

# CSCE883 Homework 2

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## Question 1

To find the maximum likelihood estimate (MLE) of  $\theta$  given the data  $(3, 0, 2, 1, 3, 2, 1, 0, 2, 1)$ , we start by considering the probability mass function (PMF) of the discrete random variable  $X$ :

$$P(X = x) = \begin{cases} \frac{2\theta}{3} & \text{if } x = 0, \\ \frac{\theta}{3} & \text{if } x = 1, \\ \frac{2(1-\theta)}{3} & \text{if } x = 2, \\ \frac{1-\theta}{3} & \text{if } x = 3. \end{cases}$$

## Counting Observations

First, we count the number of times each value appears in the data:

$$n_0 = \text{number of times } X = 0 = 2,$$

$$n_1 = \text{number of times } X = 1 = 3,$$

$$n_2 = \text{number of times } X = 2 = 3,$$

$$n_3 = \text{number of times } X = 3 = 2.$$

The total number of observations is  $N = n_0 + n_1 + n_2 + n_3 = 10$ .

## Writing the Likelihood Function

The likelihood function  $L(\theta)$  is the product of the probabilities of each observation:

$$\begin{aligned} L(\theta) &= \left(\frac{2\theta}{3}\right)^{n_0} \left(\frac{\theta}{3}\right)^{n_1} \left(\frac{2(1-\theta)}{3}\right)^{n_2} \left(\frac{1-\theta}{3}\right)^{n_3} \\ &= \left(\frac{1}{3}\right)^N (2\theta)^{n_0} (\theta)^{n_1} (2(1-\theta))^{n_2} (1-\theta)^{n_3}. \end{aligned}$$

Since the constant  $\left(\frac{1}{3}\right)^N$  does not affect the maximization, we can ignore it:

$$L(\theta) \propto (2\theta)^{n_0} (\theta)^{n_1} (2(1-\theta))^{n_2} (1-\theta)^{n_3}.$$

## Simplifying the Likelihood Function

Combine like terms and exponents:

$$L(\theta) \propto 2^{n_0+n_2} \theta^{n_0+n_1} (1-\theta)^{n_2+n_3}.$$

## Computing the Log-Likelihood Function

Taking the natural logarithm simplifies differentiation:

$$\begin{aligned} \ell(\theta) &= \ln L(\theta) \\ &= (n_0 + n_1) \ln \theta + (n_2 + n_3) \ln(1 - \theta) + (n_0 + n_2) \ln 2 + \text{constant}. \end{aligned}$$

Again, constants can be ignored during maximization:

$$\ell(\theta) = (n_0 + n_1) \ln \theta + (n_2 + n_3) \ln(1 - \theta) + \text{constant}.$$

## Differentiating and Setting the Derivative to Zero

Differentiate  $\ell(\theta)$  with respect to  $\theta$ :

$$\frac{d\ell}{d\theta} = \frac{n_0 + n_1}{\theta} - \frac{n_2 + n_3}{1 - \theta}.$$

Set the derivative equal to zero to find the maximum:

$$\frac{n_0 + n_1}{\theta} - \frac{n_2 + n_3}{1 - \theta} = 0.$$

## Solving for $\theta$

Multiply both sides by  $\theta(1 - \theta)$ :

$$(n_0 + n_1)(1 - \theta) - (n_2 + n_3)\theta = 0.$$

Simplify the equation:

$$\begin{aligned} (n_0 + n_1)(1 - \theta) - (n_2 + n_3)\theta &= 0 \\ (n_0 + n_1) - (n_0 + n_1)\theta - (n_2 + n_3)\theta &= 0 \\ (n_0 + n_1) - \theta[(n_0 + n_1) + (n_2 + n_3)] &= 0. \end{aligned}$$

Since  $(n_0 + n_1) + (n_2 + n_3) = N$ :

$$(n_0 + n_1) - \theta N = 0.$$

Solving for  $\theta$ :

$$\theta = \frac{n_0 + n_1}{N}.$$

### Computing the MLE of $\theta$

Plug in the observed values:

$$\theta_{\text{MLE}} = \frac{n_0 + n_1}{N} = \frac{2 + 3}{10} = \frac{5}{10} = 0.5.$$

### Conclusion

The maximum likelihood estimate of  $\theta$  is **0.5**.

## Question 2

We are given that  $X_1, X_2, \dots, X_n$  are independent and identically distributed (i.i.d.) with the probability density function (pdf):

$$f(x | \sigma) = \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right).$$

Our goal is to find the maximum likelihood estimate (MLE) of  $\sigma$ .

### Writing the Likelihood Function

The likelihood function  $L(\sigma)$  based on the observed data  $x_1, x_2, \dots, x_n$  is:

$$L(\sigma) = \prod_{i=1}^n f(x_i | \sigma) = \prod_{i=1}^n \left[ \frac{1}{2\sigma} \exp\left(-\frac{|x_i|}{\sigma}\right) \right].$$

Simplify the likelihood function:

$$L(\sigma) = \left(\frac{1}{2\sigma}\right)^n \exp\left(-\frac{1}{\sigma} \sum_{i=1}^n |x_i|\right).$$

### Computing the Log-Likelihood Function

Taking the natural logarithm of the likelihood function to simplify differentiation:

$$\begin{aligned} \ell(\sigma) &= \ln L(\sigma) \\ &= n \ln\left(\frac{1}{2\sigma}\right) - \frac{1}{\sigma} \sum_{i=1}^n |x_i| \\ &= -n \ln(2\sigma) - \frac{1}{\sigma} \sum_{i=1}^n |x_i|. \end{aligned}$$

### Differentiating and Setting the Derivative to Zero

Differentiate  $\ell(\sigma)$  with respect to  $\sigma$ :

$$\frac{d\ell}{d\sigma} = -n \cdot \frac{1}{\sigma} + \frac{1}{\sigma^2} \sum_{i=1}^n |x_i| = -\frac{n}{\sigma} + \frac{S}{\sigma^2},$$

where  $S = \sum_{i=1}^n |x_i|$ .

Set the derivative equal to zero to find the maximum:

$$-\frac{n}{\sigma} + \frac{S}{\sigma^2} = 0.$$

### Solving for $\sigma$

Multiply both sides by  $\sigma^2$ :

$$-n\sigma + S = 0.$$

Solve for  $\sigma$ :

$$n\sigma = S \implies \sigma = \frac{S}{n} = \frac{1}{n} \sum_{i=1}^n |x_i|.$$

### Conclusion

The maximum likelihood estimate of  $\sigma$  is:

$$\hat{\sigma} = \frac{1}{n} \sum_{i=1}^n |x_i|,$$

which is the sample mean of the absolute values of the observations.

### Question 3

We are given that  $X_1, X_2, \dots, X_n$  are independent and identically distributed (i.i.d.) with the normal distribution  $\mathcal{N}(\mu, \sigma^2)$ . Our goal is to find the maximum likelihood estimates (MLE) of  $\mu$  and  $\sigma^2$ .

#### Writing the Likelihood Function

The likelihood function  $L(\mu, \sigma^2)$  based on the observed data  $x_1, x_2, \dots, x_n$  is:

$$L(\mu, \sigma^2) = \prod_{i=1}^n f(x_i | \mu, \sigma^2) = \prod_{i=1}^n \left[ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2}\right) \right].$$

Simplify the likelihood function:

$$L(\mu, \sigma^2) = \left( \frac{1}{\sqrt{2\pi\sigma^2}} \right)^n \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2\right).$$

#### Computing the Log-Likelihood Function

Taking the natural logarithm of the likelihood function to simplify differentiation:

$$\begin{aligned} \ell(\mu, \sigma^2) &= \ln L(\mu, \sigma^2) \\ &= -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2. \end{aligned}$$

#### Estimating $\mu$

Differentiate  $\ell(\mu, \sigma^2)$  with respect to  $\mu$ :

$$\frac{\partial \ell}{\partial \mu} = -\frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu).$$

Set the derivative equal to zero to find the maximum:

$$-\frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu) = 0.$$

Simplify:

$$\sum_{i=1}^n (x_i - \mu) = 0 \implies \sum_{i=1}^n x_i - n\mu = 0.$$

Solve for  $\mu$ :

$$n\mu = \sum_{i=1}^n x_i \implies \hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i.$$

## Estimating $\sigma^2$

Differentiate  $\ell(\mu, \sigma^2)$  with respect to  $\sigma^2$ :

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2} \cdot \frac{1}{\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (x_i - \mu)^2.$$

Set the derivative equal to zero:

$$-\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^n (x_i - \mu)^2 = 0.$$

Multiply both sides by  $2\sigma^4$ :

$$-n\sigma^2 + \sum_{i=1}^n (x_i - \mu)^2 = 0.$$

Solve for  $\sigma^2$ :

$$n\sigma^2 = \sum_{i=1}^n (x_i - \mu)^2 \implies \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\mu})^2.$$

## Conclusion

The maximum likelihood estimates are:

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i,$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \hat{\mu})^2.$$

Therefore, the MLE of  $\mu$  is the sample mean, and the MLE of  $\sigma^2$  is the sample variance (unbiased by  $n$ ).