis has $\sum_i \theta_{u_i,v_i^*} = \mathrm{id}_X$ (as then taking the adjoint shows that $\sum_i \theta_{v_i^*,u_i} = \mathrm{id}_X$). In fact, [11, Lemma 2.1.6] shows that we can always suppose that $v_i = u_i^*$ and so we have a finite basis.

If we have a finite basis $(u_i)_{i=1}^n$ then $\mathcal{K}_0(X) = \mathcal{K}(X) = \mathcal{L}(X)$, and so $F$ obviously extends to $\mathcal{K}(X)$. Hence $r\text{-}I[X] = \|F(1)\|$ by Proposition 2.3.

It is hence natural to also want $\overline{X}$ to have a finite basis. See [5] for much more in this setting.

## 2.2 Imprimitivity bimodules

We depart from Rieffel's original definition in [9] and instead follow the very readable [1]. Of course, this material is by now well-covered in many sources. A bi-Hilbertian module is a *Hilbert* $A$-$B$-*bimodule* when we have the stronger compatibility between the two inner-products:

$$ {}_A(x|y) \cdot z = a \cdot (y|z)_B. $$

This automatically implies that the left $A$-action (respectively, right $B$-action) is adjointable, see [1, Remark 1.9].

Let $I_A \subseteq A$ be the closed ideal generated by $\{{}_A(x|y) : x, y \in X\}$, and $I_B \subseteq B$ the closed ideal generated by $\{(x|y)_B : x, y \in X\}$. This, contrary to Rieffel, we do not assume $I_A = A$ and $I_B = B$. By [1, Proposition 1.10] we have that $\mathcal{K}(X) \cong I_A$ and $\mathcal{K}(\overline{X}) \cong I_B$. Furthermore, [1, Corollary 1.11] shows that $\|{}_A(x|x)\| = \|(x|x)_B\|$ for each $x \in X$. Thus in our main assumed inequality we can take $\lambda = \lambda' = 1$.

See [5, Corollary 1.28] for a characterisation of Imprimitivity bimodules in the main setting of this note.

# 3 Further constructions

We recall from e.g. [6, Proposition 2.5] the notion of a $*$-homomorphism being *non-degenerate*. As argued on [6, Page 5] when $(u_i)$ is an approximate identity for $B$, we have that $x \cdot u_i \to x$ for each $x \in X_B$.

**Lemma 3.1** ([4, Proposition 2.16])**.** *When ${}_A X_B$ is bi-Hilbertian, the $*$-homomorphism $\phi\colon A \to \mathcal{L}(X_B)$ is non-degenerate.*

*Proof.* By the observation about approximate identities, $A \cdot X$ is dense in $X$ for the norm from the $A$-valued inner-product. Hence also for the norm from the $B$-valued inner product, which is exactly the definition of $\phi$ begin non-degenerate. □

We can always extend $\phi$ by weak$^*$-weak$^*$-continuity to a $*$-homomorphism $\phi^{**}\colon A^{**} \to \mathcal{K}(X)^{**}$. To do this, we first use that $\mathcal{L}(X)$ is the multiplier algebra of $\mathcal{K}(X)$, and that $M(C) \subseteq C^{**}$ for any $C^*$-algebra $C$. Thus $\phi$ can be considered as a $*$-homomorphism $A \to \mathcal{K}(X)^{**}$, and then we take the normal extension. Then $\phi^{**}$ will be unital, as $\phi$ is non-degenerate.

Let ${}_A X_B$ have finite right numerical index, so that $F\colon \mathcal{K}(X_B) \to A$ is continuous. Hence $F$ extends to $F^{**}\colon \mathcal{K}(X_B)^{**} \to A^{**}$. The *right index element* is $F^{**}(1) \in A^{**}$ denoted $\text{r-}\mathrm{Ind}[X]$. Letting $(u_\lambda)$ be a generalised basis for $X_B$, we have that $T_\lambda = \sum_{y \in u_\lambda} \theta_{y,y}$ is an increasing approximate identity for $\mathcal{K}(X_B)$ and so increases to $1$ in $\mathcal{K}(X_B)^{**}$. Hence $\text{r-}\mathrm{Ind}[X] = \lim_\lambda F(T_\lambda)$ and so by Proposition 2.33, $\text{r-I}[X] = \|\text{r-}\mathrm{Ind}[X]\|$.

# 4 Categories of bimodules

We follow [4, Section 4]. Let $\mathcal{A}$ be a class of $C^*$-algebras. Objects of the category $\mathcal{H}_{\mathcal{A}}$ are (right) Hilbert $C^*$-modules over elements $B \in \mathcal{A}$, endowed with a non-degenerate left action of $A \in \mathcal{A}$, say ${}_A X_B$. The hom space is either empty, or $\hom({}_A X_B, {}_A Y_B) = \mathcal{L}(X_B, Y_B)$ the adjointable linear maps, when $X, Y$ are over the same $A, B \in \mathcal{A}$. We use the inner tensor product, so ${}_A X_B \otimes_B {}_B Y_C$. Given $T \in \mathcal{L}(X_B, X'_B)$ we can form $T \otimes \mathrm{id}_Y$ which is adjointable.

Any $B \in \mathcal{A}$ gives $\iota_B = {}_B B_B \in \mathcal{H}_{\mathcal{A}}$ in the obvious way, and ${}_A X_B \otimes \iota_B \cong X$ canonically. We also have $\iota_A \otimes_A {}_A X_B \cong X$ as the left action is assumed non-degenerate.

Notice that $\mathcal{H}_{\mathcal{A}}$ is not quite a tensor 2-$C^*$-category, because $\mathrm{id}_X \otimes T$ need not exist. However, if we look at elements $T \in \mathcal{L}(X_B, X'_B)$ which commute with the left $A$ action, then $\mathrm{id}_X \otimes T$ will exist, see Proposition A.2. Write ${}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}}$ for this category with morphism bimodule maps.

We can also introduce a "weak-closure" of this: we have the same objects, but take the weak closure of $\mathcal{K}(X_B, Y_B)$ in a suitable bidual-like construction. See [4, Proposition 4.2] for a clearer statement.[2]

[2] But be aware that the proof of this proposition seems to end midway through the given proof: I think the last two paragraphs should not be in the proof, but rather the main body of the text.

We can now introduce the notion of a *conjugate* following [7]. We state this just for ${}_{\mathcal{A}}\mathcal{H}_{\mathcal{A}}$. Given ${}_AX_B$, a conjugate is ${}_BY_A$ if there are $R \in {}_A\mathcal{L}_B(B, Y \otimes_A X)$ and $\overline{R} \in {}_B\mathcal{L}_A(A, X \otimes_B Y)$ which satisfy the conjugate equations

$$(\overline{R}^* \otimes I_X)(I_X \otimes R) = I_X, \qquad (R^* \otimes I_Y)(I_Y \otimes \overline{R}) = I_Y.$$

To interpret these, we repeatedly use that $X \otimes_B B \cong X$, and so forth.

To avoid the weak completions, we shall just state results under slightly stronger hypotheses. Following [4, Definition 2.23] we say that a bi-Hilbertian $X$ is of *finite right index* when $X$ if of finite right numerical index, and $\text{r-Ind}[X] \in M(A)$. (Recall that by definition, this element is only in $A^{**}$ in general.) Similarly *finite left index* and *finite index* if both.

When $X$ is of finite index, we find that $\overline{X}$ is a conjugate of $X$, with intertwiners

$$\overline{R}^*(x \otimes \overline{y}) = {}_A(x|y), \qquad R^*(\overline{x} \otimes y) = (x|y)_B.$$

See [4, Theorem 4.4].

There is a converse, [4, Theorem 4.13]. Let $X_B$ be a right Hilbert $C^*$-module over $B$, and suppose there is a non-degenerate $\phi\colon A \to {}_AX_B$, so $X$ is an $A$-$B$-bimodule (but not assumed bi-Hilbertian). Suppose that $Y$ is a conjugate object, with intertwiners $R, \overline{R}$. We can then define

$${}_A(x|a \cdot y) = \overline{R}^*(\theta_{x,y} \otimes I_Y)\overline{R}(a^*) \qquad (a \in A, x, y \in X).$$

This gives an $A$-valued inner-product making $X$ bi-Hilbertian, of finite index. Once we have this structure, $Y$ is bi-unitarily equivalent to $\overline{X}$, with the intertwiners taking the above form.

So a characterisation of Imprimitivity bimodules (aka "Strong Morita Equivalence") see [4, Corollary 4.14].

# A Results on Hilbert modules

I don't know a reference for the following.

**Lemma A.1.** *Let $X_B$ be a Hilbert $C^*$-module over $B$. For $T \in \mathcal{K}_0(X_B)$ we have that $T^*T = \sum_{i=1}^n \theta_{z_i,z_i}$ for some $z_i$ in $X$.*

*Proof.* Compare https://mathoverflow.net/questions/488971. Let $T = \sum_{i=1}^n \theta_{x_i,y_i}$, so that $T^*T = \sum_{i,j} \theta_{y_i \cdot (x_i|x_j), y_j}$.

Consider the matrix $((x_i|x_j))_{i,j} \in M_n(B)$. Given $(a_i)_{i=1}^n$ in $B$,

$$\sum_{i,j} a_i^*(x_i|x_j)a_j = \Big( \sum_i x_i \cdot a_i \Big| \sum_j x_j \cdot a_j \Big) \geqslant 0,$$

so by [10, Lemma IV.3.2], the matrix is positive in $M_n(B)$. Let $R \in M_n(B)$ be the positive square-root, so $(x_i|x_j) = (R^*R)_{i,j} = \sum_k (R_{i,k})^* R_{k,j}$. Thus

$$T^*T = \sum_{i,j,k} \theta_{y_i \cdot R_{i,k}^* R_{k,j}, y_j} = \sum_{i,j,k} \theta_{y_i \cdot R_{i,k}^*, y_j \cdot R_{k,j}^*} = \sum_k \theta_{z_k,z_k},$$

here setting $z_k = \sum_i y_i \cdot R_{k,i}^*$. □

Again, I am not aware of explicit reference for the following. The idea of this proof is motivated by Lance's book [6].

**Proposition A.2.** *Let $T \in \mathcal{L}(Y_C, Y'_C)$ commute with the left action of $B$ on ${}_BY_C$ and ${}_BY'_C$. Then $\mathrm{id}_X \otimes T \in \mathcal{L}({}_AX \otimes_B Y_C, {}_AX \otimes_B Y'_C)$ with adjoint $\mathrm{id}_X \otimes T^*$.*

*Proof.* Let $x_1, \cdots, x_n \in X$ and $y_1, \cdots, y_n \in Y$, and let $x = ((x_i|x_j))_{i,j} \in M_n(B)$. By [6, Lemma 4.2], $x$ is positive; let $b \in M_n(B)$ be a square-root. Consider $u = \sum_i x_i \otimes y_i \in X \otimes_B Y$, so

$$\begin{aligned}\|(\mathrm{id}_X \otimes T)u\|^2 &= \Big\| \sum_{i,j} (x_i \otimes Ty_i | x_j \otimes Ty_j) \Big\| = \Big\| \sum_{i,j} (Ty_i | (x_i|x_j) \cdot Ty_j) \Big\| \\ &= \Big\| \sum_{i,j,k} (Ty_i | b_{k,i}^* b_{k,j} \cdot Ty_j) \Big\| = \Big\| \sum_{i,j,k} (T(b_{k,i} \cdot y_i) | T(b_{k,j} \cdot y_j)) \Big\|\end{aligned}$$

here using that $T$ commutes with the right $B$-action. This is of the form $(T\nu|T\nu)$ and by [6, Proposition 1.2] we have that $(T\nu|T\nu) \leqslant \|T\|^2(\nu|\nu)$. Hence

$$\left\|(\mathrm{id}_X \otimes T)u\right\|^2 \leqslant \|T\|^2 \left\| \sum_{i,j,k} (b_{k,i} \cdot y_i | b_{k,j} \cdot y_j) \right\| = \|T\|^2 \|u\|^2,$$

by reversing the previous calculation. Hence $\mathrm{id}_X \otimes T$ is bounded, and a routine calculation shows that it is adjointable, with adjoint $\mathrm{id}_X \otimes T^*$. ◻

### A.1 Contragradient modules

Given ${}_AX_B$ we let $\overline{X}$ be the contragradient vector space, which becomes a $B$-$A$-bimodule for actions

$$b \cdot \overline{x} \cdot a = \overline{a^* \cdot x \cdot b^*} \qquad (a \in A, b \in B, x \in X).$$

We define the inner-products by

$${}_B(\overline{x}|\overline{y}) = (x|y)_B, \quad (\overline{x}|\overline{y})_A = {}_A(x|y) \qquad (x, y \in A).$$

Notice that this definition correctly makes, for example, the right $B$-valued inner-product conjugate linear in the 1st variable.

Clearly if ${}_AX_B$ is bi-Hilbertian then so is ${}_B\overline{X}_A$, as the norms will still be equivalent.

We consider finite-rank operators on $\overline{X}$:

$$\theta_{\overline{x},\overline{y}} : \overline{z} \mapsto \overline{x} \cdot (\overline{y}|\overline{z})_A = \overline{x} \cdot {}_A(y|z) = \overline{{}_A(y|z)^* \cdot x} = \overline{{}_A(z|y) \cdot x}. \tag{6}$$

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