Multipliers

Matthew Daws

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Contents

1 Small results

1.1 Ideals in the multipliers

Let A be an *idempotent* algebra, meaning that $\operatorname{lin}\{ab : a,b \in A\} = A$ (in the topological setting, we may take the closure, and appeal to continuity in the argument below). We also suppose that the product on A is non-degenerate. In particular, these conditions hold for any Banach algebra with a bounded approximate identity, so in particular, for C^*-algebras.

Suppose that B is an ideal in $M(A)$ which contains A, so $A \subseteq B \trianglelefteq M(A)$. We claim that then $M(B) = M(A)$. As B is an ideal in $M(A)$, any $x \in M(A)$ induces a multiplier of B by left/right multiplication, and as $A \subseteq B$, this map $M(A) \to M(B)$ is injective. Our claim is that this map is a bijection.

As $B \subseteq M(A)$, clearly A is an ideal in B. As A is idempotent, it follows that $A = \operatorname{lin}\{ab : a \in A, b \in B\} = \operatorname{lin}\{ba : a \in A, b \in B\}$. Thus, for $x \in M(B)$, we see that

$$(ab)x = a(bx) \in A, \quad x(ba) = (xb)a \in A \qquad (a \in A, b \in B),$$

and so $Ax \subseteq A$ and $xA \subseteq A$. Hence x induces $y \in M(A)$. Then

$$(yb)a = y(ba) = x(ba) = (xb)a \qquad (a \in A, b \in B),$$

as $ba \in A$, so $yb = xb$ by non-degeneracy. Hence $x \in M(B)$ is given by y.

2 Automorphisms of $M(A)$

We present a careful account of the MathOverflow question [3] and the counter-example [4]. The question asked is the following:

> Let A be a (non-unital) C^*-algebra with multiplier algebra $M(A)$. Let $\phi : M(A) \to M(A)$ be a $*$-automorphism. Is it true that ϕ is automatically strictly continuous (on bounded subsets)?

As the question notes, this is true for some algebras by direct computation, e.g. when $A = \mathcal{K}(H)$ the compact operators on a Hilbert space. By [5, Proposition 1.1], when A is separable, we know that we can characterise A inside $M(A)$ as

$$A = \{x \in M(A) : xM(A) \text{ is separable}\}.$$

Then, given ϕ an automorphism of $M(A)$, if $a \in A$ then $aM(A)$ is separable, and so also $\phi(a)M(A) = \phi(aM(A))$ is separable, so that $\phi(a) \in A$. The same argument applies to ϕ^{-1} showing that ϕ restricts to an automorphism of A.

Recall now, [1, Chapter 2], that the strict extension of ϕ from A to $M(A)$, say $\overline{\phi}$, satisfies that $\overline{\phi}(x)\phi(a)b = \phi(xa)b$ for $x \in M(A), a, b \in A$. As ϕ is an automorphism, this is equivalent to $\overline{\phi}(x)\phi(a) = \phi(xa)$ for $x \in M(A), a \in A$. For $x \in M(A), a \in A$, as ϕ is a homomorphism, $\phi(x)\phi(a) = \phi(xa)$. Thus $\overline{\phi}(x)\phi(a) = \phi(x)\phi(a)$, so as ϕ is an automorphism of A, this shows that $\overline{\phi}(x)b = \phi(x)b$ for all b, so $\overline{\phi}(x) = \phi(x)$. In particular, ϕ is necessarily strictly continuous. Let us record this small argument.

Lemma 2.1. *Let ϕ be an automorphism of $M(A)$ such that ϕ restricts to an automorphism of A. Then ϕ is equal to the strict extension of ϕ restricted to A, and so ϕ is strictly continuous.*

Notice that we have dropped the condition "on bounded sets". [1, Proposition 2.5] is only stated with respect to strict continuity on the unit ball, but this holds more generally:

Proposition 2.2. *Let A, B be C^*-algebras and let E be a Hilbert B-module. For a $*$-homomorphism $\phi : A \to \mathcal{L}(E)$, the following are equivalent:*

1. *ϕ is nondegenerate, meaning that $\operatorname{lin}\{\phi(a)\xi : a \in A, \xi \in E\}$ is dense in E;*
2. *ϕ is the restriction to A of a unital $*$-homomorphism $\psi : M(A) \to \mathcal{L}(E)$ which is strictly continuous;*
3. *for some (any) approximate unit (e_i) of A, we have that $\phi(e_i) \to 1$ strictly in $\mathcal{L}(E)$.*

Proof. We turn E into a left A-module for the module action $a \cdot \xi = \phi(a)\xi$. As A has a bounded approximate identity, the Cohen–Hewitt factorisation theorem (for example, [2, Appendix A]) shows that $\{\phi(a)\xi : a \in A, \xi \in E\}$ is equal to its own closed linear span. Suppose that (1) holds, so each $\xi \in E$ is equal to $\phi(a)\eta$ for some $a \in A, \eta \in E$. Let $\overline{\phi}$ be the canonical extension of ϕ. Let $x_i \to x$ strictly in $M(A)$. With $\xi = \phi(a)\eta$, we have that $x_i a \to xa$ in norm, so

$$\|\overline{\phi}(x_i)\xi - \overline{\phi}(x)\xi\| = \|(\overline{\phi}(x_i) - \overline{\phi}(x))\phi(a)\eta\| = \|\phi(x_i a - xa)\eta\| \to 0.$$

As also $(x_i^* - x^*)a \to 0$ in A, we also have that $\overline{\phi}(x_i)^*\xi \to \overline{\phi}(x)^*\xi$, as $\overline{\phi}$ is a $*$-homomorphism. Thus $\overline{\phi}(x_i) \to \overline{\phi}(x)$ strictly. By definition, $\overline{\phi}(1)\eta = \overline{\phi}(1)\phi(a)\eta = \phi(1a)\eta = \xi$ so $\overline{\phi}(1) = 1$. So (2) holds.

If (2) holds, then as $e_i \to 1$ strictly in $M(A)$, it follows that $\phi(e_i) = \psi(e_i) \to \psi(1) = 1$ strictly in $\mathcal{L}(E)$, showing (3).

If (3) holds, then $\psi(e_i)\xi \to \xi$ in norm in E, for each $\xi \in E$. Thus certainly (1) holds. ☐

Recall that if $E = B$ then $\mathcal{L}(E) \cong M(B)$ and the associated strict topologies agree. The subtlety here occurs if we set $C = \mathcal{K}(E)$ so that $M(C) \cong \mathcal{L}(E)$, but then the strict topologies on $M(C)$ and $\mathcal{L}(E)$ only agree on bounded sets, compare [1, Chapter 8].

We finish by characterising strict continuity in terms of the original algebra, giving a converse to the above lemma.

Lemma 2.3. *Let $\phi : A \to M(B)$ and $\psi : B \to M(A)$ be non-degenerate $*$-homomorphisms with strict extensions $\overline{\phi}$ and $\overline{\psi}$. If these $*$-homomorphisms between $M(A)$ and $M(B)$ are mutual inverses, then $\phi(A) \subseteq B$ and $\psi(B) \subseteq A$, and $\psi = \phi^{-1}$.*

Proof. Let $a = \psi(b)a_1$ for some $b \in B, a_1 \in A$, so that $\phi(a) = \overline{\phi}(a) = \overline{\phi}(\psi(b))\phi(a_1) = b\phi(a_1) \in B$. As ϕ is non-degenerate, the linear span of such a are dense in A, and so we have shown that $\phi(A) \subseteq B$. Similarly $\psi(B) \subseteq A$. For $a \in A$, we see that $\psi(\phi(a)) = \overline{\psi}(\overline{\phi}(a)) = a$ and similarly $\phi\psi = \mathrm{id}$ so $\psi = \phi^{-1}$. □

Corollary 2.4. *Let θ be an automorphism of $M(A)$ which is strictly continuous, with strictly continuous inverse. Then θ restricts to an automorphism of A.*

Proof. Set ϕ to be the restriction of θ to A, so by Proposition 2.2, ϕ is non-degenerate. Let ψ be the restriction of θ^{-1} to A, which is non-degenerate. By strict density of A in $M(A)$, we have that $\theta = \overline{\phi}$ and $\theta^{-1} = \overline{\psi}$. The previous lemma shows that $\phi(A) \subseteq A, \psi(A) \subseteq A$ and $\psi = \phi^{-1}$, that is, θ restricts to an automorphism of A. □

Corollary 2.5. *Let θ be an automorphism of $M(A)$. Then θ and θ^{-1} are strictly continuous if and only if θ restricts to an automorphism of A.*

Proof. Combine Lemma 2.1, applied to both θ and θ^{-1}, and the previous corollary. □

Finally, if we only care about θ being strictly continuous, we have the following.

Proposition 2.6. *Let θ be an automorphism of $M(A)$. If $\theta(A) \supseteq A$ then θ is strictly continuous.*

Proof. Let $\phi : A \to M(A)$ be the restriction of θ, so $\phi(A) \supseteq A$, and hence $\operatorname{lin} \phi(A)A \supseteq \operatorname{lin} AA = A$ showing that ϕ is non-degenerate. Let $\overline{\phi}$ be the strict extension of ϕ, so as we argued before,

$$\overline{\phi}(x)\phi(a)b = \phi(xa)b = \theta(xa)b = \theta(x)\theta(a)b = \theta(x)\phi(a)b \qquad (a, b \in A, x \in M(A)).$$

By non-degeneracy, this shows that $\overline{\phi}(x) = \theta(x)$ for all x, so in particular, θ is strictly continuous. □

Of course, if both $A \subseteq \theta(A)$ and $A \subseteq \theta^{-1}(A)$, then $A = \theta(A)$ and θ restricts to an automorphism of A.

2.1 Stone-Cech compactifications

The counter-example [4] uses Stone-Cech compactifications of discrete spaces. We now develop this theory essentially from scratch, as it is a fun exercise to do so.

Let I be some set. A *filter* $\mathcal{F}$ on I is a collection of subsets of I such that:

1. $\emptyset \not\in \mathcal{F}$;
2. $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$;
3. $A \in \mathcal{F}$ and $B \subseteq I$ with $A \subseteq B$ implies $B \in \mathcal{F}$.

Given $i \in I$ the set $\hat{i} = \{A \subseteq I : i \in A\}$ is a filter. Filters are naturally ordered by inclusion. A maximal filter for this ordering is an *ultrafilter*. Each $\hat{i}$ is an ultrafilter, the *principle ultrafilter at i*.

Lemma 2.7 (The ultrafilter lemma)**.** *A filter $\mathcal{U}$ on I is an ultrafilter if and only if, for each $A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.*

Proof. If the condition holds, and yet $\mathcal{U}$ is not maximal, let $\mathcal{F}$ be a filter strictly containing $\mathcal{U}$. Hence there is $A \in \mathcal{F} \setminus \mathcal{U}$. By the condition, necessarily $I \setminus A \in \mathcal{U}$, so also $I \setminus A \in \mathcal{F}$, so $A \cap (I \setminus A) = \emptyset \in \mathcal{F}$, contradiction.

Conversely, let $\mathcal{U}$ be an ultrafilter, and fix $A \subseteq I$. Suppose that $A \cap B \neq \emptyset$ for all $B \in \mathcal{U}$. Then set

$$\mathcal{F} = \{C : \exists B \in \mathcal{U}, A \cap B \subseteq C\}.$$

By the assumption, $\emptyset \notin \mathcal{F}$, and the remaining axioms for $\mathcal{F}$ to be a filter are easily checked. If $B \in \mathcal{U}$ then $A \cap B \subseteq B$ so $B \in \mathcal{F}$, so $\mathcal{U} \subseteq \mathcal{F}$, so $\mathcal{U} = \mathcal{F}$ by maximality of $\mathcal{U}$. In particular, given any $B \in \mathcal{U}$, we have that $A \cap B \subseteq A$, so $A \in \mathcal{F} = \mathcal{U}$. Thus, given $A \subseteq I$, either $A \in \mathcal{U}$, or otherwise, it must be that $A \cap B = \emptyset$ for some $B \in \mathcal{U}$, in which case $B \subseteq I \setminus A$ so $I \setminus A \in \mathcal{U}$. □

Let βI be the collection of all ultrafilters on I. For $A \subseteq I$ define

$$\mathcal{O}_A = \{\mathcal{U} \in \beta I : A \in \mathcal{U}\} \subseteq \beta I.$$

Then clearly $\bigcup_A \mathcal{O}_A = \beta I$. As $A, B \in \mathcal{U}$ if and only if $A \cap B \in \mathcal{U}$, it follows that $\mathcal{O}_A \cap \mathcal{O}_B = \mathcal{O}_{A \cap B}$. Thus these sets are closed under finite intersections and cover βI, and hence form a basis for a topology on βI. We call each set $\mathcal{O}_A$ a *basic open set*. By the ultrafilter lemma, $\beta I \setminus \mathcal{O}_A = \mathcal{O}_{I \setminus A}$ and so each basic open set is also closed. Also

$$\begin{aligned} \mathcal{O}_A \cup \mathcal{O}_B &= \beta I \setminus (\beta I \setminus (\mathcal{O}_A \cup \mathcal{O}_B)) = \beta I \setminus (\mathcal{O}_{I \setminus A} \cap \mathcal{O}_{I \setminus B}) = \beta I \setminus (\mathcal{O}_{(I \setminus A) \cap (I \setminus B)}) \\ &= \beta I \setminus (\mathcal{O}_{I \setminus (A \cup B)}) = \mathcal{O}_{A \cup B}. \end{aligned}$$

We identify $I \subseteq \beta I$ be identifying $i \in I$ with the principle ultrafilter $\hat{i}$.

Lemma 2.8. *I is dense in βI.*

Proof. If not, there is a non-empty open set disjoint from I. This set must contain some $\mathcal{O}_A$ for a non-empty A, but then for each $i \in A$ we have that $\hat{i} \in \mathcal{O}_A$, contradiction. □

A topological space X is *compact* if any open cover of X has a finite subcover.

Proposition 2.9. *βI is compact*

Proof. Let $(U_j)_{j \in J}$ be an open cover of βI. Each U_j is a union of basic open sets, and so we obtain some open cover of the form $(\mathcal{O}_{A_i})_{i \in I}$. It hence suffices (and is necessary) to show that this open cover has a finite subcover. Towards a contradiction, suppose not, so for each $\{i_1, \cdots, i_n\} \subseteq I$ we have

$$\emptyset \neq \beta I \setminus (\mathcal{O}_{A_{i_1}} \cup \cdots \cup \mathcal{O}_{A_{i_n}}) = \mathcal{O}_{I \setminus A_{i_1}} \cap \cdots \cap \mathcal{O}_{I \setminus A_{i_n}} = \mathcal{O}_{(I \setminus A_{i_1}) \cap \cdots \cap (I \setminus A_{i_n})}.$$

Thus, if we set $B_i = I \setminus A_i$ for each i, then any intersection of finitely many of the B_i is non-empty. Set

$$\mathcal{F} = \{A \subseteq I : A \supseteq B_{i_1} \cap \cdots \cap B_{i_n} \text{ for some } (i_j)_{j=1}^n \subseteq I\}.$$

Then $\mathcal{F}$ does not contain the empty set, and is then easily verified to be a filter on I. Use Zorn's Lemma to refine $\mathcal{F}$ to an ultrafilter $\mathcal{U}$. For each $i \in I$, clearly $B_i \in \mathcal{F}$ so $B_i \in \mathcal{U}$ so $\mathcal{U} \in \mathcal{O}_{B_i}$ so $\mathcal{U} \notin \mathcal{O}_{A_i}$. This contradicts $(\mathcal{O}_{A_i})$ being an open cover. □

We now show that βI satisfies the universal property to be the Stone–Cech compactification of the discrete space I. Firstly we recall some topology. Let X be a topological space, and let $\mathcal{U}$ be an ultrafilter on X. We say that $\mathcal{U}$ *converges* to $x \in X$, written $x = \lim \mathcal{U}$, when for each open set U with $x \in U$, we have that $U \in \mathcal{U}$.

Lemma 2.10. *Let X be a compact Hausdorff space. Then every ultrafilter on X converges to a unique point.*

Proof. Let x, y be limits of some ultrafilter $\mathcal{U}$. If $x \neq y$ then as X is Hausdorff there are disjoint open U, V, with $x \in U$ and $y \in V$. Then clearly it is impossible for both $U \in \mathcal{U}$ and $V \in \mathcal{U}$; we conclude that limits are unique, if they exist.

Let $\mathcal{U}$ be an ultrafilter on X and towards a contradiction, suppose that $\mathcal{U}$ does not converge. This means that for each $x \in X$ there is an open set U_x with $x \in U_x$ and $U_x \notin \mathcal{U}$. By the ultrafilter lemma, the closed set $C_x = X \setminus U_x$ is in $\mathcal{U}$. It follows that for any finite subset $\{x_1, \cdots, x_n\} \subseteq X$ we have that $C_{x_1} \cap \cdots \cap C_{x_n} \in \mathcal{U}$ and so this intersection is non-zero. Equivalently, $U_{x_1} \cup \cdots \cup U_{x_n} \neq X$. However, $(U_x)_{x \in X}$ is obviously an open cover of X, so as X is compact, there is some finite subcover, contradiction. □

Theorem 2.11. *Let X be a compact Hausdorff space, and let $f : I \to X$ be a function. There is a unique continuous function $\beta f : \beta I \to X$ making the following diagram commute*

$$\begin{tikzcd} I \arrow[r, "f"] \arrow[dr] & X \\ & \beta I \arrow[u, "\exists!\ \beta f"'] \end{tikzcd}$$

where $I \to \beta I$ is the canonical inclusion.

Proof. As I is dense in βI, any continuous extension of f is unique. We show existence. For $\mathcal{U} \in \beta I$ define

$$f_*(\mathcal{U}) = \{A \subseteq X : f^{-1}(A) \in \mathcal{U}\}.$$

As inverse images commute with set-theoretic operations, it is easy to see that $f_*(\mathcal{U})$ is a filter on X. As $f^{-1}(X \setminus A) = I \setminus f^{-1}(A)$, the ultrafilter lemma shows that $F_*(\mathcal{U})$ is an ultrafilter. Set $\beta f(\mathcal{U}) = \lim f_*(\mathcal{U})$.

Given $i \in I$ we see that $f^{-1}(A) \in \hat{\imath}$ if and only if $i \in f^{-1}(A)$, that is, $f(i) \in A$. Hence $f_*(\hat{\imath}) = \widehat{f(i)}$, and it is easy to verify that $\lim \hat{x} = x$ for any $x \in X$. Thus βf extends f in the sense that $\beta f(\hat{\imath}) = f(i)$.

To show that βf is continuous, let $U \subseteq X$ be open. Let $x \in U$. As X is compact Hausdorff, it is *normal*, and so there are disjoint open sets V, W with $x \in V$ and $X \setminus U \subseteq W$, that is, $U \supseteq X \setminus W$. Set $A_x = A = f^{-1}(V)$. We claim that if $\mathcal{U} \in \mathcal{O}_A$ then $\beta f(\mathcal{U}) = \lim f_*(\mathcal{U}) \notin W$. Indeed, if not, then $x = \lim f_*(\mathcal{U}) \in W$, so by definition of the limit, $f^{-1}(W) \in \mathcal{U}$. As also $A \in \mathcal{U}$, we see that $f^{-1}(W) \cap A \in \mathcal{U}$, in particular this intersection is non-empty, so there is $a \in A$ with $f(a) \in W$. As $A = f^{-1}(V)$, we have $f(a) \in V$ which contradicts V, W being disjoint.

This shows that $\beta f(\mathcal{U}) \in U$, and thus $\mathcal{O}_{A_x} \subseteq (\beta f)^{-1}(U)$. Given now some $\mathcal{U} \in (\beta f)^{-1}(U)$, so that $x = \beta f(\mathcal{U}) = \lim f_*(\mathcal{U}) \in U$. Select V as above for x, so $x \in V$, and hence by the definition of the limit, $V \in f_*(\mathcal{U})$, so $A_x = f^{-1}(V) \in \mathcal{U}$. Thus $(\beta f)^{-1}(U) \subseteq \bigcup_{x \in U} \mathcal{O}_{A_x}$, but this is contained in $(\beta f)^{-1}(U)$, and hence we have equality, showing in particular that $(\beta f)^{-1}(U)$ is open. Hence βf is continuous. □

We shall be interested in case when $X = \beta J$ for some set J, and the map $f : I \to \beta J$ is actually given by $f_0 : I \to J$, composed with the inclusion $J \to \beta J$.

Lemma 2.12. *Let $f_0 : I \to J$ induce $f : I \to \beta J$. For $\mathcal{U} \in \beta I$, define $f_{0,*}(\mathcal{U}) = \{A \subseteq J : f_0^{-1}(A) \in \mathcal{U}\}$ which is a member of βJ. Then $\beta f : \beta I \to \beta J$ is the map $\mathcal{U} \mapsto f_{0,*}(\mathcal{U})$.*

Proof. Let $\mathcal{U} \in \beta I$. As before, $f_{0,*}(\mathcal{U})$ is indeed an ultrafilter. For $B \subseteq \beta J$, we see that $f^{-1}(B)$ depends only on $A = B \cap J$, and indeed $f^{-1}(B) = f_0^{-1}(A)$. Thus

$$f_*(\mathcal{U}) = \{B \subseteq \beta J : f^{-1}(B) \in \mathcal{U}\} = \{B \subseteq \beta J : f_0^{-1}(B \cap J) \in \mathcal{U}\} = \{B \subseteq \beta J : B \cap J \in f_{0,*}(\mathcal{U})\}.$$

Set $\mathcal{W} = f_*(\mathcal{U})$. We compute $\lim \mathcal{W} = \mathcal{V}$, say, a member of βJ. $\mathcal{V}$ is the unique point such that for any open $U \subseteq \beta J$ we have that $\mathcal{V} \in U$ implies $U \in \mathcal{W}$. Making U smaller does not affect the condition, so we may suppose that $U = \mathcal{O}_A$ for some $A \subseteq J$. Then $\mathcal{V} \in \mathcal{O}_A$ means $A \in \mathcal{V}$, while $\mathcal{O}_A \in \mathcal{W}$ means $\mathcal{O}_A \cap J \in f_{0,*}(\mathcal{U})$. As $\hat{j} \in \mathcal{O}_A$ exactly when $j \in A$, we see that $\mathcal{O}_A \in \mathcal{W}$ means that $A \in f_{0,*}(\mathcal{U})$. So $\mathcal{V} = \lim \mathcal{W}$ means $A \in \mathcal{V} \implies A \in f_{0,*}(\mathcal{U})$, that is, $\mathcal{V} = f_{0,*}(\mathcal{U})$. We conclude that $\beta f(\mathcal{U}) = f_{0,*}(\mathcal{U})$ for each $\mathcal{U} \in \beta I$. $\square$

Finally, consider the commutative C^*-algebra $c_0(I)$. A standard argument establishes that $M(c_0(I)) = \ell^\infty(I)$. Given any $f \in \ell^\infty(I)$, we can regard f as mapping into the compact space $\{z \in \mathbb{C} : |z| \leq \|f\|_\infty\}$, and so form $\beta f \in C(\beta I)$. Conversely, any $g \in C(\beta I)$ is, by continuity, determined by its restriction to I. This shows that $\ell^\infty(I) \cong C(\beta I)$ as C^*-algebras.

2.2 The construction

Let $\phi : \mathbb{N} \to \mathbb{N}$ be a bijection, view ϕ as a map $\mathbb{N} \to \beta\mathbb{N}$, and set $\theta = \beta\phi : \beta\mathbb{N} \to \beta\mathbb{N}$. By Lemma 2.12, for each $\mathcal{U} \in \beta\mathbb{N}$, we have

$$\theta(\mathcal{U}) = \phi_*(\mathcal{U}) = \{A : \phi^{-1}(A) \in \mathcal{U}\} = \{\phi(A) : A \in \mathcal{U}\}.$$

Fix a ϕ with $\phi \circ \phi = \mathrm{id}$ and such that there is an infinite set A_0 with $\phi(A_0) \cap A_0 = \emptyset$. For example, we could define $\phi(2n) = 2n - 1$ and $\phi(2n-1) = 2n$. By uniqueness of the continuous extension to $\beta\mathbb{N}$, we have that $\theta \circ \theta = \mathrm{id}$, and so θ is a homeomorphism of $\beta\mathbb{N}$.

Define

$$\mathcal{F} = \{A \subseteq \mathbb{N} : A \supseteq A_0 \cap B \text{ for some cofinite } B\}.$$

Here $B \subseteq \mathbb{N}$ is *cofinite* when $\mathbb{N} \setminus B$ is finite. As A_0 is infinite, $A_0 \cap B$ is never empty, for a cofinite B. As the intersection of two cofinite sets is again a cofinite, we see that $\mathcal{F}$ is a filter. Notice that $\mathcal{F}$ contains all cofinite sets, and $A_0 \in \mathcal{F}$. Let $\mathcal{U}$ be an ultrafilter refining $\mathcal{F}$. Then $\mathcal{U}$ can contain no finite set, so $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$. As $A_0 \in \mathcal{U}$, we see that $\phi(A_0) \in \theta(\mathcal{U})$, and so as $\phi(A_0) \cap A_0$, certainly $\theta(\mathcal{U}) \neq \mathcal{U}$. We have shown:

> There is an homeomorphism θ of $\beta\mathbb{N}$ and distinct $\mathcal{U}, \mathcal{V} \in \beta\mathbb{N} \setminus \mathbb{N}$ with $\theta(\mathcal{U}) = \mathcal{V}$ and $\theta(\mathcal{V}) = \mathcal{U}$.

Set $A = \{f \in C(\beta\mathbb{N}) : f(\mathcal{U}) = 0\}$, which is a closed ideal in $C(\beta\mathbb{N}) \cong M(c_0)$. By Section 1.1, we know that $M(A) \cong M(c_0) = C(\beta\mathbb{N})$ with $C(\beta\mathbb{N})$ acting on A in the natural way. Define $\phi : M(A) \to M(A)$ by $\phi(f) = f \circ \theta$ for $f \in C(\beta\mathbb{N})$.

Let (f_i) be some approximate identity for A, so that $f_i \to 1$ strictly in $M(A)$. Pick $g \in A$ with $g(\mathcal{V}) = 1$. Regarding now f_i, g as members of $M(A) = C(\beta\mathbb{N})$, we see that

$$\big(\phi(f_i)g\big)(\mathcal{V}) = f_i(\theta(\mathcal{V}))g(\mathcal{V}) = f_i(\mathcal{U}) = 0$$

for all i, but $(\phi(1)g)(\mathcal{V}) = 1$. Hence $\phi(f_i)g$ does not converge in norm to g in A, and hence $\phi(f_i)$ does not converge strictly to $1 = \phi(1)$ in $M(A)$.

This shows that ϕ, regarded as an automorphism of $M(A)$, is not strictly continuous.

2.3 Further thoughts

[3] further asks:

> Does a strictly continuous $*$-automorphism $\phi : M(A) \to M(A)$ preserve the subalgebra A, that is, do we have $\phi(A) \subseteq A$?

Given the results in Section 2, we want to find a strictly continuous $*$-automorphism such that ϕ^{-1} is not strictly continuous.

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