Numerical Simulations of Aquifers

Seán White Matthew Gallagher Nicola Young

Applied and Computational Mathematics
UCD

November 2024

Goal

Numerically Solve Eqns
from CH4 Porous Media

Governing Equations

Drought:

$$\hat{h}_{\hat{t}} = (\hat{h}\hat{h}_{\hat{x}})_{\hat{x}}$$

Rainfall:

$$\hat{h}_{\hat{t}} = (\hat{h}\hat{h}_{\hat{x}})_{\hat{x}} + 1$$

Drought

$$\hat{h}_{\hat{t}} = (\hat{h}\hat{h}_{\hat{x}})_{\hat{x}}$$

Solve using separation of variables:

$$\hat{h}(\hat{x}, \hat{t}) = X(\hat{x})T(\hat{t})$$

Gives:

$$\hat{h}(\hat{x}, \hat{t}) = \frac{X(\hat{x})}{\hat{t} + A}$$

$$X'^2 + XX'' + X = 0$$

Solve Numerically

ODE:

$$X'^2 + XX'' + X = 0$$

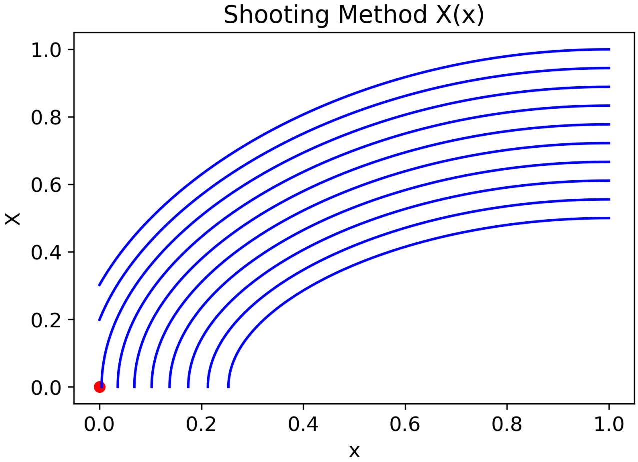
BC's:

$$X(0) = 0, \quad X'(1) = 0$$

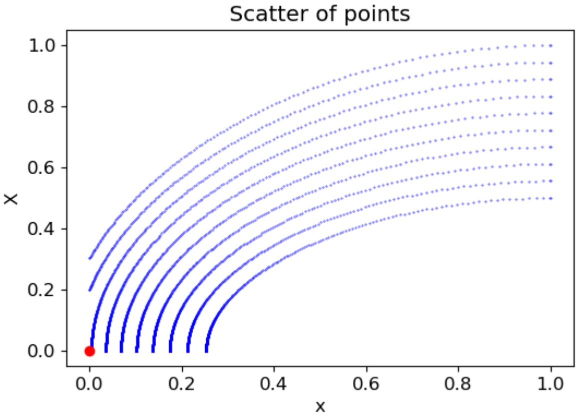
Re-arranged:

$$X'' = \frac{-X - X'^2}{X}$$

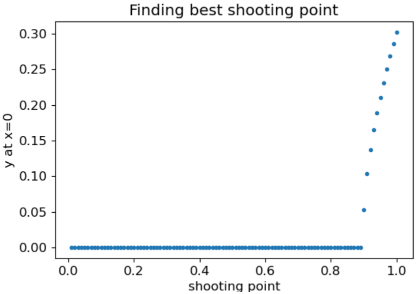
Shooting Method

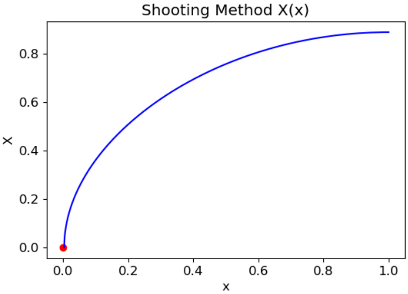


Issues: Stiff PDE

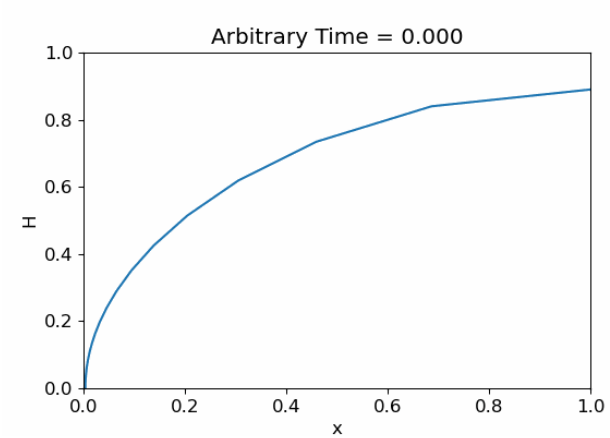


Shooting Method





Height Of Aquifer Over Time



Rainfall

$$\hat{h}_{\hat{t}} = (\hat{h}\hat{h}_{\hat{x}})_{\hat{x}} + 1$$

Solve using similarity solution:

$$\hat{h}(\hat{x}, \hat{t}) = \hat{t}^a f(\eta) \quad \text{where} \quad \eta = \frac{\hat{x}}{\hat{t}^b}$$

Gives:

$$a = b = 1$$

$$f - \eta f' = (ff')' + 1$$

Solve Numerically

ODE:

$$f - \eta f' = (ff')' + 1$$

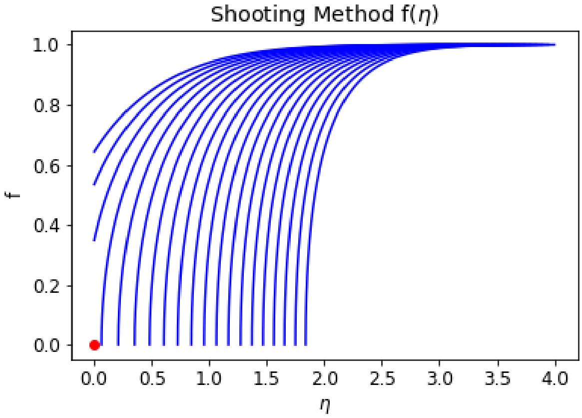
Re-arranged:

$$f'' = \frac{f - \eta f' - f'^2 - 1}{f}$$

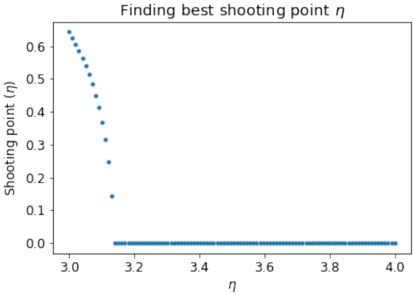
BC's:

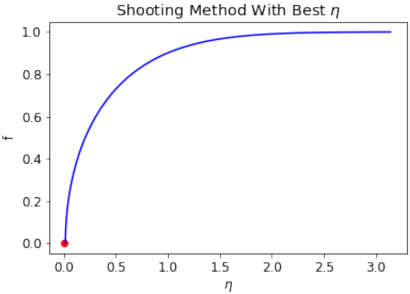
$$f(0) = 0, \quad f(\eta) = 1 \quad \text{as} \quad \eta \to \infty$$

Shooting Method



Shooting Method





Height Of Aquifer Over Time

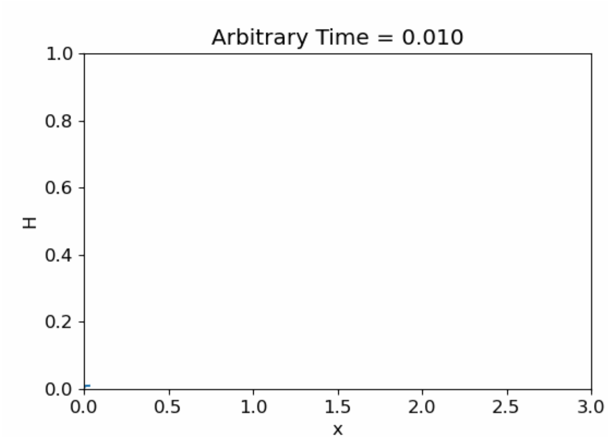


Figure: Animated sequence using the animate package.

Calculating Flux

Rainfall:

$$\hat{Q}_0 = -\hat{t} f(0) f'(0) = -\frac{C^2 \hat{t}}{2}$$

Drought:

$$\hat{Q}_0 = -\frac{X(0)X'(0)}{(\hat{t}+A)^2} = -\frac{D^2}{2(\hat{t}+A)^2}$$

C and D solved from numerical solution:

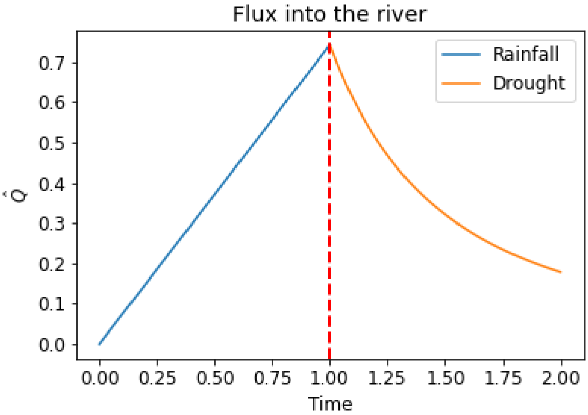
C ≈ 1.22

D ≈ 1.17

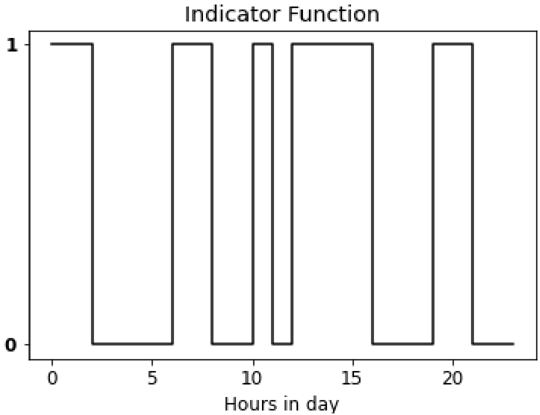
Find A:

$$A = \sqrt{\left(\frac{2D^2}{C^2 t^*}\right)} - t^* \qquad t^* = \text{Transition Time}$$

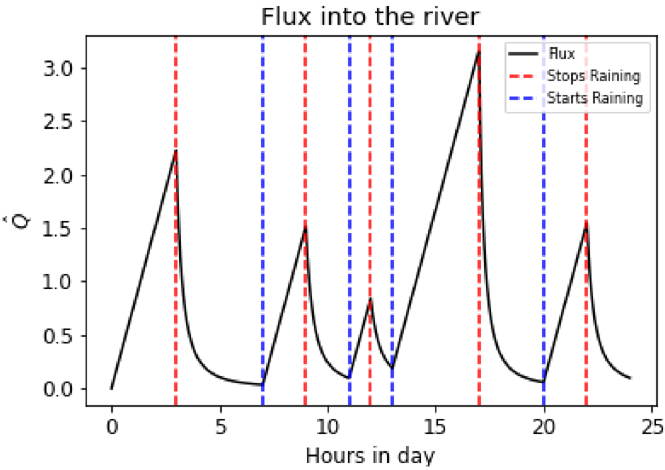
Flux


Flux into the river
Rainfall
Drought
$\hat{Q}$
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
Time

Indicator Function

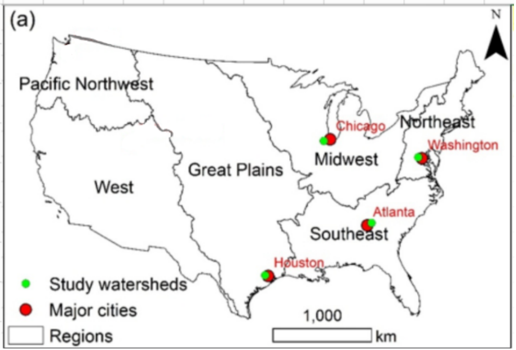


Flux Periodically

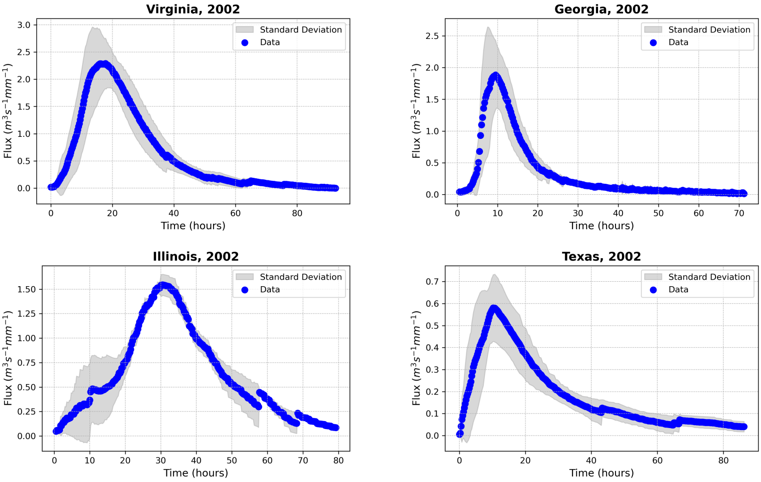


Data

We sourced data of 'Unit hydrographs of evolving urban watersheds across the United States' from the United States Geological Survey, allowing us to model hydrographs in a range of urban US regions.



Data



Virginia, 2002
Georgia, 2002
Illinois, 2002
Texas, 2002
Standard Deviation
Data
Flux ($m^3 s^{-1} mm^{-1}$)
Time (hours)
0.0 0.5 1.0 1.5 2.0 2.5 3.0
0 20 40 60 80
0.0 0.5 1.0 1.5 2.0 2.5
0 10 20 30 40 50 60 70
0.00 0.25 0.50 0.75 1.00 1.25 1.50
0 10 20 30 40 50 60 70 80
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
0 20 40 60 80

Re-Dimensionalize Flux

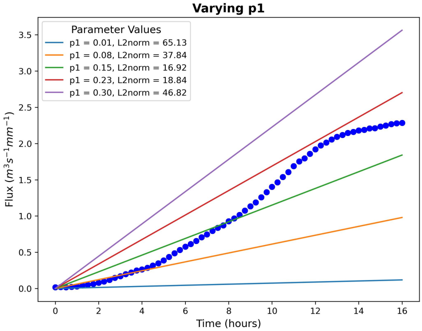
Rainfall:

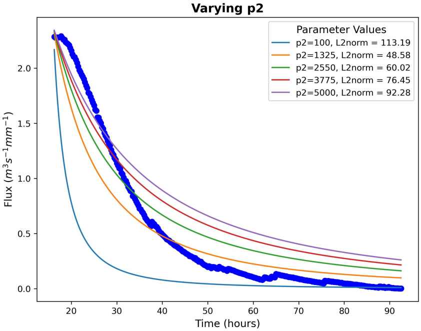
$$Q(t) = -\underbrace{\left(\frac{R^{\frac{3}{2}}K^{\frac{1}{2}}}{\phi}\right)}_{\text{Parameter 1}}\left(\frac{C^2 t}{2}\right)$$

Drought:

$$Q(t) = -\underbrace{\left(\frac{\phi^2 L^3}{K}\right)}_{\text{Parameter 2}}\left(\frac{D^2}{2(t+\theta)}\right)$$

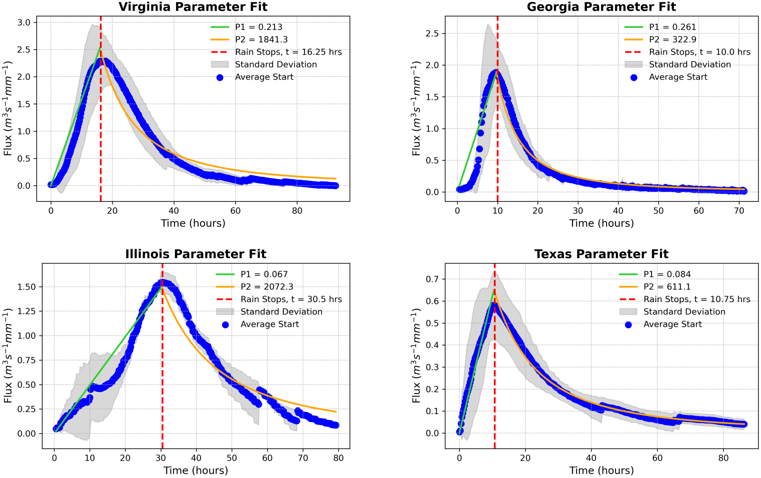
Parameter Search





$$min_{p_1,p_2}\sqrt{\sum_{i=1}^{n}|Q(t)-Data|^2}$$

Data Fitted


Virginia Parameter Fit
P1 = 0.213
P2 = 1841.3
Rain Stops, t = 16.25 hrs
Standard Deviation
Average Start
Flux ($m^3s^{-1}mm^{-1}$)
Time (hours)
Georgia Parameter Fit
P1 = 0.261
P2 = 322.9
Rain Stops, t = 10.0 hrs
Standard Deviation
Average Start
Flux ($m^3s^{-1}mm^{-1}$)
Time (hours)
Illinois Parameter Fit
P1 = 0.067
P2 = 2072.3
Rain Stops, t = 30.5 hrs
Standard Deviation
Average Start
Flux ($m^3s^{-1}mm^{-1}$)
Time (hours)
Texas Parameter Fit
P1 = 0.084
P2 = 611.1
Rain Stops, t = 10.75 hrs
Standard Deviation
Average Start
Flux ($m^3s^{-1}mm^{-1}$)
Time (hours)

$$\text{Parameter 1} = \frac{R^{\frac{3}{2}} K^{\frac{1}{2}}}{\phi} \qquad \text{Parameter 2} = \frac{\phi^2 L^3}{K}$$

	Parameter 1	Parameter 2
Virginia	0.213	1841.3
Georgia	0.261	322.9
Illinois	0.067	2027.3
Texas	0.084	611.1

Table: Parameter values in each region.

Conclusions

1. Solved for height of aquifer during Drought and Rainfall
2. Solved for flux into a river
3. Fit our parameters to real life data

References

Khand, K.B., and Senay, G.B., 2022, Unit hydrographs of evolving urban watersheds across the United States, *U.S. Geological Survey data release,* https://doi.org/10.5066/P91X5I6L.

Raka Mondal, Graham Benham, Sourav Mondal, Paul Christodoulides, Natasa Neokleous, and Katerina Kaouri. Modelling and optimisation of water management in sloping coastal aquifers with seepage, extraction and recharge, *Journal of Hydrology,* 571:471–484, 2019.