Notes of the Fell topology

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Abstract

We provide an overview of the Fell topology on the representation space of a C^*-algebra. We then, in more detail, study this topology as applied to correspondences of von Neumann algebras.

1 Introduction

Fell introduced the eponymously named topology in [8], and further refined the definitions and properties in [10]. Fix a C^*-algebra A. Given a Hilbert space H, let $\mathrm{Rep}(A, H)$ be the collection of all non-zero $*$-homomorphisms $A \to \mathcal{B}(H)$. For $\pi \in \mathrm{Rep}(A, H)$ the *essential space* of π is H^π, the closed linear span of $\{\pi(a)\xi : a \in A, \xi \in H\}$. If $H^\pi = H$ then we say that π is *non-degenerate*; otherwise π is *degenerate*, which we explicitly allow. Finally, π is *irreducible* if π restricted to H^π is irreducible in the usual sense.

We say that $\pi, \pi' \in \mathrm{Rep}(A, H)$ are *equivalent* if there is a unitary from H^π to $H^{\pi'}$ intertwining the representations. In this case we write $\pi \sim \pi'$. We say that π, π' are *unitarily equivalent* if there is a unitary U on H with $\pi'(a)U = U\pi(a)$ for each $a \in A$. We write $\pi \cong \pi'$ in this case.

Lemma 1.1. *$\pi \cong \pi'$ if and only if $\pi \sim \pi'$ and $(H^\pi)^\perp$ and $(H^{\pi'})^\perp$ have the same dimension.*

Proof. Let U intertwine π and π'. As $U\pi(a)\xi = \pi'(a)U\xi$ it follows that $U(H^\pi) \subseteq H^{\pi'}$. As U^* intertwines π' and π, also $U^*(H^{\pi'}) \subseteq H^\pi$. It follows that U restricts to a unitary between H^π and $H^{\pi'}$. Hence U also restricts to a unitary between $(H^\pi)^\perp$ and $(H^{\pi'})^\perp$.

Conversely, if U is a unitary between H^π and $H^{\pi'}$, and $(H^\pi)^\perp$ and $(H^{\pi'})^\perp$ have the same dimension, then we can extend U to a unitary on all of H. □

Let $\rho : A \to \mathcal{B}(K)$ be a (possible degenerate) representation. If $\dim K \leq \dim H$, then by choosing an isometric embedding of K into H, we may regard ρ as a member of $\mathrm{Rep}(A, H)$, say π. While ρ depends on the embedding, the equivalence class of π, with respect to $\sim$, depends only on π. Further, notice that if $\rho : A \to \mathcal{B}(K)$ is irreducible (in the usual sense) then any $\pi \in \mathrm{Rep}(A, H)$ which we associate with ρ is irreducible.

If $\rho_i : A \to \mathcal{B}(K_i)$ for $i = 1, 2$ are representations, associated to $\pi_i \in \mathrm{Rep}(A, H)$, then notice that ρ_1 and ρ_2 are unitarily equivalent (in the usual sense) if and only if $\pi_1 \sim \pi_2$. Let us record these observations in a formal way.

Proposition 1.2. *Let R_0 be the equivalence classes, for unitary equivalence, of representations of A on Hilbert spaces K of dimension $\leq \dim H$. There is a bijection between R_0 and the $\sim$ equivalence classes of* $\mathrm{Rep}(A, H)$*.*

In particular, this bijection is used without comment in [10, Section 2].

2 The Fell topology

We define a topology on $\mathrm{Rep}(A, H)$ by taking a sub-basic set about π to be sets of the form

$$M_{a,\xi,\eta,\epsilon}(\pi) = \left\{\pi' \in \mathrm{Rep}(A, H) : |(\eta|(\pi(a) - \pi'(a))\xi)| < \epsilon\right\}$$

where $a \in A$, $\xi, \eta \in H^\pi$ and $\epsilon > 0$, or of the form

$$P_{\xi,\epsilon}(T) = \{\pi' \in \mathrm{Rep}(A, H) : \|P^{\pi'}\xi - \xi\| < \epsilon\}$$

where $\xi \in H^\pi, \epsilon > 0$, where $P^{\pi'}$ is the orthogonal projection of H onto $H^{\pi'}$. Note that here and elsewhere our inner products are linear on the right. Equivalently, a net (π_i) in $\mathrm{Rep}(A, H)$ converges to π when

$$\|\pi_i(a)\xi - \pi(a)\xi\| \to 0 \qquad (a \in A, \xi \in H^\pi).$$

The equivalence between these definitions is shown in [10, page 239]. Further, by expanding norms in terms of inner products, it follows that the $M_{a,\xi,\eta,\epsilon}$ sets alone will be sub-basic.

This topology is T_0 (given $\pi \neq \pi'$ there is an open set containing exactly one of these representations) but is not T_1 (so points need not be closed). However, if we consider the subspace of non-degenerate representations, with the subspace topology, then the subspace is Hausdorff.

For $S \subseteq \mathrm{Rep}(A, H)$ define

$$S^e = \{\pi' \in \mathrm{Rep}(A, H) : \pi \sim \pi' \text{ for some } \pi \in S\},$$
$$S^u = \{\pi' \in \mathrm{Rep}(A, H) : \pi \cong \pi' \text{ for some } \pi \in S\}.$$

Then $S \subseteq S^u \subseteq S^e$. Write $\mathrm{Cl}(S)$ for the closure of S in $\mathrm{Rep}(A, H)$ for our topology.

Proposition 2.1. *Given $S \subseteq \mathrm{Rep}(A, H)$ we have that:*

1. $(\mathrm{Cl}\, S)^e \subseteq \mathrm{Cl}\, S^u$;
2. $(\mathrm{Cl}\, S)^e \subseteq \mathrm{Cl}\, S^e$ *and* $(\mathrm{Cl}\, S)^u \subseteq \mathrm{Cl}\, S^u$;
3. $\mathrm{Cl}\, S^e = \mathrm{Cl}\, S^u$;
4. *Let $S' \subseteq S$ with S' being open in the subspace topology of S. If $S = S^u$ then S'^u is open in the subspace topology of S; if $S = S^e$ then S'^e is open in the subspace topology of S';*
5. *If S is open, then S^e and S^u are open;*
6. *If $S = S^u$ then* $(\mathrm{Cl}\, S)^u = (\mathrm{Cl}\, S)^e = \mathrm{Cl}\, S$.

We work with the equivalence relation $\sim$. For $\pi \in \mathrm{Rep}(A, H)$ let $\mathrm{Eq}\,\pi$ the equivalence class of π. For $S \subseteq \mathrm{Rep}(A, H)$ let $\mathrm{Eq}\, S$ be the collection $\{\mathrm{Eq}\,\pi : \pi \in S\}$. Finally let $\mathrm{Eq}\,\mathrm{Rep}(A, H)$ be the space of equivalence classes; we give this the quotient topology, so that $U \subseteq \mathrm{Eq}\,\mathrm{Rep}(A, H)$ is open when $\{\pi \in \mathrm{Rep}(A, H) : \mathrm{Eq}\,\pi \in U\}$ is open in $\mathrm{Rep}(A, H)$.

Definition 2.2. *The* quotient topology *on* $\mathrm{Eq}\,\mathrm{Rep}(A, H)$ *is the topology just described. By Proposition 1.2, this topology is also defined on the unitary equivalence classes of representations on Hilbert spaces of dimension no greater than* $\dim H$.

Now consider $S \subseteq \mathrm{Rep}(A, H)$. There are two ways to give $\mathrm{Eq}\, S$ a topology: either the subspace topology from $\mathrm{Eq}\,\mathrm{Rep}(A, H)$, or give S the subspace topology from $\mathrm{Rep}(A, H)$, and then use the surjection $S \to \mathrm{Eq}\, S$ to give $\mathrm{Eq}\, S$ a topology. In general these are different.

Proposition 2.3. *Let $S = S^u$. Then the two topologies on* $\mathrm{Eq}\, S$ *coincide.*

For example, if S is the collection of π with $\dim H^\pi \leq \kappa$ for some cardinal κ, then $S = S^u$, for if $\pi \cong \rho$ then clearly H^ρ has the same dimension as H^π.

2.1 Spectrum of a C^*-algebra

Let $\hat{A}$ be the set of irreducible representations of A. We give $\hat{A}$ the Hull-Kernel topology. This can be equivalently described using the notion of weak containment. Here we follow [6, Chapter 3.4]. Recall that $f \in A^*$ is a *positive form associated* to a representation $\pi : A \to \mathcal{B}(K)$ when there is $\xi \in K$ with $f(a) = (\xi|\pi(a)\xi)$ for $a \in A$. Similarly we have the notion of a state associated to π.

Proposition 2.4. *Let π be a representation of A, and let S be a set of representations. The following are equivalent, and define what is means for π to be* weakly contained *in S:*

1. $\bigcap_{\rho\in S} \ker\rho \subseteq \ker\pi$*;*
2. *Every positive form on A associated to π is the weak*-limit of linear combinations of positive forms associated with S;*
3. *Every state on A associated to π is the weak*-limit of states which are sums of positive forms associated with S.*

Proof. That (3) $\implies$ (2) is clear. If (2) holds then if $\rho(a) = 0$ for all $\rho \in S$, then any positive form associated to S will annihilate a^*a. Thus every positive form associated to π annihilates a^*a, so $\pi(a) = 0$, showing (1).

The final implication is harder. □

We write $\pi \prec S$ in this case. If $S = \{\rho\}$ is a singleton, then we write $\pi \prec \rho$. By condition (1) we see that if $\pi \prec \rho$ and $\rho \prec \sigma$ then $\ker\sigma \subseteq \ker\rho \subseteq \ker\pi$ and so $\pi \prec \sigma$.

In the following, notice that we need not take sums, unlike in the previous proposition.

Theorem 2.5. *Let $\pi \in \hat{A}$ and let $S \subseteq \hat{A}$. The following are equivalent:*

1. *π is in the closure of S, with respect to the Hull-Kernel topology;*
2. *π is weakly contained in S;*
3. *at least one non-zero positive form associated with π is the weak*-limit of positive forms associated with S;*
4. *every state associated with π is the weak*-limit of states associated with S.*

Let $S \subseteq \mathrm{Rep}(A, H)$ be the collection of irreducible representations. Then $\mathrm{Eq}\, S \subseteq \mathrm{Eq}\,\mathrm{Rep}(A, H)$ can be identified with the subset of $\hat{A}$ consisting of irreducible representations $\rho : A \to \mathcal{B}(K)$ with $\dim K \leq \dim H$. Using this, give $\mathrm{Eq}\, S$ the subspace topology coming from the Hull-Kernel topology on $\hat{A}$. Alternatively, we note that clearly $S = S^u$, and so Proposition 2.3 tells us that there is a uniquely defined quotient topology on $\mathrm{Eq}\, S$ coming from the topology on $\mathrm{Rep}(A, H)$.

Theorem 2.6. *The Hull-Kernel, and quotient, topologies on S coincide.*

There is a useful link between the quotient topology and weak containment which works for any representation (not necessarily irreducible).

Theorem 2.7. *Let $\pi \in \mathrm{Rep}(A, H)$ and let $T \subseteq \mathrm{Rep}(A, H)$. Then $\pi \prec T$ if and only if $\mathrm{Eq}\,\pi$ is in the closure of $\mathrm{Eq}\, T_0$ in the quotient topology, where T_0 is the set of finite direct sums of members of T.*

2.2 The inner hull kernel topology

Let X be any topological space, and let $C(X)$ be the family of closed subsets of X. We define a topology on $C(X)$ by taking as basic open sets

$$U(A_1, \cdots, A_n) = \{E \in C(X) : E \cap A_i \neq \emptyset \ (1 \leq i \leq n)\},$$

where $n \geq 1$ and each A_i is open and non-empty in X. This is the *inner topology* on $C(X)$.

For our C^*-algebra A, as before we consider $\operatorname{Eq}\operatorname{Rep}(A,H)$ to be (bijective with) the collection of all unitary equivalence classes of non-degenerate representations of A on a Hilbert space of dimension $\leq \dim H$. For $\rho \in \operatorname{Rep}(A,H)$ the *spectrum*, $\operatorname{Sp}(\rho)$, is the collection of those $\pi \in \hat{A}$ with π weakly contained in ρ. Here we consider ρ as a non-degenerate representation of A on H^ρ. Clearly $\operatorname{Sp}(\rho)$ does indeed only depend upon the equivalence class $\operatorname{Eq}\rho$.

Lemma 2.8. $\operatorname{Sp}(\rho)$ *is closed in* $\hat{A}$.

Proof. For any $\sigma \in \operatorname{Sp}(\rho)$, by definition, $\sigma \prec \rho$, so by Proposition 2.4, $\ker\rho \subseteq \ker\sigma$. It follows that if $X = \bigcap_{\sigma\in\operatorname{Sp}(\rho)} \ker\sigma$, then $\ker\rho \subseteq X$. Now let $\pi \in \hat{A}$ with $\pi \prec \operatorname{Sp}(\rho)$. Then $X \subseteq \ker\pi$ so $\ker\rho \subseteq \ker\pi$ so $\pi \prec \rho$ so $\pi \in \operatorname{Sp}(\rho)$. We have shown that $\pi \prec \operatorname{Sp}(\rho) \implies \pi \in \operatorname{Sp}(\rho)$. By Theorem 2.5, we conclude that $\operatorname{Sp}(\rho)$ is closed. □

In fact, more is true:

Theorem 2.9 ([9, Theorem 1.6])**.** *Sp*(ρ) *is the unique closed subset of* $\hat{A}$ *weakly equivalent to* ρ.

Sketch. That $\operatorname{Sp}(\rho) \prec \rho$ is easy to see; that $\rho \prec \operatorname{Sp}(\rho)$ requires considering extreme points of states (in order to obtain members of $\hat{A}$). □

Lemma 2.10. *For* $\pi \in \hat{A}$*, we have that* $\operatorname{Sp}(\pi)$ *equals the closure of* $\{\pi\}$ *in* $\hat{A}$.

Proof. Follows directly from Theorem 2.5. □

We define a closure operation on $\operatorname{Eq}\operatorname{Rep}(A,H)$ as follows. For $S \subseteq \operatorname{Eq}\operatorname{Rep}(A,H)$ define $\rho \in \overline{S}$ exactly when $\operatorname{Sp}(\rho)$ is in the closure of $\{\operatorname{Sp}(\pi) : \pi \in S\}$ in $C(\hat{A})$. This defines the *inner hull-kernel topology* on $\operatorname{Eq}\operatorname{Rep}(A,H)$. That the Kuratowski closure axioms hold follows easily because they must hold for the closure operation in $C(\hat{A})$.

If $\pi, \rho \in \operatorname{Rep}(A,H)$ with $\pi \prec \rho$ and $\rho \prec \pi$ (that is, π and ρ are *weakly equivalent*) then clearly $\operatorname{Sp}(\pi) = \operatorname{Sp}(\rho)$. It follows that the inner hull-kernel topology will not distinguish $\operatorname{Eq}\pi$ and $\operatorname{Eq}\rho$. In particular, it does not distinguish π from a multiple of π.

This definition seems terribly complicated to me, and Fell makes no further discussion. There is an alternative presentation in [12, Chapter 5], and in particular a very different definition in the form of [12, Definition 5.5]. We now show that these definitions actually agree.

Proposition 2.11. *The inner hull-kernel topology on* $\operatorname{Eq}\operatorname{Rep}(A,H)$ *has as basic open sets collections of the form*
$$\{\operatorname{Eq}\pi \in \operatorname{Eq}\operatorname{Rep}(A,H) : \operatorname{Sp}(\pi) \cap A_i \neq \emptyset\}$$
where $A_1, \cdots, A_n \subseteq \hat{A}$ *are open and non-empty.*

Proof. Let us abstract the definition of the closure operation of $\operatorname{Eq}\operatorname{Rep}(A,H)$. Let Y be a set, X a topological space, and $\theta : Y \to X$ a map. Define a closure operation on Y as follows: for $S \subseteq Y$ define $t \in \overline{S}$ exactly when $\theta(t) \in \overline{\theta(S)}$. This gives the inner hull-kernel topology when we let $Y = \operatorname{Eq}\operatorname{Rep}(A,H), X = C(\hat{A})$ and let θ be the map $\operatorname{Eq}\pi \mapsto \operatorname{Sp}(\pi)$.

Let $\tau = \{\theta^{-1}(U) : U \subseteq X \text{ is open}\}$ which is a topology on Y (in fact the coarsest topology making θ continuous). For $S \subseteq Y$ we have that $t \in \overline{S}$ when, for any open $U \subseteq X$ with $\theta(t) \in U$, we have that $\theta(S) \cap U \neq \emptyset$. Equivalently, for any $V \in \tau$ with $t \in V$, we have that $S \cap V \neq \emptyset$. However, this is just the closure operation given by the topology τ.

Applied to $\operatorname{Eq}\operatorname{Rep}(A,H)$, we just pull back the topology on $C(\hat{A})$ using Sp. But pulling back the basic open sets in $C(\hat{A})$ gives exactly the sets in the statement of the proposition. □

Restrict the inner hull-kernel topology to the irreducibles in $\operatorname{Eq}\operatorname{Rep}(A,H)$. By Lemma 2.10, the basic open sets are of the form $\{\operatorname{Eq}\pi : \overline{\{\pi\}} \cap A_i \neq \emptyset\}$ for non-empty, open A_i. But $\overline{\{\pi\}} \cap A_i \neq \emptyset$ exactly when $\pi \in A_i$, and so we just recover the (subspace) topology of $\hat{A}$. Thus, restricted to the irreducibles, the inner hull-kernel topology agrees with the hull-kernel topology on $\hat{A}$.

We can describe the inner hull-kernel topology using positive forms. Notice that if $f \in A^*$ is a positive form associated with $\pi \in \operatorname{Rep}(A,H)$, then if $\pi \sim \rho$, also f is associated to ρ. Thus we may speak of f being associated with an equivalence class $\operatorname{Eq}\pi$.

For $\pi \in \operatorname{Eq}\operatorname{Rep}(A,H)$, we describe some sets containing π. Let $\epsilon > 0$, let $(f_i)_{i=1}^n$ be positive forms associated to π, and let $(a_i)_{i=1}^m \subseteq A$. Define U to be the collection of $\rho \in \operatorname{Eq}\operatorname{Rep}(A,H)$ such that there are $(g_i)_{i=1}^n$, each a *sum* of positive forms associated to ρ, with

$$|f_i(a_j) - g_i(a_j)| < \epsilon \qquad (1 \leq i \leq n, 1 \leq j \leq m),$$
$$\big| \|f_i\| - \|g_i\| \big| < \epsilon \qquad (1 \leq i \leq n).$$

Theorem 2.12 ([10, Theorem 2.1], [12, Theorem 5.9]). *Such U forms a basis of open*[1] *neighbourhoods of π for the inner hull-kernel topology.*

Every open inner hull-kernel topology set is also open in the quotient topology on $\operatorname{Eq}\operatorname{Rep}(A,H)$, but the converse is not true. Indeed, the inner hull-kernel topology cannot distinguish between a representation π and multiples of π, while of course the quotient topology can.

Proposition 2.13 ([10, Lemma 2.4]). *Let S be the set of $\pi \in \operatorname{Rep}(A,H)$ such that π is equivalent to a countably-infinite multiple of π. Then the inner hull-kernel topology and the quotient topology agree on $\operatorname{Eq} S$.*

We now turn attention to the presentation of the inner hull-kernel topology given in [12, Chapter 5]. This is in many ways easier to follow than Fell's original presentation; however, we must be a little careful as here we fix H of sufficiently large size: the cardinality of $\dim(H)$ should be greater than or equal to the cardinality of A. It is not clear to me how much this assumption is really used.

Proposition 2.14 ([12, Proposition 5.11]). *Let $\pi \in \operatorname{Rep}(A,H)$ and $T \subseteq \operatorname{Rep}(A,H)$, and consider the following conditions:*

1. *π is weakly contained in T;*
2. *$\operatorname{Sp}(\pi)$ is contained in the closure of $\bigcup\{\operatorname{Sp}(\tau) : \tau \in T\}$ in $\hat{A}$;*
3. *$\operatorname{Eq}\pi$ belongs to the inner hull-kernel topology closure of $\operatorname{Eq} T$ in $\operatorname{Eq}\operatorname{Rep}(A,H)$.*

Then (3) $\Rightarrow$(2) $\Leftrightarrow$ (1), and if π is irreducible, then also (2) $\Rightarrow$ (3).

We can apply this abstract work to the C^*-algebra $C^*(G)$; one can describe the topology in terms of the group alone. Of note is the result that the operation of taking the tensor product is jointly continuous for the inner hull-kernel topology.

We now consider [11]. This is mostly a paper about the representation theory of groups, but there is a useful summary of results about inner hull-kernel topology.

Theorem 2.15. *Consider the inner hull-kernel topology on $\operatorname{Eq}\operatorname{Rep}(A,H)$, and by an abuse of notation, pull this back to a topology on $\operatorname{Rep}(A,H)$. Then:*

1. *[11, Proposition 1.2] A net (π_i) in $\operatorname{Rep}(A,H)$ converges to π if and only if, for every subnet $(\pi_{i(j)})$, we have that $\pi \prec \{\pi_{i(j)}\}$.*
2. *[11, Proposition 1.3] If $\pi_i \to \pi$ and $\rho \prec \pi$, then also $\pi_i \to \rho$.*
3. *[11, Proposition 1.4] We have that $\pi_i \to \pi$ if and only if, for each $a \in A$, we have $\|\pi(a)\| \leq \liminf_i \|\pi_i(a)\|$.*

[11] offers no proofs. However, [12, Lemma 5.7] proves (1) and (2) (notice that once we know (1), then (2) follows easily.) See Appendix B.

Another useful reference is [3, Appendix F]. While this reference only considers group representations, it is relatively easy to translate results back to C^*-algebras. In particular, [3, Definition F.2.1] defines *Fell's Topology* as follows. Let $\mathcal{R}$ be the unitary equivalence classes of representations of A, on a Hilbert space of cardinality less than some fixed cardinal. A basic open set W about $\pi \in \mathcal{R}$ is of the form $W = W(\pi, (\varphi_i)_{i=1}^n, Q, \epsilon)$, where $\epsilon > 0, Q \subseteq A$ is finite, and the φ_i are each positive forms

[1] Fell does not say "open" in [10] but I think such U are open

associated to π. There $\rho \in W$ exactly when for each φ_i there is ψ, a sum of positive forms associated to ρ, with $|\varphi_i(a) - \psi(a)| < \epsilon$ for each $a \in Q$. This is very close to the basic open sets considered above in Theorem 2.12 but without the 2nd condition: this might be an artifact of me incorrectly translating from the group to the C^*-algebra case.[2]

In the following, as usual, we identify $\mathcal{R}$ and $\operatorname{Eq}\operatorname{Rep}(A, H)$ for a sufficiently large H.

Proposition 2.16. *Let $\pi \in \mathcal{R}$ and let (π_i) be a net in $\mathcal{R}$. The following are equivalent:*

1. *$\pi_i \to \pi$ for Fell's topology on $\mathcal{R}$;*
2. *for any subnet $(\pi_{i(j)})$ of (π_i), we have that $\pi \prec \bigoplus_j \pi_{i(j)}$;*
3. *for any subnet $(\pi_{i(j)})$ of (π_i), and letting T_0 be the collection of finite direct sums of $\{\pi_j\}$, we have that $\pi \prec T_0$;*
4. *for any subnet $(\pi_{i(j)})$ of (π_i), we have that π is in the closure of $\{\pi_{i(j)}\}$ for Fell's topology on $\mathcal{R}$;*
5. *for any subnet $(\pi_{i(j)})$ of (π_i), and letting T_0 be as before, we have that $\operatorname{Eq}\pi$ is in the closure of $\operatorname{Eq} T_0$, for the quotient topology.*

Proof. That (1) and (2) are equivalent follows from the definition of Fell's topology on $\mathcal{R}$, see [3, Proposition F.2.2]. That (1) and (4) are equivalent is a standard fact from point-set toplogy, see Appendix A. That (2) and (3) are equivalent follows from the definition of weak containment (we can approximate a positive form associated with an infinite direct sum of representations by a form associated with a finite direct sum). That (3) and (5) are equivalent follows immediately from Theorem 2.7. $\square$

3 Correspondences

We now apply the above to the study of correspondences of von Neumann algebras. There is a nice summary of results in [1, page 316]. There is more discussion in [2, Section 1.12], which references the original [4]. [3]

A *correspondence* between von Neumann algebras M and N is a Hilbert space H with commuting normal representations of M and N^{op}. Here we work with N^{op} so as to regard N as a *right* action of N on H. We hence write $x\xi y$ for $x \in M, y \in N, \xi \in H$. We can linearise this bilinear definition by using the *binormal* tensor product, as defined in [7]. This gives a C^*-algebra norm on the algebraic tensor product $M \odot N^{\mathrm{op}}$, leading to the completion $M \otimes_{\mathrm{bin}} N^{\mathrm{op}}$. Of interest is [7, Theorem 4.1] which shows that M is semidiscrete if and only if $M \otimes_{\mathrm{bin}} N^{\mathrm{op}} = M \otimes_{\mathrm{min}} N^{\mathrm{op}}$ for all N, where $M \otimes_{\mathrm{min}} N^{\mathrm{op}}$ is the C^*-algebraic spatial tensor product.

Write $\operatorname{Corr}(M, N)$ for the collection of all correspondences H from M to N. Then $\operatorname{Corr}(M, N)$ bijects with the representations of $A = M \otimes_{\mathrm{bin}} N^{\mathrm{op}}$, say $\pi : A \to \mathcal{B}(H)$ such that the restriction of π to M and N^{op} are normal (that is, π is separately normal). Let $S \subseteq \operatorname{Rep}(A, H)$ be the binormal representation. It is easy to see that $S = S^e = S^u$, and so the quotient topology on $\operatorname{Eq} S$ is well-defined.

Now restrict $\operatorname{Corr}(M, N)$ to be the correspondences whose Hilbert space dimension does not exceed some fixed cardinal. The definition from [4] and [2, Section 1.12] is that a basic open neighbourhood of $H_0 \in \operatorname{Corr}(M, N)$ is of the form

$$U = U(H_0; \epsilon, E, F, S) = \big\{H : \exists (k_i)_{i=1}^n \subseteq H,\ |(k_i|xk_jy) - (h_i|xh_jy)| < \epsilon\ (x \in E, y \in F, 1 \le i, j \le n)\big\}.$$

Here $E \subseteq M, F \subseteq N$ are finite subsets, $\epsilon > 0$, and $S = (h_i)_{i=1}^n \subseteq H_0$.

It is stated without justification that this is the Fell topology on $M \otimes_{\mathrm{bin}} N^{\mathrm{op}}$. It seems to me that there is some work here!

Alternatively, we can think of correspondences as self-dual Hilbert C^*-bimodules, see [1, 2]. That is, self-dual Hilbert C^*-modules X over N (so X is a right module over N, and X has an N-valued

[2]If A is unital then one of our test elements a_j could be the unit, which immediately implies the 2nd condition.
[3]More brief lit review here?

inner product) together with a normal $*$-homomorphism from M to the adjointable operators on X, turning X into a left M-module.

The associated topology on bimodules is as follows (following [2, Section 1.12] although we note that the proof presented in [2] assumes that N and M are σ-finite). Write $C(M,N)$ for the M-N-bimodules. A basic open neighbourhood of $X_0 \in C(M,N)$ is of the form

$$V = V(X_0; \mathcal{V}, E, S) = \big\{ X : \exists (\eta_i)_{i=1}^n \subseteq X,\ (\eta_i | x\eta_j) - (\xi_i | x\xi_j) \in \mathcal{V}\ (1 \leq i,j \leq n, x \in E) \big\}.$$

Here $\mathcal{V}$ is weak*-open neighbourhood of 0 in N, $E \subseteq M$ is finite, and $(\xi_i)_{i=1}^n \subseteq X_0$.

3.1 Equivalence with the Fell topology

We first translate the above topology on correspondences into a statement about representations. Let H be a Hilbert space of dimension our fixed cardinal, let $A = M \otimes_{\mathrm{bin}} N^{\mathrm{op}}$, and consider $\mathrm{Rep}(A,H)$. Given a correspondence K from M to N, we can equivalently view K as a representation $\rho : A \to \mathcal{B}(K)$. As $\dim K \leq \dim H$ by our standing assumption, there is an isometry $v : K \to H$. Set $\pi(a) = v\rho(a)v^*$ for $a \in A$, so that $\pi \in \mathrm{Rep}(A,H)$ with $H^\pi = v(K)$. If $\pi \cong \sigma$ then composing the unitary implementing this equivalence with v, we see that σ, restricted to H^σ, is unitarily equivalent to ρ. A similar remark replies in the case when $\pi \sim \sigma$. Thus $\mathrm{Eq}\,\pi$ depends only on ρ, and not the choice of v.

The topology on $\mathrm{Corr}(M,N)$, viewed as representations of A, has as basic open sets about π_0 sets of the form

$$U' = U'(\pi_0; \epsilon, E, S) = \big\{ \pi : \exists (k_i)_{i=1}^n \subseteq H,\ |(k_i|\pi(a)k_j) - (h_i|\pi_0(a)h_j)| < \epsilon\ (a \in E, 1 \leq i,j \leq n) \big\}.$$

Here $E \subseteq M \odot N^{\mathrm{op}}$ is finite, $\epsilon > 0$, and $S = (h_i)_{i=1}^n \subseteq H_0$. A tedious but routine argument (compare to Section A.2.1) shows that if we allow $E \subseteq A$ finite, then we obtain the same topology. Notice also that these sets satisfy the conditions of Theorem A.4 and so do indeed define a unique topology.

[[4]]

4 A potted bibliography

In this section[5] we look at examples of the use of the "Fell Topology" in the literature, and see which of the competing definitions are really used.

We start by noting that the "Fell Topology" is often used for what we call the "inner topology" (or close variants) in Section 2.2, that is, purely a topological space notion. We shall not explore instances of this. Below, we shall use "fell topology" without capitalisation in the generic sense, before referring to a section above to clarify which definition is actually used.

- The "quotient topology", Definition 2.2, was defined by Fell in [8], and also studied in [10]. Theorem 2.7 gives a link between weak containment and the quotient topology.
- The "inner hull-kernel topology", Proposition 2.11, was defined by Fell in [8]. A quick summary (and useful extra results) are given in [11]. See Theorem 2.15.
- The book [3, Appendix F] also considers what turns out to be the inner hull-kernel topology, for groups.
- The book [5] considers the fell topology in Section 2.5.6. This source references [10], but really the summary in [11] is meant, that is, the results of Theorem 2.15. So this is the inner hull-kernel topology.

[4]To be finished.
[5]Very much a work in progress

A Point-set topology

A.1 Nets

Definition A.1. *A* directed set *is a set I with a relation $\leq$ such that: $i \leq j$ and $j \leq k$ implies $i \leq k$; if $i \leq j$ and $j \leq i$ then $i = j$; for any $i, j \in I$ there is $k \in I$ with $i \leq k$ and $j \leq k$.*

A net *in a set X is a function $I \to X$ from some directed set I. We typically write $(x_i)_{i \in I}$ for a net.*

A subnet *of a net $(x_i)_{i\in I}$ is a function $J \to I$, with J a directed set, written as $j \mapsto i(j)$, and the subnet denoted by $(x_{i(j)})_{j \in J}$, with: $j_1 \leq j_2$ implies that $i(j_1) \leq i(j_2)$, and for each $i' \in I$ there is $j \in J$ with $i' \leq i(j)$.*

We recall that in a topological space X we say that a net (x_i) converges to x if for each open set $x \in U$ there is i_0 so that $i \leq i_0$ implies that $x_i \in U$.

Proposition A.2. *If a net (x_i) converges to x then any subnet $(x_{i(j)})_{j\in J}$ also converges to x.*

Proof. Exercise. □

Proposition A.3. *A net (x_i) converges to x if and only if, for any subnet $(x_{i(j)})_{j\in J}$, the closure of $\{x_{i(j)} : j \in J\}$ contains x.*

Proof. "only if" follows from the previous proposition. We show the contrapositive of "if", so suppose that (x_i) does not converge to x. So there is an open set $x \in U$ so that for each $i_0 \in I$ there is $i \geq i_0$ with $x_i \notin U$. Let $J = \{i \in I : x_i \notin U\}$, and give J the relative ordering from I. Given $i, j \in J$ there is $k_0 \in I$ with $k_0 \geq i, k_0 \geq j$. By assumption, there is $k \in A$ with $k \geq k_0$, so that $k \geq i, k \geq j$. Thus J is a directed set, and furthermore, the inclusion $J \to I$ defines a subnet. As $U \cap \{x_j : j \in J\}$ is empty, the closure of $\{x_j : j \in J\}$ is disjoint from x, as required. □

A.2 Bases

The notion of a *base* for a topology is standard: a collection $\mathcal{U}$ of subsets of X so that every open set in X is a union of members of $\mathcal{U}$. A *subbase* is a collection $\mathcal{V}$ such that if $\mathcal{U}$ is the collection of finite intersections of members of $\mathcal{V}$, then $\mathcal{U}$ is a base.

A less common notion is to, for each $x \in X$, define a *base of open neighbourhoods of* x. This is, for each $x \in X$, a collection $\mathcal{U}_x$ of subsets of X such that:

1. For each $x \in X$ and $A \in \mathcal{U}_x$, we have that $x \in A$;

2. Each $\mathcal{U}_x$ should be a filter-base for the collection of all open neighbourhoods of x. That is, each $A \in \mathcal{U}_x$ is open, and if $U \subseteq X$ is open and contains x, there is $A \in \mathcal{U}_x$ with $A \subseteq U$.

Theorem A.4. *Let X be a set and $(\mathcal{U}_x)_{x\in X}$ satisfy (1). There is a topology on X with (2) holding if and only if:*

3. *For $x \in X$ and $A_1, A_2, \cdots, A_n \in \mathcal{U}_x$, there is $A \in \mathcal{U}_x$ with $A \subseteq A_1 \cap \cdots \cap A_n$;*

4. *For $x, y \in X$ and $B \in \mathcal{U}_y$, if $x \in B$, there is $A \in \mathcal{U}_x$ with $A \subseteq B$.*

In this case, the resulting topology on X is unique.

Proof. Suppose a topology τ exists on X satisfying (2). Then (3) holds as each A_i will be open, so we can apply (2) with $U = A_1 \cap \cdots \cap A_n$. Then (4) will hold as B is open and contains x.

Conversely, let τ_0 be the topology generated by the $\mathcal{U}_x$, that is, members of τ_0 are arbitrary unions of sets of the form $A_1 \cap \cdots \cap A_n$ where $A_i \in \mathcal{U}_{x_i}$ for some family $(x_i)_{i=1}^n$ (we allow repeats: each A_i could be from the same $\mathcal{U}_x$, for example). Then each $A \in \mathcal{U}_x$ is in τ_0, and if $x \in U \in \tau_0$ then there are $A_i \in \mathcal{U}_{x_i}$ with $x \in A_1 \cap \cdots \cap A_n \subseteq U$. If $x_i \neq x$ then apply (4) to find $A_i' \in \mathcal{U}_x$ with $A_i' \subseteq A_i$. Thus we may assume that each A_i is actually a member of $\mathcal{U}_x$, and then (3) provides $A \in \mathcal{U}_x$ with $A \subseteq A_1 \cap \cdots \cap A_n \subseteq U$. We have hence verified that (2) holds for τ_0.

Now let

$$\tau_1 = \big\{B \subseteq X : x \in B \implies \exists\, A \in \mathcal{U}_x, A \subseteq B \ (x \in X)\big\}.$$

Then $\emptyset, X \in \tau_1$ and τ_1 is closed under arbitrary unions. Condition (3) readily implies that τ_1 is closed under finite intersections, and so τ_1 is a topology. If τ is any topology on X with (2) holding, then clearly any $U \in \tau$ is also a member of τ_1. Hence $\tau_0 \subseteq \tau \subseteq \tau_1$.

Finally, let $B \in \tau_1$, so if $x \in B$ there is $A_x \in \mathcal{U}_x$ with $A_x \subseteq B$. Thus $B = \bigcup\{A_x : x \in B\}$ and so $B \in \tau_0$. Hence $\tau_0 = \tau_1$ and so the resulting topology is unique. $\square$

Let $\mathcal{V}_x$ be a collection of open sets containing x, with the property that for each $A \in \mathcal{U}_x$ there is $B \in \mathcal{V}_x$ with $B \subseteq A$. In the following, we shall refer to this as "passing to a cofinal family". The preceding discussion now easily shows that we can replace $\mathcal{U}_x$ by $\mathcal{V}_x$ and still generate the same topology (for example, the τ_1 generated will clearly be the same).

A.2.1 Application to the Fell topology

We defined the topology on $\mathrm{Rep}(A, H)$ by specifying as sub-basic sets $\{\pi' \in \mathrm{Rep}(A, H) : |(\eta|(\pi(a) - \pi'(a))\xi)| < \epsilon\}$. That is, the basic open sets are

$$\{\pi' \in \mathrm{Rep}(A, H) : |(\eta_i|(\pi_i(a_i) - \pi'(a_i))\xi_i)| < \epsilon_i \ (1 \le i \le n)\}.$$

However, we might instead define the basic open sets about π to be of the form

$$\{\pi' \in \mathrm{Rep}(A, H) : |(\eta_i|(\pi(a_i) - \pi'(a_i))\xi_i)| < \epsilon_i \ (1 \le i \le n)\}.$$

These sets clearly satisfy condition (3); we now verify (4). Suppose that $\pi_0 \in B \in \mathcal{U}_{\pi_1}$, say $|(\eta_i|(\pi_1(a_i) - \pi_0(a_i))\xi_i)| < \epsilon_i$ for $1 \le i \le n$. For each i, set $\delta_i = |(\eta_i|(\pi_1(a) - \pi_0(a))\xi_i)|$ and consider the open neighbourhood A of π_0 defined by

$$A = \{\pi' : |(\eta_i|(\pi_0(a_i) - \pi'(a_i))\xi_i)| < \epsilon_i - \delta_i (1 \le i \le n)\}.$$

Then, if $\pi' \in A$, the triangle inequality shows that $\pi' \in B$, so that $A \subseteq B$ as required.

Thus the Fell topology on $\mathrm{Rep}(A, H)$ is generated by these basic open neighbourhoods. As remarked above, we can replace these sets by a "cofinal family" and maintain the topology. For example, we could define a basic open neighbourhood of π to be a set of the form

$$\big\{\pi' \in \mathrm{Rep}(A, H) : |(\eta|(\pi(a) - \pi'(a))\xi)| < \epsilon \ (\xi, \eta \in E, a \in F)\big\},$$

where $\epsilon > 0$ and $E \subseteq H, F \subseteq A$ are finite.

Alternatively, we could keep the same form, but restrict E to be an orthonormal set $E = \{e_1, \cdots, e_n\}$. Indeed, for any $E' \subseteq H$ finite, let E be an orthonormal basis for the linear span of E'. Choose $\epsilon > 0$. Suppose that $|(e_i|(\pi(a) - \pi'(a))e_j)| < \delta$ for all $1 \le i, j \le n$ and $a \in F$. Then for $\xi = \sum_i \xi_i e_i$ and $\eta = \sum_j \eta_j e_j$ in E', we have that

$$|(\eta|(\pi(a) - \pi'(a))\xi)| \le \sum_{i,j=1}^{n} |\eta_j||\xi_i|\delta \le n\delta\Big(\sum_i |\xi_i|^2\Big)^{1/2}\Big(\sum_j |\xi_j|^2\Big)^{1/2} = n\delta\|\xi\|\|\eta\|.$$

Set $M = \max\{\|\xi\| : \xi \in E'\}$ so that if $nM^2\delta < \epsilon$, we may conclude that π' is a member of the basic open neighbourhood defined by (ϵ, E', F).

Similarly, we could suppose that each member of E was either a member of H^π, or a member of $(H^\pi)^\perp$, as by adjusting ϵ, this again constitutes passing to a cofinal family.

A.2.2 Application to quotient topologies

Let X be a topological space, the topology given by specifying basic open neighbourhoods $(\mathcal{U}_x)_{x\in X}$, and let $f : X \to Y$ be a surjection, used to give Y the quotient topology. This is the same as defining an equivalence relation on X by $x \sim x'$ if and only if $\phi(x) = \phi(x')$. In particular, if $A \subseteq X$ then $\{x \in X : x \sim x' \text{ for some } x \in A\} = \phi^{-1}(\phi(A))$.

We shall assume that if $A \subseteq X$ is open, then $\phi^{-1}(\phi(X))$ is open; equivalently, ϕ is an open map. By Proposition 2.1(5) the Fell topology on $\mathrm{Rep}(A, H)$ and the map Eq satisfy this condition.

Proposition A.5. *Let $\psi : Y \to X$ be some function with $\phi(\psi(y)) = y$ for each $y \in Y$. Let $\mathcal{V}_y = \{\phi(A) : A \in \mathcal{U}_{\psi(y)}\}$. Then each $\mathcal{V}_y$ is a collection of basic open neighbourhoods of y which together generate the quotient topology on Y.*

Proof. For $B = \phi(A) \in \mathcal{V}_y$ we have that $x = \psi(y) \in A$ so $\phi(x) = y \in B$. As ϕ is open, each member of $\mathcal{V}_y$ is open. If $U \subseteq Y$ is open and $y \in U$, then $\phi^{-1}(U)$ is open and $x = \psi(y) \in \phi^{-1}(U)$. Thus there is $A \in \mathcal{U}_x$ with $A \subseteq \phi^{-1}(U)$, that is, $\phi(A) \subseteq U$. So there is $B \in \mathcal{V}_y$ with $B \subseteq U$. Hence the conditions (1) and (2) hold, and so the $(\mathcal{V}_y)$ generate a unique topology, which must be the quotient topology. □

Let $\mathcal{R}$ be the collection of unitary equivalence classes of representations of A on Hilbert spaces of dimension no more than $\dim H$. Then $\mathcal{R}$ is naturally identified with the quotient space $\mathrm{Eq}\,\mathrm{Rep}(A, H)$. In this case we can define $\psi : \mathcal{R} \to \mathrm{Rep}(A, H)$ by, for each representation $\pi : A \to \mathcal{B}(K)$, picking an isometry $v : K \to H$, and setting $\psi(\pi) = v\pi(\cdot)v^* \in \mathrm{Rep}(A, H)$. The resulting basic open neighbourhoods of π are of the form

$$\big\{\pi' : A \to \mathcal{B}(K') : \exists v : K' \to H,\ |(v^*\eta|\pi'(a)v^*\xi) - (u^*\eta|\pi(a)u^*\xi)| < \epsilon\ (\xi, \eta \in E, a \in F)\big\},$$

where v is meant to be an isometry depending on π', and $\epsilon > 0$ and $E \subseteq H, F \subseteq A$ are finite.

B More about the inner hull-kernel topology

In this section, we provide some proofs of results from Section 2.2. Compare also to [3, 12].

The key starting point is Theorem 2.9 which says that for any representation π, the spectrum $\mathrm{Sp}(\pi)$ is weakly equivalent to π. This provide a key link with π, weak containment, and $\hat{A}$.

From now on, for notational convenience, we shall pull the inner hull-kernel topology back from $\mathrm{Eq}\,\mathrm{Rep}(A, H)$ to $\mathrm{Rep}(A, H)$. Proposition 2.11 tells us that a subbasic open set about π is $\{\rho : \mathrm{Sp}(\rho) \cap U \neq \emptyset\}$ where $U \subseteq \hat{A}$ is open and intersects $\mathrm{Sp}(\pi)$.

Lemma B.1. *Let (π_i) be a net in $\mathrm{Rep}(A, H)$. Then $\pi_i \to \pi$ if and only if, for each open $U \subseteq \hat{A}$ which contains some $\phi \in \mathrm{Sp}(\pi)$, there is i_0 such that for $i \geq i_0$ there is $\phi_i \in \mathrm{Sp}(\pi_i)$ with $\phi_i \in U$.*

Proof. As (π_i) is indexed by a directed set, we have that $\pi_i \to \pi$ if and only if, for any subbasic open set V containing π, there is i_0 so that $i \geq i_0 \implies \pi_i \in V$. We may take V to be $\{\rho : \exists \phi \in \mathrm{Sp}(\rho),\ \phi \in U\}$ where $U \subseteq \hat{A}$ is open, and there is $\phi \in \mathrm{Sp}(\pi)$ with $\phi \in U$. The claim now follows. □

By Theorem 2.9, as $\pi \prec \mathrm{Sp}(\pi) \prec \pi$, given $S \subseteq \mathrm{Rep}(A, H)$, set $S_0 = \{\phi \in \hat{A} : \phi \in \mathrm{Sp}(\rho) \text{ for some } \rho \in S\}$, and then we see that $\pi \prec S$ if and only if $\phi \prec S_0$ for any $\phi \in \mathrm{Sp}(\pi)$. Thus we have reduced the question of $\pi \prec S$ to a question inside $\hat{A}$, where weak containment is equivalent to closure, Theorem 2.5.

Proposition B.2. *Let $\pi_i \to \pi$. Then $\pi \prec \{\pi_i\}$.*

Proof. Given the above discussion, we need to show that if $\phi \in \mathrm{Sp}(\pi)$ then and $S_0 = \bigcup \mathrm{Sp}(\pi_i)$, then $\phi \prec S_0$, that is, $\phi \in \overline{S_0}$. Towards a contradiction, suppose this is not so, so there is an open set $U \subseteq \hat{A}$ with $\phi \in U$ but $U \cap \mathrm{Sp}(\pi_i) = \emptyset$ for each i. However, this contradicts $\pi_i \to \pi$ by the above lemma, as required. □

Proposition B.3. *Suppose that $\pi \prec \{\pi_{i(j)}\}$ for each subset of (π_i). Then $\pi_i \to \pi$.*

Proof. Towards a contradiction, suppose not, so by the lemma, there is $\phi \in \mathrm{Sp}(\pi)$ and an open $U \subseteq \hat{A}$ with $\phi \in U$, such that we can extract a subnet $(\pi_{i(j)})$ with $\mathrm{Sp}(\pi_{i(j)}) \cap U = \emptyset$ for each j. With $S_0 = \bigcup \mathrm{Sp}(\pi_i)$, we see that $S_0 \cap U = \emptyset$ and so $\phi \notin \overline{S_0}$, equivalently, $\phi \not\prec S_0$. Thus $\pi \not\prec \{\pi_{i(j)}\}$ giving the required contradiction. □

Thus we have shown Theorem 2.15(1). Then Theorem 2.15(2) follows easily. To show Theorem 2.15(3) we will use:

Proposition B.4 ([9, Lemma 2.1]). *For $S \subseteq \hat{A}$ and $\phi \in \hat{A}$, we have that $\phi \in \overline{S}$ if and only if, for each $a \in A$, we have that $\|\phi(a)\| \leq \sup\{\|\rho(a)\| : \rho \in S\}$.*

Again, we need to relate this result in $\hat{A}$ to a result about general representations. For $\pi \in \mathrm{Rep}(A, H)$, by Theorem 2.9 again, we have that $\pi \prec \mathrm{Sp}(\pi) \prec \pi$, equivalently, $K = \ker(\pi)$ where $K = \bigcap\{\ker \phi : \phi \in \mathrm{Sp}(\pi)\}$. Now, K is equal to the kernel of representation

$$\theta : A \to \mathcal{B}\Big(\oplus_{\phi \in \mathrm{Sp}(\pi)} H_\phi \Big); \quad a \mapsto \big(\phi(a)\big)_{\phi \in \mathrm{Sp}(\pi)}.$$

By properties of C^*-algebras, we know that A/K is isometric to $\theta(A)$, and similarly for $A/\ker(\pi)$ and $\pi(A)$. That is,

$$\|\pi(a)\| = \|a + \ker(\pi)\|_{A/\ker(\pi)} = \|a + K\|_{A/K} = \sup_{\rho \in \mathrm{Sp}(\pi)} \|\rho(a)\| \qquad (a \in A).$$

Lemma B.5. *For $\pi \in \mathrm{Rep}(A, H)$ and $S \subseteq \mathrm{Rep}(A, H)$, we have that $\pi \prec S$ if and only if, $\|\pi(a)\| \leq \sup_{\rho \in S} \|\rho(a)\|$ for each $a \in A$.*

Proof. Set $S_0 = \bigcup\{\mathrm{Sp}(\rho) : \rho \in S\}$ We know that $\pi \prec S$ if and only if $\phi \prec S_0$ for each $\phi \in \mathrm{Sp}(\pi)$, equivalently, $\phi \in \overline{S_0}$ for each $\phi \in \mathrm{Sp}(\pi)$. From the above proposition, this is equivalent to $\|\phi(a)\| \leq \sup\{\|\rho(a)\| : \rho \in S_0\}$, for each $a \in A$. From the above discussion, this is equivalent to $\|\phi(a)\| \leq \sup\{\|\rho(a)\| : \rho \in S\}$, for each a, which in turn is equivalent to the stated claim. □

Proposition B.6. *For a net (π_i), we have that $\pi \prec \{\pi_{i(j)}\}$ for any subnet, if and only if $\|\pi(a)\| \leq \liminf_i \|\pi_i(a)\|$ for each $a \in A$.*

Proof. This follows from the above lemma, as for a net of real numbers (t_i), we have that $t \leq \sup_j t_{i(j)}$ for all subnets if and only if $t \leq \liminf_i t_i$. □

Corollary B.7. *We have that $\pi_i \to \pi$ if and only if $\|\pi(a)\| \leq \liminf_i \|\pi_i(a)\|$ for $a \in A$.*

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