Quantum Graphs: using diagrams to move from adjacency matrices to subspaces

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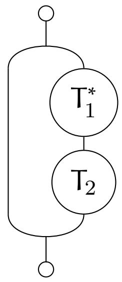
1 Introduction

These are some brief notes which follow ideas from [4], making links with [1, Section 5.4] and [3]. These notes are not self-contained, but can be read following [1]. In particular, we use the notation of [1, Section 5.1], with the exception that we write $|\xi\rangle\langle\eta|$ for the rank-one operator $\theta_{\eta,\xi} : \alpha \mapsto (\eta|\alpha)\xi$.

1.1 An inner product

Let B be a finite-dimensional C^*-algebra and ψ a faithful state on B. We use the diagrammatical calculus, see [2] for example. Introduce an inner-product on $\mathcal{B}(L^2(B))$ by

$(T_1|T_2) =$  $= \eta^* m(1 \otimes T_1^* T_2) m^* \eta = \psi(m(1 \otimes T_1^* T_2) m^*(1)) \qquad (T_1, T_2 \in \mathcal{B}(L^2(B))).$

Let's compute this more explicitly. Let $m^*(1) = \sum_i e_i \otimes f_i$, so that $\psi(b^* a^*) = (ab|1) = (a \otimes b|m^*(1)) = \sum_i (a|e_i)(b|f_i) = \sum_i \psi(a^* e_i)\psi(b^* f_i)$ for each $a, b \in B$. Let $T_j = |t_j\rangle\langle s_j|$ for $j = 1, 2$ be rank-one operators. Then

$$(T_1|T_2) = \psi\Big(\sum_i e_i T_1^* T_2(f_i) \Big) = \sum_i \psi(e_i s_1)(t_1|t_2)(s_2|f_i).$$

With ψ having density Q, we see that $\psi(e_i s_1) = \mathrm{Tr}(Q e_i s_1) = \mathrm{Tr}(Q Q^{-1} s_1 Q e_i) = (Q s_1^* Q^{-1}|e_i)$. Thus

$$\begin{aligned} (T_1|T_2) &= (t_1|t_2) \sum_i (Q s_1^* Q^{-1}|e_i)(s_2|f_i) = (t_1|t_2)(Q s_1^* Q^{-1} s_2|1) = (t_1|t_2)\psi(s_2^* Q^{-1} s_1 Q) \\ &= (t_1|t_2)\psi(s_2^* \sigma_i(s_1)) = (t_1|t_2)(s_2|\sigma_i(s_1)), \end{aligned}$$

where (σ_t) is the modular automorphism group. We briefly stop suppressing the GNS map Λ, and recall that the modular operator is given by $\nabla\Lambda(a) = \Lambda(\sigma_{-i}(a))$. Thus

$$(T_1|T_2) = (\Lambda(t_1)|\Lambda(t_2))(\Lambda(s_2)|\nabla^{-1}\Lambda(s_1)) = (\Lambda(t_1)|\Lambda(t_2))(\nabla^{-1/2}\Lambda(s_2)|\nabla^{-1/2}\Lambda(s_1)).$$

Consequently, with $T_j = |\xi_j\rangle\langle\eta_j|$ for $j = 1, 2$, we find that

$$(T_1|T_2) = (\xi_1|\xi_2)(\nabla^{-1/2}\eta_2|\nabla^{-1/2}\eta_1) = \big(\xi_1 \otimes \overline{\nabla^{-1/2}\eta_1}\big|\xi_2 \otimes \overline{\nabla^{-1/2}\eta_2}\big),$$

the final inner-product being the natural one on $L^2(B) \otimes \overline{L^2(B)}$, where $\overline{L^2(B)}$ is the conjugate Hilbert space to $L^2(B)$. Thus the map

$$\mathcal{B}(L^2(B)) \to L^2(B) \otimes \overline{L^2(B)}, \quad |\xi\rangle\langle\eta| \mapsto \xi \otimes \overline{\nabla^{-1/2}\eta} \tag{1}$$

extends linearly to a unitary.

In [1], we always identify $\mathcal{B}(L^2(B))$ with $L^2(B)\otimes\overline{L^2(B)}$, but for the "untwisted map" $|\xi\rangle\langle\eta|\mapsto\xi\otimes\overline{\eta}$. We continue to let $B\otimes B^{op}$ act on $\mathcal{B}(L^2(B))$ as $(a\otimes b)\cdot T = aTb$, for $a\in B, b\in B^{op}, T\in\mathcal{B}(L^2(B))$; similarly for $B'\otimes(B')^{op}$. We now see what this action becomes on $L^2(B)\otimes\overline{L^2(B)}$.

For $a,b\in B$, when $T=|\xi\rangle\langle\eta|$ we have that $aTb=|a\xi\rangle\langle b^*\eta|$, and so

$$(a\otimes b)\cdot(\xi\otimes\overline{\nabla^{-1/2}\eta})\cong aTb=|a\xi\rangle\langle b^*\eta|\cong a\xi\otimes\overline{\nabla^{-1/2}b^*\eta}.$$

Equivalently,

$$(a\otimes b)\cdot(\xi\otimes\overline{\eta}) = a\xi\otimes\overline{\nabla^{-1/2}b^*\nabla^{1/2}\eta} = a\xi\otimes\sigma_{-i/2}(b)^\top\overline{\eta}\qquad(a\otimes b\in B\otimes B^{op},\ \xi,\eta\in L^2(B)).$$

This uses that $\sigma_{-i/2}(b)=\nabla^{1/2}b\nabla^{-1/2}$. Exactly the same calculations shows that

$$(a\otimes b)\cdot(\xi\otimes\overline{\eta}) = a\xi\otimes(\nabla^{1/2}b\nabla^{-1/2})^\top\overline{\eta}\qquad(a,b\in B').$$

Hence the natural actions of $B\otimes B^{op}$, and $B'\otimes(B')^{op}$, on $L^2(B)\otimes\overline{L^2(B)}$ are "twisted". The following lemma shows that in both cases, we can think of this as a twist on the algebra ($B\otimes B^{op}$ or $B'\otimes(B')^{op}$ respectively) followed by the "natural" action on $L^2(B)\otimes\overline{L^2(B)}$.

Lemma 1.1. *For any $b\in B'\subseteq\mathcal{B}(L^2(B))$ and $z\in\mathbb{C}$, we have that $\nabla^z b\nabla^{-z}\in B'$, and so $b\mapsto\nabla^z b\nabla^{-z}$ defines an automorphism of B' (in general not $*$-preserving).*

Proof. This is related to identifing B' and B^{op}, compare [1, Lemma 5.32]. However, a simple calculation suffices. As $\nabla^z a\nabla^{-z}=\sigma_{-iz}(a)\in B$ for each $a\in B$, for $a\in B, b\in B'$ we see that

$$\nabla^z b\nabla^{-z}a=\nabla^z b\nabla^{-z}a\nabla^z\nabla^{-z}=\nabla^z b\sigma_{iz}(a)\nabla^{-z}=\nabla^z\sigma_{iz}(a)b\nabla^{-z}=\nabla^z\nabla^{-z}a\nabla^z b\nabla^{-z}=a\nabla^z b\nabla^{-z},$$

and so $\nabla^z b\nabla^{-z}$ commutes with a, for each $a\in B$, and hence is in B'. □

While it might seem odd to twist the action, notice that by doing so we maintain the bijections we used in [1], namely between:

(1) B'-bimodules $\mathcal{S}\subseteq\mathcal{B}(L^2(B))$;

(2) $B'\otimes(B')^{op}$-invariant subspaces $V\subseteq L^2(B)\otimes\overline{L^2(B)}$;

(3) projections $e\in B\otimes B^{op}$.

To be explicit, here we use the unitary from equation (1) to link $\mathcal{S}$ and V. The twisted action of $B'\otimes(B')^{op}$ on $L^2(B)\otimes\overline{L^2(B)}$ is not a $*$-homomorphism, but that does not matter when we are showing that (1) and (2) biject; all that matters is the compatibility of the B' actions.

When showing that (2) and (3) biject, we do need a $*$-homomorphism, and so here we consider the usual action of $B\otimes B^{op}$ on $L^2(B)\otimes\overline{L^2(B)}$. We now have two actions of $B'\otimes(B')^{op}$ on $L^2(B)\otimes\overline{L^2(B)}$, but they have the same invariant subspaces. Henceforth, we shall only consider the "natural" action.

1.2 Linking adjacency operators and projections

We follow [1, Section 5.4]. Let $A\in\mathcal{B}(L^2(B))$ be a quantum adjacency operator, and here we consider this to mean that the underlying linear map $A\colon B\to B$ is completely positive, and that A is idempotent for the Schur product. Then it is natural to consider the map $\Psi'=\Psi'_{0,1/2}\colon\mathcal{B}(L^2(B))\to B\otimes B^{op}$, as [1, Theorem 5.36] shows that $f=\Psi'(A)$ is positive, while (the proof of) [1, Theorem 5.37] shows that $f=f^2$, that is, f is a projection. (Here we write f not e to follow the notation of [1, Theorem 5.37].)

Henceforth, we continue with the notation that A is linked with $e=\Psi'_{0,1/2}(A)$. To be explicit, if $A=|b\rangle\langle a|\in\mathcal{B}(L^2(B))$, then $e=b\otimes\sigma_{i/2}(a)^*$. Letting $B\otimes B^{op}$ act "naturally" on $L^2(B)\otimes\overline{L^2(B)}$, we obtain the operator $b\otimes(\sigma_{i/2}(a)^*)^\top$. Now considering the unitary given by formula (1), we have

$$\mathcal{B}(L^2(B))\ni|\xi\rangle\langle\eta|\mapsto\xi\otimes\overline{\nabla^{-1/2}\eta}\xrightarrow{e}b\xi\otimes\overline{\sigma_{i/2}(a)\nabla^{-1/2}\eta}=b\xi\otimes\overline{\nabla^{-1/2}a\eta}\mapsto|b\xi\rangle\langle a\eta|=b|\xi\rangle\langle\eta|a^*.$$

Thus the associated action on $\mathcal{B}(L^2(B))$ is simply $T \mapsto bTa^*$; compare with Proposition 1.3 below.

We now consider $A = \sum_{j=1}^k |b_j\rangle\langle a_j|$ assumed to be Schur idempotent and completely positive, so that $e = \Psi'_{0,1/2}(A) = \sum_j b_j \otimes \sigma_{i/2}(a_j)^*$ is a (self-adjoint) projection. Hence

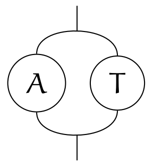
$$V = \mathrm{Im}(e) = \mathrm{lin}\left\{ \sum_j b_j\xi \otimes (\sigma_{i/2}(a_j)^*)^\top\overline{\eta} : \xi, \eta \in L^2(B)\right\} \subseteq L^2(B) \otimes \overline{L^2(B)},$$

and given the bijections just established, we have that

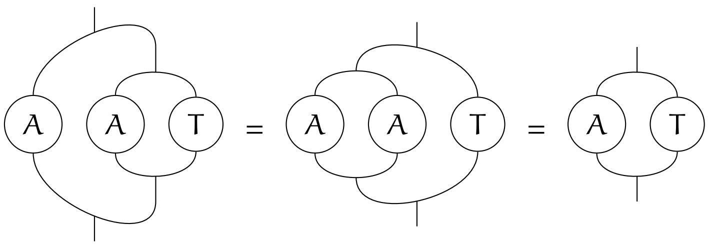
$$\mathcal{S} = \left\{ \sum_j b_j T a_j^* : T \in \mathcal{B}(L^2(B))\right\}. \tag{2}$$

Remark 1.2. One way to think here is that the "twist" introduced by $\Psi'_{0,1/2}$ is cancelled out by the unitary given by (1), and so the relation between A, and for the formula for $\mathcal{S}$, looks "untwisted". △

We can nicely write much of this using diagrams. Consider the map

$$\theta_A : \mathcal{B}(L^2(B)) \to \mathcal{B}(L^2(B)); \qquad T \mapsto$$

$$= A \bullet T. \tag{3}$$

This defines an idempotent, which can be seen diagrammatically, using that the multiplication on B is associative and that A is Schur idempotent:

$\theta_A(\theta_A(T)) =$


$= \theta_A(T).$

Proposition 1.3. *With $A = \sum_{j=1}^k |b_j\rangle\langle a_j|$, we have that $\theta_A(T) = \sum_j b_j T a_j^*$ for $T \in \mathcal{B}(L^2(B))$. Hence, with $\mathcal{S}$ given by (2) using A, we have that $\mathcal{S}$ is the image of the idempotent θ_A.*

Proof. Let $a \in B$ and let $m^*(a) = \sum_k c_k \otimes d_k$, so $\sum_k (a_1|c_k)(a_2|d_k) = (a_1a_2|a)$ for each $a_1, a_2 \in B$. Then, for $b \in A$, and with $T = |c\rangle\langle d|$, we have

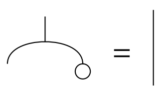
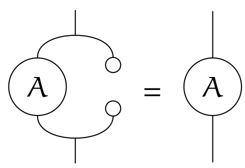
$$\begin{aligned}(b|\theta_A(T)a) = (m^*(b)|(A\otimes T)m^*(a)) &= \sum_{j,k}(m^*(b)|b_j \otimes T(d_k))(a_j|c_k) = \sum_{j,k}(m^*(b)|b_j\otimes c)(a_j|c_k)(d|d_k) \\ &= \sum_{j,k}(b|b_jc)(a_j|c_k)(d|d_k) = \sum_j (b|b_jc)(a_jd|a),\end{aligned}$$

from which it follows that

$$\theta_A(T) = \sum_j |b_jc\rangle\langle a_jd| = \sum_j b_jTa_j^*.$$

By linearity, this holds for all T, as claimed. It is now clear that $\mathcal{S}$ is the image of θ_A. □

Diagrammatically, we see that


$$\implies \qquad \theta_A(|1\rangle\langle 1|) =$$


$= A$

Thus we can recover A from θ_A.

Proposition 1.4. *We have that $\mathcal{S} = \mathrm{lin}\, B'AB'$, the B'-bimodule generated by the operator A.*

Proof. The previous proposition shows that $\mathcal{S} = \{\sum_j b_j T a_j^* : T \in \mathcal{B}(L^2(B))\}$ where $A = \sum_{j=1}^k |b_j\rangle\langle a_j|$. Let J be the modular conjugation, which satisfies $JaJ\Lambda(b) = \Lambda(b\sigma_{-i/2}(a^*))$, for $a, b \in B$, where Λ is the GNS map, and we have $JBJ = B'$. Thus

$$\begin{aligned}(Ja^*J)A(Jb^*J) &= \sum_j |Ja^*J\Lambda(b_j)\rangle\langle JbJ\Lambda(a_j)| = \sum_j |\Lambda(b_j\sigma_{-i/2}(a))\rangle\langle\Lambda(a_j\sigma_{-i/2}(b^*))| \\ &= \sum_j b_j|\sigma_{-i/2}(a)\rangle\langle\sigma_{-i/2}(b^*)|a_j^* = \theta_A(|\sigma_{-i/2}(a)\rangle\langle\sigma_{-i/2}(b^*)|).\end{aligned}$$

Taking the linear span as a, b vary, it follows that $B'AB' = \theta_A(\mathcal{B}(L^2(B))) = \mathcal{S}$, as claimed. □

2 KMS Inner products

We now consider [3]. Here Wasilewski works with the "KMS inner-product", which we shall denote by

$$(a|b)_K = \psi(a^*\sigma_{-i/2}(b)) \qquad (a, b \in B). \tag{4}$$

A simple calculation shows that $(a|b)_K = (\sigma_{-i/4}(a)|\sigma_{-i/4}(b))$. We continue to identify (or "confuse") B with $L^2(B)$, and so the rank-one operator $|a\rangle\langle b|$ can be considered as the map $B \to B; c \mapsto (b|c)a = \psi(b^*c)a$. Similarly, we define $|a\rangle\langle b|_K$ using the KMS inner-product, so

$$|a\rangle\langle b|_K : B \to B; \quad c \mapsto (b|c)_K a = \psi(b^*\sigma_{-i/2}(c))a.$$

Hence $|a\rangle\langle b|_K = |a\rangle\langle b| \circ \sigma_{-i/2}$. As $\psi(b^*\sigma_{-i/2}(c)) = \psi(\sigma_{-i/2}(b)^*c)$, also $|a\rangle\langle b|_K = |a\rangle\langle\sigma_{-i/2}(b)|$.

Then [3, Lemma 3.3] defines a map $\Psi^{\mathrm{KMS}} : \mathcal{B}(L^2(B)) \to B \otimes B^{\mathrm{op}}$ by, in our notation,

$$\Psi^{\mathrm{KMS}} : |a\rangle\langle b|_K \mapsto a \otimes b^*.$$

Consequently $\Psi^{\mathrm{KMS}}(|a\rangle\langle b|) = \Psi^{\mathrm{KMS}}(|a\rangle\langle\sigma_{i/2}(b)|_K) = a \otimes \sigma_{i/2}(b)^* = \Psi'_{0,1/2}(|a\rangle\langle b|)$. Thus $\Psi^{\mathrm{KMS}} = \Psi'_{0,1/2}$ in our notation.

Note that [3, Proposition 3.7] nicely separates out the correspondence between properties of $\Psi^{\mathrm{KMS}}(A)$ to those of A.

In Section 1.2, we established bijections between $A \in \mathcal{B}(L^2(B))$ with $e \in B \otimes B^{\mathrm{op}}$, and with $\theta_A \in \mathcal{B}(\mathcal{B}(L^2(B)))$. These are

$$A = |b\rangle\langle a| \qquad \leftrightarrow \qquad e = b \otimes \sigma_{i/2}(a)^* \qquad \leftrightarrow \qquad \theta_A : T \mapsto bTa^*.$$

We can write this alternatively as

$$A = |b\rangle\langle\sigma_{-i/2}(c^*)| \qquad \leftrightarrow \qquad e = b \otimes c \qquad \leftrightarrow \qquad \theta_A : T \mapsto bT\sigma_{i/2}(c). \tag{5}$$

This does not agree verbatim with [3, Proposition 3.14], but private communication with Wasilewski verifies that this is a typo (compare with the paragraph before [3, Proposition 3.14], where $\sigma_{-i/2}^{\psi^{-1}}$ is used: when restricted to B, as is occurring in this context, this map agrees with $\sigma_{i/2}$). Thus [3, Proposition 3.14] uses the same map θ_A.

2.1 A difference

An important difference between the conventions in this note and [3] is that the relation between $\mathcal{S}$ and A (and/or e) is different, compare [3, Theorem 3.15]. (Again, there is a sign convention here, with this note henceforth making the choice which seems to work, given our other conventions.)

For $z \in \mathbb{C}$ and a B'-bimodule $\mathcal{S} \subseteq \mathcal{B}(L^2(B))$, define $\mathcal{S}_z = Q^{-iz}\mathcal{S}Q^{iz}$ where $Q \in B$ is the density of ψ as in Section 1.1.

Lemma 2.1. *We have that $\mathcal{S}_z$ is a B'-bimodule.*

Proof. This is similar to the proof of Lemma 1.1. As $\sigma_z(a) = Q^{iz}aQ^{-iz}$ for each $a \in B$, exactly the same argument as before shows that $Q^{iz}xQ^{-iz} \in B'$ for each $x \in B'$. Hence, given $T \in \mathcal{S}$, so that $Q^{-iz}TQ^{iz} \in \mathcal{S}_z$, and given $x \in B'$, we see that

$$Q^{-iz}TQ^{iz}x = Q^{-iz}T(Q^{iz}xQ^{-iz})Q^{iz} \in Q^{-iz}\mathcal{S}Q^{iz} = \mathcal{S}_z,$$

as $Q^{iz}xQ^{-iz} \in B'$ and using that $\mathcal{S}$ is a B'-bimodule. Similarly $B'\mathcal{S}_z \subseteq \mathcal{S}_z$. □

Given A and hence θ_A as before, we now define $\mathcal{T} = \mathcal{S}_{i/4}$ where $\mathcal{S}$ is the image of the idempotent θ_A. From Proposition 1.3, we see that

$$\begin{aligned}
T \in \mathcal{T} = \mathcal{S}_{i/4} \quad &\Leftrightarrow \quad T \in Q^{1/4}\mathcal{S}Q^{-1/4} \quad \Leftrightarrow \quad Q^{-1/4}TQ^{1/4} \in \mathcal{S} \\
&\Leftrightarrow \quad \sum_j b_j Q^{-1/4}TQ^{1/4}a_j^* = Q^{-1/4}TQ^{1/4} \\
&\Leftrightarrow \quad \sum_j Q^{1/4}b_jQ^{-1/4}TQ^{1/4}a_j^*Q^{-1/4} = T \\
&\Leftrightarrow \quad \sum_j \sigma_{-i/4}(b_j)T\sigma_{i/4}(a_j)^* = T
\end{aligned} \tag{6}$$

This may look strange, but we'll see below in Section 3.1 that in some senses it is quite natural.

Remark 2.2. We could also use ∇ in place of Q, when defining $\mathcal{S}_z$. Remember that, with $\Lambda\colon B \to L^2(B)$ the GNS map, we have that $\nabla^z\Lambda(a) = \Lambda(Q^zaQ^{-z})$ for $a \in B$. Hence $\nabla^zQ^{-z}\Lambda(a) = \Lambda(aQ^{-z}) = Q^{-z}\nabla^z\Lambda(a)$ for $a \in B$, and hence $\nabla^zQ^{-z} = Q^{-z}\nabla^z \in B'$. So $\nabla^zQ^{-z}\mathcal{S}Q^z\nabla^{-z} \subseteq B'\mathcal{S}B' \subseteq \mathcal{S}$ for all z, and hence $\mathcal{S}_z = Q^{-iz}\mathcal{S}Q^{-iz} \subseteq \nabla^{-iz}\mathcal{S}\nabla^{iz}$. Reversing the roles of Q and ∇ shows the other inclusion, and so we have equality. △

Using Lemma 1.1 (and the above remark) and Proposition 1.4, the following is an easy calculation.

Proposition 2.3. *For $z \in \mathbb{C}$ we have that $\mathcal{S}_z = B'(Q^{-iz}AQ^{iz})B' = B'(\nabla^{-iz}A\nabla^{iz})B'$.*

3 One wrinkle: operator systems

We have not talked about operator systems $\mathcal{S}$, only B'-bimodules. Here we consider the missing properties. As in [1, Definition 5.11], we write

$$J_0\colon L^2(B) \otimes \overline{L^2(B)} \to L^2(B) \otimes \overline{L^2(B)};\, \xi \otimes \overline{\eta} \mapsto \eta \otimes \overline{\xi}$$

for the anti-linear unitary "tensor swap map".

When is $\mathcal{S}$ self-adjoint? Due to the relation given by formula (1), if $\mathcal{S}$ and V are related, then as $|\xi\rangle\langle\eta| \mapsto \xi \otimes \overline{\nabla^{-1/2}\eta}$, we see that

$$|\eta\rangle\langle\xi| \mapsto \eta \otimes \overline{\nabla^{-1/2}\xi} = (\nabla^{1/2} \otimes (\nabla^{-1/2})^\top)(\nabla^{-1/2}\eta \otimes \overline{\xi}) = (\nabla^{1/2} \otimes (\nabla^{-1/2})^\top)J_0(\xi \otimes \overline{\nabla^{-1/2}\eta}).$$

Hence $\mathcal{S}^*$ corresponds to $(\nabla^{1/2} \otimes (\nabla^{-1/2})^\top)J_0(V) = V_a$ (where "a" is chosen for "adjoint"). Let e be the orthogonal projection onto V, and e_a the orthogonal projection onto V_a. It seems hard to write down e_a in terms of e, as $V_a^\perp = J_0(\nabla^{1/2} \otimes (\nabla^{-1/2})^\top)(V^\perp)$.

Set $J_1 = (\nabla^{1/2} \otimes (\nabla^{-1/2})^\top)J_0$ and notice that

$$J_0(\nabla^{-1/2} \otimes (\nabla^{1/2})^\top)(\xi \otimes \overline{\eta}) = J_0(\nabla^{-1/2}\xi \otimes \overline{\nabla^{1/2}\eta}) = \nabla^{1/2}\eta \otimes \overline{\nabla^{-1/2}\xi} = (\nabla^{1/2} \otimes (\nabla^{-1/2})^\top)J_0(\xi \otimes \overline{\eta}).$$

Thus $J_1^2 = (\nabla^{1/2} \otimes (\nabla^{-1/2})^\top)J_0(\nabla^{1/2} \otimes (\nabla^{-1/2})^\top)J_0 = J_0(\nabla^{-1/2} \otimes (\nabla^{1/2})^\top)(\nabla^{1/2} \otimes (\nabla^{-1/2})^\top)J_0 = J_0^2 = 1$.

Consider when $\mathcal{S} = \mathcal{S}^*$; equivalently, when $V = V_a$, equivalently, $e = e_a$. We have that $V = V_a$ exactly when

$$J_1(V) = V \quad \Leftrightarrow \quad J_1^{-1}(V) = V \quad \Leftrightarrow \quad J_1(V) \subseteq V \quad \Leftrightarrow \quad J_1^{-1}(V) \subseteq V,$$

because, for example, if $J_1(V) \subseteq V$ then $V = J_1^2(V) \subseteq J_1(V) \subseteq V$ and so we have equality throughout. As V is the image of the projection e, we see immediately that $J_1(V) \subseteq V$ if and only if $eJ_1e = J_1e$, if and only if $J_1eJ_1e = e$. Set $e' = J_1eJ_1$, so $e'^2 = e'$ and the image of e' is exactly V_a, but note that in general e' is not self-adjoint. So $V = V_a$ if and only if $e'e = e$ (equivalently, $ee' = e'$).

As in Section 1.2, let $A = \sum_{j=1}^k |b_j\rangle\langle a_j|$ so $e = \Psi'_{0,1/2}(A) = \sum_j b_j \otimes \sigma_{i/2}(a_j)^*$, and hence

$$e'(\xi \otimes \overline{\eta}) = J_1 \sum_j b_j \nabla^{1/2}\eta \otimes \overline{\sigma_{i/2}(a_j)\nabla^{-1/2}\xi} = (\nabla^{1/2} \otimes (\nabla^{-1/2})^\top) \sum_j \nabla^{-1/2} a_j \xi \otimes \overline{b_j \nabla^{1/2}\eta}$$
$$= \sum_j a_j \xi \otimes \overline{\nabla^{-1/2} b_j \nabla^{1/2}\eta} = \Big(\sum_j a_j \otimes (\sigma_{i/2}(b_j)^*)^\top\Big)(\xi \otimes \overline{\eta}).$$

Hence $e' = \sum_j a_j \otimes \sigma_{i/2}(b_j)^*$ and so $A' = A^*$.

The following conclusion is perhaps not very satisfying.

Proposition 3.1. *We have that $\mathcal{S} = \mathcal{S}^*$ if and only if $A \bullet A^* = A^*$, if and only if $A^* \in \mathcal{S}$.*

Proof. As above, $\mathcal{S} = \mathcal{S}^*$ is equivalent to $e'e = e$, which we now see is equivalent to $A^* \bullet A = A$. Equivalently, $ee' = e'$, the same as $A \bullet A^* = A^*$, is the same as $\theta_A(A^*) = A^*$, recall (3), and as θ_A is an idempotent with image $\mathcal{S}$, this is the same as $A^* \in \mathcal{S}$. □

The second condition to be an operator system, that $1 \in \mathcal{S}$, is easier to study. We have that $1 \in \mathcal{S}$ if and only if $\theta_A(1) = 1$ if and only if $m(A \otimes 1)m^* = 1$, which is the usual axiom.

3.1 For the twisted correspondence

We finally make links with the idea explored in Section 2.1. Again let $A, e, \mathcal{S}$ be linked, and set $\mathcal{T} = \mathcal{S}_{i/4}$, as before. To avoid notational clashes, we write $\tau\colon B \otimes B^{op} \to B \otimes B^{op}$ for the tensor swap map, an anti-$*$-homomorphism.

The expression $A_K^* = \nabla^{-1/2} A^* \nabla^{1/2}$ occurring in the following is the KMS adjoint, that is, satisfies

$$(a|A_K^*(b))_K = (A(a)|b)_K \qquad (a, b \in B),$$

a fact easily verified from the definition of the KMS inner-product (4). For the following, compare [3, Theorem A].

Proposition 3.2. *Let $\mathcal{T}$ correspond to e and A. Then $\mathcal{T}^*$ corresponds with $\tau(e)$ and $A_K^* = \nabla^{-1/2} A^* \nabla^{1/2}$.*

Proof. Recall (6), so with $A = \sum_{j=1}^k |b_j\rangle\langle a_j|$ we have that $T \in \mathcal{T}$ if and only if $\sum_j \sigma_{-i/4}(b_j) T \sigma_{i/4}(a_j)^* = T$. Hence $T \in \mathcal{T}^*$ if and only if $T^* = \sum_j \sigma_{-i/4}(b_j) T^* \sigma_{i/4}(a_j)^*$, equivalently,

$$T = \sum_j \sigma_{i/4}(a_j) T \sigma_{-i/4}(b_j)^* = \sum_j \sigma_{-i/4}(\sigma_{i/2}(a_j)) T \sigma_{i/4}(\sigma_{-i/2}(b_j))^*.$$

Thus $\mathcal{T}^*$ is associated to the operator

$$\sum_{j=1}^k |\sigma_{i/2}(a_j)\rangle\langle \sigma_{-i/2}(b_j)| = \sum_{j=1}^k \nabla^{-1/2} |a_j\rangle\langle b_j| \nabla^{1/2} = \nabla^{-1/2} A^* \nabla^{1/2} = A_K^*,$$

as claimed. In turn, this corresponds to

$$\Psi'_{0,1/2}(A_K^*) = \sum_j \sigma_{i/2}(a_j) \otimes \sigma_{i/2}(\sigma_{-i/2}(b_j))^* = \sum_j \sigma_{i/2}(a_j) \otimes b_j^* = \tau\Big(\sum_j b_j \otimes \sigma_{i/2}(a_j)^*\Big)^* = \tau(e)^*,$$

which of course equals $\tau(e)$, as $e = e^*$. □

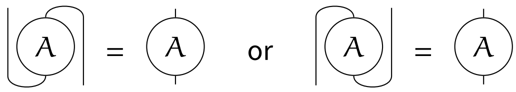
This gives the somewhat more transparent condition that $\mathcal{T}$ is self-adjoint if and only if $e = \tau(e)$. Notice that $1 \in \mathcal{T}$ if and only if $1 \in \mathcal{S}$; see the discussion at the end of the last section.

This gives some motivation for considering the B'-bimodule $\mathcal{T} = \mathcal{S}_{i/4}$ and not $\mathcal{S}$ directly; in [3], motivation for considering $\mathcal{S}_{i/4}$ was given from considerations of the KMS inner-product.

4 The other axioms

Analogously, we now consider the other axioms on A, coming to the same conclusions as [4] (and [1] though there we did not previously explicitly consider $\Psi'_{0,1/2}$). The conditions on A which we will consider are that A is self-adjoint, $A = A^*$, and that A is "self-transpose", which can be either of the conditions:

- $(1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^*\eta \otimes 1) = A$ or $(\eta^* m \otimes 1)(1 \otimes A \otimes 1)(1 \otimes m^*\eta) = A$;
- which as diagrams are written as



The following is immediate from [1, Proposition 5.5, Lemma 5.7]; again we now write $\tau \colon B \otimes B^{op} \to B \otimes B^{op}$ for the tensor swap map.

Proposition 4.1. *Let $A \in \mathcal{B}(L^2(B))$ and set $e = \Psi'_{0,1/2}(A)$. Then:*

1. $\Psi'_{0,1/2}(A^*) = (\sigma_{-i/2} \otimes \sigma_{-i/2})\tau(e^*)$;
2. $\Psi'_{0,1/2}((1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^*\eta \otimes 1)) = (\sigma_{-i/2} \otimes \sigma_{-i/2})\tau(e)$;
3. $\Psi'_{0,1/2}((\eta^* m \otimes 1)(1 \otimes A \otimes 1)(1 \otimes m^*\eta)) = (\sigma_{i/2} \otimes \sigma_{i/2})\tau(e)$;
4. *define $A_r(a) = A(a^*)^*$ for $a \in B$. Then $\Psi'_{0,1/2}(A_r) = e^*$.*

Define a one-parameter (complex) automorphism group on $\mathcal{B}(L^2(B))$ by setting $\tilde{\sigma}_z(T) = Q^{iz}TQ^{-iz}$ for $T \in \mathcal{B}(L^2(B)), z \in \mathbb{C}$. Notice that $\tilde{\sigma}_z$ restricts to σ_z on B.

Recall that [2] shows that of the three conditions (i) A is real; (ii) $A = A^*$; (iii) A is self-transpose, any two of the conditions imply the third. Compare also [1, Theorem 5.17].

Corollary 4.2. *Let A be Schur idempotent and real (equivalently completely positive), so $e = \Psi'_{0,1/2}(A)$ is a self-adjoint projection. Then the following are equivalent:*

1. *A is self-adjoint;*
2. *A satisfies one or both of the self-transpose conditions;*
3. *A commutes with ∇;*
4. *$e = (\sigma_z \otimes \sigma_z)(e)$ for all $z \in \mathbb{C}$;*
5. *$\tilde{\sigma}_z \circ \theta_A = \theta_A \circ \tilde{\sigma}_z$ for each $z \in \mathbb{C}$;*
6. *$\mathcal{S} = Q^{iz}\mathcal{S}Q^{-iz}$ for each $z \in \mathbb{C}$.*

Proof. From the previous proposition, that A is self-adjoint is that $(\sigma_{-i/2} \otimes \sigma_{-i/2})\tau(e) = e$ (as $e^* = e$) and that A satisfies one of the self-transpose conditions is that $(\sigma_{-i/2} \otimes \sigma_{-i/2})\tau(e) = e$ or $(\sigma_{i/2} \otimes \sigma_{i/2})\tau(e) = e$. These are now seen to be the same condition. When they hold, we see that

$$e = (\sigma_{i/2} \otimes \sigma_{i/2})\tau(e) = (\sigma_{i/2} \otimes \sigma_{i/2})\tau(e^*) = (\sigma_{-i/2} \otimes \sigma_{-i/2})\tau(e)^*,$$

using that $e = e^*$. Hence $(\sigma_{i/2} \otimes \sigma_{i/2})\tau(e) = e = e^* = (\sigma_{-i/2} \otimes \sigma_{-i/2})\tau(e)$ and hence $(\sigma_i \otimes \sigma_i)e = e$. So e commutes with $Q \otimes Q^{-1}$ (remember that the second tensor factor of e is B^{op} where multiplication is reversed), and hence with any power of this strictly positive operator, and so $e = (\sigma_z \otimes \sigma_z)(e)$ for all $z \in \mathbb{C}$.

The correspondence between A, e and θ_A is given by (5) above, which gives

$$A = |\sigma_z(b)\rangle\langle\sigma_{z+i/2}(c)^*| \quad \leftrightarrow \quad e = \sigma_z(b) \otimes \sigma_z(c) \quad \leftrightarrow \quad \theta_A : T \mapsto \sigma_z(b)T\sigma_{z+i/2}(c).$$

For $a, c \in B$ we see that $(\sigma_z(c)^*|a) = \psi(\sigma_z(c)a) = \psi(c\sigma_{-z}(a)) = (c^*|\sigma_{-z}(a))$ by invariance of ψ under the modular automorphism group. As $\Lambda(\sigma_z(a)) = \nabla^{-iz}\Lambda(a)$, we see that

$$|\sigma_z(b)\rangle\langle\sigma_{z+i/2}(c)^*| = \nabla^{iz}|b\rangle\langle\sigma_{i/2}(c)^*|\nabla^{-iz}.$$

So $e = (\sigma_z \otimes \sigma_z)(e)$ is equivalent to $\nabla^{iz}A = A\nabla^{iz}$, and this holding for all z is equivalent to $\nabla A = A\nabla$.

Similarly, $\sigma_z(b)T\sigma_{z+i/2}(c) = \tilde{\sigma}_z(b\tilde{\sigma}_{-z}(T)\sigma_{i/2}(c))$ and so $e = (\sigma_z \otimes \sigma_z)(e)$ is equivalent to $\tilde{\sigma}_z\theta_A\tilde{\sigma}_{-z} = \theta_A$. When this condition holds, $\mathcal{S} = \theta_A(\mathcal{B}(L^2(B))) = \theta_A(\tilde{\sigma}_z(\mathcal{B}(L^2(B)))) = \tilde{\sigma}_z(\theta_A(\mathcal{B}(L^2(B))) = \tilde{\sigma}_z(\mathcal{S})$ for each z. For the converse, notice that using the unitary from (1), we have that

$$Q^{iz}|\xi\rangle\langle\eta|Q^{-iz} = |Q^{iz}\xi\rangle\langle Q^{i\overline{z}}\eta| \mapsto Q^{iz}\xi \otimes \overline{Q^{i\overline{z}}\nabla^{-1/2}\eta} = Q^{iz}\xi \otimes (Q^{-iz})^\top\overline{\nabla^{-1/2}\eta}.$$

So that $\mathcal{S} = Q^{iz}\mathcal{S}Q^{-iz}$ means that $V = (Q^{iz} \otimes (Q^{-iz})^\top)V$. As e is the orthogonal projection onto V, and for $z = t \in \mathbb{R}$, this is equivalent to e commuting with $Q^{it} \otimes (Q^{-it})^\top$, as this is a unitary operator. This holding for all t is again equivalent to e commuting with $Q \otimes Q^{-1}$. □

Again, we could replace Q by ∇ in the above conditions. Notice that when these conditions hold, $\mathcal{T} = \mathcal{S}_{i/4} = Q^{1/4}\mathcal{S}Q^{-1/4} = \mathcal{S}$. It follows that $\mathcal{S}^* = \mathcal{S}$ if and only if $e = \tau(e)$.

5 Conclusion

This is subjective, but my opinion is that this gives further weight to the belief that the "nicest" axioms are to suppose that A is completely positive and Schur idempotent, equivalently, A is Schur idempotent and "real". (That is, do not suppose that A is self-adjoint or "symmetric" / "self-transpose", though these might be imposed as further, optional, conditions.)

Taking this axiomatic approach means that we link A with $e \in B \otimes B^{op}$ using $\Psi' = \Psi'_{0,1/2}$. We find that this is "natural", either from considering the graphical calculus, or from considering KMS inner-products. One drawback is that $\mathcal{S}$ being self-adjoint becomes hard to express in a natural way at the level of the operator A or the projection e. However, the "twisting" considered in [3] solves this.

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