A Universal Performance Measure

A Study by Simulation

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Outline

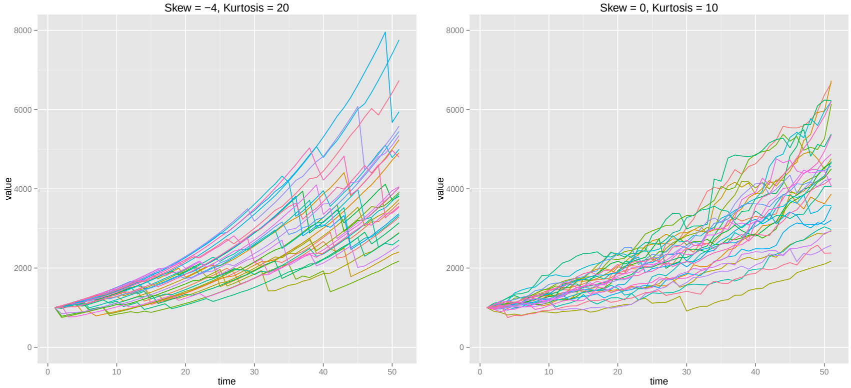


Figure : Sample Dollar Value Paths for Distributions with Mean 3 and Sigma 5

Omega (also refered to as Gamma in the early literature)[?] is a performance measure defined as

$$\Omega(L) = \frac{\int_L^b (1 - F(x))dx}{\int_a^L F(x)dx}$$

where L denotes the level at which we differentiate between a loss and a gain

Pros

- No parametric assumptions
- Invariant under linear transformation:

$$\varphi(x) = ax + b$$
$$\Omega(\varphi(L)) = \Omega(L) \text{ if } a > 0$$
$$\Omega(\varphi(L)) = \frac{1}{\Omega(L)} \text{ if } a < 0$$

- $\frac{d\Omega}{dL} < 0$ everywhere and is as smooth as $F(L)$
- Nice economic intuition, one can adjust the L depending on the state of macro-economy
- Sub-additive

Cons

- Ω takes a value of 1 when $L = \mu$, i.e. does not distinguish between distribution with same mean at the mean point
- Possibly unbounded given an infinite interval
- Carries downside-type characteristic, so difficult to use if no sample returns are below the threshold

The Johnson family of distributions[?] are a set of densities where the first four central moments can be specified given appropriate choice of parameters. They consists of continuous random variables z such that when appropriately transformed become standard normal[?], i.e.

$$y = a + b \times g\left(\frac{z - c}{d}\right), \quad y \sim N(0, 1) \tag{1}$$

Where a and b in Equation (1) are shape parameters, c is a location parameter, d is a scale parameter and $g(\cdot)$ is one of the following four functions

$$g(u) = \begin{cases} \ln(u) & \text{lognormal family} \\ \ln(u + \sqrt{u^2 + 1}) & \text{unbounded family} \\ \ln\left(\frac{u}{1 - u}\right) & \text{bounded family} \\ u & \text{normal family} \end{cases} \tag{2}$$

Given the first for moments, **?** discussed an algorithm for determining the associated Johnson family parameters. Denoting S_L, S_U and S_B as the lognormal, unbounded and bounded cases of Equation (2) respectively, the alorithm is as follows:

Letting $\sqrt{\beta_1} = \frac{\mu_3}{\sigma^3}$, $\beta_2 = \frac{\mu_4}{\sigma^4}$ and $\omega = e^{-b^2}$, first solve

$$(\omega - 1)(\omega + 2)^2 = \beta_1$$

Then

$$\beta_2 < \omega^4 + 2\omega^3 + 3\omega^2 - 3 \Longrightarrow g(\cdot) = S_B$$
$$\beta_2 > \omega^4 + 2\omega^3 + 3\omega^2 - 3 \Longrightarrow g(\cdot) = S_U$$
$$\beta_2 = \omega^4 + 2\omega^3 + 3\omega^2 - 3 \Longrightarrow g(\cdot) = S_L$$

For S_L

$$b = \ln(\omega)^{-\frac{1}{2}} \qquad a = \frac{1}{2} b \times \ln\left(\frac{\omega(\omega-1)}{\sigma}\right)$$

$$c = sign(\mu_3) \cdot \mu - e^{\frac{\frac{1}{2}b - a}{b}} \qquad d = sign(\mu_3)$$

For S_U

If $\beta_1 = 0$

$$\omega = [(2\beta_2 - 2)^{\frac{1}{2}} - 1]^{\frac{1}{2}}, \quad b = (\ln \omega)^{-\frac{1}{2}}, \quad a = 0$$

If $\beta_1 \neq 0$

$$\omega_1 = [(2\beta_2 - 2.8\beta_1 - 1)^{\frac{1}{2}} - 2]^{\frac{1}{2}}$$

as an initial estimate and ω, a and b are found using the **?** iterative method

Then c and d are found using

$$\sigma^2 = \frac{1}{2}d^2(\omega - 1)(\omega cosh\left(\frac{2a}{b}\right) + 1)$$
$$\mu = c - d\omega^{\frac{1}{2}} sinh\left(\frac{a}{b}\right)$$

For S_B

$$b = \frac{0.626\beta_2 - 0.408}{(3-\beta_2)^{0.479}} \qquad \text{if } \beta_2 \geq 1.8$$
$$b = 0.8(\beta_2 - 1) \qquad \text{otherwise}$$

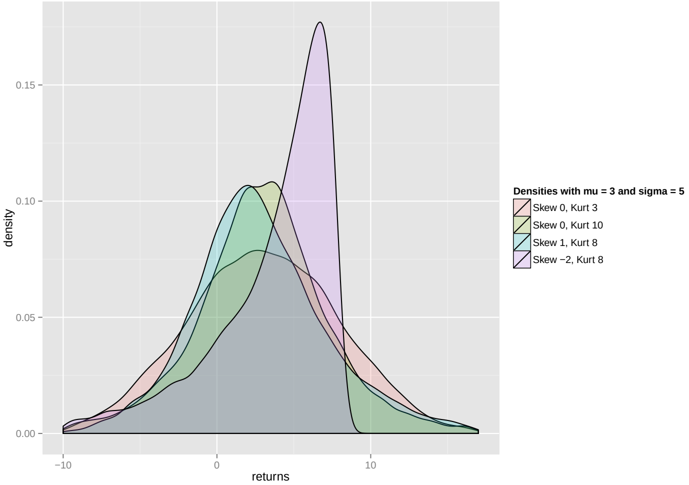
Using **?**, a is calculated. From initial estimates of a and b the first 6 moments are calculated using **?** and then Newton-Rhapson is used to solve for a and b, and the first two moments are then used to determine c and d

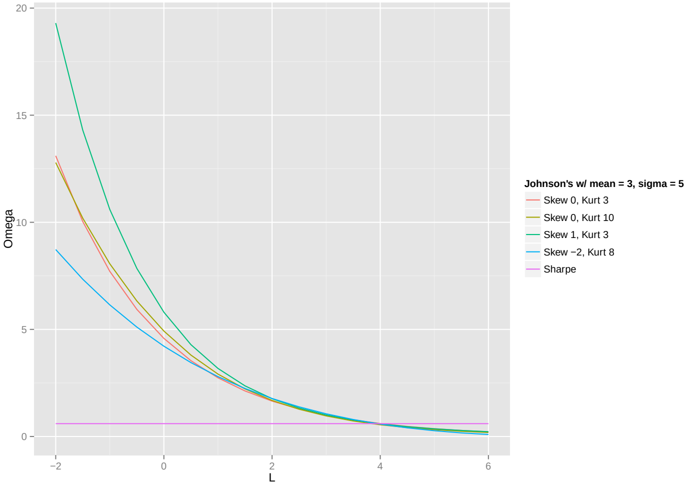
Then we can simulate from the Johnson random variable using

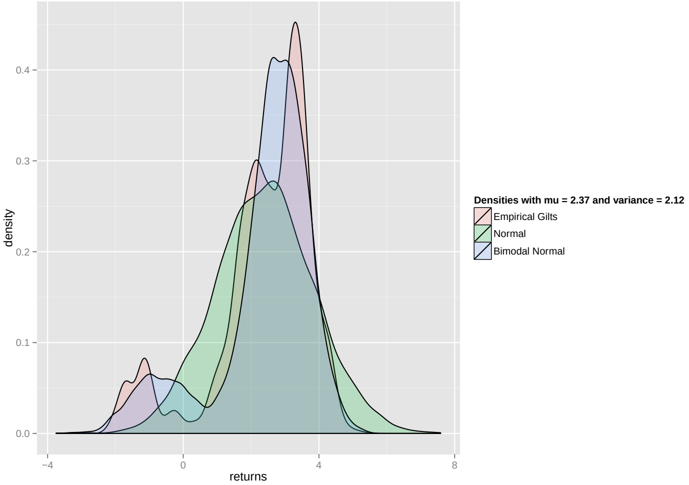
$$z = c + d \times g^{-1}\left(\frac{y-a}{b}\right) \tag{3}$$

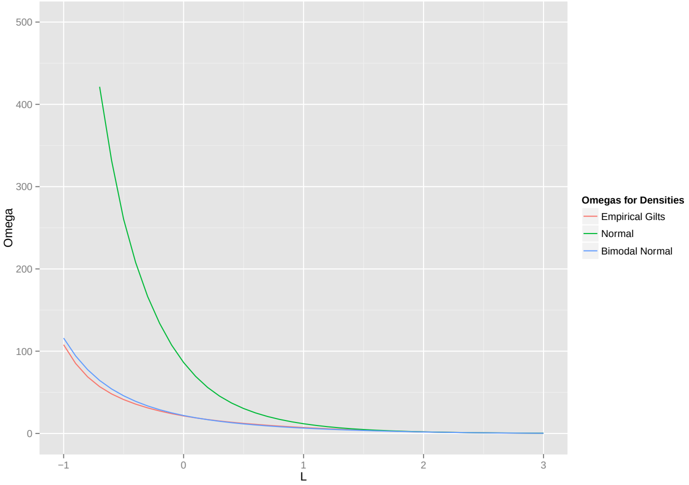
with

$$g^{-1}(u) = \begin{cases} e^{u} & \text{lognormal family} \\ (e^{u} - e^{-u})/2 & \text{unbounded family} \\ 1/(1+e^{-u}) & \text{bounded family} \\ u & \text{normal family} \end{cases} \tag{4}$$


0.15
0.10
0.05
0.00
density
−10
0
10
returns
Densities with mu = 3 and sigma = 5
Skew 0, Kurt 3
Skew 0, Kurt 10
Skew 1, Kurt 8
Skew −2, Kurt 8


Omega
20
15
10
5
0
−2
0
2
4
6
L
Johnson's w/ mean = 3, sigma = 5
Skew 0, Kurt 3
Skew 0, Kurt 10
Skew 1, Kurt 3
Skew −2, Kurt 8
Sharpe


0.4
0.3
0.2
0.1
0.0
density
−4
0
4
8
returns
Densities with mu = 2.37 and variance = 2.12
Empirical Gilts
Normal
Bimodal Normal


Omega
500
400
300
200
100
0
−1
0
1
2
3
L
Omegas for Densities
Empirical Gilts
Normal
Bimodal Normal

- The Omega performance measure provides a useful alternative to a Sharpe performance measure since it is well known financial returns are not normal and does not require a specific choice of utility
- Investigate real world performance of hedge fund data using Sharpe and Omega
- Determine the effects of sampling frequency on the estimation error of Omega