Inductive limits of Banach algebras

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Abstract

We provide an overview of inductive limits of Banach algebras, correcting a few misleading statements in the literature.

Contents

1 Introduction

The inductive (or direct) limit of Banach algebras have not been widely studied, perhaps reflecting a lack of applications in the literature. Inductive limits are widely studied for locally convex spaces – often when the space appearing in the inductive system are Banach spaces, but the limit is not in general a Banach space, for example, [4, Chapter IV, Section 5]. They have also been a useful tool in the theory of C^*-algebras, see references below. For Banach algebras, there are a number of misleading statements in standard textbooks, which we wish to correct. We do not pretend to make a literature review, instead giving such references as we are aware of.

Let C be a general category, let I be a directed set (often $\mathbb{N}$ with the usual ordering), for each $i \in I$ let A_i be an object in C, and for $i \leq j$ let $\varphi_{j,i} : A_i \to A_j$ be morphism. We require that for $i \leq j \leq k$ that $\varphi_{k,j} \circ \varphi_{j,i} = \varphi_{k,i}$. We warn the reader that some authors write $\varphi_{i,j}$ instead. This is an *inductive system*, written as $((A_i), (\varphi_{j,i}))$ or sometimes just (A_i), the latter being an abuse of notation, as the "connecting morphisms" $\varphi_{j,i}$ are vitally important.

The *inductive limit* of the system, if it exists, is an object A and morphisms $\phi_i : A_i \to A$ such that:

- For each $i \leq j$ the following diagram commutes:

$$\begin{tikzcd} A_i \arrow[rr, "\varphi_{j,i}"] \arrow[dr, "\phi_i"'] & & A_j \arrow[dl, "\phi_j"] \\ & A & \end{tikzcd}$$

- Let B is an object and suppose we have morphisms $\psi_i : A_i \to B$ with, as above, $\psi_i = \psi_j \circ \varphi_{j,i}$ for all $i \leq j$. We call $(B, (\psi_i))$ a *coherent system*. Then there is a unique morphism $\phi : A \to B$

making the following diagram commutes:

$$\begin{tikzcd} & A_i \arrow[dl, "\phi_i"'] \arrow[dr, "\psi_i"] & \\ A \arrow[rr, "\phi"'] & & B \end{tikzcd}$$

A standard argument, using the "unique" clause, shows that if the limit exists, it is unique, up to isomorphism in the category. We write $\varinjlim A_i$ for the inductive limit (again, with an abuse of notation, as we have suppressed the connecting morphisms).

This presentation follows [7, Section 6.2]. In category theory, the construction can be much extended to the notion of a *colimit*, see [5, Chapter 5] for example. For a combined category theory and functional analytic take, see [3]. A very readable introduction, which ultimately focuses on C^*-algebras, is [8, Appendix L]. For C^*-algebras, see also [2, Section II.8.2].

2 Banach algebras

Having now set the scene using this language of categories, we will now perform a u-turn and present the standard construction, which unfortunately does not have a simple interpretation in terms of category theory. Below we return to the more general setting. We follow [6, Definition 1.3.4] and [1, Section 3.3], both of which make slightly misleading claims which we note below. We are not aware of any other standard textbooks covering this material.[1]

For the remainder of this section, $((A_i),(\varphi_{j,i}))$ shall be an inductive system of Banach algebras. Thus each A_i is a Banach algebra (to be sure, this means that A_i has a contractive product) and each $\varphi_{j,i}$ is a bounded algebra homomorphism. Essentially identical arguments will give analogous results for Banach space, Banach modules over Banach algebras, and so forth.

Definition 2.1. Suppose that for each $i \in I$

$$\limsup_{j\geq i} \|\varphi_{j,i}\| = \lim_{j\geq i} \sup_{k\geq j} \|\varphi_{k,i}\| = \inf_{j\geq i} \sup_{k\geq j} \|\varphi_{k,i}\|$$

is finite. Then we say that our system is a *normed inductive system* of Banach algebras.

Given a normed inductive system, the classical construction of $\varinjlim A_i$ is as follows. Consider the disjoint union of the A_i with the equivalence relation that, for $a_i \in A_i, a_j \in A_j$, we have $a_i \sim a_j$ if and only if there is $k \geq i, k \geq j$ with $\varphi_{k,i}(a_i) = \varphi_{k,j}(a_j)$. A slightly tedious check shows that the quotient becomes a vector space. Given an equivalence class $[a]$ with representative $a \in A_i$, it is easy to see that $\limsup_{j\geq i} \|\varphi_{j,i}(a)\|$ defines a seminorm which is submultiplicative. The quotient by the null ideal of this seminorm gives a normed algebra, and the completion is $\varinjlim A_i$. We define $\phi_i : A_i \to \varinjlim A_i$ by setting $\phi_i(a)$ to be the equivalence class of a.

An alternative construction is the following. Let $\ell^\infty(A_i)$ be the Banach algebra of all bounded families (a_i), where $a_i \in A_i$ for each i, with pointwise operations. Let $c_0(A_i)$ be the collection of families $(a_i) \in \ell^\infty(A)$ such that, for each $\epsilon > 0$, there is $i_0 \in I$ so that $\|a_i\| < \epsilon$ for $i \geq i_0$. Then $c_0(A_i)$ is a closed ideal in $\ell^\infty(A_i)$. Denote by $\mathfrak{A}$ the quotient algebra $\ell^\infty(A_i)/c_0(A_i)$. We shall abuse notation, and write $(a_i) \in \ell^\infty(A_i)$ for the equivalence class of $\mathfrak{A}$ which it defines.

Given $i \in I$, by assumption, there is $j \geq i$ so that $K = \sup_{k\geq j} \|\varphi_{k,i}\| < \infty$. Then, for $a \in A_i$, we define

$$a_k = \begin{cases} \varphi_{k,i}(a) & : k \geq j, \\ 0 & : \text{otherwise.} \end{cases}$$

Thus $\|a_k\| \leq K\|a\|$ for each k, and so $(a_k) \in \mathfrak{A}$. Suppose we chose $j' \in I$ instead, also with $\sup_{k\geq j} \|\varphi_{k,i}\| < \infty$, and form (a'_k) using j'. There is $k_0 \in I$ with $k_0 \geq j$ and $k_0 \geq j'$, so for $k \geq k_0$ we have that $a_k = a'_k$, and so $(a_k) = (a'_k)$ in $\mathfrak{A}$. It follows that we have a well-defined map

[1] But we have not even tried to make a literature review.

$\phi_i : A_i \to \mathfrak{A}$ which is a homomorphism, and is seen to satisfy that $\|\phi_i\| \leq \limsup_{j\geq i} \|\varphi_{j,i}\|$. We introduce the notation that $\phi_i(a) = (\varphi_{k,i}(a))_{k\geq j}$ where $j \geq i$ is any suitable choice.

Given $j \geq i$ there is $k \geq j$ so that both $\sup_{l\geq k} \|\varphi_{l,i}\| < \infty$ and $\sup_{l\geq k} \|\varphi_{l,j}\| < \infty$. For $a \in A_i$, it follows that $\phi_i(a) = (\varphi_{l,i}(a))_{l\geq k}$ and that $\phi_j(\varphi_{j,i}(a)) = (\varphi_{l,j}(\varphi_{j,i}(a)))_{l\geq k} = (\varphi_{l,i}(a))_{l\geq k}$. Hence $\phi_i = \phi_j \circ \varphi_{j,i}$. Set $\varinjlim A_i$ to be the closure of $\{\phi_i(a) : i \in I, a \in A_i\}$. For $i \leq j$, as $\phi_i = \phi_j \circ \varphi_{j,i}$, it follows that $\phi_i(A_i) \subseteq \phi_j(A_j)$. Thus $\varinjlim A_i$ is the closure of an increasing union of algebras, and hence is a Banach algebra.

Here is an alternative description of a dense subalgebra of $\varinjlim A_i$. Define $\mathfrak{A}_0$ to be the equivalence classes in $\mathfrak{A}$ with representatives (a_i) such that, for some i_0, we have that $\varphi_{k,j}(a_j) = a_k$ for $i_0 \leq j \leq k$. We claim that $\mathfrak{A}_0$ is equal to the union of the images of the ϕ_i, and hence is dense in $\varinjlim A_i$. If $a \in A_i$ and $j \geq i$ is suitable, then $\phi_i(a) = (\varphi_{k,i}(a))_{k\geq j}$, and so with $i_0 = j$, if $i_0 \leq k \leq l$ then $\varphi_{l,k}(\varphi_{k,i}(a)) = \varphi_{l,i}(a)$ and so $\phi_i(a) \in \mathfrak{A}_0$. Conversely, given (a_i) with i_0 such that $\varphi_{k,j}(a_j) = a_k$ for $i_0 \leq j \leq k$, let $a = a_{i_0} \in A_{i_0}$, so that $\varphi_{k,i_0}(a) = a_k$ for $k \geq i_0$, and hence $(a_i) = \phi_{i_0}(a)$ in $\mathfrak{A}$.

Let us compare this construction with the "classical construction" sketched above. Clearly we define the same equivalence relation on the disjoint union of the A_i. Given $a \in A_i$ we have that $\|\phi_i(a)\| = \|(\varphi_{k,i}(a))_{k\geq j}\|_{\mathfrak{A}}$ for a suitable $j \geq i$, and by the definition of $\mathfrak{A}$, this is equal to $\limsup_{k\geq j} \|\varphi_{k,i}(a)\|$ which is equal to $\limsup_{k\geq i} \|\varphi_{k,i}(a)\|$. Thus our construction agrees with the classical one.

We now turn to verifying the universal property. In fact, this is not considered in [1], while we believe that [6] is wrong to say that the universal property always holds. Suppose that B is a Banach algebra with bounded homomorphisms $\psi_i : A_i \to B$ such that $\psi_i = \psi_j \circ \varphi_{j,i}$ for each $i \leq j$. For $a \in A_i$ we must define $\phi(\phi_i(a)) = \psi_i(a)$, which motivates the following definition. Given $(a_i) \in \mathfrak{A}_0$ with $\varphi_{k,j}(a_j) = a_k$ for $i_0 \leq j \leq k$, define $\phi((a_i)) = \psi_j(a_j)$ for any $j \geq i_0$. This is well-defined, for $\psi_j(a_j) = \psi_j(\varphi_{j,i_0}(a_{i_0})) = \psi_{i_0}(a_{i_0})$, and so the definition does not depend on j, and hence also does not depend on the choice of i_0. To ensure that ϕ is bounded, there must exist a constant K with

$$\|\psi_i(a)\| \leq K\|\phi_i(a)\| = K \limsup_{j\geq i} \|\varphi_{j,i}(a)\| \qquad (i \in I, a \in A_i).$$

When such a K exists, ϕ extends by continuity to $\varinjlim A_i$, with $\|\phi\| \leq K$.

We summarise the construction as follows.

Theorem 2.2. *Let $(A_i)_{i\in I}$ be a family of Banach algebras indexed by a direct set I. For each $i \leq j$ suppose there is a bounded homomorphism $\varphi_{j,i} : A_i \to A_j$ with $\varphi_{k,j} \circ \varphi_{j,i} = \varphi_{k,i}$ for $i \leq j \leq k$, and such that $\limsup_{j\geq i} \|\varphi_{j,i}\| < \infty$ for each $i \in I$.*

There exists a Banach algebra $A = \varinjlim A_i$ and bounded homomorphisms $\phi_i : A_i \to A$ with $\|\phi_i(a)\| = \limsup_{j\geq i} \|\varphi_{j,i}(a)\|$ for $a \in A_i$, and such that $\phi_j \circ \varphi_{j,i} = \phi_i$ for $i \leq j$, and with the property that if B is a Banach algebra and $\psi_i : A_i \to B$ are homomorphisms with $\psi_j \circ \varphi_{j,i} = \psi_i$ for $i \leq j$, and $\|\psi_i(a)\| \leq \limsup_{j\geq i} \|\varphi_{j,i}(a)\|$ for $i \in I, a \in A_i$, then there is a unique contractive homomorphism $\phi : A \to B$ with $\phi \circ \phi_i = \psi_i$.

In the statement of the theorem, we could weaken the norm condition on ϕ_i to $\|\phi_i(a)\| \leq \limsup_{j\geq i} \|\varphi_{j,i}(a)\|$ for $a \in A_i$. Then the universal property would imply that actually we must have equality. There is a certain appeal to stating the condition in this weaker way, as it would make the conditions on A and B become similar: see Theorem 2.4 below for a variant where the conditions become fully symmetric.

A standard diagram chase shows that all such A satisfying the conclusions of the theorem are isometrically isomorphic. Furthermore, the "uniqueness' clause in the existence of ϕ can be used to show that the union of the images of the A_i must be dense in A.

For completeness, we now give the details of the claims in the previous two paragraphs.

Proof. Firstly, suppose we have a Banach algebra A_0 and a coherent family of homomorphisms $\phi_i^0 : A_i \to A_0$ such that $\|\phi_i^0(a)\| \leq \limsup_{j\geq i} \|\varphi_{j,i}(a)\|$ for each i and $a \in A_i$, and suppose further that A_0 satisfies the universal property. In particular, we can apply the universal property to $B = \varinjlim A_i$ to find that there is a (unique) contractive homomorphism $\phi : A_0 \to \varinjlim A_i$ with $\phi \circ \phi_i^0 = \phi_i$ for each i. Then

$$\limsup_{j\geq i} \|\varphi_{j,i}(a)\| = \|\phi_i(a)\| = \|\phi(\phi_i^0(a))\| \leq \|\phi_i^0(a)\| \leq \limsup_{j\geq i} \|\varphi_{j,i}(a)\|$$

and so we have equality throughout, as claimed.

Let A satisfying the stated universal property, and now let B be the closure of the union of the images of the ϕ_i. We aim to show that $B = A$. We can apply the universal property with $\psi_i : A_i \to B$ being the corestriction of ϕ_i. Thus there is a unique contractive homomorphism $\phi : A \to B$ with $\phi \circ \phi_i = \psi_i$, that is, ϕ is the identity on B, and is hence a projection. We now apply the universal property to A itself, so there is a unique contractive homomorphism $A \to A$ intertwining the maps ϕ_i. However, both the identity homomorphism and ϕ satisfy this, so ϕ is the identity, and $A = B$ as claimed.

Finally, if A satisfies the stated properties, then applying the universal property to $B = \varinjlim A_i$ shows that there is a contractive homomorphism $A \to \varinjlim A_i$ intertwining the coherent homomorphisms. Reversing the roles of A and $\varinjlim A_i$ gives a homomorphism $\varinjlim A_i \to A$, and by the density of the images of the ϕ_i, these two homomorphisms must be mutual inverses. As they are contractive, they must actually be isometric, as claimed. $\square$

It is too much to expect that the "universal property" holds without norm control, as the next example shows.

Example 2.3. Let A be a Banach algebra, and let our directed set be $\mathbb{N}$ with the usual ordering. Let $A_n = A$ for each $n \in \mathbb{N}$, and define $\varphi_{n+1,n}$ to be 1/2 of the identity, which we denote by $\varphi_{n+1,n} = \frac{1}{2}$. Then $\varphi_{n+k,n} = 2^{-k}$ for $k \geq 1$. This defines a normed inductive system of Banach algebras, as for each n we have that $\limsup_{m\geq n} \|\varphi_{m,n}\| = \limsup_{k\geq 1} 2^{-k} = 0$. Indeed, it is now easy to check that $\varinjlim A_n = \{0\}$.

Now set $B = A$ and define $\psi_n = 2^{n-1}$ as a map $A_n \to B$. For $n \leq m$, we see that $\psi_m \circ \varphi_{m,n} = 2^{m-1}2^{n-m} = \psi_n$. So while $(B, (\psi_n))$ is a system, there can be no map $\phi : \varinjlim A_n \to B$ with $\phi\circ\phi_n = \psi_n$. Thus, without some norm condition on the family (ψ_n), we cannot hope to factor through $\varinjlim A_n$.

2.1 Weaker conditions

Rather than requiring that $\limsup_{j\geq i} \|\varphi_{j,i}\| < \infty$ for each i, we could instead ask that $\limsup_{j\geq i} \|\varphi_{j,i}(a)\| < \infty$ for each i and $a \in A_i$. Let us follow through the above construction. Given $i \in I$ and $a \in A_i$, there is $j \geq i$ so that $K = \sup_{k\geq j} \|\varphi_{j,i}(a)\| < \infty$. Then again define

$$a_k = \begin{cases} \varphi_{k,i}(a) & : k \geq j, \\ 0 & : \text{otherwise.} \end{cases}$$

Thus $\|a_k\| \leq K$ for each all k and so $(a_k) \in \mathfrak{A}$. Again, the equivalence class which (a_k) defines is independent of the choice of j. Hence we obtain a well-defined homomorphism $\phi_i : A_i \to \mathfrak{A}$, with

$$\|\phi_i(a)\| = \limsup_{j\geq i} \|\varphi_{j,i}(a)\| \qquad (a \in A_i).$$

However, there seems to be no reason why ϕ_i need be bounded, in general, and Example 2.6 below shows that ϕ_i can indeed fail to be bounded.

To obtain a bounded ϕ_i we need to impose the condition that for each i there is K_i with $\limsup_{j\geq i} \|\varphi_{j,i}(a)\| \leq K_i\|a\|$ for each $a \in A_i$. Then $\|\phi_i\| \leq K_i$. Given this, the remainder of the previous construction still works, and we may again define $\varinjlim A_i$ as a closed subalgebra of $\ell^\infty(A_i)/c_0(A_i)$. The same universal property will hold. We state this as a theorem.

Theorem 2.4. *Let $(A_i)_{i\in I}$ be a family of Banach algebras indexed by a direct set I. For each $i \leq j$ suppose there is a bounded homomorphism $\varphi_{j,i} : A_i \to A_j$ with $\varphi_{k,j}\circ\varphi_{j,i} = \varphi_{k,i}$ for $i \leq j \leq k$. Suppose further that for each $i \in I$ there is K_i with $\limsup_{j\geq i} \|\varphi_{j,i}(a)\| \leq K_i\|a\|$ for each $a \in A_i$.*

There exists a Banach algebra $A = \varinjlim A_i$ and bounded homomorphisms $\phi_i : A_i \to A$ with $\|\phi_i(a)\| \leq \limsup_{j\geq i} \|\varphi_{j,i}(a)\|$ for $a \in A_i$, and such that $\phi_j \circ \varphi_{j,i} = \phi_i$ for $i \leq j$, and with the property that if B is a Banach algebra and $\psi_i : A_i \to B$ are bounded homomorphisms with $\psi_j \circ \varphi_{j,i} = \psi_i$ for $i \leq j$, and $\|\psi_i(a)\| \leq \limsup_{j\geq i} \|\varphi_{j,i}(a)\|$ for $i \in I, a \in A_i$, then there is a unique contractive homomorphism $\phi : A \to B$ with $\phi \circ \phi_i = \psi_i$.

Again, the universal property implies that we actually have equality: $\|\phi_i(a)\| = \limsup_{j\geq i} \|\varphi_{j,i}(a)\|$ for $a \in A_i$ and $i \in I$.

It is stated in [1, 6] that this condition implies that $\limsup_{j\geq i} \|\varphi_{j,i}\| < \infty$ for each i, by the Uniform Boundedness Principle. When $I = \mathbb{N}$ with the usual ordering, this is correct, as here $\limsup_{j\geq i} \|\varphi_{j,i}\| < \infty$ is equivalent to $\sup_{j\geq i} \|\varphi_{j,i}\| < \infty$, and similarly for the case of $\varphi_{j,i}(a)$. For more general directed sets I this is no longer true, as the following counter-example shows.

Example 2.5. The following studies an inductive system of Banach spaces, but these can be turned into Banach algebras by equipping each space with the zero product. Let I be the collection of pairs (F, n) where $F \subseteq \ell^1$ is finite, and $n \in \mathbb{N}$, turned into a directed set for the partial order $(F, n) \leq (F', n')$ where $F \subseteq F'$ and $n \leq n'$. Let $0 \in I$ be an extra element with $0 \leq i$ for all $i \in I$. For each $i \in I$ set $A_i = \ell^1$.

Let ℓ^1 have standard unit vector basis (e_t). For $a, n \in \mathbb{N}$ define $T_{a,n} : \ell^1 \to \ell^1$ by $T_{a,n}(e_a) = ne_a$ and $T_{a,n}(e_t) = e_t$ for $t \neq a$, extending $T_{a,n}$ to ℓ^1 by linearity and continuity. Then $\|T_{a,n}\| = n$.

For $i = (F, n) \in I$ define $\varphi_{i,0} = T_{a,n}$ where a satisfies that

$$\sum_{t\in\mathbb{N}} |x_t| + (n-1)|x_a| < \|x\| + n^{-1} \qquad (x = (x_n) \in F).$$

As $F \subseteq \ell^1$ is finite, such a a does exist. Indeed, the inequality is equivalent to $(n-1)|x_a| < n^{-1}$, that is, $|x_a| < 1/(n(n-1))$, for each $x \in F$. Then $\|\varphi_{i,0}(x)\| = \|T_{a,n}(x)\| < \|x\| + n^{-1}$ for any $x \in F$. Due to the definition of the partial order, if $j \geq i$ then also $\|\varphi_{j,0}(x)\| < \|x\| + n^{-1}$ for any $x \in F$.

For each $i \in I$ define $\varphi_{i,i}$ to be the identity. To obtain a system, we require that for $k \geq j \neq 0$ that

$$\varphi_{k,j} = \varphi_{k,0} \circ \varphi_{j,0}^{-1}.$$

Notice that $\varphi_{j,0}^{-1}$ does exist, and is a contraction. With this definition, we do have an inductive system.

Finally, for $x \in \ell^1$ and $i \in I$, set $y = \varphi_{i,0}^{-1}(x)$. There is some $j = (F, n) \in I$ with $y \in F$ and $j \geq i$. Then, for $k \geq j$,

$$\|\varphi_{k,i}(x)\| = \|\varphi_{k,0}(y)\| < \|y\| + n^{-1} \leq \|x\| + n^{-1}.$$

Thus $\limsup_{k\geq i} \|\varphi_{k,i}(x)\| \leq \limsup_{k\geq j} \|\varphi_{k,i}(x)\| < \|x\| + n^{-1}$, and so the "weaker" condition holds.

However, given $i \in I$, choose $i \leq j$ with $j = (F, n) \in I$. Let $k = (F', m)$ for some $m \geq n$ and $F' \supseteq F$, so that $k \geq j$. Then $\varphi_{i,0} = T_{a,p}$ for some $a, p \in \mathbb{N}$. By choosing F' finite but sufficiently large, we can ensure that $\varphi_{k,0} = T_{b,m}$ where b cannot equal a. Then

$$\varphi_{k,i}(e_b) = \varphi_{k,0}(\varphi_{i,0}^{-1}(e_b)) = T_{b,m}(T_{a,p}^{-1}(e_b)) = T_{b,m}(e_b) = me_b,$$

and so $\|\varphi_{k,i}\| \geq m$. Thus $\limsup_{j\geq i} \|\varphi_{j,i}\| = \infty$, and so we do not have a normed inductive system.

We now present a further counter-example (again, we work just with Banach spaces, algebras been obtained for the zero product) to show that it is possible to have that $\limsup_{j\geq i} \|\varphi_{j,i}(a)\| < \infty$ for each i and $a \in A_i$, while there is no constant K_i with $\limsup_{j\geq i} \|\varphi_{j,i}(a)\| \leq K_i\|a\|$ for $a \in A_i$. Again, we would think that such an example would be impossible because of the Principle of Uniform Boundedness, but using lim sup and not sup changes things.

Example 2.6. Let E be some infinite-dimensional Banach space, and pick an everywhere defined but unbounded functional $\phi : E \to \mathbb{C}$. We can find ϕ using the axiom of choice to pick a Hamel basis for E, for example. Let I be the collection of finite-dimensional subspaces of E partially ordered by inclusion. Let $0 \in I$ be another element with $0 \leq F$ for each $F \in I$.

Given $F \in I$, so that F is a finite-dimensional subspace of E, the functional ϕ restricted to F is bounded, and so admits a Hahn-Banach extension to E, say $\mu_F \in E^*$. Pick $x_F \in F$ with $\mu_F(x_F) = 0$ and $\|x_F\| = 1$ (here we need to suppose that F has dimension at least 2). Let $T_F : E \to E$ be the operator defined by $T_F(x) = x - \mu_F(x)x_F$ for $x \in E$. Then T_F is bounded, with inverse $T_F^{-1}(x) = x + \mu_F(x)x_F$. Indeed,

$$T_F(x + \mu_F(x)x_F) = x + \mu_F(x)x_F - \mu_F(x)x_F - \mu_F(x)\mu_F(x_F)x_F = x,$$
$$T_F^{-1}(x - \mu_F(x)x_F) = x - \mu_F(x)x_F + \mu_F(x)x_F - \mu_F(x)\mu_F(x_F)x_F = x$$

For $F \in I$ let $E_F = E$. Define $\varphi_{F,0} = T_F$ and as before, given $0 \neq F \leq G$ define $\varphi_{G,F} = T_G T_F^{-1}$, and finally define $\varphi_{F,F}$ to be the identity. Then $((E_F), (\varphi_{G,F}))$ is an inductive system.

Fix $F \in I$ and $x \in E$. For $G \geq F$,

$$\varphi_{G,F}(x) = T_G(T_F^{-1}(x)) = T_G(x + \mu_F(x)x_F) = x + \mu_F(x)x_F - \mu_G(x)x_G - \mu_F(x)\mu_G(x_F)x_G.$$

As $F \leq G$ we see that $x_F \in G$ and so $\mu_G(x_F) = \phi(x_F) = \mu_F(x_F) = 0$. If G is sufficiently large so that $x \in G$ then $\mu_G(x) = \phi(x)$, and so

$$\varphi_{G,F}(x) = x + \mu_F(x)x_F - \phi(x)x_G.$$

Thus

$$\|\varphi_{G,F}(x)\| \leq \|x + \mu_F(x)x_F\| + |\phi(x)|, \qquad \|\varphi_{G,F}(x)\| \geq |\phi(x)| - \|x + \mu_F(x)x_F\|.$$

Hence

$$|\phi(x)| - \|x + \mu_F(x)x_F\| \leq \limsup_{G \geq F} \|\varphi_{G,F}(x)\| \leq \|x + \mu_F(x)x_F\| + |\phi(x)|.$$

In particular, $\limsup_{G\geq F} \|\varphi_{G,F}(x)\| < \infty$ for each $x \in E_F$. However, we also see that there is no constant K so that $\limsup_{G\geq F} \|\varphi_{G,F}(x)\| \leq K\|x\|$ for every $x \in E_F$, because ϕ is unbounded.

3 From a categorical perspective

If the reader has encountered Banach spaces in a category theory setting before, they will know that there are two main choices for which morphisms to consider:

- all bounded linear maps. From an analytic perspective, this seems natural, but the resulting category is somewhat difficult to manage;
- just contractive linear maps. This might seem slightly artificial, but it leads to a somewhat better behaved category.

Let BA_1 be the category of Banach algebras with contractive homomorphisms. Given an inductive system $((A_i), (\varphi_{j,i}))$ in BA_1, it is immediate that this is a normed inductive system, in the sense of Definition 2.1. By Theorem 2.2 we can form $\varinjlim A_i$. Now let $(B, (\psi_i))$ be any coherent system. As each ψ_j is a contraction, for $i \in I$ and $a \in A_i$, we have that

$$\|\psi_i(a)\| = \|\psi_j(\varphi_{j,i}(a))\| \leq \|\varphi_{j,i}(a)\|$$

for any $j \geq i$, and so $\|\psi_i(a)\| \leq \limsup_{j\geq i} \|\varphi_{j,i}(a)\|$. Thus $(B, (\psi_i))$ satisfies the conditions in Theorem 2.2, and so we conclude that $\varinjlim A_i$ is the inductive limit in BA_1. Recall that isomorphisms in BA_1 are *isometric* isomorphisms.

Indeed, abstract category theoretic ideas work for BA_1. We recall that the inductive limit is really an example of a *colimit*, [5, Chapter 5]. If we have all coproducts, and coequalisers, then all colimits exists; although we remark that in this particular case, this does not lead to a very efficient construction of $\varinjlim A_i$. Quickly, we recall that coproducts in BA_1 are simply direct sums normed with the ℓ^1 norm (which also works for Banach algebras). Coequalisers are given as certain quotient maps. Indeed, given Banach spaces X, Y and contractions $f, g : X \to Y$, let Y_0 be the closed linear span of $\{f(x) - g(x) : x \in X\}$ in Y. Then the quotient map $Y \to Y/Y_0$ is the coequaliser of f, g. For Banach algebras, take instead the closed ideal generated by $\{f(x) - g(x) : x \in X\}$.

3.1 For bounded linear maps

We consider now BA, the category of Banach algebra and bounded homomorphisms. The situation here seems much more complicated.

Example 3.1. We return to Example 2.3, where $A_n = A$ for each $n \in \mathbb{N}$ and $\varphi_{n+1,n} = \frac{1}{2}$. Then $\varinjlim A_n = \{0\}$, but we noticed that we could define $B = A$ and $\psi_n = 2^{n-1} : A_n \to B$, and then $(B, (\psi_n))$ is a coherent system, but of course there is no suitable homomorphism $\varinjlim A_n \to B$. Hence $\varinjlim A_n$ is not an inductive limit in BA.

However, B actually satisfies the universal property. Indeed, let $(C, (\phi_n))$ be a coherent system. Then $\phi_1 = \phi_n \circ \varphi_{n,1} = 2^{1-n}\phi_n$ and so $\phi_n = 2^{n-1}\phi_1$ for each n. We can then define $\phi : B \to C$ by $\phi = \phi_1$ (recalling that $B = A = A_n$). Then $\phi \circ \psi_n = 2^{n-1}\phi_1 = \phi_n$, and so we have factored through the coherent system $(B, (\psi_n))$. We conclude that B is the inductive limit in BA.

Example 3.2. Consider the construction in Example 2.5. We shall show that $\varinjlim A_i$ is the inductive limit in BA.

We first compute $\varinjlim A_i$, in the sense of Theorem 2.4. We can take $K_i = 1$ for each i. Furthermore, for $i \in I$, let $x \in \ell^1$ and set $y = \varphi_{i,0}^{-1}(x)$. We showed that there is $j \geq i$ so that for $k \geq j$ we have $\varphi_{k,i}(x) = \varphi_{k,0}(y)$ has norm close to $\|y\| \leq \|x\|$. Then $\phi_i(a) = (\varphi_{k,i}(x)) = (\varphi_{k,0}(y)) \in \mathfrak{A}$. The construction of $\varphi_{k,0}$ shows that in fact $\|\varphi_{k,0}(y) - y\|$ is arbitrarily small for large k. Thus $\phi_i(a) = (y) = (\varphi_{i,0}^{-1}(a)) \in \mathfrak{A}$. We conclude that $\varinjlim A_i$ is isomorphic to ℓ^1, and ϕ_i is identified with $\varphi_{i,0}^{-1}$.

Let $(B, (\psi_i))$ be a coherent system. Then $\psi_0 = \psi_i \circ \varphi_{i,0}$ and so $\psi_i = \psi_0 \circ \varphi_{i,0}^{-1}$. We may hence define $\phi : \varinjlim A_i = \ell^1 \to B$ by $\phi = \psi_0$. Then $\phi \circ \phi_i = \psi_0 \circ \varphi_{i,0}^{-1} = \psi_i$. Thus we have factored our system, and we conclude that $\varinjlim A_i$ is the inductive limit in BA.

Example 3.3. Now consider the construction in Example 2.6. Let $(B, (\psi_F))$ be a coherent system. Then $\psi_0 = \psi_F \circ \varphi_{F,0} = \psi_F \circ T_F$ so that $\psi_F = \psi_0 \circ T_F^{-1}$. Given $x \in F$ we have that $T_F^{-1}(x) = x + \phi(x)x_F$ and so $\psi_F(x) = \psi_0(x) + \phi(x)\psi_0(x_F)$. Thus

$$\|\psi_F\|\|x\| \geq \|\psi_F(x)\| \geq |\phi(x)|\|\psi_0(x_F)\| - \|\psi_0(x)\| \geq |\phi(x)|\|\psi_0(x_F)\| - \|\psi_0\|\|x\|.$$

As ϕ is unbounded, this inequality can only hold when $\psi_0(x_F) = 0$. We hence conclude that $\psi_0(x_F) = 0$ for all $F \neq 0$, and that hence $\psi_F = \psi_0$ for each F.

Let E_0 be the closed linear span of $\{x_F : F \in I, F \neq 0\}$, and let $q : E \to E/E_0$ be the quotient map. Let $\phi_F = q : E_F = E \to E/E_0$. For $F \leq G$ and $x \in E$

$$\begin{aligned} \phi_G \circ \varphi_{G,F}(x) &= q \circ T_G \circ T_F^{-1}(x) = q(T_G(x + \mu_F(x)x_F)) \\ &= q(x - \mu_G(x)x_G + \mu_F(x)x_F - \mu_F(x)\mu_G(x_F)x_G) = q(x) = \phi_F(x). \end{aligned}$$

Thus $(E/E_0, (\phi_F))$ is a coherent system.

Furthermore, $\psi_0 = \psi_F$ must factor through q as ψ_0 sends E_0 to $\{0\}$. Hence any coherent system factors, and we conclude that $(E/E_0, (\phi_F))$ is the inductive limit in BA.

Example 3.4. Let's explore a very simple example. Let I be some directed set, and set $A_i = \mathbb{C}$ for each $i \in I$. Suppose we have an inductive system $(\varphi_{j,i})$, so each $\varphi_{j,i}$ can be identified with a complex number, and $\varphi_{k,j}\varphi_{j,i} = \varphi_{k,i}$ for all $i \leq j \leq k$.

For simplicity set $I = \mathbb{N}$ so we need only specify $\varphi_{n+1,n} = \alpha_n$ say. Then $\varphi_{j,i} = \alpha_i\alpha_{i+1}\cdots\alpha_{j-1}$. There are then exactly two cases. Firstly, if $\alpha_n = 0$ for arbitrarily large n, then for each i there is some $j > i$ with $\varphi_{j,i} = 0$. It follows that if $(B, (\psi_i))$ is any coherent system, then $\psi_i = 0$ for all i. Thus 0 is the category theoretic inductive limit.

Now assume that α_n is eventually non-zero, in which case we can identify the (category theoretic) inductive limit as $A = \mathbb{C}$. Choose n_0 so that $\alpha_n \neq 0$ for $n \geq n_0$, set $\phi_{n_0} = 1$, and set

$$\phi_n = \alpha_{n_0}^{-1}\alpha_{n_0+1}^{-1}\cdots\alpha_{n-1}^{-1} \qquad (n > n_0).$$

Finally set $\phi_n = \alpha_n \alpha_{n+1} \cdots \alpha_{n_0 - 1}$ for $n < n_0$. For $i \leq j$ we have that

$$\begin{aligned}
\phi_j \varphi_{j,i} &= \phi_j \alpha_i \alpha_{i+1} \cdots \alpha_{j-1} \\
&= \begin{cases} \alpha_{n_0}^{-1} \cdots \alpha_{j-1}^{-1} \times \alpha_i \alpha_{i+1} \cdots \alpha_{j-1} & : j \geq n_0 \\ \alpha_j \cdots \alpha_{n_0 - 1} \times \alpha_i \alpha_{i+1} \cdots \alpha_{j-1} & : j \leq n_0 \end{cases} \\
&= \begin{cases} \alpha_{n_0}^{-1} \cdots \alpha_{i-1}^{-1} & : j \geq n_0, i \geq n_0 \\ \alpha_i \alpha_{i+1} \cdots \alpha_{n_0 - 1} & : j \geq n_0, i \leq n_0 \\ \alpha_i \alpha_{i+1} \cdots \alpha_{n_0 - 1} & : j \leq n_0 \end{cases} \\
&= \phi_i,
\end{aligned}$$

and so we have a coherent system.

Let $(B, (\psi_i))$ be a coherent system, and set $p_i = \psi_1(1)$ an idempotent in B. Then for $n > n_0$ we have $p_{n_0} = \alpha_{n_0} \alpha_{n_0+1} \cdots \alpha_{n-1} p_n = \phi_n p_n$ so $p_n = \phi_n p_{n_0}$, while for $n < n_0$ we have $p_n = \alpha_n \alpha_{n+1} \cdots \alpha_{n_0-1} p_{n_0} = \phi_n p_{n_0}$. Thus, if we define $\phi : A = \mathbb{C} \to B$ by $\phi(1) = p_{n_0}$, then $\psi_n(1) = p_n = \phi_n \phi(1)$ and so $\psi_n = \phi \phi_n$ for each n. Thus A is universal.

Example 3.5. We again work with Banach spaces for simplicity. Let $I = \mathbb{N}$, set $E_1 = \mathbb{C}$ and $E_{n+1} = E_n \oplus_1 \mathbb{C} = \ell^1_{n+1}$. Define $\varphi_{n+1,n} : E_n \to E_n \oplus_1 \mathbb{C}$ to be inclusion on the first component. Thus we obtain a normed inductive system, and so $\varinjlim A_i$ exists.

Let $(B, (\psi_n))$ be some coherent system, and let $y_1 = \psi_1(1) \in B$. Then a typical element of E_2 is (a, b) for some $a, b \in \mathbb{C}$, and coherence ensures that $\psi_2((a, b)) = a y_1 + b y_2$ for some $y_2 \in B$. Similarly we find a sequence (y_n) in B so that for $a = (a_i)_{i=1}^n \in E_n$ we have that $\psi_n(x) = \sum_{i=1}^n a_i y_i$. Notice that every ψ_n is bounded, regardless of the norms of the (y_n). We can reverse this argument, and so we conclude that coherent systems biject with sequences (y_n) in B.

From Theorem 2.2 or Theorem 2.4, $\varinjlim A_i$ is universal for coherent systems $(B, (\psi_n))$ where each $\psi_n : \ell^1_n \to B$ is a contraction. This corresponds to the sequence (y_n) in B satisfying $\|y_n\| \leq 1$ for each n. It follows that $\varinjlim A_n$ is actually (isometrically isomorphic to) ℓ^1.

We claim that there is no inductive limit in the category theory sense. Indeed, if (E, ϕ_n) were a universal coherent system, then as in the previous paragraph, we find a sequence (x_n) in E. By universality, for any Banach space B and any sequence (y_n) in B, there is a (unique) bounded linear map $\phi : E \to B$ with $\phi(x_n) = y_n$ for each n. To obtain a contradiction, let (α_n) be some sequence of positive numbers, let $B = \ell^1$ with unit vector basis (δ_n), and let $y_n = \alpha_n \delta_n$. If ϕ exists then

$$|\alpha_n| = \|y_n\| = \|\phi(x_n)\| \leq \|\phi\| \|x_n\| \qquad (n \in \mathbb{N}),$$

and so $\sup_n |\alpha_n| \|x_n\|^{-1} < \infty$. As (x_n) is fixed, we can easily find a sequence (α_n) to give a contradiction.

Example 3.6. We now adapt the previous example. Have (E_n) as before, but now let $\varphi_{n+1,n}$ be twice the inclusion on the first component. Then $\varinjlim A_i$ is not defined. A coherent system $(B, (\psi_n))$ still bijects with a sequence (y_n) in B, but now for $x = (x_i) \in E_n$ we have that $\psi_n(x) = \sum_{i=1}^n 2^{i-n} x_i y_i$. However, the same argument will show that there is no inductive limit in the category theory sense.

We hence see that in BA the category theoretical inductive limit may or may not exist, and that the existence of $\varinjlim A_i$ (in either sense) seems unrelated. This raises two questions:

- Can we characterise those inductive systems in BA which have an inductive limit in BA?
- What relation does $\varinjlim A_i$ have to category theoretical constructions?

We do not know the answer to either of these questions.

3.2 Algebraic inductive limits

Let us think a little about the purely algebraic inductive limit. Let Alg be the category of complex algebras with algebra homomorphisms. It is well-known that there are always inductive limits: we give a slightly unusual construction.

Let $((A_i), (\varphi_{j,i}))$ be an inductive system in Alg. Let $\prod_i A_i$ be the algebraic direct sum of the (A_i), that is, families (a_i) where $a_i \in A_i$ for each i, with coordinate wise operations. Let A_0 be the collection of those $(a_i) \in \prod_i A_i$ such that there is i_0 so that $\varphi_{j,i}(a_i) = a_j$ for any $i_0 \leq i \leq j$. Then A_0 is a subalgebra.

Let I be the collection of those $(a_i) \in \prod_i A_i$ such that there is i_0 with $a_i = 0$ for $i_0 \leq i$. Then I is an ideal, so we may consider the quotient algebra $\prod_i A_i/I$. Notice that if $a = (a_i) \in A_0$ and $b = (b_i) \in I$ then there are i_0, i_1 with $b_i = 0$ for $i \leq i_1$ and $\varphi_{j,i}(a_i) = a_j$ for $i_0 \leq i \leq j$. Then there is i_2 greater than both i_0, i_1, and for $i_2 \leq i \leq j$ we have both $b_i = 0$ and $\varphi_{j,i}(a_i) = a_j$. Thus $a + b = (a_i + b_i) \in A_0$. We conclude that $A_0 + I = A_0$, in particular, $I \subseteq A_0$.

Let A be the algebra A_0/I. Notice that alternatively, A is (canonically isomorphic to) the image of A_0 in $\prod_i A_i/I$. For each k and $a \in A_k$ define (a_i) by $a_i = \varphi_{i,k}(a)$ if $i \geq k$, and $a_i = 0$ otherwise. Set $\phi_k(a) = (a_i)$. Then ϕ_k is a homomorphism, and it is easy to see that the resulting family (ϕ_i) is coherent.

Let B be an algebra with a coherent system $\psi_i : A_i \to B$. Given $a = (a_i) + I \in A$ there is i_0 with $\varphi_{j,i}(a_i) = a_j$ for $i_0 \leq i \leq j$. Define $\phi(a) = \psi_i(a_i) \in B$ for any $i \geq i_0$. If also $j \geq i_0$ there is k greater than both i, j and so $\psi_k(a_k) = \psi_k(\varphi_{k,i}(a_i)) = \psi_i(a_i)$ and similarly $\psi_k(a_k) = \psi_j(a_j)$. It follows that ϕ is well-defined. Clearly $\phi \circ \phi_i = \psi_i$ for each i. It follows that A is the (algebraic) inductive limit.

Supposing each A_i is a Banach algebra and each $\varphi_{j,i}$ is bounded, we might seek an algebra norm on A making the maps ϕ_i bounded. We could then take the norm completion of A to obtain a Banach algebra. Let us use this idea to make a few, somewhat inconclusive, observations.

Suppose $\overline{A}$ is an inductive limit of the Banach algebras (A_i); here we are being deliberately vague as to what this might mean. Let $\overline{\phi}_i$ be the coherent maps $A_i \to \overline{A}$. In particular, this means that $(\overline{A}, (\overline{\phi}_i))$ is a coherent system of algebras, and so there is a unique $\overline{\phi} : A \to \overline{A}$. If $(B, (\psi_i))$ is a coherent system of Banach algebras, then we have a unique $\psi : A \to B$, and we seek a bounded homomorphism $\overline{\psi} : \overline{A} \to B$, making the following diagram commute:

$$\begin{tikzcd}
A \arrow[rr, "!\psi"] \arrow[dd, "!\overline{\phi}"'] & & B \\
& A_i \arrow[ul, "\phi_i"'] \arrow[ur, "\psi_i"] \arrow[dl, "\overline{\phi}_i"] & \\
\overline{A} \arrow[uurr, bend right, "\exists ?\,\overline{\psi}"'] & &
\end{tikzcd}$$

If $\overline{\psi}$ exists then $\ker \overline{\phi} \subseteq \ker \psi$, and conversely, if this holds, then $\overline{\psi}$ will exist, at least as a linear map. Let's explore this in a couple of examples.

Example 3.7. Let's consider Example 2.3 again. As each connecting map is an isomorphism of algebras, we see that A, the algebraic inductive limit, is just the algebra we started with. With $\overline{A} = \varinjlim A_n$ we have $\overline{A} = \{0\}$, and so we see that we need $\psi = 0$. However, this is actually not too restrictive. Let $(B, (\psi_n))$ be a coherent system, so that $\psi_1 = \psi_n \circ \varphi_{n,1} = 2^{1-n}\psi_n$ and hence $\psi_n = 2^{n-1}\psi_1$, for each n. It is not hard to see that when A is identified with A_1, then ψ is identified with ψ_1. Thus we recover the previous result that only when $\psi_1 = 0$, that is, the coherent system is trivial, does $\overline{\psi}$ exist.

If, however, we take $\overline{A}$ as the inductive limit in BA_1, that is, $\overline{A} = A_1$, as in Example 3.1, then $\overline{\phi}$ is just the identity. From the computation in the previous paragraph, if $(B, (\psi_n))$ is a coherent system in BA_1, then in particular ψ_1 is bounded, and so ψ is bounded, and we may take $\overline{\psi} = \psi$.

Example 3.8. Now consider the example from Example 3.5, where we work with vector spaces, or Banach spaces, and not algebras, for simplicity. Notice that the argument shows that coherent systems of vector spaces are just sequences (y_n) in B. By universality, it follows that A is the algebraic direct sum of $\mathbb{N}$ many copies of $\mathbb{C}$, that is, the vector space of all functions $f : \mathbb{N} \to \mathbb{C}$ with $f(n) = 0$ for all but finitely many n. Let $(\delta_n) \subseteq A$ be the obvious basis. Then $\psi : A \to B$ is the unique linear map with $\psi(\delta_n) = y_n$ for each n.

Any norm on A will lead to some choice of $\overline{A}$. However $\overline{\phi}$ is always injective, and so $\overline{\psi}$ will always exists as a linear map. However, we learn nothing about whether $\overline{\psi}$ is bounded.

3.3 Uses

There are perhaps two possible uses of inductive limits of Banach spaces:

- One naturally comes across a (normed) inductive system when working on some problem, and identifying the inductive limit will be useful;
- We are interested in constructing examples of Banach algebras, using inductive limits.

We are not actually aware of any example in the literature, but surely the 2nd must occur somewhere.

We now study the second case, that of constructing examples using inductive limits.

Proposition 3.9. *Let A be a Banach algebra, and let $\mathcal{S}$ be a collection of closed subalgebras of A which when ordered by inclusion becomes a directed set (that is, given $B, C \in \mathcal{S}$ there is $D \in \mathcal{S}$ with $B \subseteq D$ and $C \subseteq D$). For $B, C \in \mathcal{S}$ with $B \subseteq C$ let $\varphi_{C,B} : B \to C$ be the inclusion map. Then $((B)_{B\in\mathcal{S}}, (\varphi_{C,B}))$ is a normed inductive system, and $\varinjlim \mathcal{S}$ is equal to the closure of the union of $\mathcal{S}$.*

Proof. We need only prove that $\varinjlim \mathcal{S}$ has the stated form; we may suppose that the union of $\mathcal{S}$ is dense in A, and we shall verify the universal property from Theorem 2.2. The maps $\phi_B : B \to A$, for $B \in \mathcal{S}$, are simply the inclusion maps. Let A_1 be a Banach algebra with coherent maps $\psi_B : B \to A_1$ for $B \in \mathcal{S}$ such that $\|\psi_B(a)\| \leq \limsup_{C \supseteq B} \|\varphi_{C,B}(a)\| = \|a\|$ for each $a \in B$, that is, ψ_B is a contraction. We shall show there is a unique contractive homomorphism $\phi : A \to A_1$ with $\phi \circ \phi_B = \phi|_B = \psi_B$ for $B \in \mathcal{S}$.

Given $a \in A$ in the union of the $\mathcal{S}$ define $\phi(a) = \psi_B(a)$ for any $B \in \mathcal{S}$ with $a \in B$. If also $C \in \mathcal{S}$ with $a \in C$ then given $D \in \mathcal{S}$ containing B and C, by coherence, we have that $\psi_B(a) = \psi_D(\varphi_{D,B}(a)) = \psi_D(a)$ and similarly $\psi_C(a) = \psi_D(a)$, so $\psi_B(a) = \psi_C(a)$. Thus ϕ is well-defined. It remains to show that ϕ is a contraction (and so extends by continuity to all of A) but this is immediate as $\|\phi(a)\| = \|\psi_B(a)\| \leq \|a\|$ for each $a \in B$, for any $B \in \mathcal{S}$. □

Let $(A_i, (\varphi_{j,i}))$ be a normed inductive system (or, indeed, an inductive system satisfying the weaker condition of Theorem 2.4), and let $A = \varinjlim A_i$. For each i let B_i be the closure of $\phi_i(A_i)$ in A, so that $B_i \subseteq B_j$ when $i \leq j$. Then $\mathcal{S} = \{B_i : i \in I\}$ is a family of closed subalgebras of A which becomes a directed set under inclusion, and so the above proposition shows that $A = \varinjlim \mathcal{S}$. Thus, if we are only interested in the construction of A, then it is no loss of generality to work with normed inductive systems in which each connecting morphism is an isometry. The same argument applies to the category theoretic inductive limit in BA.

References

[1] B. Blackadar, *K-theory for operator algebras*, second edition, Mathematical Sciences Research Institute Publications, 5, Cambridge University Press, Cambridge, 1998. MR1656031

[2] B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, 122, Springer-Verlag, Berlin, 2006. MR2188261

[3] J. M. F. Castillo, The hitchhiker guide to categorical Banach space theory. Part I, Extracta Math. **25** (2010), no. 2, 103–149. MR2814472

[4] J. B. Conway, *A course in functional analysis*, second edition, Graduate Texts in Mathematics, 96, Springer-Verlag, New York, 1990. MR1070713

[5] T. Leinster, *Basic category theory*, Cambridge Studies in Advanced Mathematics, 143, Cambridge University Press, Cambridge, 2014. MR3307165

[6] T. W. Palmer, *Banach algebras and the general theory of *-algebras. Vol. I*, Encyclopedia of Mathematics and its Applications, 49, Cambridge University Press, Cambridge, 1994. MR1270014

[7] M. Rørdam, F. Larsen and N. Laustsen, *An introduction to K-theory for C^*-algebras*, London Mathematical Society Student Texts, 49, Cambridge University Press, Cambridge, 2000. MR1783408

[8] N. E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993. MR1222415