1 Introduction

A compact quantum group is a unital C*-algebra A together with a coassociative map $\Delta : A \to A \otimes A$ such that $\Delta(A)(A \otimes 1)$ and $\Delta(A)(1 \otimes A)$ are linearly dense in $A \otimes A$. We get the Haar measure φ which is the unique state on A with $(\varphi \otimes \iota)\Delta(a) = (\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all $a \in A$.

As argued in my PAMS paper, we can find a maximal family of irreducible unitary corepresentations $\{v^\alpha = (v^\alpha_{ij})_{i,j=1}^{n_\alpha} : \alpha \in \mathbb{A}\}$ such that the associated "F-matrices" are all diagonal.

Firstly, if $\mathcal{A}$ is the linear span of $\{v^\alpha_{ij}\}$, then $\mathcal{A}$ is a Hopf-$*$-algebra and is dense in A. We have that

$$\Delta(v^\alpha_{ij}) = \sum_k v^\alpha_{ik} \otimes v^\alpha_{kj}, \quad S(v^\alpha_{ij}) = (v^\alpha_{ji})^*, \quad \epsilon(v^\alpha_{ij}) = \delta_{ij}, \quad \varphi(v^\alpha_{ij}) = \delta_{\alpha,\alpha_0},$$

where α_0 is the unique member of $\mathbb{A}$ with $v^\alpha_0 = 1$.

Then we have positive numbers $(\lambda^\alpha_i)_{i=1}^{n_\alpha}$ such that $\sum_i \lambda^\alpha_i = \sum_i (\lambda^\alpha_i)^{-1} = \Lambda_\alpha$ say. We have that

$$\varphi\big((v^\alpha_{ij})^* v^\beta_{kl}\big) = \delta_{\alpha,\beta}\delta_{i,k}\delta_{j,l}\frac{1}{\Lambda_\alpha \lambda^\alpha_i}, \quad \varphi\big(v^\alpha_{ij}(v^\beta_{kl})^*\big) = \delta_{\alpha,\beta}\delta_{i,k}\delta_{j,l}\frac{\lambda^\alpha_j}{\Lambda_\alpha}.$$

We define characters f_z, for $z \in \mathbb{C}$, on $\mathcal{A}$ by

$$f_z\big(v^\alpha_{ij}\big) = \delta_{i,j}(\lambda^\alpha_i)^z,$$

where of course $t^z = \exp(z \log t)$ for $t > 0$. Then the modular automorphism group for φ, restricted to $\mathcal{A}$, is given by

$$\sigma_z : v^\alpha_{ij} \mapsto \sum_{k,l} f_{iz}(v^\alpha_{ik}) v^\alpha_{kl} f_{iz}(v^\alpha_{lj}) = (\lambda^\alpha_i)^{iz}(\lambda^\alpha_j)^{iz} v^\alpha_{ij}.$$

For example, we can show that $\varphi(ba) = \varphi(a\sigma_{-i}(b))$ for all $a, b \in \mathcal{A}$. Also, as $J\Lambda(a) = \Lambda(\sigma_{i/2}(a)^*)$ for $a \in \mathcal{A}$, we see that

$$J\Lambda(v^\alpha_{ij}) = (\lambda^\alpha_i \lambda^\alpha_j)^{-1/2}\Lambda((v^\alpha_{ij})^*).$$

Similarly, the scaling group on $\mathcal{A}$ is given by

$$\tau_z : v^\alpha_{ij} \mapsto (\lambda^\alpha_i)^{iz}(\lambda^\alpha_j)^{-iz} v^\alpha_{ij}.$$

Thus in particular,

$$S(v^\alpha_{ij}) = (v^\alpha_{ji})^* = R\tau_{-i/2}(v^\alpha_{ij}) = (\lambda^\alpha_i)^{1/2}(\lambda^\alpha_j)^{-1/2} R(v^\alpha_{ij}) \implies R(v^\alpha_{ij}) = \sqrt{\frac{\lambda^\alpha_j}{\lambda^\alpha_i}}(v^\alpha_{ji})^*.$$

However, also $R(x) = \hat{J}x^*\hat{J}$, and so

$$\hat{J} v^\alpha_{ij} \hat{J} = \sqrt{\frac{\lambda^\alpha_j}{\lambda^\alpha_i}} v^\alpha_{ji}.$$

2 Reduced case and duality

Now suppose that φ is faithful. Let (H, Λ) be the GNS construction for φ.

For each $\alpha \in \mathbb{A}$, let H_α be the finite-dimensional subspace of H spanned by $\{\Lambda((v^\alpha_{ij})^*) : 1 \leq i, j \leq n_\alpha\}$. Notice that H_α is orthogonal to H_β for $\alpha \neq \beta$. As $\mathcal{A}$ is dense in H, it follows that