University of Waterloo

Computer Intensive Methods for Stochastic Models in Finance

Statistics 906

A Universal Performance Measure: A Study by Simulation

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1. Introduction

Typical performance measures are based on the over-simplification that mean and variance fully capture the distribution of portfolio returns. One such measure is the much celebrated Sharpe ratio. The paper of interest, "A Universal Performance Measure" by Shadwick and Keating (2002), introduces a new performance measure named Ω and applies it to simulated returns and hedge fund returns. In our implementation we test this new measure on both simulated returns and empirical observations. We simulate returns based off the family of distributions proposed by Johnson (1949). The family of distributions, referred to as the Johnson family, can be transformed into standard normal distributions and has the property that the first four moments can be specified given an appropriate choice of parameters. An algorithm by Hill et al. (1976) was developed to infer the parameters for the Johnson family distributions given the first four moments. With these results, we generate different Johnson family distributions specifying the first four moments and compare the Ω measures of the returns generated by independent samples from these distributions.

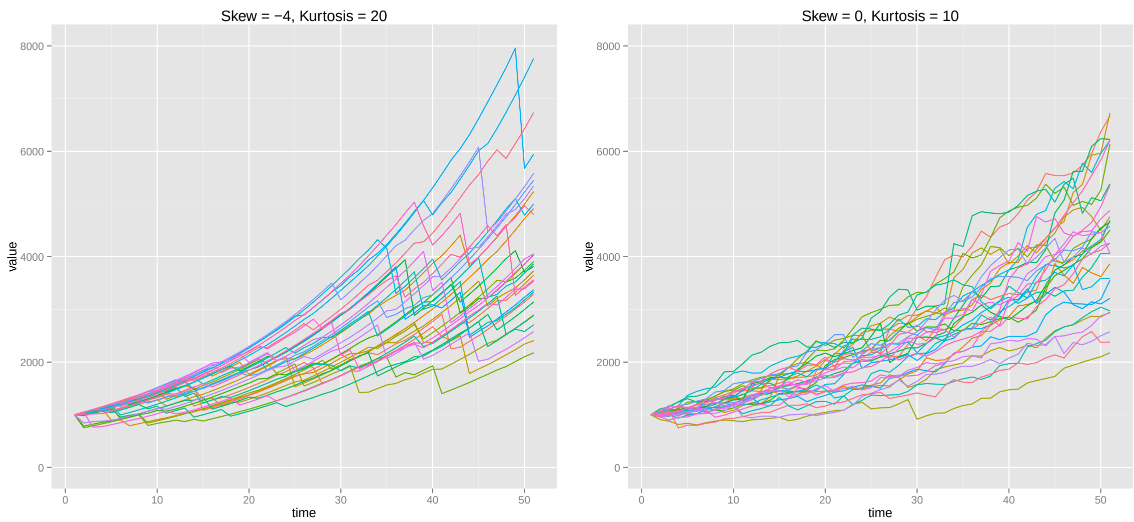

Figure 1: Dollar value of portfolio under different return distributions

In Figure 2, the simulated paths on the left and right graphs are generated from two different distributions that have the same mean and variance, but different skewness and kurtosis. They have distinctly different return profiles. The return paths on the left have frequent positive returns, but large losses, sometimes more than half of portfolio values. This is not a risk a typical pensioner should take on their pension investments. However, Sharpe ratio does not distinguish between the two portfolio return profiles. There is a necessecity for a new measure that captures the higher moments in portfolio returns such as skewness and kurtosis.

Suppose a distribution is defined on the interval [a,b]. Let F(x) be its cumulative distribution function for the distribution of interest. Then for any given loss threshold L, the $\Omega(\cdot)$ of the distribution is defined as:

$$\Omega(L) = \frac{\int_L^b [1 - F(r)]\mathrm{d}r}{\int_a^L F(r)\mathrm{d}r} \tag{1}$$

This definition makes no parametric assumption and is applicable to almost all distributions. It is also highly intuitive because the loss threshold L can be adjusted to fit the current investment conditions. During economic boom, L can be set to the market return to assess an investment's return against the market. During economic downturn, L can be set to 0 to determine an investment's ability to limit loss and preserve capital. Unlike some risk measures, such as value-at-risk, Ω is also subadditive. In addition the Ω measure is invariant under linear transformation of the underlying random variable X:

$$\begin{aligned} \varphi(X) &= aX + b \\ \Omega(\varphi(L)) &= \Omega(L) \text{ if } a > 0 \\ \Omega(\varphi(L)) &= \frac{1}{\Omega(L)} \text{ if } a < 0 \end{aligned} \tag{2}$$

In order to illustrate the ability of Ω measure in capturing higher moments, we use the set of Johnson distributions to find distributions with the same mean and variance, but different skewness and kurtosis. The Johnson family of distributions is a set of distributions where the first four moments can be specified given appropriate choice of parameters. The family consists of random variables z such that z becomes normal when it is appropriately transformed.

$$y = a + b \times g(\frac{z-c}{d}), \quad y \sim N(0,1) \tag{3}$$

Where a and b are shape parameters, c is a location parameter, d is a scale parameter and g is one of the following four functions:

$$g(u) = \begin{cases} \ln(u) & \text{lognormal family} \\ \ln(u + \sqrt{u^2+1}) & \text{unbounded family} \\ \ln\big(\frac{u}{1-u}\big) & \text{bounded family} \\ u & \text{normal family} \end{cases} \tag{4}$$

Given the first four moments, Hill et al. (1976) discussed an algorithm for determining the associated Johnson family parameters. Denoting S_L, S_U and S_B as the lognormal, unbounded and bounded cases of Equation (3) respectively, the algorithm is as follows:

Letting $\sqrt{\beta_1} = \frac{\mu_3}{\sigma^3}$, $\beta_2 = \frac{\mu_4}{\sigma^4}$ and $\omega = e^{-b^2}$, first solve

$$(\omega - 1)(\omega + 2)^2 = \beta_1$$

Then

$$\begin{aligned}
\beta_2 &< \omega^4 + 2\omega^3 + 3\omega^2 - 3 \Longrightarrow g(\cdot) = S_B \\
\beta_2 &> \omega^4 + 2\omega^3 + 3\omega^2 - 3 \Longrightarrow g(\cdot) = S_U \\
\beta_2 &= \omega^4 + 2\omega^3 + 3\omega^2 - 3 \Longrightarrow g(\cdot) = S_L
\end{aligned}$$

<u>For S_L</u>

$$\begin{aligned}
b &= \ln(\omega)^{-\frac{1}{2}} & a &= \frac{1}{2}b \times \ln\left(\frac{\omega(\omega-1)}{\sigma}\right) \\
c &= sign(\mu_3) \cdot \mu - e^{\frac{\frac{1}{2}b - a}{b}} & d &= sign(\mu_3)
\end{aligned}$$

<u>For S_U</u>

If $\beta_1 = 0$

$$\omega = [(2\beta_2 - 2)^{\frac{1}{2}} - 1]^{\frac{1}{2}}, \quad b = (\ln \omega)^{-\frac{1}{2}}, \quad a = 0$$

If $\beta_1 \neq 0$

$$\omega_1 = [(2\beta_2 - 2.8\beta_1 - 1)^{\frac{1}{2}} - 2]^{\frac{1}{2}}$$

as an initial estimate and ω, a and b are found using the Elderton and Johnson (1969) iterative method

Then c and d are found using

$$\begin{aligned}
\sigma^2 &= \frac{1}{2}d^2(\omega - 1)(\omega cosh\left(\frac{2a}{b}\right) + 1) \\
\mu &= c - d\omega^{\frac{1}{2}} sinh\left(\frac{a}{b}\right)
\end{aligned}$$

<u>For S_B</u>

$$\begin{aligned}
b &= \frac{0.626\beta_2 - 0.408}{(3 - \beta_2)^{0.479}} & &\text{if } \beta_2 \geq 1.8 \\
b &= 0.8(\beta_2 - 1) & &\text{otherwise}
\end{aligned}$$

Using Draper (1951), a is calculated. From initial estimates of a and b the first 6 moments are calculated using Draper (1952) and then Newton-Rhapson is used to solve for a and b, and the first two moments are then used to determine c and d.

Then we can simulate from the Johnson random variable using

$$z = c + d \times g^{-1}\left(\frac{y-a}{b}\right) \tag{5}$$

with

$$g^{-1}(u) = \begin{cases} e^u & \text{lognormal family} \\ (e^u - e^{-u})/2 & \text{unbounded family} \\ 1/(1+e^{-u}) & \text{bounded family} \\ u & \text{normal family} \end{cases} \tag{6}$$

2. Results and Discussion

2.1. Simulations

To analyze the effects of upper moments on Ω we simulated returns from Johnson distributions with constrained mean and variance. The empirical return distributions are shown in Figure 2. Code for generating these figures in included in the Appendix, which makes use of the R package contributed by McLeod and King (2012).

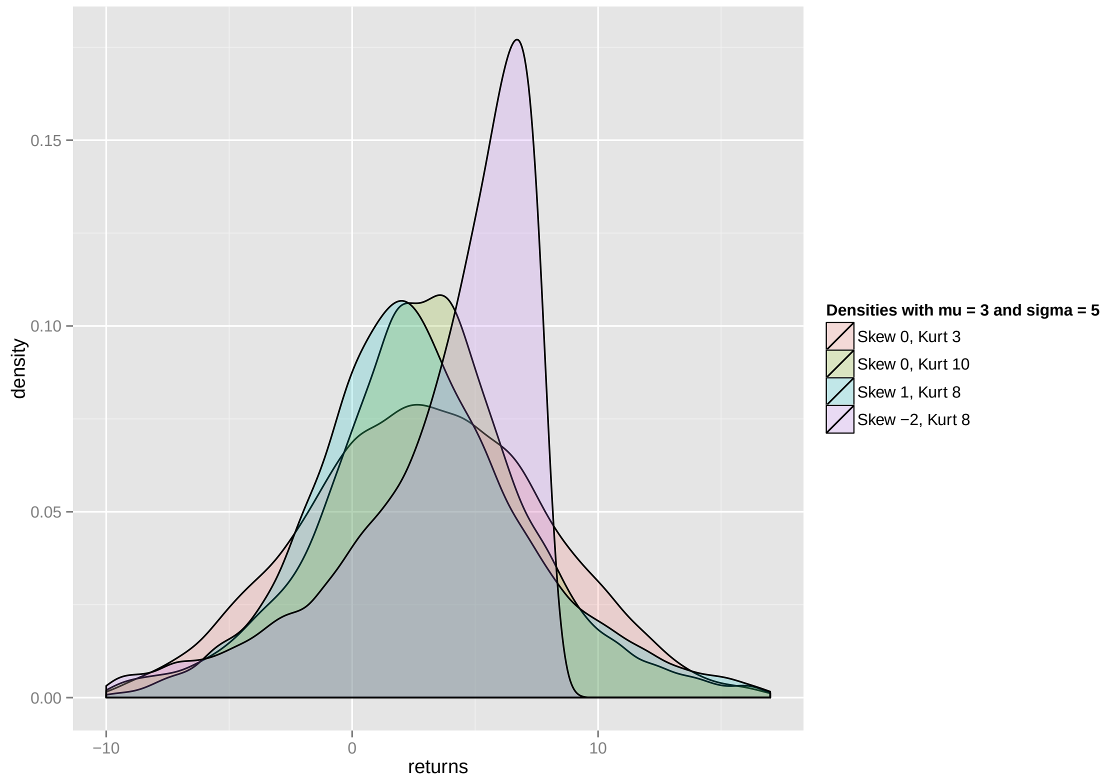

Figure 2: Empirical densities for returns generated from Johnson distributions

The Ω values for various loss levels for these return distributions are shown in Figures 3 and 4. As seen in Figure 3, Ω measure differentiates these four different distributions, particularly for loss threshold L smaller than the mean μ. Skewness plays a larger role in differentiating the Ω measure and higher positive skewness is preferred. Though kurtosis' role is small compared with skewness, larger kurtosis seems to be preferred when skewness is 0 for low levels of L while less kurtosis is preferred for higher values of L.

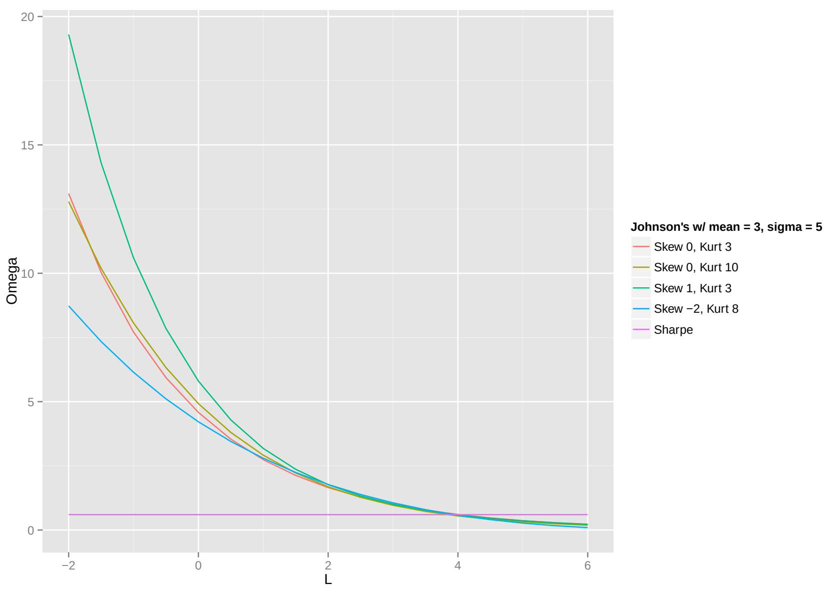

Figure 3: Omegas for the simulated Johnson returns

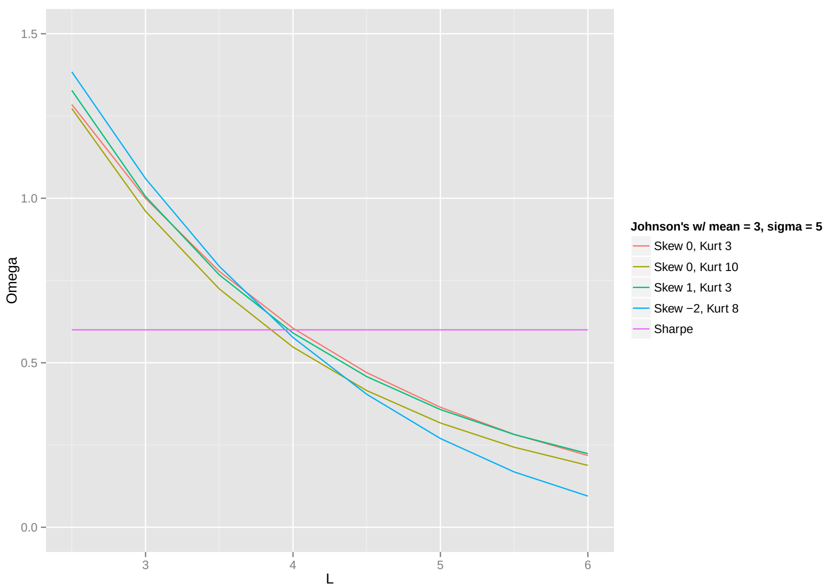

Figure 4: Omegas for the simulated Johnson returns

2.2. Empirical

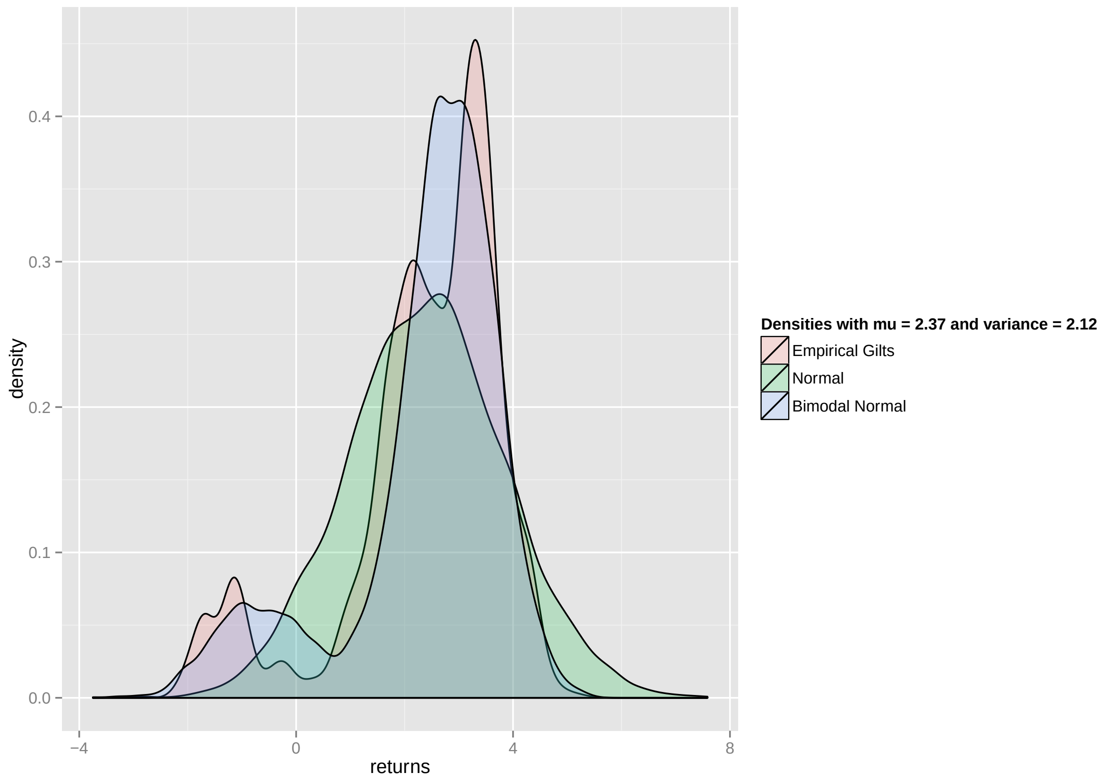

Figure 5: Densities for empirical Gilt returns and normal and bimodal returns with equivalent first two central moments

Many assets do not follow normal distributions and may possess higher moments that Sharpe ratio cannot capture. The British treasury bonds, called gilts, are one example. We obtained 25 years of daily returns of 1-year-maturity inflation-indexed gilts from Bank of England. The pink region in Figure 5 demonstrates the empirical density. It is highly non-Gaussian and appears to have two peaks - implying a bimodal density structure. We attempt to model this empirical distribution. A normal distribution, shown as the blue region, is first fitted. A mixture normal distribution, which is a linear combination of two normal densities, is also fitted and shown as the green region. By construction of fitted normal and mixture normal distributions, all three distributions have the same mean of 2.37 and same variance of 2.12. Therefore, they cannot be distinguished by the Sharpe ratio. It is apparent that the mixture normal distribution models the empirical data much better than the normal distribution.

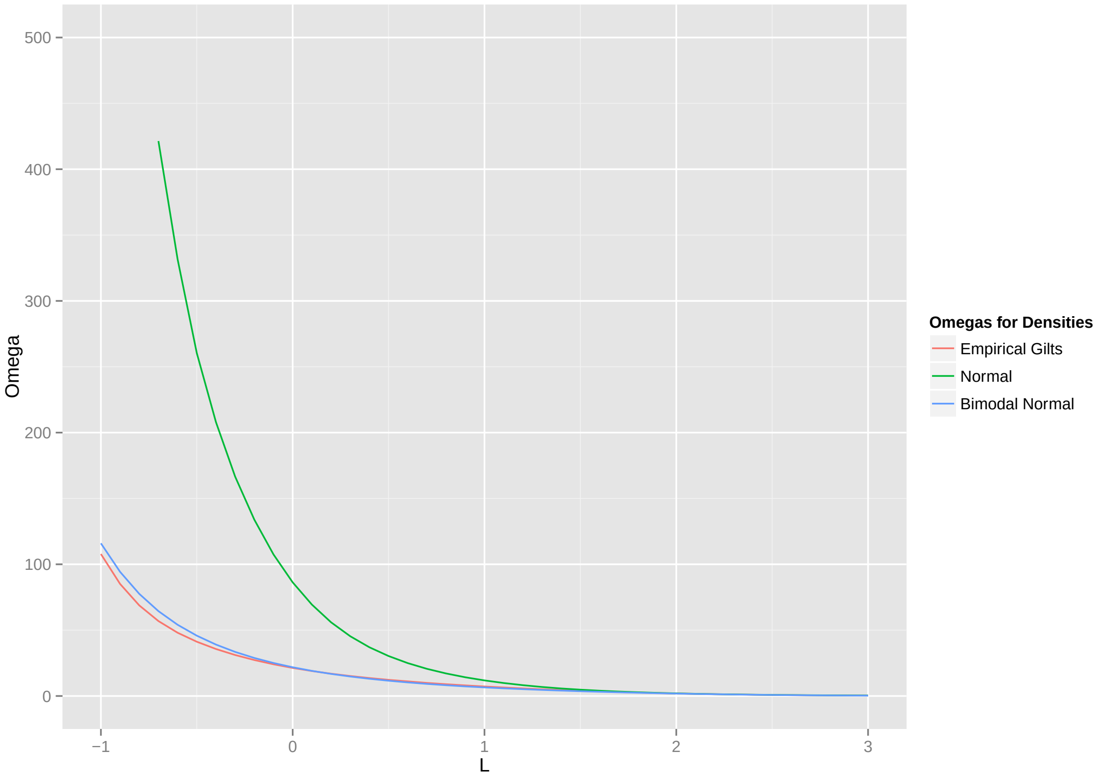

Figure 6: Omegas for empirical Gilt returns and normal and bimodal returns with equivalent first two central moments

The Ω measure is applied to the empirical data itself as well as simulations generated from the fitted normal and mixture normal distributions. The results are shown in Figure 6. Ω measure is able to capture the differences in higher moments between the three distributions. As seen in Figure 6, normal distribution fits the data poorly as the Ω measure of the normal distribution is far from the empirical observation. Mixture normal distribution fits the empirical data much closer. As indicated by theory, the Ω measures tend to one as the loss threshold L tends to the mean 2.37. It is interesting to note that the mixture normal distribution is fitted to have the same first five moments as the empirical data. The slight difference in Ω measures between the mixture normal distribution and the empirical data indicates the Ω's ability to detect differences in even higher moments than the 5th moment.

3. Summary

Ω measure, unlike Sharpe ratio, is designed to capture differences in higher moments between distributions. Its capability has been shown in both simulated and empirical settings. In addition, it has some attractive features, such as non-parametric definition and the flexibility of the loss threshold. Non-parametric definition removes estimation errors and allows quick estimation of the Ω measure even for complex portfolios - using the empirical quantiles as the CDF. Flexibility of the loss threshold allows the investor to choose a threshold that best fits the investor's risk apetite.

The Ω measure is not without drawbacks. It is possibly unbounded for a distribution defined on an infinite interval. It carries similar estimation issues as other risk measures that focus on the tail. This means Ω measure does not make sense unless we observe sufficient amount data below or ablove the risk threshold L. In addition, when the L is set to the mean of the distribution, i.e. L $= \mu$, Ω is equal to 1 and therefore does not distinguish any distributions with the same mean.

Despite its drawbacks, Ω measure still provides a useful alternative to the Sharpe ratio, particularly for complex and non-Gaussian portfolios. This measure extends naturally to hedge funds, who by nature take large bets in unconventional and complex strategies. It would be of particular interest to look at how rankings of hedge fund returns would change using the Ω measure instead of the Sharpe ratio. It is also clear that the Ω measure becomes more stable as more empirical data becomes available. It is thus interesting to examine its convergence rate with respect to the amount of data. In addition, volatility estimates are known to become more accurate when the sampling frequency is increased. Whether Ω measure has this property remains to be tested. These areas pave a way to new areas of research into the Ω measure and further tests its applicability in practice.

References

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Draper, J. (1952). Properties distributions of resulting from certain simple transformations the normal of distribution. *Biometrika*, 39:290–301.

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Shadwick, W. and Keating, C. (2002). Performance measurement. *Journal of Performance Measurement.*

A.

```r
#function to calculate the performance metric Omega
#inputs:
#    -a realization of returns
#    -L (can be a vector), which specifies the point at which we define losses, e.g. 0,
     risk free rate or some benchmark
Omega <- function(returns,L) {

    library(pracma)
    sudoCDF = ecdf(returns)
    a = min(returns)
    b = max(returns)

    Omega = matrix(nrow=length(L),ncol=2)
    colnames(Omega) <- c('L','Omega')
    Omega[,1] = L
    j = 1
    for (i in L) {
        Omega[j,2] = ( (b - i) - quad(sudoCDF,i,b) ) / quad(sudoCDF,a,i)
        j = j+1
    }

    return(Omega)
}
```

Omega.r

```r
# R wrapper for Generating Johnson Distributions with desired first 4 moments
#Inputs:
#    First 4 moments and the desired number of simulations
JohnsonGenerate <- function(mu,sigma,skew,kurt, N) {

    library(JohnsonDistribution)

    #calculate the johnson family parameters
    params <- FitJohnsonDistribution(mu, sigma, skew, kurt)
    ITYPE <- params[1]
    GAMMA <- params[2]
    DELTA <- params[3]
    XLAM <- params[4]
    XI <- params[5]
    IFAULT <- params[6]

    if (ITYPE == 5)
        stop("Invalid parameters lead to unsupported distribution boundary case (5)")
    if (IFAULT != 0)
        stop("WARNING: Error exit, JNSN. IFAULT !=  0")

    output <- yJohnsonDistribution(rnorm(N), ITYPE, GAMMA, DELTA, XLAM, XI)
}
```

JohnsonGenerate.r

```
##############################################
### Script to Compare Various Distributions ###
##############################################
set.seed(1989)
source("JohnsonGenerate.r")
source("Omega.r")
source("multiplot.r")
library('ggplot2')
library('reshape')
N = 10000


#first two moments set constant
MU = 3
SIGMA = 5
#try the params mu = 2, std = 1.6, skew = +/- 0.398, kurt = 3.84
#try normal(3,2.76) and mu = 3, std = 2.76, skew = 0, kurt = 9.6
#normal from johnson
z1 <- JohnsonGenerate(MU,SIGMA,0,3,N)
#excel kurtosis
z2 <- JohnsonGenerate(MU,SIGMA,0,10,N)
#some skewness
z3a <- JohnsonGenerate(MU,SIGMA,1,8,N)
z3b <- JohnsonGenerate(MU,SIGMA,-2,8,N)

#plot densities of the distributions
densities = data.frame(z1,z2,z3a,z3b)
colnames(densities) <- c("Skew 0, Kurt 3", "Skew 0, Kurt 10", "Skew 1, Kurt 8", "Skew
    -2, Kurt 8")
densities = melt(densities, measure = c("Skew 0, Kurt 3", "Skew 0, Kurt 10", "Skew 1,
    Kurt 8", "Skew -2, Kurt 8"))
colnames(densities) <- c("variable","returns")
pdf("plots/Densities.pdf", width=10)
qplot(returns, data=densities, geom="density", fill = variable, alpha = I(0.2), xlim = c
    (-10,17)) + labs(fill='Densities with mu = 3 and sigma = 5')
dev.off()

#plot DVA of 1000$ THIS NEEDS WORK IT IS BLOWING UP
initWealth = 1000
M = 50
paths = 30
z1Paths = matrix(nrow = M+1, ncol = paths)
z2Paths = matrix(nrow = M+1, ncol = paths)
z3aPaths = matrix(nrow = M+1, ncol = paths)
z3bPaths = matrix(nrow = M+1, ncol = paths)
z4Paths = matrix(nrow = M+1, ncol = paths)
z5Paths = matrix(nrow = M+1, ncol = paths)

z1Paths[1,] = z2Paths[1,] = z3aPaths[1,] = z3bPaths[1,] = z4Paths[1,] = z5Paths[1,] =
    initWealth
for (i in 1:paths) {
    z1Paths[-1,i] = initWealth*cumprod((JohnsonGenerate(3,5,0,3,M) + 100)/100)
```

```
    z2Paths[-1,i] = initWealth*cumprod((JohnsonGenerate(3,5,0,10,M) + 100)/100)
    z3aPaths[-1,i] = initWealth*cumprod((JohnsonGenerate(3,5,1,8,M) + 100)/100)
    z3bPaths[-1,i] = initWealth*cumprod((JohnsonGenerate(3,5,-2,8,M) + 100)/100)
    z4Paths[-1,i] = initWealth*cumprod((JohnsonGenerate(3,5,-4,20,M) + 100)/100)
    z5Paths[-1,i] = initWealth*cumprod((JohnsonGenerate(3,5,0,10,M) + 100)/100)
}
time = 1:nrow(z1Paths)
z1Paths = data.frame(time,z1Paths)
z1Paths = melt(z1Paths, id = "time")#, measure = c("X1", "X2", "X3", "X4", "X5","X6","X7
    ","X8","X9","X10"))
z2Paths = data.frame(time,z2Paths)
z2Paths = melt(z2Paths, id = "time")#, measure = c("X1", "X2", "X3", "X4", "X5","X6","X7
    ","X8","X9","X10"))
z3aPaths = data.frame(time,z3aPaths)
z3aPaths = melt(z3aPaths, id = "time")#, measure = c("X1", "X2", "X3", "X4", "X5","X6","
    X7","X8","X9","X10"))
z3bPaths = data.frame(time,z3bPaths)
z3bPaths = melt(z3bPaths, id = "time")#, measure = c("X1", "X2", "X3", "X4", "X5","X6","
    X7","X8","X9","X10"))
z4Paths = data.frame(time,z4Paths)
z4Paths = melt(z4Paths, id = "time")
z5Paths = data.frame(time,z5Paths)
z5Paths = melt(z5Paths, id = "time")

#set ylim so it is the max of all paths
p1 <- qplot(time, value, data = z1Paths, geom = "line", colour = variable, ylim = c
    (0,8000)) + theme(legend.position = "none") + ggtitle("Skew = 0, Kurtosis = 3")
p2 <- qplot(time, value, data = z2Paths, geom = "line", colour = variable, ylim = c
    (0,8000)) + theme(legend.position = "none") + ggtitle("Skew = 0, Kurtosis = 10")
p3 <- qplot(time, value, data = z3aPaths, geom = "line", colour = variable, ylim = c
    (0,8000)) + theme(legend.position = "none") + ggtitle("Skew = 1, Kurtosis = 8")
p4 <- qplot(time, value, data = z3bPaths, geom = "line", colour = variable, ylim = c
    (0,8000)) + theme(legend.position = "none") + ggtitle("Skew = -2, Kurtosis = 8")
p5 <- qplot(time, value, data = z4Paths, geom = "line", colour = variable, ylim = c
    (0,8000)) + theme(legend.position = "none") + ggtitle("Skew = -4, Kurtosis = 20")
p6 <- qplot(time, value, data = z5Paths, geom = "line", colour = variable, ylim = c
    (0,8000)) + theme(legend.position = "none") + ggtitle("Skew = 0, Kurtosis = 10")
pdf("plots/DVApaths.pdf")
multiplot(p1,p2,p3,p4, cols=2)
dev.off()

pdf("plots/DVApaths2.pdf",width=15)
multiplot(p5,p6, cols=2)
dev.off()

#investigate omegas for the above distributions
L = seq(-2,6,0.5)
OmegaZ1 = Omega(z1,L)
OmegaZ2 = Omega(z2,L)
OmegaZ3a = Omega(z3a,L)
OmegaZ3b = Omega(z3b,L)
sharpe = cbind(L, 3/5)
```

```
omegas = data.frame(OmegaZ1,OmegaZ2[,2],OmegaZ3a[,2],OmegaZ3b[,2],sharpe[,2])
colnames(omegas) <- c("L","Skew 0, Kurt 3","Skew 0, Kurt 10","Skew 1, Kurt 3","Skew -2,
    Kurt 8","Sharpe")
omegas = melt(omegas, measure = c("Skew 0, Kurt 3","Skew 0, Kurt 10","Skew 1, Kurt 3","
    Skew -2, Kurt 8","Sharpe"))
colnames(omegas) <- c("L","variable","Omega")
pdf("plots/Preferences.pdf", width=10)
qplot(x=L, y=Omega, data=omegas, geom="line", colour = variable) + labs(colour="Johnson'
    s w/ mean = 3, sigma = 5")
dev.off()

pdf("plots/PreferencesZoom.pdf", width=10)
qplot(x=L, y=Omega, data=omegas, geom="line", colour = variable) + labs(colour="Johnson'
    s w/ mean = 3, sigma = 5") + xlim(2.5,6) + ylim(0,1.5)
dev.off()


#pdf("Preferences.pdf")
#ggplot(omegas, aes(L)) +
#geom_line(aes(y = Omega, colour = "Skew 0, Kurt 3")) +
#geom_line(aes(y = OmegaZ2, colour = "Skew 0, Kurt 10")) +
#geom_line(aes(y = OmegaZ3a, colour = "Skew 1, Kurt 3")) +
#geom_line(aes(y = OmegaZ3b, colour = "Skew -2, Kurt 8")) +
#geom_line(aes(y = sharpe, colour = "sharpe ratio")) +
#scale_colour_hue("3rd and 4th Moments")
#dev.off()

#analysis of empirical data
load('giltData.dat')
giltsDf = data.frame(gilt,gilt.rand.normal,gilt.rand.bimodal)
colnames(giltsDf) <- c("Empirical Gilts", "Normal", "Bimodal Normal")
giltDensities = melt(giltsDf, measure = c("Empirical Gilts", "Normal", "Bimodal Normal")
    )
colnames(giltDensities) <- c("variable","returns")
pdf("plots/EmpiricalDensities.pdf",width=10)
qplot(returns, data=giltDensities, geom="density", fill = variable, alpha = I(0.2)) +
    labs(fill='Densities with mu = 2.37 and variance = 2.12')
dev.off()

L = seq(-1,3,0.1)
sharpe = cbind(L, 3/5)
OmegaGilt = Omega(gilt, L)
OmegaNorm = Omega(gilt.rand.normal, L)
OmegaBimodal = Omega(gilt.rand.bimodal, L)

omegasEmpirical = data.frame(OmegaGilt, OmegaNorm[,2], OmegaBimodal[,2])
colnames(omegasEmpirical) <- c("L", "Empirical Gilts", "Normal", "Bimodal Normal")
omegasEmpirical = melt(omegasEmpirical, measure = c("Empirical Gilts", "Normal", "
    Bimodal Normal"))
colnames(omegasEmpirical) <- c("L","variable","Omega")
pdf("plots/EmpiricalOmegas.pdf", width=10)
```

```r
qplot(x=L, y=Omega, data=omegasEmpirical, geom="line", colour = variable, ylim= c(0,500)) + labs(colour='Omegas for Densities')
dev.off()
```

DistributionComparison.r