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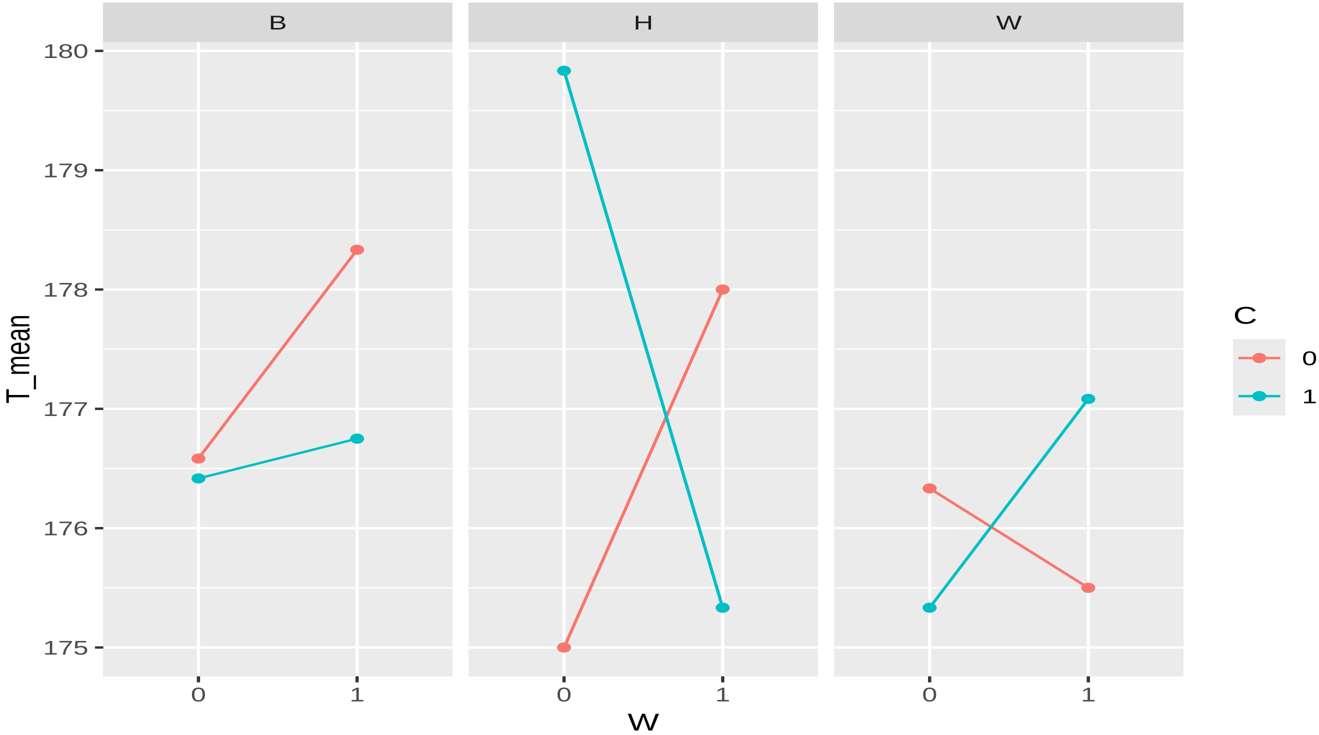
Description of assignment: Twelve adults were randomly selected from each of the Black, Hispanic and White populations. The adults read two types of words (Form and Colour) under two cue conditions (Normal and Congruent) and the order in which the four reading tasks were carried out was randomized. The time in milliseconds for completing each reading task was recorded, on each adult. The variables of interest for this Assignment are:

- *Adult*: This is a factor variable that identifies the adult.
- T: Measures the time in milliseconds to complete a reading task.
- W: This is a fixed factor (categorical variable) that identifies the type of word used in the reading task. It has two levels (0 = *Form*, 1 = *Colour*).
- C: This is a fixed factor (categorical variable) that identifies the type of cue condition used in the reading task. It has two levels (0 = *Normal*, 1 = *Congruent*).
- $R2$: This is a fixed factor (categorical variable) that identifies whether the adult is Hispanic or not. It has 2 levels (0 = *Not-Hispanic*, 1 = *Hispanic*).
- $R3$: This is a fixed factor (categorical variable) that identifies whether the adult is White or not. It has 2 levels (0 = *Not-White*, 1 = *White*).
- *Race*: This is a fixed factor (categorical variable) that identifies the race of the adult. It has 3 levels (B = *Black*, H = *Hispanic*, W = *White*).

1 Graphical analysis

1. The three-way interaction effect between W, C and $Race$ should be included in the linear mixed model. It is evident in *Figure 1* that the fixed intercept for $[W = 0, C = 0]$ would start at different points for each of the three races, therefore $Race$ has an effect. *Figure 1* also shows that $Black$ participants recorded faster reading times, on average, when using $Form$ type of words $[W = 0]$ compared to $Colour$ type of words $[W = 1]$. Slower reading times for $Black$ adults using $Colour$ words were minimised when coupled with the $Congruent$ cue condition $[W = 1, C = 1]$ whereas the $Normal$ cue condition $[W = 1, C = 0]$ amplified this result. In contrast, Hispanic participants recorded faster times times when using $Form$ type of words in the $Normal$ cue condition $[W = 0, C = 0]$ and when using $Colour$ type of words in the $Congruent$ cue condition $[W = 1, C = 1]$, yet the other two combinations produced very different results. This thereby shows that the combination of conditions affects Black participants and Hispanic participants differently. Different effects are also evident in $White$ participants, as faster times were recorded, on average, for the $[W = 0, C = 1]$ and $[W = 1, C = 0]$ conditions compared to the $[W = 1, C = 1]$ and $[W = 0, C = 0]$ conditions. These differences show that the results cannot simply be explained by simple fixed effects, as the conditions interact to produce different effects.

Figure 1: Plot of mean T as a function of W grouped by C and $Race$



2 Describing the model

The following linear mixed model was set up to analyze the research questions.

$$\begin{aligned}
T_{ti} = {} & \beta_0 + \beta_1 W_{ti} + \beta_2 C_{ti} + \beta_3 R2_i + \beta_4 R3_i \\
& + \beta_5 W_{ti} \times C_{ti} + \beta_6 W_{ti} \times R2_i + \beta_7 W_{ti} \times R3_i + \beta_8 C_{ti} \times R2_i + \beta_9 C_{ti} \times R3_i \\
& + \beta_{10} W_{ti} \times C_{ti} \times R2_i + \beta_{11} W_{ti} \times C_{ti} \times R3_i \\
& + \mu_{0i} + \mu_{1i} C_{ti} + \varepsilon_{ti}, \qquad (1)
\end{aligned}$$

- where T_{ti} is the time in milliseconds to complete a reading task for adult i ($i = 1, \ldots, 36$) at occasion t ($t = 1, 2, 3, 4$),
- $W_{ti} = 1$ if the type of word used in the reading task for adult i at occasion t is *Colour*, and 0 otherwise,
- $C_{ti} = 1$ if the type of cue condition used in the reading task for adult i at occasion t is *Congruent*, and 0 otherwise.
- $R2_i = 1$ if the race of adult i is *Hispanic*, and 0 otherwise,
- $R3_i = 1$ if the race of adult i is *White*, and 0 otherwise,
- Note that if $R2_i = 0$ and $R3_i = 0$ for adult i, then the race of adult i is *Black*.
- β_0 is the fixed intercept,
- β_1, β_2, β_3 and β_4 are the the fixed simple effects of W, C, $R2$ and $R3$, respectively,
- β_5, β_6, β_7, β_8 and β_9 are the fixed two-way interaction effects of $W \times C$, $W \times R2$, $W \times R3$, $C \times R2$ and $C \times R3$, respectively,
- β_{10} and β_{11} are the fixed three-way interaction effects of $W \times C \times R2$ and $W \times C \times R3$, respectively,
- μ_{0i} is the random intercept specific to adult i,
- μ_{1i} is the random effect of C on T specific to adult i,
- ε_{ti} is the random error associated with measuring T at occasion t, for adult i.

2. In matrix form, the full linear mixed model can be expressed as: $\boldsymbol{Y}_i = \boldsymbol{X}_i \boldsymbol{\beta} + \boldsymbol{Z}_i \boldsymbol{\mu}_i + \boldsymbol{\varepsilon}_i$, where

   (a) $\boldsymbol{Y}_i$ represents a $\mathbf{4 \times 1}$ response vector of reading times for adult i. That is,

$$\boldsymbol{Y}_i = \begin{bmatrix} T_{1i} \\ T_{2i} \\ T_{3i} \\ T_{4i} \end{bmatrix},$$

   (b) $\boldsymbol{X}_i$ represents a $\mathbf{4 \times 12}$ matrix which represents a column of 1s and the known values of the predictors associated with the fixed effects. That is, for adult i,

$$\boldsymbol{X}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

   (c) $\boldsymbol{\beta}$ is a $\mathbf{12 \times 1}$ vector of twelve unknown fixed-effects. That is,

$$\boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \\ \beta_{11} \end{bmatrix},$$

(d) $\boldsymbol{Z}_i$ is a $\mathbf{4} \times \mathbf{2}$ matrix that contains the values of the predictors associated with the random effects. That is, for adult i,

$$\boldsymbol{Z}_i = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},$$

(e) $\boldsymbol{\mu}_i$ is a $\mathbf{2} \times \mathbf{1}$ random effect vector containing the random intercept and the random effect of C on T, for adult i. That is,

$$\boldsymbol{\mu}_i = \begin{bmatrix} \mu_{0i} \\ \mu_{1i} \end{bmatrix},$$

(f) $\boldsymbol{\varepsilon}_i$ is a $\mathbf{4} \times \mathbf{1}$ random error vector of four random errors associated with four reading time measurements for adult i. That is,

$$\boldsymbol{\varepsilon}_i = \begin{bmatrix} \varepsilon_{1i} \\ \varepsilon_{2i} \\ \varepsilon_{3i} \\ \varepsilon_{4i} \end{bmatrix},$$

(g) An *unstructured* structure for the variance-covariance matrix of the random effect vector, $\boldsymbol{\mu}_i$ was chosen. That is, the variance-covariance matrix of the random effect vector, $\boldsymbol{\mu}_i$, is

$$\boldsymbol{D} = \begin{bmatrix} \tau_0 & \tau_{01} \\ \tau_{01} & \tau_1 \end{bmatrix},$$

- where τ_0 and τ_1 denotes the variance of the random effects μ_{0i} and μ_{1i}, respectively,
- τ_{01} denotes the covariance between the random effects μ_{0i} and μ_{1i}.

The following structure for the variance-covariance matrix of the random error vector was also chosen, $\boldsymbol{\varepsilon}_i$,

$$\boldsymbol{R}_i = \begin{bmatrix} \theta_1 & 0 & 0 & 0 \\ 0 & \theta_2 & 0 & 0 \\ 0 & 0 & \theta_1 & 0 \\ 0 & 0 & 0 & \theta_2 \end{bmatrix}$$

- where $\theta_1 = Var(\varepsilon_{1i}) = Var(\varepsilon_{3i})$ and $\theta_2 = Var(\varepsilon_{2i}) = Var(\varepsilon_{4i})$.

The variance-covariance matrix of the response vector, $\boldsymbol{Y}_i$, for adult i, can be derived using the following formula,

$$\begin{aligned} Var(Y_i) &= Var(X_i\beta + Z_i\mu_i + \varepsilon_i) \\ &= Var(Z_i\mu_i)Z_i^T + Var(\varepsilon_i) \ X_i\beta \text{ is fixed and } \mu_i,\ \varepsilon_i \text{ ind} \\ &= Z_iVar(\mu_i)Z_i^T + Var(\varepsilon_i) \\ &= Z_iDZ_i^T + R_i \end{aligned}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \tau_0 & \tau_{01} \\ \tau_{01} & \tau_1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \theta_1 & 0 & 0 & 0 \\ 0 & \theta_2 & 0 & 0 \\ 0 & 0 & \theta_1 & 0 \\ 0 & 0 & 0 & \theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \tau_0 & \tau_{01} \\ \tau_0 + \tau_{01} & \tau_{01} + \tau_1 \\ \tau_0 & \tau_{01} \\ \tau_0 + \tau_{01} & \tau_{01} + \tau_1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \theta_1 & 0 & 0 & 0 \\ 0 & \theta_2 & 0 & 0 \\ 0 & 0 & \theta_1 & 0 \\ 0 & 0 & 0 & \theta_2 \end{bmatrix}$$

$$= \begin{bmatrix} \tau_0 & \tau_0 + \tau_{01} & \tau_0 & \tau_0 + \tau_{01} \\ \tau_0 + \tau_{01} & \tau_0 + 2\tau_{01} + \tau_1 & \tau_0 + \tau_{01} & \tau_0 + 2\tau_{01} + \tau_1 \\ \tau_0 & \tau_0 + \tau_{01} & \tau_0 & \tau_0 + \tau_{01} \\ \tau_0 + \tau_{01} & \tau_0 + 2\tau_{01} + \tau_1 & \tau_0 + \tau_{01} & \tau_0 + 2\tau_{01} + \tau_1 \end{bmatrix} + \begin{bmatrix} \theta_1 & 0 & 0 & 0 \\ 0 & \theta_2 & 0 & 0 \\ 0 & 0 & \theta_1 & 0 \\ 0 & 0 & 0 & \theta_2 \end{bmatrix}$$

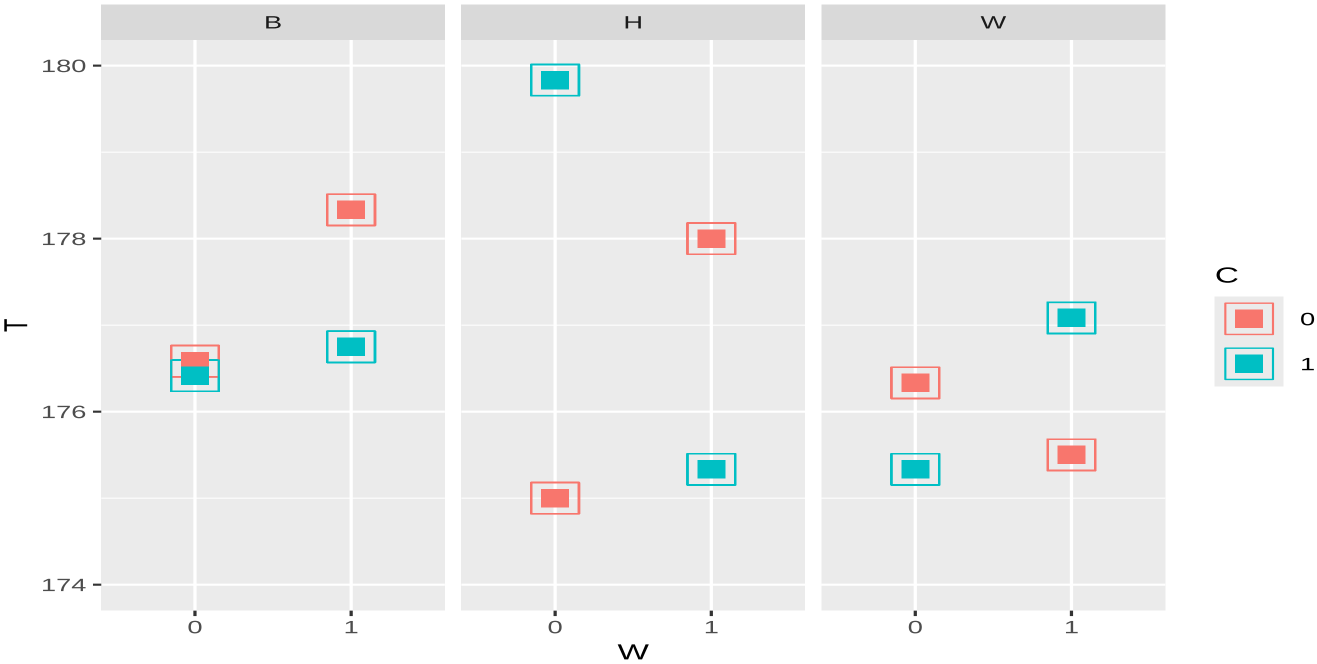
$$= \begin{bmatrix} \tau_0 + \theta_1 & \tau_0 + \tau_{01} & \tau_0 & \tau_0 + \tau_{01} \\ \tau_0 + \tau_{01} & \tau_0 + 2\tau_{01} + \tau_1 + \theta_2 & \tau_0 + \tau_{01} & \tau_0 + 2\tau_{01} + \tau_1 \\ \tau_0 & \tau_0 + \tau_{01} & \tau_0 + \theta_1 & \tau_0 + \tau_{01} \\ \tau_0 + \tau_{01} & \tau_0 + 2\tau_{01} + \tau_1 & \tau_0 + \tau_{01} & \tau_0 + 2\tau_{01} + \tau_1 + \theta_2 \end{bmatrix}$$

3 Diagnostics of the final linear mixed model

The model above was used as the final linear mixed model.

3. The R computer package was used to produce Figure 2 below that checks the agreement between the predicted marginal values of T (that come from fitting the model to the data) and the observed mean values of T, as a function of W, grouped by C and *Race*. The large unshaded squares correspond to the predicted marginal values of T and the shaded squares correspond to the observed mean values of T.

Figure 2: Predicted marginal and observed mean values of T as a function of W, grouped by C and *Race*



4 Variance-covariance estimates of the final linear mixed model

4. The R computer package was used to calculate an estimate of the $\boldsymbol{D}$ matrix of the final linear mixed model:

$$\hat{\boldsymbol{D}} = \begin{bmatrix} 12.41 & 14.87 \\ 14.87 & 111.51 \end{bmatrix}$$

5. The R computer package was used to calculate an estimate of the $\boldsymbol{R}$ matrix of the final linear mixed model:

$$\hat{\boldsymbol{R}} = \begin{bmatrix} 228.21 & 0 & 0 & 0 \\ 0 & 351.79 & 0 & 0 \\ 0 & 0 & 228.21 & 0 \\ 0 & 0 & 0 & 351.79 \end{bmatrix}$$

6. An estimate of the variance-covariance matrix of the response vector of the final linear mixed model was calculated using the formula

$$Var(Y_i) = Z_i D Z_i^T + R_i$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 12.41 & 14.87 \\ 14.87 & 111.51 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 228.21 & 0 & 0 & 0 \\ 0 & 351.79 & 0 & 0 \\ 0 & 0 & 228.21 & 0 \\ 0 & 0 & 0 & 351.79 \end{bmatrix}$$

$$= \begin{bmatrix} 12.41 & 14.87 \\ 27.28 & 126.38 \\ 12.41 & 14.87 \\ 27.28 & 126.38 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 228.21 & 0 & 0 & 0 \\ 0 & 351.79 & 0 & 0 \\ 0 & 0 & 228.21 & 0 \\ 0 & 0 & 0 & 351.79 \end{bmatrix}$$

$$= \begin{bmatrix} 12.41 & 27.28 & 12.41 & 27.28 \\ 27.28 & 153.66 & 27.28 & 153.66 \\ 12.41 & 27.28 & 12.41 & 27.28 \\ 27.28 & 153.66 & 27.28 & 153.66 \end{bmatrix} + \begin{bmatrix} 228.21 & 0 & 0 & 0 \\ 0 & 351.79 & 0 & 0 \\ 0 & 0 & 228.21 & 0 \\ 0 & 0 & 0 & 351.79 \end{bmatrix}$$

$$= \begin{bmatrix} 240.62 & 27.28 & 12.41 & 27.28 \\ 27.28 & 505.45 & 27.28 & 153.66 \\ 12.41 & 27.28 & 240.62 & 27.28 \\ 27.28 & 153.66 & 27.28 & 505.45 \end{bmatrix}$$

5 Fixed effect estimates of the final linear mixed model

7. The R computer package was used to produce a table that lists the estimates of the fixed effects in the model, together with their corresponding standard errors, degrees of freedom, observed test statistics and p-values.

Table 1: Estimates of fixed effects

	Value	Std.Error	DF	t-value	p-value
β_0	176.58	4.48	99	39.43	0.00
β_1	1.75	6.17	99	0.28	0.78
β_2	-0.17	7.59	99	-0.02	0.98
β_3	-1.58	6.33	33	-0.25	0.80
β_4	-0.25	6.33	33	-0.04	0.97
β_5	-1.42	9.83	99	-0.14	0.89
β_6	1.25	8.72	99	0.14	0.89
β_7	-2.58	8.72	99	-0.30	0.77
β_8	5.00	10.74	99	0.47	0.64
β_9	-0.83	10.74	99	-0.08	0.94
β_{10}	-6.08	13.90	99	-0.44	0.66
β_{11}	4.00	13.90	99	0.29	0.77

8. The estimates of β_0, β_2 and β_7 can be interpreted as follows.

   - The estimate of β_0 is 176.58. We estimate that the mean time for *Black* adults to complete a reading task using *Form* type of words and *Normal* cues is 176.58 and is significantly different from 0 (p-value < 0.001).
   - The estimate of β_2 is -0.17. When *Form* words are used, we estimate that the difference in mean time between *Black* adults completing a reading task using *Congruent* cues and *Black* adults completing a reading task using *Normal* cues is -0.17 and this difference is insignificant (p-value $= 0.98$).
   - The estimate of β_7 is -2.58. We estimate that the change in the difference in mean reading time between *Black* and *White* adults when changing from *Form* words to *Colour* words is -2.58 and this change is significant (p-value $= 0.77$).

9. Letting η_1 denote the mean time to complete a reading task for *White* adults where the word is *Form* and cue condition is *Congruent*, can otherwise interpreted as the difference associated with β_9. Letting η_2 denote the mean time to complete a reading task for *Hispanic* adults where the word is *Colour* and cue condition is *Normal*, can otherwise interpreted as the difference associated with β_6. Using the `glht` command in R to calculate $\eta = \eta_1 - \eta_2$, η was estimated to be -2.083. At the 5% level of significance ($p < 0.05$), there is insufficient statistical evidence to suggest that $\eta_1 \neq \eta_2$ (p-value $= 0.87$). It should be noted that this is a relatively small sample size, therefore a significant difference may be found in a larger sample. However, at this stage, it is not clear that there is a significant change in the difference between reading times for *White* adults using *Congruent* cues and *Form* words, and *Hispanic* adults using *Normal* cues and *Colour* words.

10. Letting ω_1 denote the difference in mean time to complete a reading task between *Congruent* and *Normal* cue conditions for *White* adults and *Colour* words, can otherwise be interpreted as the difference associated with β_{11}. Letting ω_2 denote the difference in mean time to complete a reading task between *Congruent* and *Normal* cue conditions for *Hispanic* adults and *Colour* words, can otherwise be interpreted as the difference associated with β_{10}. Using the `glht` command in R to calculate $\omega = \omega_1 - \omega_2$, ω was estimated as 10.08. At the 5% level of significance ($p < 0.05$), there is insufficient statistical evidence to suggest that $\omega_1 \neq \omega_2$ (p-value $= 0.47$). Again, it should be noted that this is a relatively small sample size, therefore a significant difference may be found in a larger sample. However, at this stage, when using *Colour* words, it is not clear that there is a significant change in the differences in reading times between *White* adults and *Hispanic* adults when changing from *Normal* cues to *Congruent cues.*

11. The marginal residual for adult $i = 14$, when the type of word is *Colour* and the cue condition is *Congruent* is given by the formula

$$\varepsilon^*_{ti} = T_{ti} - \hat{E}(T_{ti})$$

where $t = 4$, $i = 14$, $T_{414} = 215$ and

$$\begin{aligned}\hat{E}(T_{414}) &= \beta_0 + \beta_1 W_{414} + \beta_2 C_{414} + \beta_3 R2_{414} \\ &+ \beta_5 W_{414} \times C_{414} + \beta_6 W_{414} \times R2_{414} + \beta_8 C_{414} \times R2_{414} \\ &+ \beta_{10} W_{414} \times C_{414} \times R2_{414} \\ &= 176.58 + 1.75 - 0.17 - 1.58 - 1.42 + 1.25 + 5 - 6.08 \\ &= 175.33\end{aligned}$$

therefore

$$\begin{aligned}\varepsilon^*_{414} &= 215 - 175.33 \\ &= 39.67\end{aligned}$$

12. Figure 3 presents the random effect predictions of the final linear mixed model, for the first 5 adults in the study. These predictions were obtained by using the `ranef()` command in R. The conditional

Figure 3: Random Effect Predictions

	(Intercept)	C
1	2.94	6.29
2	-0.85	4.12
3	-2.21	-8.63
4	-1.37	-9.31
5	-0.62	-2.56

residual for adult $i = 2$, when the type of word is *Form* and the cue condition is *Congruent* is given by the formula

$$\varepsilon_{ti} = T_{ti} - \hat{E}(T_{ti} | \mu_{0i}, \mu_{1i})$$

where $t = 2$, $i = 2$, $T_{22} = 196$ and

$$\begin{aligned}\hat{E}(T_{22} | \mu_{0i}, \mu_{1i}) &= \beta_0 + \beta_2 C_{22} + \mu_{02} + \mu_{12} \\ &= 176.58 - 0.17 - 0.85 + 4.12 \\ &= 179.68\end{aligned}$$

therefore

$$\begin{aligned}\varepsilon_{22} &= 196 - 179.68 \\ &= 16.32\end{aligned}$$

6 Testing for random effects

13. The researchers would like to test whether the random effect of C should be included in model. They decide to test, at the 5% significance level, the null hypothesis $H_0 : \tau_1 = 0$ vs the alternative hypothesis $H_1 : \tau_1 > 0$ using the REML-based likelihood ratio test p-value.

    (a) The reference model (1) for this test is

$$\begin{aligned} T_{ti} =& \beta_0 + \beta_1 W_{ti} + \beta_2 C_{ti} + \beta_3 R2_i + \beta_4 R3_i \\ & + \beta_5 W_{ti} \times C_{ti} + \beta_6 W_{ti} \times R2_i + \beta_7 W_{ti} \times R3_i + \beta_8 C_{ti} \times R2_i + \beta_9 C_{ti} \times R3_i \\ & + \beta_{10} W_{ti} \times C_{ti} \times R2_i + \beta_{11} W_{ti} \times C_{ti} \times R3_i \\ & + \mu_{0i} + \mu_{1i} C_{ti} + \varepsilon_{ti} \end{aligned}$$

    (b) The nested model (2) for this test, that does not include the random effect of C, is

$$\begin{aligned} T_{ti} =& \beta_0 + \beta_1 W_{ti} + \beta_2 C_{ti} + \beta_3 R2_i + \beta_4 R3_i \\ & + \beta_5 W_{ti} \times C_{ti} + \beta_6 W_{ti} \times R2_i + \beta_7 W_{ti} \times R3_i + \beta_8 C_{ti} \times R2_i + \beta_9 C_{ti} \times R3_i \\ & + \beta_{10} W_{ti} \times C_{ti} \times R2_i + \beta_{11} W_{ti} \times C_{ti} \times R3_i \\ & + \mu_{0i} + \varepsilon_{ti} \end{aligned}$$

    (c) The R computer package was used to perform the REML-based likelihood ratio test (RLRT). The RLRT p-value for testing the null hypothesis $H_0 : \tau_1 = 0$ that the variance of the random effect C is zero is

$$p\text{-}value = 0.5 \times P(X_1^2 > 2.33) + 0.5 \times P(X_2^2 > 2.33) \approx 0.22$$

    (d) The p-value is greater than 0.05, therefore we cannot reject the null hypothesis that Model 2 is sufficient ($H_0 : \tau_1 = 0$). The alternative hypothesis, that Model 1 is a better fit to the data ($H_1 : \tau_1 > 0$), is not preferred given this information. Model 2 is chosen to continue the analysis of this data.

7 Testing for fixed effects

14. The researchers would like to test whether the three-way interaction effects should be included in the model chosen. They decide to test, at the 5% significance level, the null hypothesis $H_0 : \beta_{10} = \beta_{11} = 0$ vs the alternative hypothesis $H_1 : \beta_{10} \neq 0$ or $\beta_{11} \neq 0$ using the ML-based likelihood ratio test p-value.

    (a) The reference model (2) for this test is

    $$\begin{aligned} T_{ti} =& \beta_0 + \beta_1 W_{ti} + \beta_2 C_{ti} + \beta_3 R2_i + \beta_4 R3_i \\ &+ \beta_5 W_{ti} \times C_{ti} + \beta_6 W_{ti} \times R2_i + \beta_7 W_{ti} \times R3_i + \beta_8 C_{ti} \times R2_i + \beta_9 C_{ti} \times R3_i \\ &+ \beta_{10} W_{ti} \times C_{ti} \times R2_i + \beta_{11} W_{ti} \times C_{ti} \times R3_i \\ &+ \mu_{0i} + \varepsilon_{ti} \end{aligned}$$

    (b) The nested model (3) for this test is

    $$\begin{aligned} T_{ti} =& \beta_0 + \beta_1 W_{ti} + \beta_2 C_{ti} + \beta_3 R2_i + \beta_4 R3_i \\ &+ \beta_5 W_{ti} \times C_{ti} + \beta_6 W_{ti} \times R2_i + \beta_7 W_{ti} \times R3_i + \beta_8 C_{ti} \times R2_i + \beta_9 C_{ti} \times R3_i \\ &+ \mu_{0i} + \varepsilon_{ti} \end{aligned}$$

    (c) The R computer package was used to perform the ML-based likelihood ratio test (MLRT). The MLRT p-value for testing the null hypothesis $H_0 : \beta_{10} = \beta_{11} = 0$, that the fixed interaction effects $W \times C \times R2$ and $W \times C \times R3$ are equal to zero, is

    $$p\text{-}value = P(X_2^2 > 0.5) \approx 0.78$$

    (d) Although Model 2 indicates a slightly better fit to the data, the p-value is greater than 0.05, therefore there is insufficient statistical evidence to suggest that the alternative hypothesis, that either $\beta_{10} \neq 0$ or $\beta_{11} \neq 0$, is true. Therefore, the null hypothesis $H_0 : \beta_{10} = \beta_{11} = 0$ is preferred. Model 3 is chosen to continue the analysis of this data.

15. The researchers would like to test whether the two-way interaction effects should be included in the model chosen. They decide to test, at the 5% significance level, the null hypothesis $H_0 : \beta_5 = \beta_6 = \beta_7 = \beta_8 = \beta_9 = 0$ vs the alternative hypothesis $H_1 : \beta_5 \neq 0$ or $\beta_6 \neq 0$ or $\beta_7 \neq 0$ or $\beta_8 \neq 0$ or $\beta_9 \neq 0$ using the ML-based likelihood ratio test p-value.

    (a) The reference model (3) for this test is

    $$\begin{aligned} T_{ti} =& \beta_0 + \beta_1 W_{ti} + \beta_2 C_{ti} + \beta_3 R2_i + \beta_4 R3_i \\ &+ \beta_5 W_{ti} \times C_{ti} + \beta_6 W_{ti} \times R2_i + \beta_7 W_{ti} \times R3_i + \beta_8 C_{ti} \times R2_i + \beta_9 C_{ti} \times R3_i \\ &+ \mu_{0i} + \varepsilon_{ti} \end{aligned}$$

    (b) The nested model (4) for this test is

    $$\begin{aligned} T_{ti} =& \beta_0 + \beta_1 W_{ti} + \beta_2 C_{ti} + \beta_3 R2_i + \beta_4 R3_i \\ &+ \mu_{0i} + \varepsilon_{ti} \end{aligned}$$

    (c) The R computer package was used to perform the ML-based likelihood ratio test (MLRT). The MLRT p-value for testing the null hypothesis $H_0 : \beta_5 = \beta_6 = \beta_7 = \beta_8 = \beta_9 = 0$, that the two-way interaction effects are equal to zero, is

    $$p\text{-}value = P(X_2^2 > 0.24) \approx 0.99$$

    (d) The p-value is greater than 0.05, therefore there is insufficient statistical evidence to suggest that the alternative hypothesis, $\beta_5 \neq 0$ or $\beta_6 \neq 0$ or $\beta_7 \neq 0$ or $\beta_8 \neq 0$ or $\beta_9 \neq 0$, is true. We therefore cannot accept the alternative hypothesis. Therefore, the null hypothesis $H_0 : \beta_5 = \beta_6 = \beta_7 = \beta_8 = \beta_9 = 0$ is preferred. Model 4 is chosen to continue the analysis of this data.

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[1]There are interaction effects in the data (refer to *Figure* 1 on p. 2), however the final Model (4) stated here does not include these effects, as a result of statistical testing used above. It should be noted that the large p-values yielded by the statistical tests may be a result of the small sample size. Statistical tests should be used again with a larger sample size as more elaborate models are generally preferred to estimate effects and find relationships in linear mixed models.