Failure of Kaplansky Density

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1 Introduction

We give some examples to show how (analogues of) the Kaplansky Density theorem fails, and make some remarks about the Krein–Smulian Theorem.

The Kaplansky Density Theorem is the following.

Theorem 1.1. *Let H be a Hilbert space, and endow $\mathcal{B}(H)$ with the weak*-topology coming from regarding $\mathcal{B}(H)$ as the dual space of the trace-class operators $\mathcal{B}_*(H)$. Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra, by which we mean a self-adjoint, unital, weak*-closed subalgebra of $\mathcal{B}(H)$, and let $A \subseteq M$ be a C^*-algebra which generates M, by which we mean that the weak*-closure of A is all of M.*

Then the unit ball of A is weak*-dense in the unit ball of M.

We seek counter examples in the setting of dual Banach algebras.

Definition 1.2. Let A be a Banach algebra, and suppose there is a Banach space E and a bounded linear isomorphism $A \to E^*$, which we use the induce a weak*-topology on A. We say that A is a *dual Banach algebra* when the algebra product on A is separately weak*-continuous.

If $\theta : A \to E^*$ is the isomorphism, then the composition

$$E \xhookrightarrow{\kappa_E} E^{**} \xrightarrow{\theta^*} A^*$$

is bounded below. Letting $F \subseteq A^*$ be the image, the product on A^* is separately weak*-continuous if and only if F is an A-submodule of A^*.

Thus, we are interested in cases when A is a dual Banach algebra with weak*-dense subalgebra $B \subseteq A$, but such that the unit ball of B is not weak*-dense in the unit ball of A. As we are not necessarily interested only in the "isometric" setting (for example, we do not assume θ is an isometric isomorphism) it is also interesting to seek cases when no ball (of finite radius) of B has weak*-closure containing the unit ball of A.

A final remark is that we can always give a Banach space the zero product, to turn it into a Banach algebra, we may first seek examples working with Banach spaces. Later we make remarks on more "natural" algebra examples.

2 Counter-examples

The following construction is from [3], but we work with ℓ^1 instead of ℓ^∞, for variety, and to give a separable example. This construction must be folklore, but we do not know, for example, a textbook reference.

Consider ℓ^1, a generic element of which we denote as $a = (a_n)_{n\geq 1}$, so $\|a\| = \sum_{n=1}^\infty |a_n| < \infty$. We give ℓ^1 the predual c_0 in the usual way. Pick $r > 0$ and let

$$X = X_r = \big\{a = (a_n) \in \ell^1 : ra_1 = \sum_{n=2}^\infty a_n\big\}.$$

Then X is a closed subspace. We claim that X is weak*-dense in ℓ^1, which (by Hahn-Banach) is equivalent to showing that if $x = (x_n) \in c_0$ with $\langle a, x\rangle = 0$ for each $a \in X$, then $x = 0$. Fix such an x, and firstly choose $N > 1$ and set $a = (a_n)$ with $a_1 = 1, a_N = r$ and $a_n = 0$ otherwise. Then $a \in X$ and

$$0 = \langle a, x\rangle = x_1 + rx_N.$$

Letting $N \to \infty$ shows that $x_1 = 0$, because $x \in c_0$. However, we then see that $rX_N = 0$ for any $N > 1$, and so $x = 0$, as required.

Now let (a_α) be a net in X converging weak* to $e_1 \in \ell^1$. Thus $a_1^{(\alpha)} \to 1$ and so

$$\liminf_\alpha \|a_\alpha\| = \liminf_\alpha |a_1^{(\alpha)}| + \sum_{n\geq 2} |a_n^{(\alpha)}| \geq \liminf_\alpha |a_1^{(\alpha)}| + r|a_1^{(\alpha)}| = 1 + r.$$

In particular, the unit ball of X is not weak*-dense in the unit ball of ℓ^1.

Now let E be the ℓ^1-direct sum of infinitely many copies of ℓ^1 (so E is itself isometrically isomorphic to ℓ^1), which has predual the c_0-direct sum of c_0. Now let X be the direct sum of the subspaces X_n for $n \in \mathbb{N}$. X is weak*-dense in E, as if $x = (x_n) \in \bigoplus_n c_0$ is annihilated by X, then each $x_n = 0$ in c_0, as X_n is weak*-dense in ℓ^1.

Let $e_1^{(n)}$ be the first unit vector basis in the nth copy of ℓ^1. Let (k_n) be a rapidly increasing sequence of integers. Set

$$a = \sum_n 2^{-n} e_1^{(k_n)},$$

so $\|a\| = 1$. Let (a_α) be a net in X converging weak* to a. For each α, let $a_\alpha = a_n^{(\alpha)} \in E = \bigoplus_n \ell^1$, and let $a_n^{(\alpha)} = (a_m^{(\alpha,n)}) \in \ell^1$. Thus

$$\lim_\alpha a_1^{(\alpha,k_n)} \to 2^{-n}.$$

For any M there exists α_0 so that, if $\alpha \geq \alpha_0$, then

$$|a_1^{(\alpha,k_n)}| > 2^{-n}(1 - M^{-1}) \qquad (1 \leq n \leq M).$$

Thus

$$\|a_{k_n}^{(\alpha)}\| \geq |a_1^{(\alpha,k_n)}|(1 + k_n) > 2^{-n}(1 - M^{-1})(1 + k_n) \qquad (1 \leq n \leq M),$$

and so

$$\|a_\alpha\| > \sum_{n=1}^M 2^{-n}(1 - M^{-1})(1 + k_n).$$

It follows that

$$\liminf_\alpha \|a_\alpha\| \geq \sum_{n=1}^M 2^{-n} k_n,$$

for any M. By choosing k_n suitably, this shows that (a_α) cannot be bounded.

We have hence constructed a (separable) Banach space E which is isometrically a dual space, and found a subspace X of E which is weak*-dense, but such that there is a unit vector $x_0 \in E$ such that no bounded net in X converges weak* to x_0.

2.1 A more abstract construction

Let F be any Banach space which is not reflexive, and let $E = F^*$. Fix $x_0 \in F$ a unit vector, regard F as a closed subspace of F^{**} as usual, and pick $M \in F^{***} = E^{**}$ a unit vector which annihilates F. For $r > 0$ choose $F \in F^{**} = E^*$ a unit vector with $|\langle M, F\rangle| > 1/2$, and define

$$X = X_r = \{\mu \in E : r\langle \mu, x_0\rangle = \langle F, \mu\rangle\}.$$

We claim that X is weak*-dense in E. A quick way to see this is the following. Notice that if we define

$$Y = \operatorname{lin}\{F - r\kappa_F(x_0)\} = \mathbb{C}(F - r\kappa_F(x_0)) \subseteq F^{**},$$

then ${}^\perp Y = X$ and so $X^\perp = Y$ as Y is weak*-closed in F^{**}. In particular,

$$\{0\} = Y \cap \kappa_F(F) = X^\perp \cap \kappa_F(F) = \kappa_F\big({}^\perp X\big).$$

It follows that ${}^\perp X = \{0\}$ and so X is weak*-dense in $E = F^*$.

By Hahn-Banach, pick a unit vector $\mu_0 \in E$ with $\langle \mu_0, x_0\rangle = 1$. Now let (μ_α) be a net in X converging weak* to μ_0, so

$$r = \lim_\alpha r\langle \mu_\alpha, x_0\rangle = \lim_\alpha \langle F, \mu_\alpha\rangle.$$

We conclude that $\liminf_\alpha \|\mu_\alpha\| \geq r$, because $\|F\| = 1$.

Thus, arguing as before, $\ell^1(E)$ admits a weak*-dense subspace X such that there is $\mu \in \ell^1(E)$ which is not the weak*-limit of any bounded net in X. I have been unable to decide if this construction is possible in E itself.

3 For algebras

Taking the Banach space example from the previous section and giving the zero product yields a Banach algebra example.

A much more sophisticated argument given by Dowson in [2] constructs a normal operator T on a separable Hilbert space H such that if A denotes the algebra generated by T inside $\mathcal{B}(H)$, then T^* is in the weak*-closure of A, but there is no bounded net in A converging weak* to T^*. Thus, the weak*-closure of A is a von Neumann algebra M, but there is no Kaplansky Density type result for A.

4 The Krein–Smulian Theorem

The following can be found in, for example, [1, Chapter 5, Section 12].

Theorem 4.1 (The Krein–Smulian Theorem)**.** *Let E be a Banach space and let $X \subseteq E^*$ be convex. If $X \cap \{\mu \in E^* : \|\mu\| \leq r\}$ is weak*-closed for each $r > 0$, then X is weak*-closed.*

After the proof of the theorem, Conway warns the reader thus:

> There is a misinterpretation of the Krein–Smulian Theorem that the reader should be warned about. If X is a weak-star closed convex balanced subset of the unit ball of E^*, let $M = \bigcup\{rX : r > 0\}$. It is easy to see that M is a linear manifold, but it does not follow that M is weak-star closed.

We believe that a possible way to reach this erroneous conclusion is as follows. Denote by $E^*_{[r]}$ the closed ball $\{\mu \in E^* : \|\mu\| \leq r\}$. To apply Krein–Smulian, we would need to show that $M \cap E^*_{[r]}$ were weak*-closed. It is tempting to think that $M \cap E^*_{[r]} = rX$, but of course this need not hold.

Let E be the example from before, so there is $\mu_0 \in E^*$ a unit vector, and $Y \subseteq E$ a weak*-dense subspace, such that no bounded net in Y converges weak* to μ_0. Let X be the weak*-closure of $Y \cap E^*_{[1]}$, and form M as above. Towards a contradiction, suppose that M is weak*-closed. Clearly $X \subseteq M$ and so $M = E^*$. Thus $\mu_0 \in M$ and so there is some $r > 0$ with $\mu_0 \in rX$, but as rX is the weak*-closure of $Y \cap E^*_{[r]}$ we obtain a bounded net in Y converging weak* to μ_0, contradiction.

References

[1] J. B. Conway, *A course in functional analysis*, second edition, Graduate Texts in Mathematics, 96, Springer-Verlag, New York, 1990. MR1070713

[2] H. R. Dowson, On an unstarred operator algebra, J. London Math. Soc. (2) **5** (1972), 489–492. MR0315467

[3] N. Ozawa (`https://mathoverflow.net/users/7591/narutaka-ozawa`), Ultraweak closure inside a closed ball, URL (version: 2012-07-17): `https://mathoverflow.net/q/102411`