Conditional exceptations

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February 13, 2026

Abstract

We quickly show how Conditional Expectations arise on von Neumann algebras with traces.

I have been unable to find a really simple proof that given a tracial von Neumann algebra N with a sub-von Neumann algebra $B \subseteq N$, there is always a conditional expectation $N \to B$. This note shows this, a theorem first shown by Umegaki in [7].

1 Conditional Expectations

Definition 1.1. Let A be a C^*-algebra and $B \subseteq A$ a C^*-subalgebra. A *conditional expectation* from A to B is a contractive projection. That is, a linear map $E\colon A \to B$ with $E(a) = a$ for $a \in B$, and with $\|E(x)\| \leq \|x\|$ for each $x \in A$.

Theorem 1.2. *Let $E\colon A \to B$ be a conditional expectation. Then:*

1. *E is positive, so $E(x^*x) \geq 0$ for each $x \in A$;*
2. *E is a B-bimodule map, so $E(axb) = aE(x)b$ for $a, b \in B$ and $x \in A$;*
3. *E satisfies the Schwarz inequality, $E(x)^*E(x) \leq E(x^*x)$ for $x \in A$.*

Proof. This is standard, see [5, Theorem 3.4, Chapter III], [2, Theorem 1.5.10], [1, Section II.6.10] for example. □

Corollary 1.3. *Let $E\colon A \to B$ be a conditional expectation. Then E is a contractive completely poisitve map.*

Proof. We adopt [2, Theorem 1.5.10]. Let $(x_{i,j}) \in M_n(A)$ be positive, let $\phi\colon B \to \mathcal{B}(H)$ be a faithful $*$-representation with cyclic vector ξ, and let $(b_k) \subseteq B$. Then, using the bimodule property of E,

$$\sum_{i,j} \big(\phi(b_i)\xi \big| \phi(E(x_{i,j}))\phi(b_j)\xi\big) = \Big(\xi \Big| \pi\Big(E\Big(\sum_{i,j} b_i^* x_{i,j} b_j\Big)\Big)\xi\Big) \geq 0,$$

because $\sum_{i,j} b_i^* x_{i,j} b_j \geq 0$ (see [5, Lemma 3.2, Chapter IV] for example). It follows that $(E(x_{i,j})) \in M_n(B)$ is positive. So E is completely positive. As E extends to unitisations (see the proof of [5, Theorem 3.4, Chapter III], or pass to biduals) and becomes a unital map, it follows that E is contractive. □

The "Schwarz inequality" is also sometimes called the "Kadison inequality" after [3].

Corollary 1.4. *Let $E\colon A \to B$ be an idempotent, positive, B-bimodule map. Then E is contractive, so a conditional expectation.*

Proof. We follow [1, Corollary II.6.10.3]. For $x \in A$ we have

$$\begin{aligned} 0 \leq E((x - E(x))^*(x - E(x))) &= E(x^*x - E(x)^*x - x^*E(x) + E(x)^*E(x)) \\ &= E(x^*x) - E(x)^*E(x) - E(x^*)E(x) + E(x)^*E(x) = E(x^*x) - E(x^*)E(x), \end{aligned}$$

so Kadison's inequality holds. If E is not contractive there is $x \in A$ with $\|x\| = 1$ and $1 < \|E(x)\|$ and $\|E\| < \|E(x)\|^2$. Then $\|E(x)\|^2 = \|E(x)^*E(x)\| \leq \|E(x^*x)\| \leq \|E\|$, as $\|x^*x\| = 1$, which is a contradiction. □

2 Invariant states

We follow [1, Section II.6.10]. Again let $B \subseteq A$. Let ϕ be a faithful state on A and let $\psi = \phi|_B$. The GNS space H_ψ can naturally be identified with a closed subspace of the GNS space H_ϕ. Let $\xi_\phi = \xi_\psi$ be the cyclic vectors, which are identified under $H_\psi \subseteq H_\phi$.

The following shows that when a conditional expectation leaves a faithful state invariant, then the conditional expectation gives the orthogonal projection onto H_ψ under the GNS map.

Proposition 2.1. *Let $E\colon A \to B$ be a conditional expectation with $\phi = \phi \circ E$. The orthogonal projection $e\colon H_\phi \to H_\psi$ satisfies that $e(x\xi_\phi) = E(x)\xi_\psi$ for $x \in A$.*

Proof. Define $f\colon A\xi_\phi \to H_\psi$ by $f(x\xi_\phi) = E(x)\xi_\psi$ which is well-defined as ψ is faithful, so $A \to H_\phi; x \mapsto x\xi_phi$ is injective. As E is idempotent, f has dense range, and as $\xi_\psi = \xi_\phi$ it follows that $f^2 = f$. For $x \in A$ we have

$$\|f(x\xi_\phi)\|^2 = \|E(x)\xi_\psi\|^2 = \phi\big(E(x)^*E(x)\big) \leq \phi(E(x^*x)) = \phi(x^*x) = \|x\xi_\phi\|^2,$$

so f is contractive, and so extends to an idempotent $H_\phi \to H_\psi$ which is now seen to be surjective. A contractive idempotent on a Hilbert space is an orthogonal projection, and so $f = e$ as claimed. □

Corollary 2.2. *Let $B \subseteq A$ and let ϕ be a faithful state on A. There is at most one conditional expectation $E\colon A \to B$ with $\phi = \phi \circ E$.*

Notice that the same argument would apply to a weight.

3 Tracial von Neumann algebras

We recall some of the basics, [5, Section 2, Chapter V]. A *trace* on a von Neumann algebra M is a map $\tau\colon M_+ \to [0, \infty]$ which is:

1. additive, so $\tau(x + y) = \tau(x) + \tau(y)$ for $x, y \in M_+$;
2. positive homogeneous, so $\tau(\lambda x) = \lambda\tau(x)$ for $\lambda \geq 0, x \in M_+$;
3. tracial, so $\tau(x^*x) = \tau(xx^*)$ for $x \in M$.

As usual, $0 \cdot \infty = 0$. We say τ is *faithful* when $\tau(x) > 0$ for each $0 \neq x \in M_+$, is *semifinite* when for each $0 \neq x \in M_+$ there is $0 \neq y \leq x$ with $\tau(y) < \infty$, is *finite* when $\tau(1) < \infty$, and is *normal* when $\tau(\sup_i x_i) = \sup_i \tau(x_i)$ for increasing bounded nets (x_i) in M_+.

We will always assume our traces are faithful, semifinite and normal (an "nsf trace"). A finite normal trace is simply a normal functional $\tau \in M_*$ with $\tau(xy) = \tau(yx)$ for $x, y \in M$.

Given an nsf trace τ define

$$\mathfrak{p}_\tau = \{x \in M_+ : \tau(x) < \infty\}, \qquad \mathfrak{n}_\tau = \{x \in M : \tau(x^*x) < \infty\}, \qquad \mathfrak{m}_\tau = \Big\{\sum_{i=1}^n : x_i y_i \in \mathfrak{n}_\tau\Big\}.$$

We have that $\mathfrak{n}_\tau$ is an ideal in M, and so $\mathfrak{m}_\tau$ is also an ideal. We have that $\mathfrak{p}_\tau = \mathfrak{m}_\tau \cap M_+$ and $\mathfrak{m}_\tau = \operatorname{lin}\mathfrak{p}_\tau$. The map τ can be extended to a linear functional, still denoted by τ, on $\mathfrak{m}_\tau$ which is self-adjoint, and with

$$\tau(ax) = \tau(xa) \quad (a \in M, x \in \mathfrak{m}_\tau), \qquad \tau(xy) = \tau(yx) \quad (x, y \in \mathfrak{n}_\tau).$$

For fun, we follow Martin Argerami[1] and show some equivalences to semifiniteness.

Proposition 3.1. *The following are equivalent:*

1. *τ is semifinite;*

[1] https://math.stackexchange.com/q/1840578

2. *for each $x \in M_+$ there is an increasing net $(x_i) \subseteq \mathfrak{p}_\tau$ with $x_i \to x$ in SOT.*

3. *for each $x \in M_+$ we have $\tau(x) = \{\tau(y) : y \in \mathfrak{p}_\tau, y \leq x\}$.*

4. *the σ-weak closure of $\mathfrak{p}_\tau$ is M_+.*

Proof. When (1) holds, given $x \in M_+$, we use Zorn's Lemma to find a maximal chain $(x_i)_{i\in I}$ in $\mathfrak{p}_\tau$ with $0 \neq x_i \leq x$ for each i. Then $\|x_i\| \leq \|x\|$ so the ordered net in bounded, and hence $y = \sup x_i$ exists in the SOT, and $y \leq x$. If $z = x - y \geq 0$ is not 0, then there is $0 \neq z_0 \in \mathfrak{p}_\tau$ with $z_0 \leq z$, and so $y + z_0 \leq y + z = x$ and $y + z_0 \geq x_i + z_0 \geq x_i$ for each i, and so we conclude that the family (x_i) is not maximal, contradiction. So (2) holds.

When (2) holds, as τ is normal, (3) follows immediately. (3) implies (1) is clear, as τ is faithful.

Clearly (2) implies (4). For the converse, I follow ideas from [5].[2] As $\mathfrak{m}_\tau$ is an ideal, there is a unique central projection $z \in M$ with the σ-weak closure $\overline{\mathfrak{m}_\tau}$ equal to Mz. Furthermore, there is an increasing net (e_i) of positive elements in $\mathfrak{m}_\tau$, that is, in $\mathfrak{p}_\tau$, with $e_i \to z$ σ-strongly. Then, for $x \in (Mz)_+ = M_+z$ we have that $x = \lim_i x^{1/2} e_i x^{1/2}$, an increasing net in $M_+ \cap \mathfrak{m}_\tau = \mathfrak{p}_\tau$. Under hypothesis (4), $\mathfrak{p}_\tau$ is σ-weakly dense in M_+, so taking linear combinations, $\mathfrak{m}_\tau$ is σ-weakly dense in M, thus $z = 1$. Hence (2) holds. $\square$

Using polar decomposition arguments, $\|x\|_1 = \tau(|x|)$ for $x \in \mathfrak{m}_\tau$ defines a norm on $\mathfrak{m}_\tau$, and the map

$$\mathfrak{m}_\tau \to M_*; \quad x \mapsto \tau(\cdot x)$$

is an isometry onto a dense subspace of the predual M_*. We denote by $L^1(M, \tau)$ the completion of $(\mathfrak{m}_\tau, \|\cdot\|_1)$, a Banach space isometric with M_*.

We turn $\mathfrak{n}_\tau$ into an inner-product space via $(x|y) = \tau(x^*y)$, the completion being a generalised GNS space $H_\tau = L^2(M, \tau)$, with GNS map $\Lambda\colon \mathfrak{n}_\tau \to L^2(M, \tau)$ and $*$-representation $\pi\colon M \to \mathcal{B}(H_\tau)$. As τ is a trace, there is also an anti-$*$-homomorphism $\pi'\colon M \to \mathcal{B}(H_\tau)$ given by $\pi'(x)\Lambda(y) = \Lambda(yx)$. The map $J\colon \Lambda(x) \mapsto \Lambda(x^*)$ extends to a conjugate-linear isometry on H_τ with $J^2 = 1$. Finally, $J\pi(x)J = \pi'(x^*)$ for each $x \in M$, and $\pi(M)' = \pi'(M)$.

We now follow an unpublished book by C. Anantharaman and S. Popa[3] adapted to the semifinite case. Given $\xi \in L^2(M, \tau)$ there is a linear map $L^0_\xi\colon \Lambda(\mathfrak{n}_\tau) \to H_\tau; \Lambda(x) \mapsto \xi x = \pi'(x)(\xi)$. Similarly, define $R^0_\xi\colon \Lambda(\mathfrak{n}_\tau) \to H_\tau; \Lambda(x) \mapsto x\xi = \pi(x)(\xi)$.

Lemma 3.2. *The maps L^0_ξ and R^0_ξ are closable as densely defined operators on $L^2(M, \tau)$.*

Proof. Suppose that $R^0_\xi\Lambda(x_n) \to \eta$ while $\Lambda(x_n) \to 0$. For $y \in M$ we have

$$\begin{aligned}(\Lambda(y)|R^0_\xi\Lambda(x_n)) &= (\Lambda(y)|\pi(x_n)\xi) = (\pi(x_n)^*\Lambda(y)|\xi) = (\Lambda(x_n^*y)|\xi) \\ &= (\pi'(y)\Lambda(x_n^*)|\xi) = (J\Lambda(x_n)|\pi'(y)^*\xi) \to 0.\end{aligned}$$

Thus $(\Lambda(y)|\eta) = 0$, for each $y \in M$, as $\eta = 0$. A similar argument holds for L^0_ξ. $\square$

Denote by L_ξ the closure of L^0_ξ. When L_ξ is bounded, we say that ξ is *left bounded*. Similarly define R_ξ and the notion of being *right bounded*.

Theorem 3.3. *A vector $\xi \in L^2(M, \tau)$ is left bounded if and only if $\xi \in \Lambda(\mathfrak{n}_\tau)$, and similarly for right bounded vectors. Furthermore, if this case, $\xi = \Lambda(x)$ for $x \in \mathfrak{n}_\tau$ with $\|x\| = \|L_\xi\|$, and analogously on the right.*

Proof. When $\xi = \Lambda(x)$ for some $x \in \mathfrak{n}_\varphi$, see that $L_\xi\Lambda(a) = \pi'(a)\Lambda(x) = \Lambda(xa) = \pi(x)\Lambda(a)$ for $a \in \mathfrak{n}_\tau$, and so $L_\xi = \pi(x)$, and $\|x\| = \|L_\xi\|$. Similarly, $R_\xi = \pi'(x)$.

We have that [5, Chapter V, Lemma 2.21] shows that $\xi \in \Lambda(\mathfrak{n}_\tau)$ if and only if $\sup\{\|a\xi\| : a \in \mathfrak{n}_\tau, \|\Lambda(a)\| \leq 1\} < \infty$, and if so, then the supremum equals $\|x\|$ for the $x \in \mathfrak{n}_\tau$ with $\xi = \Lambda(x)$. That is, R^0_ξ is bounded if and only if $\xi \in \Lambda(\mathfrak{n}_\tau)$.

[2] See Lemma 2.13, Chapter V, but I do not see how working with $\mathfrak{n}_\tau$ works.

[3] "An introduction to II_1 factors" available at https://www.math.ucla.edu/~popa/Books/IIun.pdf August 2024.

We now compute that for any $\xi \in H_\tau$ and $x \in \mathfrak{n}_\tau$, we have $JR^0_\xi J\Lambda(x) = J(\pi(x^*)\xi) = J\pi(x^*)JJ\xi = \pi'(x)J\xi = L^0_{J\xi}\Lambda(x)$ so $JR^0_\xi J = L^0_{J\xi}$, and hence also $JL^0_\xi J = R^0_{J\xi}$. This is ξ is left-bounded, that $R_{J\xi} = JL_\xi J$ is bounded, so $J\xi$ is right-bounded, hence $J\xi \in \Lambda(\mathfrak{n}_\tau)$ so also $\xi \in \Lambda(\mathfrak{n}_\tau)$. □

4 Approaches to existance of conditional expectations

We aim to give various approaches to proving the following result.

Theorem 4.1. *Let τ be an nsf trace on M, and let $N \subseteq M$ be a sub-von Neumann algebra such that $\tau|_N$ is semifinite. There is a unique conditional expectation $E\colon M \to N$ with $\tau \circ E = \tau$.*

When $\tau|_N$ is semifinite, it is of course normal and faithful, so we can form $L^2(N,\varphi)$ which is naturally identified with a closed subspace of $L^2(M,\varphi)$. Directly adapting the proof of Proposition 2.1 shows that if such an E exists, then $e\Lambda(x) = \Lambda(Ex)$ for each $x \in \mathfrak{n}_\varphi$ where $e\colon L^2(M,\varphi) \to L^2(N,\varphi)$ is the orthogonal projection. Thus E is unique.

Proof 1 of Theorem 4.1. Define $E\colon M \to \mathcal{B}(L^2(N,\tau))$ by $E(x) = p\pi_M(x)\iota$ where $p\colon L^2(M,\tau) \to L^2(N,\tau)$ is the orthogonal projection, though of as having codomain $L^2(N,\tau)$, and $\iota\colon L^2(N,\tau) \to L^2(M,\tau)$ is the inclusion.[4] Then $\iota^* = p$, and $\pi_M(a)\iota = \iota\pi_N(a)$ for $a \in N$, so also $p\pi_M(a) = \pi_N(a)p$. In particular, $E(a) = \pi_N(a)p\iota = \pi_N(a)$ for each $a \in N$. For J_N the conjugation operator on $L^2(N,\tau)$, and similarly J_M, we easily see that $\iota J_N = J_M\iota$, and so taking adjoints gives $J_Np = pJ_M$.[5] For $a \in N$ and $x \in M$ we hence have $E(x)J_N\pi_N(a)J_N = p\pi_M(x)J_M\pi_M(a)J_M\iota = pJ_M\pi_M(a)J_M\pi_M(x)\iota = J_N\pi_N(a)J_NE(x)$. Thus $E(x) \in (J_N\pi_N(N)J_N)' = \pi_N(N)$ and hence we can consider E as a linear map $M \to N$. As $E(x) = \iota^*\pi_M(x)\iota$, we see that E is normal, unital completely positive. Thus E is a conditional expectation.

Finally, we argue as in the proof of Proposition 3.1. As $\mathfrak{m}_\tau \cap N$ is a σ-weakly dense ideal in N, there is an increasing net (e_i) in $\mathfrak{p}_\tau \cap N$ which converges to $1_N = 1_M$. Then each $e_i^{1/2} \in \mathfrak{n}_\tau \cap N$, and for $x \in M_+$, the net $(x^{1/2}e_ix^{1/2})$ is in $\mathfrak{m}_\tau$ and increases to x. Then $x^{1/2}e_i^{1/2}$ and $e_i^{1/2}x^{1/2}$ are in $\mathfrak{n}_\tau$, and so

$$\begin{aligned}\tau(x) &= \lim_i \tau(x^{1/2}e_ix^{1/2}) = \lim_i \tau(e_i^{1/2}x^{1/2}x^{1/2}e_i^{1/2}) = \lim_i \tau(e_i^{1/2}xe_i^{1/2}) \\ &= \lim_i \big(\Lambda_M(e_i^{1/2})\big|\pi_M(x)\Lambda_M(e_i^{1/2})\big) = \lim_i \big(\Lambda_N(e_i^{1/2})\big|\pi_N(E(x))\Lambda_N(e_i^{1/2})\big) = \tau_N(E(x)),\end{aligned}$$

where in the final equality we reverse the initial calculation, now working with N and τ_N. Hence $\tau = \tau \circ E$ on M_+, as required. □

Corollary 4.2. *Any τ-invariant conditional expectation $E\colon M \to N$ is normal and faithful (meaning that if $x \in M_+$ is non-zero, then $E(x) \neq 0$.)*

Proof. The conditional expectation is unique, and the E just constructed is normal. As τ is faithful, given a non-zero $x \in M_+$ we have $\tau(E(x)) = \tau(x) \neq 0$ and so $E(x) \neq 0$. Thus E is faithful. □

It is also possible to use the relation between E and e to define E; this is the approach taken in [4, Section 3.6], though we do not quite follow their argument.

Proof 2 of Theorem 4.1. We again use ι and p, to be careful about (co)domains. Let $x \in \mathfrak{n}_\tau$ and $a, b \in \mathfrak{n}_\tau \cap N$, so

$$\begin{aligned}\big(\Lambda_N(a)\big|J_N\pi_N(b^*)J_Np\Lambda_M(x)\big) &= \big(J_N\pi_N(b)J_N\Lambda_N(a)\big|p\Lambda_M(x)\big) = \big(\Lambda_N(ab^*)\big|p\Lambda_M(x)\big) \\ &= \big(\Lambda_M(ab^*)\big|\Lambda_M(x)\big) = \big(J_M\pi_M(b)J_M\Lambda_M(a)\big|\Lambda_M(x)\big) \\ &= \big(\Lambda_M(a)\big|J_M\pi_M(b^*)J_M\Lambda_M(x)\big) = \big(\Lambda_M(a)\big|\Lambda_M(xb)\big) = \big(\Lambda_M(a)\big|\pi_M(x)\Lambda_M(b)\big).\end{aligned}$$

[4]We could also write $E(x) = exe$ but we find it clear to make rather explicit domains and codomains.

[5]If you are unhappy with the adjoint of a conjugate-linear operator, perform the calculation: let $a \in N, x \in M$, and consider $(\Lambda_N(a)|J_Np\Lambda_M(x)) = (p\Lambda_M(x)|J_N\Lambda_N(a)) = (p\Lambda_M(x)|\Lambda_N(a^*)) = (\Lambda_M(x)|\iota\Lambda_N(a^*)) = (\Lambda_M(x)|\Lambda_M(a^*)) = (\Lambda_M(x)|J_M\Lambda_M(a)) = (\Lambda_M(a)|J_M\Lambda_M(x)) = (\Lambda_N(a)|J_M\Lambda_M(x)) = (\Lambda_N(a)|pJ_M\Lambda_M(x))$ and so indeed $J_Np = pJ_M$.

Set $\xi = p\Lambda_M(x) \in L^2(N,\tau)$, so this calculation shows that

$$\begin{aligned}|(\Lambda_N(a)|L_\xi\Lambda_N(b))| &= |(\Lambda_N(a)|\pi'_N(b)\xi)| = |(\Lambda_N(a)|J_N\pi_N(b^*)J_Np\Lambda_M(x))| \\ &= |(\Lambda_M(a)|\pi_M(x)\Lambda_M(b))| \leq \|x\|\|\Lambda_N(a)\|\|\Lambda_N(b)\|.\end{aligned}$$

Thus L_ξ is bounded, so by Theorem 3.3, $\xi = \Lambda_N(c)$ for some $c \in \mathfrak{n}_\tau \cap N$ with $\|c\| = \|L_\xi\| \leq \|x\|$. There is hence a contractive map $E\colon \mathfrak{n}_\tau \to \mathfrak{n}_\tau \cap N$ with $\Lambda_N \circ E = p \circ \Lambda_M$. It is easy to see that $E(x) = x$ for $x \in \mathfrak{n}_\tau \cap N$, and so E is idempotent.[6]

We now basically copy the previous proof. For $a, b \in \mathfrak{n}_\tau \cap N$ and $x \in \mathfrak{n}_\tau$, we have

$$\begin{aligned}(\Lambda_N(a)|p\pi_M(x)\iota\Lambda_N(b)) &= (\Lambda_M(a)|\pi_M(x)\Lambda_M(b)) = \tau(a^*xb), \\ &= (\Lambda_M(ab^*)|\Lambda_M(x)) = (\Lambda_N(ab^*)|p\Lambda_M(x)) = (\pi'_N(b^*)\Lambda_N(a)|p\Lambda_M(x)) \\ &= (\Lambda_N(a)|\pi'_N(b)\Lambda_N(E(x))) = (\Lambda_N(a)|\Lambda_N(E(x)b)) = (\Lambda_N(a)|\pi_N(E(x))\Lambda_N(b))\end{aligned}$$

As τ is a trace, these agree, and so $\pi_N(E(x)) = p\pi_M(x)\iota$ for $x \in \mathfrak{n}_\tau$. By normality, it follows that E extends to a normal map $M \to N$, necessarily idempotent, with $\pi_N(E(x)) = p\pi_M(x)\iota$ for each $x \in M$. Thus E is a normal UCP map, so a conditional expectation. □

We have established that $E(x) = exe$, but this is only true when regarded as operators on $L^2(N,\tau)$, which motivates our careful notation in the next result.

Proposition 4.3. *Let $N \subseteq M$ be an inclusion of von Neumann algebra, with τ an nsf trace on M which restricts to a semifinite trace on N. Let $e\colon L^2(M,\tau) \to L^2(N,\tau)$ be the orthogonal projection. The unique conditional expectation $E\colon M \to N$ satisfies:*

1. *$\Lambda_N(E(x)) = e\Lambda_M(x)$ for $x \in \mathfrak{n}_\tau$;*
2. *for $x \in M$ we have $e\pi_M(x)e = \pi_N(E(x))$ as operators on $L^2(N,\tau)$;*
3. *for $x \in M$ we have $e\pi_M(x)e = e\pi_M(E(x)) = \pi_M(E(x))e$ as operators on $L^2(M,\tau)$;*
4. *for $x, y \in \mathfrak{n}_\tau$ we have $\tau(xE(y)) = \tau(E(x)E(y)) = \tau(E(x)y)$;*
5. *$\{e\}' \cap \pi_M(M) = \pi_M(N)$.*

Proof. We established (1) in the 2nd proof, or adapting Proposition 2.1 to the semifinite case. (2) is shown in both proofs above. For $a \in N$ we have $\pi_M(a)e = \pi_N(a)e = e\pi_N(a)e = e\pi_M(a)e$ and so taking the adjoint shows also $e\pi_M(a) = e\pi_M(a)e$, and so (4) follows from (2).

Given $x, y \in \mathfrak{n}_\tau$, as $e^2 = e = e^*$, we have

$$\begin{aligned}\tau(xE(y)) &= (\Lambda_M(x^*)|\Lambda_M(E(y))) = (\Lambda_M(x^*)|e\Lambda_M(y)) = (e\Lambda_M(x^*)|e\Lambda_M(y)) \\ &= (\Lambda_M(E(x)^*)|\Lambda_M(E(y))) = \tau(E(x)E(y)),\end{aligned}$$

using that E is self-adjoint, as it is positive. Similarly, $\tau(E(x)y) = \tau(E(x)E(y))$, and so (4) holds.

Clearly $\pi_M(N) \subseteq \{e\}' \cap \pi_M(M)$. Given $x \in \{e\}' \cap \pi_M(M)$, we write $E(x)$ for what might more properly be denoted $(\pi_M \circ E \circ \pi_M^{-1})(x)$. For $\xi \in L^2(N,\tau)$, we have

$$(x - E(x))\xi = (x - E(x))e\xi = e(x - E(x))e\xi = 0,$$

by (3), and using $xe = ex$. As in the first proof, there is a net (e_i) in $\mathfrak{p}_\tau \cap N$ increasing to 1, so $(e_i^{1/2})$ is a net in $\mathfrak{n}_\tau \cap N$, and for each $y \in M_+$, the net $(y^{1/2}e_iy^{1/2})$ is in $\mathfrak{m}_\tau$ and increases to y. Given $x \in \{e\}' \cap \pi_M(M)$, choose $y \in M_+$ with $\pi_M(y) = (x - E(x))^*(x - E(x))$, so again $y^{1/2}e_i^{1/2}, e_i^{1/2}y^{1/2} \in \mathfrak{n}_\tau$, and hence as $x - E(x)$ vanishes on $L^2(N,\tau)$,

$$\begin{aligned}0 = (x - E(x))\Lambda_N(e_i^{1/2}) &\implies \big(\Lambda_N(e_i^{1/2})\big|(x - E(x))^*(x - E(x))\Lambda_N(e_i^{1/2})\big) = 0 \\ &\implies \tau\big(e_i^{1/2}ye_i^{1/2}\big) = 0 \\ &\implies \tau\big(y^{1/2}e_iy^{1/2}\big) = 0,\end{aligned}$$

for each i. Taking the limit shows that $\tau(y) = 0$, so $y = 0$ as τ is faithful, and hence $x = E(x) \in \pi_M(N)$ as required to show (5). □

[6] When τ is finite, E is now defined on all of M, which gives the proof from [4, Section 3.6].

We finish with a third way to establish existance, following [5, Chapter V, Proposition 2.36].

Proof 3 of Theorem 4.1. Clearly the inclusion $\mathfrak{m}_\tau \cap N \to \mathfrak{m}_\tau$ induces an isometric inclusion $L^1(N,\tau) \to L^1(M,\tau)$, and hence an isometry $E_*\colon N_* \to M_*$. Let $E\colon M \to N$ be the Banach space adjoint, so by construction E is normal and contractive. For $x \in \mathfrak{m}_\tau$ let $\theta_x \in M_*$ be the functional $\theta_x(y) = \tau(xy)$ for $y \in M$, so $E_*(\theta_a) = \theta_a$ for $a \in N \subseteq M$, and hence

$$\tau(E(x)a) = \theta_a(E(x)) = E_*(\theta_a)(x) = \theta_a(x) = \tau(xa) \qquad (x \in M, a \in \mathfrak{m}_\tau \cap N).$$

So $\tau(E^2(x)a) = \tau(E(x)a)$ for each $x \in M, a \in \mathfrak{m}_\tau \cap N$, while implies that $(\xi|E^2(x)\eta) = (\xi|E(x)\eta)$ for each $\xi, \eta \in L^2(N,\tau)$, and hence $E^2 = E$. So E is a conditional expectation.

Yet again, we chose an increasing net (e_i) in $\mathfrak{p}_\tau \cap N$ converging to 1, so for $x \in M_+$, the net $(x^{1/2}e_ix^{1/2})$ is in $\mathfrak{m}_\tau$ and converges to x. Then $e_ix^{1/2}$ and $x^{1/2}e_i$ are both in $\mathfrak{m}_\tau$, and so

$$\tau(x) = \lim_i \tau(x^{1/2}e_ix^{1/2}) = \lim_i \tau(xe_i) = \lim_i \tau(E(x)e_i) = \tau(E(x)).$$

Thus $\tau = \tau \circ E$, completing the proof. □

5 For nsf weights

We just remark that much the same idea works for nsf *weights* instead of traces, with of course more care. See [1, Theorem III.4.7.7] or [6, Chapter , Theorem 4.2].

Let $N \subseteq M$ and φ an nsf weight on M such that $\varphi|_N$ is semifinite. There is a φ-preserving conditional expectation $E\colon M \to N$ if and only if the modular automorphism group (τ_t) of φ restricts to N. In this case, we can again regard $L^2(N,\varphi)$ as a subspace of $L^2(M,\varphi)$, and then with e the orthogonal projection, we again have that $E(x) = exe$ as operators on $L^2(N,\varphi)$.

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