# Isomorphism of von Neumann algebras

Matt Daws

September 2024

**Abstract**

We give a simple proof of the fact that isomorphic von Neumann algebras have unitarily equivalent amplifications.

## 1 Introduction

At a number of places in the literature, it is claimed that when $M \subseteq \mathcal{B}(H_M)$ and $N \subseteq \mathcal{B}(H_N)$ are von Neumann algebras which are $*$-isomorphic, then there is an amplification which is unitarily equivalent: there is some Hilbert space $L$ and a unitary $u\colon H_M \otimes L \to H_N \otimes L$ with $u(x \otimes 1) = (\theta(x) \otimes 1)u$ where $\theta\colon M \to N$ is the $*$-isomorphism.

This is stated without proof in [1, Theorem III.2.2.8], for example, and I have seen references to e.g. [6, Theorem IV.5.5], which doesn't even state this result explicitly. A rather minimal proof is given in [2, Theorem 2.4.26]. The implication is that this result should be an obvious corollary from the "structure theorem" for $*$-homomorphism between von Neumann algebras, but my belief is that some extra ingredients are needed. This note spells out these extra steps.

I had forgotten that I asked this on `math.stackexchange` [3], and provided my own answer, using the same technique as here. The strategy is that given by Jones in his lecture notes [4, Theorem 7.2.1].

## 2 The structure theorem

What I refer to as the "structure theorem" for normal unital $*$-homomorphisms between von Neumann algebras is the following (see [1, Theorem III.2.2.8], [6, Theorem IV.5.5] for example).

**Theorem 2.1.** *Let $M \subseteq \mathcal{B}(H_M)$ be a von Neumann algebra, and let $\theta\colon M \to \mathcal{B}(K)$ be a normal unital $*$-homomorphism. There is a Hilbert space $K'$ and an isometry $u\colon K \to H \otimes K'$ such that $u\theta(x) = (x \otimes 1)u$ for $x \in M$. As such, $\theta(x) = u^*(x \otimes 1)u$ for $x \in M$, we have that $e = uu^*$ is a projection in $(M \otimes 1)' = M' \bar{\otimes} \mathcal{B}(K')$, and so $u$ gives a spatial isomorphism between $\theta(M)$ and the induced von Neumann algebra $(M \otimes 1)_e$.*

*Proof.* Given a unit vector $\xi \in K$, the functional $\omega_\xi \circ \theta$ is a normal state on $M$, and so there is a square-summable sequence $(\eta_n)$ in $H$ with $\omega_\xi \circ \theta = \sum_n \omega_{\eta_n}$. Let $\eta = \sum_n \eta_n \otimes \delta_n \in H \otimes \ell^2$ with $\delta_n$ the standard orthonormal basis of $\ell^2$, so

$$(\eta|(x \otimes 1)\eta) = \sum_n (\eta_n|x\eta_n) = (\xi|\theta(x)\xi) \qquad (x \in M).$$

By Zorn's Lemma, we may choose $(\xi_i)_{i \in I}$ a maximal family of unit vectors in $K$ such that $(\xi_i|\theta(x)\xi_j) = 0$ for $i \neq j, x \in M$. As $\theta$ is a $*$-homomorphism, this is equivalent to the subspaces $K_i = \{\theta(x)\xi_i : x \in M\} \subseteq K$ being pairwise orthogonal. For each $i$ choose $\eta_i \in H \otimes \ell^2$ associated to $\xi_i$. We define

$$u\colon \sum_i \theta(x_i)\xi_i \mapsto \sum_i (x_i \otimes 1)\eta_i \otimes \delta_i,$$

for any finitely supported family $(x_i)$ in $M$, where $(\delta_i)$ is the canonical basis of $\ell^2(I)$, and then claim that $u$ is an isometry. Indeed,

$$\Big\| \sum_i (x_i \otimes 1)\eta_i \otimes \delta_i \Big\|^2 = \sum_i \|(x_i \otimes 1)\eta_i\|^2 = \sum_i (\eta_i | (x_i^* x_i \otimes 1)\eta_i) = \sum_i (\xi_i | \theta(x_i^* x_i)\xi_i)$$
$$= \Big( \sum_i \theta(x_i)\xi_i \Big| \sum_j \theta(x_j)\xi_j \Big) = \Big\| \sum_i \theta(x_i)\xi_i \Big\|^2,$$

in the last step using that $K_i \perp K_j$ for $i \neq j$. So $u$ extends to an isometry on the closure of its domain. If this is not all of $K$, then there is a unit vector $\xi_0$ orthogonal to each $K_i$, which contradicts the maximality of our family $(\xi_i)$.

Set $K' = \ell^2 \otimes \ell^2(I)$ so $u\colon K \to H \otimes K'$. Almost by definition, $u\theta(x) = (x \otimes 1)u$ for $x \in M$, so as $u$ is an isometry, $\theta(x) = u^*(x \otimes 1)u$. Then also $\theta(x)u^* = u^*(x \otimes 1)$ for each $x \in M$, and hence $uu^*(x \otimes 1) = (x \otimes 1)uu^*$ so $e = uu^* \in (M \otimes 1)' = M'\bar{\otimes}\mathcal{B}(K')$. Then $u$ is a unitary between $K$ and the image of $e$, and so $u$ gives a spatial isomorphism between $(M \otimes 1)_e$ and $\theta(M)$. □

Notice that when $K$ is separable, we can choose the family $(\xi_i)$ by using a version of the Gram–Schmidt procedure, starting from some dense subset of $K$ say. Thus in this case $I$ is countable, and we can choose $K'$ to be separable.

## 3 The case of isomorphisms

When $\theta$ is an isomorphism (equivalently, $\theta$ is injective and $N = \theta(M)$) we hence obtain $u\colon H_N \to H_M \otimes K$ and $v\colon H_M \to H_N \otimes K'$ both isometries, with $u\theta(x) = (x \otimes 1)u$ and $vx = (\theta(x) \otimes 1)v$ for $x \in M$. However, there seems no particular reason for a direct relationship between $u$ and $v$ and $K$ and $K'$.

We proceed by using the comparison theory of projections, [6, Section V.1], [5, Chapter 6]. Recall that projections $e, f \in M$ are equivalent, written $e \sim f$, when there is a partial isometry $u \in M$ with $u^*u = e, uu^* = f$. We further write $e \preceq f$ when $e$ is equivalent to a subprojection of $f$, that is, there is $u \in M$ with $u^*u = e$ and $uu^* \leqslant f$, that is, $fuu^* = uu^*$. A variant of the Cantor–Bernstein theorem shows that if $e \preceq f$ and $f \preceq e$ then $e \sim f$, see [6, Proposition V.1.3] for example. We shall make use of the result in an essential way.

Define

$$M_1 = \left\{ \begin{pmatrix} x & 0 \\ 0 & \theta(x) \end{pmatrix} : x \in M \right\} \subseteq \mathcal{B}(H_M \oplus H_N),$$

a von Neumann algebra. We identify

$$\mathcal{B}(H_M \oplus H_N) \cong \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \in \mathcal{B}(H_M), d \in \mathcal{B}(H_N), b, c^* \in \mathcal{B}(H_N, H_M) \right\}. \tag{1}$$

A simple calculation shows that

$$M_1' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a \in M', d \in \theta(M)', xb = b\theta(x), cx = \theta(x)c \ (x \in M) \right\}.$$

Given any Hilbert space $L$, we identify $(H_M \oplus H_N) \otimes L$ with $(H_M \otimes L) \oplus (H_N \otimes L)$ and so have a similar identification as (1). Let $w \in M_1'\bar{\otimes}\mathcal{B}(L)$ be a partial isometry with

$$w^*w = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad ww^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \implies \quad w = w\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} w \quad \implies \quad w = \begin{pmatrix} 0 & 0 \\ u & 0 \end{pmatrix},$$

where $u\colon H_M \otimes L \to H_N \otimes L$ satisfies $u^*u = 1_{H_M} \otimes 1_L$ and $uu^* = 1_{H_N} \otimes 1_L$, that is, $u$ is unitary. As $w \in M_1'\bar{\otimes}\mathcal{B}(L)$, we have that $u(x \otimes 1) = (\theta(x) \otimes 1)u$. Conversely, any such $u$ gives rise to such a $w$.

So we exactly wish to show that such a $w$ exists. By the theorem quoted above, it is enough to show that inside $M_1'\bar{\otimes}\mathcal{B}(L)$ we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Applying the structure theorem to $\theta$, as above, we have $u\colon H_N \to H_M \otimes K$ with $u^*u = 1$ and $u\theta(x) = (x \otimes 1)u$. For a set $I$ with sufficiently large cardinality, we have $K \otimes \ell^2(I) \cong \ell^2(I)$ (for example, if $K$ is separable, $I = \mathbb{N}$ suffices). Then $u \otimes 1\colon H_N \otimes \ell^2(I) \to H_M \otimes K \otimes \ell^2(I) \cong H_M \otimes \ell^2(I)$ and $(u \otimes 1)(\theta(x) \otimes 1) = (x \otimes 1)u \otimes 1 \cong (x \otimes 1)(u \otimes 1)$. So $u_1 = u \otimes 1$ satisfies $u_1^*u_1 = 1$ and $u_1u_1^* \leqslant 1$, showing that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

inside $M_1'\bar{\otimes}\mathcal{B}(\ell^2(I))$. Applying the same argument to $\theta^{-1}$ shows the other partial order, but possibly for a different index set $I$. Of course, we can then simply choose the $I$ with the larger cardinality, and we have verified both partial orders, as required.

## 4 A worked example

We give an example to show that things can be a little complicated. It is a fun exercise to show that if $H, K$ are Hilbert spaces, then a normal unital $*$-homomorphism $\theta\colon \mathcal{B}(H) \to \mathcal{B}(K)$ must take the following form: $K \cong H \otimes K'$ for some $K'$, and under this isomorphism, $\theta(x) = x \otimes 1$. (For example, use the structure theorem, and note that $(\mathcal{B}(H) \otimes 1)' = 1 \otimes \mathcal{B}(K')$.)

Now let $M$ be the von Neumann algebraic direct sum of finite-dimensional matrix algebras $\mathbb{M}_{n_\alpha}$. Let $\theta\colon M \to \mathcal{B}(K)$ be an injective unital normal $*$-homomorphism. Let $1_\alpha \in \mathbb{M}_{n_\alpha}$ be the units of the matrix blocks, so $(1_\alpha)$ are minimal central projections in $M$, each mutually orthogonal. The same is hence true of $(\theta(1_\alpha))$ in $\mathcal{B}(K)$, and so $K$ decomposes as the direct sum of orthogonal subspaces $K_\alpha = \theta(1_\alpha)H$. The restriction of $\theta$ to the matrix block $\alpha$ gives a non-zero $*$-homomorphism $\mathbb{M}_{n_\alpha} \to \mathcal{B}(K_\alpha)$ and so $K_\alpha \cong \mathbb{C}^{n_\alpha} \otimes H_\alpha$ say, with $\mathbb{M}_{n_\alpha}$ acting as $x \otimes 1$. Any family $(H_\alpha)$ could occur in this way.

So, given two embeddings $M \subseteq \mathcal{B}(H_1)$ and $M \subseteq \mathcal{B}(H_2)$, we have

$$H_k = \bigoplus_\alpha \mathbb{C}^{n_\alpha} \otimes H_\alpha^{(k)},$$

for $k = 1, 2$. Tensoring $H_k$ with $L$ is equivalent to tensoring $H_\alpha^{(k)}$ with $L$, for each $\alpha$. Any unitary equivalence between these representations must restrict to unitaries $H_\alpha^{(1)} \to H_\alpha^{(2)}$ for each $\alpha$.

It is now clear that $H_1 \otimes L$ and $H_2 \otimes L$ will admit a unitary intertwiner when $L$ is sufficiently large, but that not just any choice of $L$ will work.

We now consider when there is an isometry $u\colon H_2 \to H_1 \otimes K'$ intertwining the representations. Again, $u$ must restrict to the components

$$u_\alpha\colon \mathbb{C}^{n_\alpha} \otimes H_\alpha^{(1)} \to \mathbb{C}^{n_\alpha} \otimes (H_\alpha^{(2)} \otimes K'), \quad u_\alpha(x \otimes 1) = (x \otimes 1 \otimes 1)u_\alpha \qquad (x \in \mathbb{M}_{n_\alpha}),$$

for each $\alpha$. It follows that $u_\alpha = 1 \otimes v_\alpha$ for some isometry $u_\alpha'\colon H_\alpha^{(1)} \to H_\alpha^{(2)} \otimes K'$, which may be arbitrary. Again, this puts constraints on the size of $K'$. It also shows that if $v\colon H_1 \to H_2 \otimes K''$ is an intertwiner going the other way, there need indeed be little relation between $u$ and $v$, as the associated isometries $v_\alpha'\colon H_\alpha^{(2)} \to H_\alpha^{(1)} \otimes K''$ are again arbitrary.

## References

[1] B. Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of C*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[2] Ola Bratteli and Derek W. Robinson. *Operator algebras and quantum statistical mechanics. 1*. Texts and Monographs in Physics. Springer-Verlag, New York, second edition, 1987. $C^*$- and $W^*$-algebras, symmetry groups, decomposition of states.

[3] Matthew Daws (https://math.stackexchange.com/users/1438/matthew daws). Isomorphic von neumann algebras have spatially equivalent amplifications. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/4728018 (version: 2023-06-30).

[4] Vaughan F. R. Jones. Von Neumann Algebras. https://my.vanderbilt.edu/jonesvf/files/2020/10/vonNeumann2

[5] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. II*, volume 16 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997. Advanced theory, Corrected reprint of the 1986 original.

[6] M. Takesaki. *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002. Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.