

An Exploration in the Properties of the Lorenz Equations

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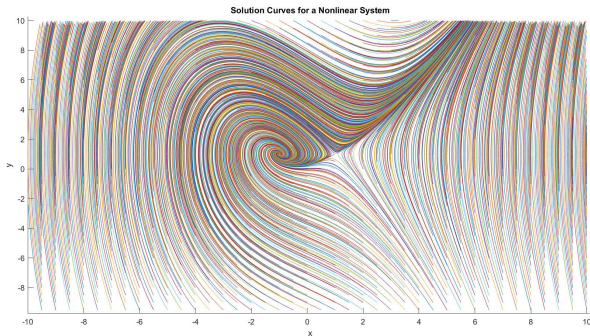


Nonlinear Dynamics

- There do not exist techniques to solve nonlinear systems of differential equations analytically.
- Studying nonlinear systems of differential equations often relies on linearizing the system around certain points, as in the example below.
- However, some systems are more difficult (or impossible) to linearize.

$$\begin{aligned}x' &= y - 1 \\ y' &= x^2 - y\end{aligned}$$

These equations can be linearized about $(-1, 1)$ and $(1, 1)$.



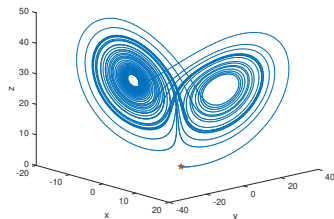
The Lorenz System

- The Lorenz system is a three-dimensional system of nonlinear differential equations, defined below:

$$\begin{aligned}x' &= \sigma(y - x) \\ y' &= \rho x - y - xz \\ z' &= xy - \beta z\end{aligned}\tag{1}$$

with the restrictions that $\sigma, \rho, \beta > 0$ and $\sigma > \beta + 1$.

- These equations roughly model 2-dimensional fluid convection.



Basic properties of the Lorenz system

A few properties are quickly apparent from the Lorenz equations.

- The Lorenz equations only have two nonlinearities, but can still exhibit complex, chaotic behavior.
- The system has symmetry about the z -axis. Therefore, every solution $x(t)$, $y(t)$, $z(t)$ is either symmetric itself, or has a symmetric partner.
- The system is invariant about the z -axis.



Methods

Finite difference methods

- Finite difference methods involve approximating the solutions to differential equations by taking small, discrete steps forward from some initial condition.
- These methods rely on information about a function's derivatives to approximate the original function.
- For example, Euler's method is defined as $y_{n+1} = y_n + \Delta x \cdot f_n(x, t)$.
- We used numerical simulation in MATLAB, using the 4th order Runge-Kutta method.



Fixed Points

A *fixed point* of the Lorenz system is a point (x, y, z) such that $x' = y' = z' = 0$.

- If the system reaches one of these states, it will remain there for all time.
- These fixed points can be shown to be $(0, 0, 0)$ and $(\pm\sqrt{\beta(\rho-1)}, \pm\sqrt{\beta(\rho-1)}, \rho-1)$.
- We can categorize these fixed points as *stable* and *unstable*, depending on whether trajectories are attracted to or repelled from the fixed point.



- As we change the values of parameters σ , ρ , and β , what happens to the dynamics of the system?
- We can see from the previous slide that the coordinates of the fixed points are dependent on the system parameters.
- Changes in number or type of fixed points and limit cycles are called *bifurcations*.
- We were interested in what happens as the parameter ρ increases. As it increases from 0 to a critical value ρ^* , the behavior goes from stable to chaotic.



Stable Lorenz Configuration

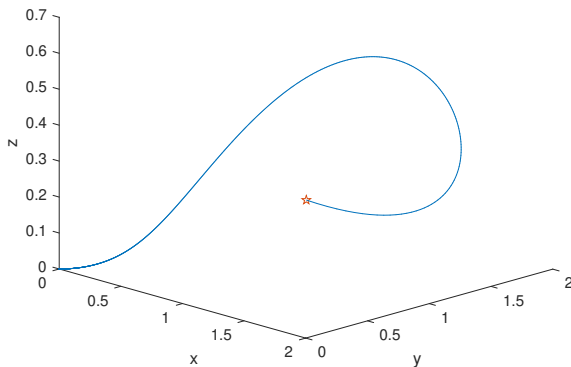


Figure: $\sigma = 10$, $\rho = 0.5$, $\beta = 8/3$



Bifurcation at $\rho = 1$

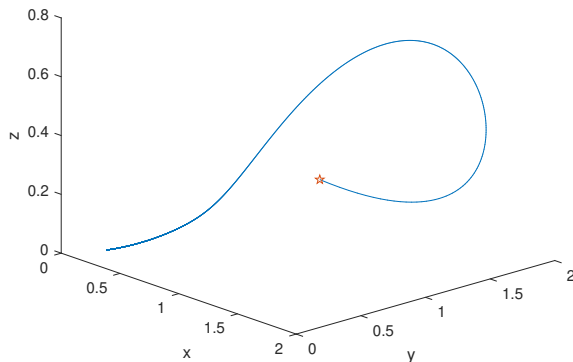


Figure: $\sigma = 10$, $\rho = 1$, $\beta = 8/3$



Changing dynamics

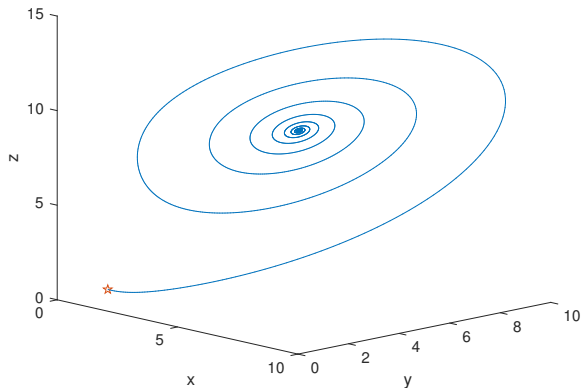


Figure: $\sigma = 10$, $\rho = 10$, $\beta = 8/3$



Changing dynamics

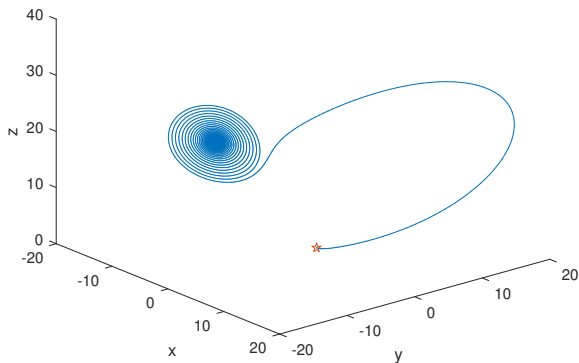


Figure: $\sigma = 10$, $\rho = 20$, $\beta = 8/3$



Just under threshold ρ

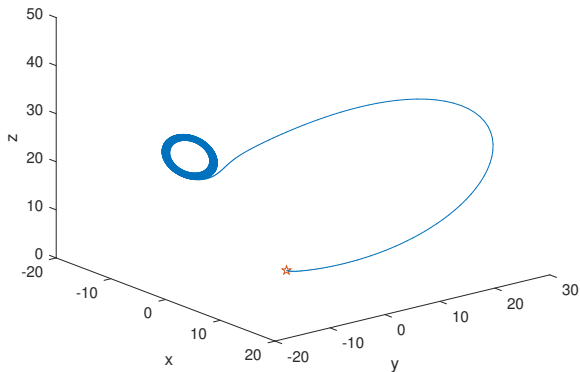


Figure: $\sigma = 10$, $\rho = 24$, $\beta = 8/3$



Just over threshold ρ

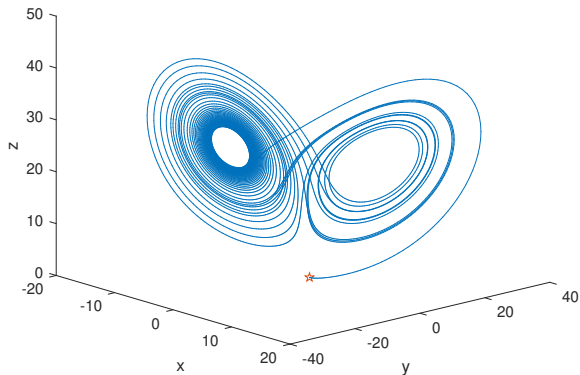


Figure: $\sigma = 10$, $\rho = 25$, $\beta = 8/3$



Lorenz's original parameters

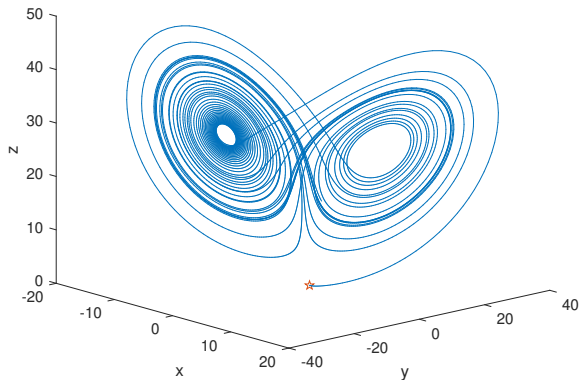


Figure: $\sigma = 10$, $\rho = 28$, $\beta = 8/3$

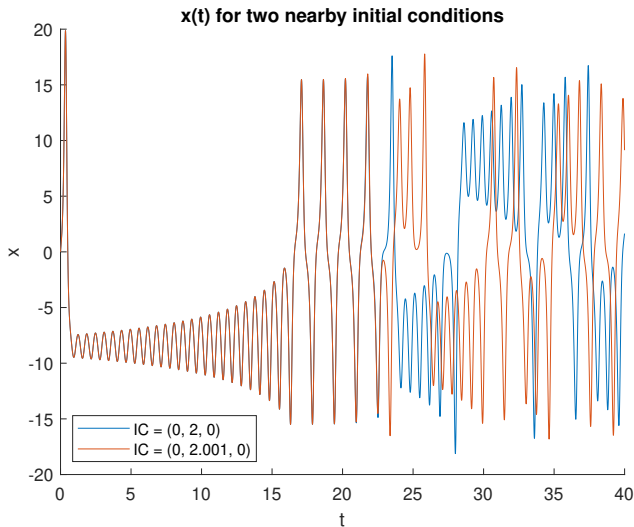


Sensitive Dependence on Initial Conditions

- Two different initial conditions in a chaotic regime of the Lorenz system will always diverge onto unrelated trajectories in finite time, even if they start extremely close.
- This makes long-term prediction of the system's behavior impossible without direct simulation.



Sensitive Dependence on Initial Conditions



Chaos and the Lorenz System

Most definitions of chaos in the context of dynamical systems include the following three parts:

- 1 A deterministic system
- 2 Aperiodic long-term behavior
- 3 Sensitive dependence on initial conditions

The Lorenz system exhibits all three of these properties for certain values of σ , ρ , and β , so we can characterize its behavior as chaotic.



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- Everyone here today



References



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The End

