2 Positive maps

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1 An elementary characterisation of 2 positive maps

We follow the presentation of [5] but draw further conclusions.

lem:elem **Lemma 1.1.** *Let $x \geq 0, y \geq 0$ and $z \in \mathbb{C}$. Then*

$$x|t|^2 + y|s|^2 + 2\Re(zs\bar{t}) \geq 0$$

for all $s, t \in \mathbb{C}$ if and only if $|z|^2 \leq xy$.

Proof. By choosing the argument of $s\bar{t}$ appropriately, this is equivalent to

$$0 \leq xt^2 + ys^2 - 2|z|st = \big(t\sqrt{x} - s\sqrt{y}\big)^2 + 2st(\sqrt{xy} - |z|)$$

for all $s, t \geq 0$. This is clearly equivalent to $\sqrt{xy} - |z| \geq 0$, as claimed. $\square$

Lemma 1.2. *Consider the scalar matrix*

$$x = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$$

in $\mathbb{M}_2(\mathbb{C})$ acting on $\mathbb{C}^2$. Then $x \geq 0$ if and only if $a \geq 0, c \geq 0$ and $|b|^2 \leq ac$.

Proof. Let $\alpha = (\xi, \eta)^t \in \mathbb{C}^2$ and consider when $(x\alpha|\alpha) \geq 0$. It's immediate that we need $a \geq 0$ and $c \geq 0$. We then need

$$a|\xi|^2 + c|\eta|^2 + 2\Re(b\eta\bar{\xi}) \geq 0$$

for all choices of ξ, η. By Lemma 1.1 the result follows. $\square$

Lemma 1.3. *Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space H, and consider*

$$x = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in \mathbb{M}_2(\mathfrak{A}).$$

Then $x \geq 0$ if and only if $a \geq 0, c \geq 0$ and $|(b\eta|\xi)|^2 \leq (a\xi|\xi)(c\eta|\eta)$ for all $\xi, \eta \in H$.

Proof. Clearly we need $a \geq 0$ and $c \geq 0$. Proceeding as before, we now consider $(x\alpha|\alpha)$ for $\alpha = (\xi, \eta) \in H^2$ where $\mathfrak{A}$ acts on H. Thus $x \geq 0$ if and only if

$$(a\xi|\xi) + (c\eta|\eta) + 2\Re(b\eta|\xi) \geq 0$$

for all $\xi, \eta \in H$. By scaling ξ by $t \in \mathbb{C}$ and η by $s \in \mathbb{C}$ this is equivalent to

$$|t|^2(a\xi|\xi) + |s|^2(c\eta|\eta) + 2\Re\big(s\bar{t}(b\eta|\xi)\big) \geq 0$$

for all $\xi, \eta \in H$ and $s, t \in \mathbb{C}$. By Lemma 1.1 this is equivalent to

$$\big|(b\eta|\xi)\big|^2 \leq (a\xi|\xi)(c\eta|\eta) \qquad (\xi, \eta \in H),$$

as claimed. $\square$