A dynamic approach to input-output modeling

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Abstract

Previous frameworks for input-output modeling have made the assumption that flows into and out of each economic sector balance, such that there is no accumulation of economic factors or embodied energy within any of the sectors. This may be an adequate assumption for a sector of the economy operating at 'steady-state', however the assumption introduces errors for example in analysis of sectors that are growing rapidly, where a non-negligible proportion of input factors may be invested in accumulation within the sector. This paper presents an extension to the traditional input-output framework, wherein accumulation is incorporated explicitly using a dynamic (transient) analysis method. This new approach gives new insight into macro-economics including an alternative metric for social development. It also raises issues for input-output-based methods for net energy analysis. The alternative perspective offered by this new method is used to explore the implications of extraction of declining quality resources from the environment.

Keywords:

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input-output modeling, physical resource modeling, dynamic economic modeling, net energy analysis (NEA), energy return on investment (EROI)

1. Introduction

2   BLAH BLAH BLAH

1.1. Brief history of input-output (I-O) modeling

4   Input-output analysis, developed by Wassilly Leontief in the 1930's as an extension to the work of Quesnay and Walras ?, is of primary importance in national accounting, allowing determination of the structure of an economy as well as, among other things, calculation of a nation's gross domestic product (GDP), the predominant measure of economic activity.

1.2. Basic I-O method

10   The basic premise of the I-O method, as outlined in Figure 1A, is that each economic sector takes in factors of production from other sectors (and possibly itself) to produce an economic good at some rate. E.g., the automotive sector takes in steel, rubber, glass, etc. and produces a number of cars per year. In contrast to high-level economic growth models that include only a few factors of production (such as land, capital, and labor), the I-O analysis technique allows many differentiated factors of production and raw material feedstocks. ? In I-O frameworks, each factor of production is considered to be the output from a sector of the economy. As will be discussed later [MAKE SURE TO DISCUSS THIS LATER!], the traditional primary factors of production (land, capital, and labor) are not *flows* into the production processes. Rather, they are *stocks* that, when present, allow factors

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22 of production (steel, rubber, and glass) to be transformed into final products

23 (automobiles).

24   In addition to stocks of land, capital, and labor, a flow of energy (or

25 more precisely, the degradation of an exergetic gradient/destruction of ex-

26 ergy) is also required for economic activity. These energy flows originate from

27 the natural environment, recognition of which has provoked researchers from

28 fields of net energy analysis (NEA), material flow analysis (MFA), industrial

29 ecology (IE) and life-cycle assessment (LCA) to extend the traditional (Leon-

30 tief) input-output framework to include important material and energy flows

31 to and from the environment, as depicted in Figure 1B ????????. While the

32 Leontief I-O approach relies exclusively on monetary units to represent value

33 flows among sectors of an economy, the key insight of these extensions of

34 the Leontief I-O framework is to rely upon physical units (especially energy

35 units of joules) to represent some of the value flows among economic sectors.

36 In doing so, energy and material intensities of value flows can be estimated.

37 Their approaches are similar to Figure 1B.

38   Both the original Leontief I-O framework and the extensions cited above

39 assume steady-state conditions in an economy, i.e., flows of value and mate-

40 rial into and out of each economic sector are in balance. Dynamic or transient

41 behavior of the economic system is not considered. Thus, there is no accumu-

42 lation of economic factors or embodied energy within any of the sectors. The

43 analysis techniques provide "snapshots" of economic activity at an instant

44 in time.

45   [MIK'S NEW ADDITION]

46   Assuming no accumulation of materials, within economic sectors or so-

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ciety itself, is tantamount to assuming that *all* material flows through the economy are directed toward the production of non-durable goods. However, evidence of the durability of goods and the accumulation of materials surrounds us. Furthermore, energy was required to both fabricate and emplace the durable goods and infrastructure of modern economies. (The energy it took to create the durable goods and infrastructure can be considered "embodied" within the built environment, a point to which we will return in detail later). As Georgescu-Roegen [REFERENCE NEEDED –MKH] notes, "in the everyday world one cannot possibly cross a river only on the flow of main maintenance materials of a non-existent bridge." [I DON'T HAVE THE SOURCE WITH ME, BUT "MAIN MAINTENANCE MATERIALS" SOUNDS WEIRD. SHOULD "MAIN" BE DELETED? SHOULD "OF A NON-" BE REPLACED WITH "FOR A NON-"? –MKH]

Analysis methods that neglect the accumulation of materials and embodied energy in the durable goods and infrastructure of the everyday world lack explanatory power. Such models can tell us how at what rates materials and energy are required to *use* our built environment. But, such models cannot tell us *how* the built environment came to be (and how much energy was required to construct it) or *why* flows of non-durable goods are needed. [I DON'T GET THIS SECOND PART: I.E., WHY FLOWS OF NON-DURABLE GOODS ARE NEEDED. LET'S DISCUSS. –MKH] To use Georgescu-Roegen's imagery, models that neglect accumulation fail to explain why we need any material flows to maintain a non-existent bridge. [IF I UNDERSTOOD THE PREVIOUS SENTENCE, I WOULD PROBABLY UNDERSTAND THE EXTENSION OF THE METAPHOR. BUT,

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72 AT THIS POINT, I DON'T. –MKH] The stocks of accumulated materials
73 (capital, appliances, even people) are the drivers of demand. It is to service
74 their needs and wants that we put the economy to work. [THESE LAST
75 TWO SENTENCES ARE EXCELLENT! –MKH]

76 Because economic activity requires energy, we need to understand the way
77 energy flows through economies. The steady-state I-O techniques of Bullard,
78 Herendeen, and others [REFERENCES NEEDED –MKH] offer a means to
79 that end. We contend, however, that these techniques need to be extended
80 and modified to include transient effects that arise when durability of goods
81 and infrastructure (and associated embodied energy) are considered. This
82 paper attempts to address that need.

83 *1.3. Issues of resource quality*

84 [DO WE NEED THIS SECTION? IT FEELS LIKE WE SHOULD MOVE
85 FROM THE PREVIOUS SECTION WHEREIN WE CLEARLY DEMON-
86 STRATE THE NEED FOR OUR WORK TO BEGINNING THE PROCESS
87 OF DEVELOPING THE METHOD. IF WE WANT TO KEEP THIS SEC-
88 TION, WE SHOULD DO A BETTER JOB AT SHOWING WHY IT IS
89 NEEDED. WE SHOULD POINT BACK TO BULLARD AND HEREN-
90 DEEN AND SAY THAT THEIR METHOD DOESN'T ACCOUNT FOR
91 DECREASING RESOURCE QUALITY AND WHY A METHOD IS NEEDED
92 THAT DOES ACCOUNT FOR DECLINING RESOURCE QUALITY. –
93 MKH] Raw material and energy resources must first be extracted from the
94 natural environment before they may be utilized in the economy to pro-
95 vide goods and service to society. Despite increasing levels of technological
96 efficiency, for example in consumer goods such as refrigerators and cars, ev-

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idence shows that the energy intensity of primary resource extraction, i.e. the energy required to extract raw materials from the environment, has been steadily increasing over the last fifty years ???. This increasing energy requirement for primary extraction means that less *net energy* is available for downstream uses. If this decline in net energy availability outpaces technological advances in energy efficiency, there may be deleterious impacts on the economic output of the economy.

1.4. An I-O method for dynamic (transient) economic analysis

In this paper, we develop a physical input-output, matrix-based method for modeling multi-sector economies, in the tradition of Georgescu-Roegen's "flow-fund" model ??. The method presented in this paper takes a decidedly engineering approach to extend the techniques of Bullard, Herendeen, and others to account for durability of goods and embodied energy. This method allows us to see how energy and materials flow through the economy, where embodied energy accumulates in the economy, and how declining resource quality may affect these dynamics. [NEED TO MAKE SURE WE ACHIEVE THIS LAST POINT]

This paper is organized as follows. We first discuss methodology and the model economy. Thereafter, we present three examples, each with increasing levels of disaggregation among society, the energy sector, and goods and services sectors, culminating with a matrix formulation of the new method. The examples leverage the First Law of Thermodynamics, account for total energy ((T), and develop accounting relationships for embodied energy (B). Within the examples, we develop a precise definition for embodied energy and a matrix formulation of the method that can be extended to an arbitrarily

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122 large number of economic sectors. Finally, we draw several implications from
123 the development of the new method.

2. Methodology

2.1. Model economy

126 The model economy employed herein consists of sectors that produce a
127 single product, either an energy product (energy sectors) or other goods and
128 services (non-energy sectors). Economic sectors receive as inputs direct en-
129 ergy (E) and materials in which energy is embodied (B).[1] Economic sectors
130 emit waste heat (Q).

2.2. Direct energy (E), indirect (embodied) energy (B), and waste heat (Q)

132 We distinguish between direct energy resources (E), such as coal or oil,
133 and indirect energy (B) "embodied" in outputs from economic sectors. E
134 represents the energetic value of an energy resource (measured as heating
135 value, chemical potential energy, or exergy). In contrast, B represents the
136 energy expended in the production and delivery of goods in the economy,
137 and, as such, measures accumulated upstream energy consumption from the
138 network of economic sectors within the economy. 'Indirect' energy and 'em-
139 bodied' energy are synonyms. Both E and B are measured in energy units
140 (joules or BTUs). The flow rates of direct energy ($\dot{E}$) and indirect energy
141 ($\dot{B}$) among sectors of the economy, the Earth, and society are in units of
142 power (energy per unit time, J/time or BTU/time).

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[1]A formal definition for embodied energy (B) is presented in Section 6.4.

143   Waste heat ($\dot{Q}$) flows from sectors of the economy and society to the

144   Earth and its atmosphere, the necessary result of inefficient consumption of

145   direct energy E. Like $\dot{E}$ and $\dot{B}$, the units of $\dot{Q}$ are energy per unit time.
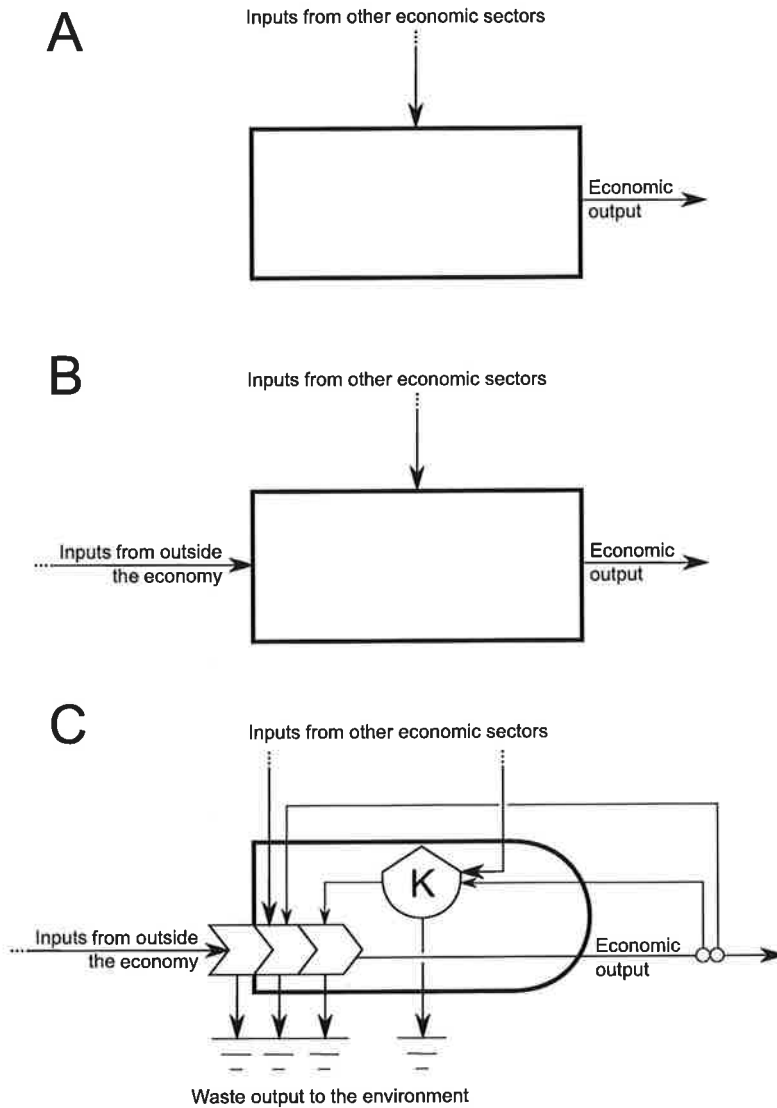
Figure 1: The basic unit of input-output modeling: A) the standard economic approach includes only transactions among sectors of the economy; B) the ecological economics approach models inputs from the natural environment outside the economy as factors of production [MIK: REPLACE 'OUTSIDE THE ECONOMY' WITH 'THE NATURAL ENVIRONMENT' IN THE FIGURE.] and; C) the method presented here accounts also for accumulation, K, of embodied energy within materials in economic sectors.

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[Handwritten annotation at top: "I like S & L as it is reminiscent of entropy & pressure ~~in thermo?~~"]

2.3. Total energy (T)

Total energy (T) is the sum of the direct and indirect (embodied) energy.

$$T \equiv E + B \tag{1}$$

[Handwritten: $T \equiv E + B = E + B = E + S + L$; "non-persistent"; "per"; $L \equiv$ long-lived; $S \equiv$ short lived]

In general, the flow rate of total energy among sectors in the economy, the earth, and society is given by

$$\dot{T} = \dot{E} + \dot{B}. \tag{2}$$

[Handwritten: $= \dot{E} + \dot{S} + \dot{L}$]

In some cases, total energy flows may consist of direct energy ($\dot{E}$) or embodied energy ($\dot{B}$) exclusively. For example, the flow of extracted crude oil from the earth consists of direct energy only ($\dot{B} = 0$ and $\dot{T} = \dot{E}$), because, in this method, no embodied energy (B) has been added to the crude oil until it reaches the downstream side of the pump. The flow of goods produced by a non-energy sector of the economy consists of indirect energy only ($\dot{E} = 0$, and therefore $\dot{T} = \dot{B}$), because no direct energy (E) is produced by a non-energy sector in this model economy.

[Handwritten: $\dot{T} = \dot{B} = \dot{S} + \dot{L}$]

In other cases, total energy flows may have both direct *and* indirect components. For example, the flow of refined petroleum from the energy sector has both a direct energy ($\dot{E}$, the energy content of the oil product, usually represented by chemical potential energy) and embodied energy ($\dot{B}$, which accounts for the energy consumed in upstream processes to extract and refine the crude oil).[2]

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[2]Outputs from agricultural sectors will be similar: both the direct energy component (comprising chemical potential energy) and the embodied energy component will be non-

164   Single subscripts on T, E, or B can mean one of two things: $\dot{T}_i$ indicates

165 the outflow of total energy from sector i, whereas T_i denotes the total energy

166 content of sector i. Double subscripts on T, E, or B (e.g., $\dot{T}_{ij}$) indicate a

167 flow from sector i to sector j,[3] in this case for total energy (T).

168   The I-O literature [REF TO BULLARD AND HERENDEEN, ETC.

169 HERE –MKH] assumes (a) that steady state conditions exist (i.e., no ac-

170 cumulation of total energy in economic sectors) and (b) that flows of total

171 energy $(\dot{T})$ are *conserved*, where by *conserved*, it is meant that total energy

172 can be neither created nor destroyed. Like the literature, we assume that

173 total energy is conserved. However, we depart from the literature to allow

174 durability of goods as represented by total energy accumulation in economic

175 sectors. Steady state, this approach is not.

176   Total energy may accumulate within an economic sector as stocks of di-

177 rect energy materials (piles of coal or tanks of oil) but also as embodied

178 energy in stocks of capital goods (e.g. machinery or buildings). The rate of

179 accumulation of total energy $(\frac{dT}{dt})$ in a sector of the economy, the Earth, or

180 society is given by the time derivative of total energy:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\mathrm{d}B}{\mathrm{d}t}. \; = \frac{dE}{dt} + \frac{dS}{dt} + \frac{dP}{dt} \quad (3)$$

181   We note that the definition of total energy (Equation 1) includes direct en-

182 ergy (E) and embodied energy (B) terms. On the other hand, the First Law

183 of Thermodynamics includes direct energy (E) and waste heat (Q) terms.

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zero.

    [3]In the following discussion, the first index always indicates the sector *from* which a quantity flows, and the second index indicates the sector *to* which a quantity flows.

184 The consequence of the foregoing difference is that an interesting relationship
185 exists between embodied energy (B) and waste heat (Q). We shall see in the
186 following example that waste heat from an economic sector can be considered
187 to contribute to energy embodied within the products of that sector.

3. Example A: single sector economy

189   In this section, we present an example economic analysis using a single-
190 sector economy wherein the economy and society are merged together.
191   Figure 2 shows a single-sector Economy (represented by "economy/society,"
192 2) that extracts direct energy from the earth $(\dot{E}_{12})$. Direct energy and waste
193 heat flows are identified by vectors. No direct energy flows from the economy
194 (2) to the earth (1), only waste heat $(\dot{Q}_{21})$.

3.1. First Law of Thermodynamics

196   Both direct energy ($\dot{E}$, such as the energy content of coal, oil, and electric-
197 ity), and waste heat ($\dot{Q}$) are accounted by the First Law of Thermodynamics.
198 Accounting for possible accumulation of direct energy in the economy, the
199 First Law of Thermodynamics indicates that

$$\frac{\mathrm{d}E_2}{\mathrm{d}t} = \dot{E}_{12} - \dot{Q}_{21}. \tag{4}$$

200   Aside from, for example, the U.S. Strategic Petroleum Reserve, we are not
201 stockpiling oil and coal at any meaningful rate, i.e. we consume fossil fuels at
202 a rate equal to the extraction rate. Thus, the world is not accumulating direct
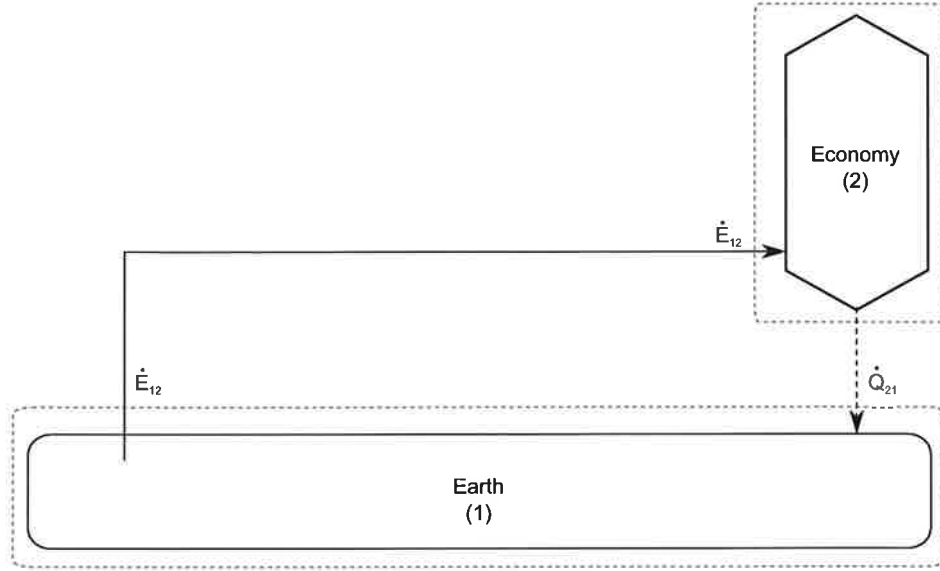
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Figure 2: Direct energy ($\dot{E}$) and waste heat ($\dot{Q}$) flows for a single-sector economy. [CHANGE "ECONOMY" TO "ECONOMY/SOCIETY" TO MATCH TEXT. OR CHANGE THE TEXT ABOVE. –MKH]

energy in the economy.[4] (The world *is*, however, accumulating embodied energy in the economy as we shall see shortly.) Thus, the accumulation rate for direct energy $\left(\frac{\mathrm{d}E_2}{\mathrm{d}t}\right)$ in the above equation can be set to zero to obtain

$$0 = \dot{E}_{12} - \dot{Q}_{21}. \tag{5}$$

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[4]A counter-example could be made for nuclear fuels where 'spent' fuel represents a large exergetic stockpile, however, this reserve is not (presently) economically useful.

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3.2. Total energy accounting

207   Figure 3 shows the flows of total energy ($\dot{T}$) through the single-sector
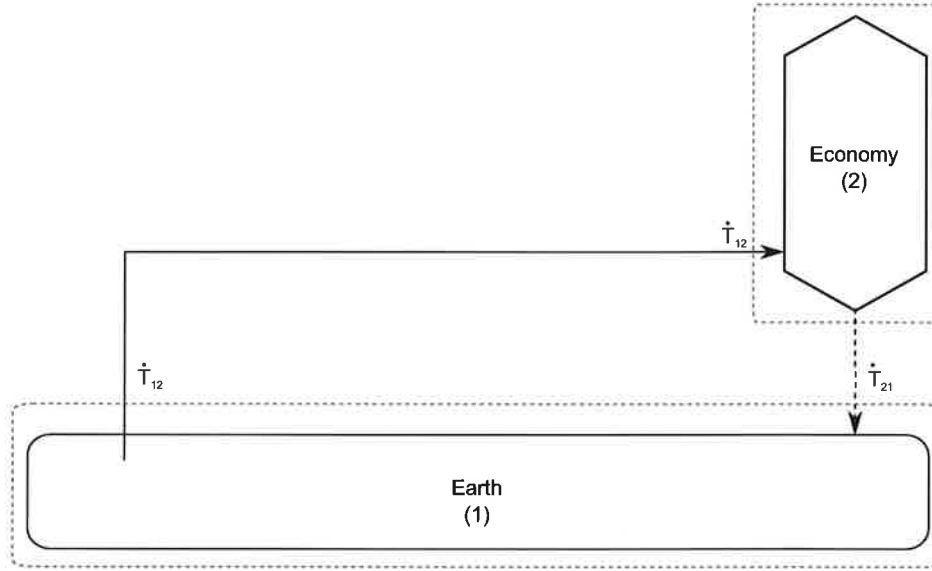
208 economy.

Figure 3: Total Energy Flows ($\dot{T}$) in a Single-sector Economy.

209   We follow the I-O literature in assuming that total energy (T) is con-

210 served. The I-O literature assumes steady-state operation of the economy

211 with no accumulation of embodied energy in the economic sectors. (We will

212 see later how the assumption in the literature introduces errors into I-O anal-

213 yses.) We depart from the I-O literature by accounting for both accumulation

214 and depreciation of energy embodied in sectors of the economy and society.

215 By doing so, the present analysis does *not* assume a steady-state economy.

216 A total energy accounting around the single-sector economy (2) gives

14

$$\frac{dT_2}{dt} = \dot{T}_{12} - \dot{T}_{21}. \tag{6}$$

3.3. Embodied energy accounting

The First Law of Thermodynamics accounts for both direct energy (E) and waste heat (Q), whereas total energy (T) accounting tracks direct energy (E) and embodied energy (B). If we substitute the First Law into the total energy accounting equation, we can eliminate direct energy (E) to arrive at an embodied energy accounting equation. We begin by expanding the T terms in Equation 6 using Equations 1 and 2 to obtain

$$\frac{dE_2}{dt} + \frac{dB_2}{dt} = \dot{E}_{12} + \dot{B}_{12} - \dot{E}_{21} - \dot{B}_{21}. \tag{7}$$

Realizing that $\frac{dE_2}{dt} = 0$ (because direct energy does not accumulate in meaningful amounts in the economy) and $\dot{E}_{21} = 0$ (because energy is returned to the earth as waste heat, see Figure 2) yields

$$\frac{dB_2}{dt} = \dot{E}_{12} + \dot{B}_{12} - \dot{B}_{21}. \tag{8}$$

Equation 8 shows that the accumulation rate of embodied energy in the economy is a function of the inflows of direct and embodied energy less the outflow of embodied energy.

In this example, we substitute[5] Equation 5 into Equation 8 to obtain an embodied energy accounting equation:

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[5] We shall encounter this move to substitute the First Law of Thermodynamics into the total energy accounting equation repeatedly below.

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$$\frac{dB_2}{dt} = \frac{dL_2}{dt} = \dot{Q}_{21} + \dot{L}_{12} + \dot{S}_{12} - \dot{L}_{21} - \dot{S}_{21}$$

$$\frac{\mathrm{d}B_2}{\mathrm{d}t} = \dot{Q}_{21} + \dot{B}_{12} - \dot{B}_{21}. \qquad (9)$$

232 An important result of Bullard-Herendeen-style I-O analyses, historically,

233 has been the quantification of the embodied energy content of economic sector

234 outputs, in this case $\dot{B}_{21}$. Equation 8 can be rearranged to give

$$\dot{S}_{21} + \dot{L}_{21} = \dot{Q}_{21} + \dot{S}_{12} + \dot{L}_{12} - \frac{dL_2}{dt}$$

$$\dot{B}_{21} = \dot{Q}_{21} + \dot{B}_{12} - \frac{\mathrm{d}B_2}{\mathrm{d}t}. \qquad (10)$$

Isn't $\dot{B}_{12} = \dot{S}_{12} + \dot{L}_{12} = 0$? in this case

235 Equation 10 indicates that the embodied energy content of the product

236 of an economic sector (in this case $\dot{B}_{21}$) can be thought of as the sum of the

237 embodied energy inputs to the sector (in this case $\dot{B}_{12}$) and the waste heat

238 from the sector (in this case $\dot{Q}_{21}$) less the accumulation rate of embodied

239 energy in the sector (in this case $\frac{\mathrm{d}B_2}{\mathrm{d}t}$). This derivation indicates that waste

240 heat ($\dot{Q}$) plays an important role[6] in Bullard-Herendeen-style I-O analyses:

241 the accumulation of waste heat along a production path leads to energy being

242 'embodied' in the output of an economic sector.

243 In Equation 10 we also see the first indication that the traditional ap-

244 proach of neglecting dynamic effects in I-O analyses may lead to errors. If

245 $\frac{\mathrm{d}B_2}{\mathrm{d}t}$ is both neglected and nonzero, calculation of the embodied energy out-

246 flow rate ($\dot{B}_{21}$) will be in error.

247 *3.4. Depreciation*

248 It is worthwhile to note that $\dot{B}_{21}$ represents the disposal rate of embodied

249 energy from the economy back to the earth, akin to depreciation of physical

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[6]To our knowledge, there has been no prior identification of the role of waste heat in Bullard-Herendeen-style I-O analyses.

assets. This physical depreciation is different from, but related to, financial depreciation, as financial depreciation is usually faster than physical depreciation. Embodied energy depreciation ($\dot{B}_{21}$ in this example) can be represented by a depreciation term such as

$$\dot{B}_{21} = \gamma_2 B_2, \tag{11}$$

where γ represents the depreciation rate in units of inverse time (e.g., 1/year) with $\gamma > 0$. The depreciation rate (γ) indicates that a fraction of the total stock of embodied energy is disposed over a period of time (e.g, $\gamma = 0.05$/year). In the absence of other inputs or outputs, this depreciation function provides exponential decay of embodied energy (B). γ is, in general, a function of time.

Equation 11 can be substituted into Equation 9 and rearranged to obtain

$$\frac{dB_2}{dt} = \dot{Q}_{21} + \dot{B}_{12} - \gamma_2 B_2 \tag{12}$$

which indicates that the accumulation rate of embodied energy in an economic sector (in this case $\frac{dB_2}{dt}$) is equal to the sum of the waste heat rate from the economic sector ($\dot{Q}_{21}$) and the inflow rate of embodied energy to the sector ($\dot{B}_{12}$) less the embodied energy disposal rate ($\gamma_2 B_2$).

4. Value (X), energy intensity (ε), and the input-output ratio (a)

We now turn to defining flows of value ($\dot{X}$), energy intensity (ε), and input-output ratios (a).

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4.1. Value flows $(\dot{X})$

Among sectors of the economy and society, value $(\dot{X})$ flows in the same direction as goods, services, and energy, but in the opposite direction from currency payments. Typical of the Bullard-Herendeen I-O analyses technique [NEED REFERENCE HERE –MKH], we allow value flows to be in either monetary units or physical units. For non-energy sectors of the economy, value outflows are in currency units per time (\$/time). For energy-producing sectors, value outflows are in units of J/time or BTU/time.

4.2. Energy intensity (ε)

Energy intensity (ε) is the ratio of total energy and value outflow rates from an economic sector, such that for the jth economic sector,

$$\varepsilon_j \equiv \frac{\dot{T}_j}{\dot{X}_j}. \tag{13}$$

For goods and services sectors of the economy, ε is in units of J/\$, but for energy-producing sectors of the economy, the units of ε are J/J. For inter-sector flows, we have

$$\varepsilon_{ij} = \frac{\dot{T}_{ij}}{\dot{X}_{ij}}. \tag{14}$$

Furthermore, we note that

$$\varepsilon_i = \varepsilon_{ij} \tag{15}$$

for all j, because the energy intensity of a sector's output is the same regardless of its destination. I.e., we assume that all goods produced within a

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Table 1: Units for input-output ratios (a).

		Output of	
		Non-energy sector	Energy sector
Inputs from	Non-energy sector	$\frac{\$}{\$}$	$\frac{\$}{J}$
	Energy sector	$\frac{J}{\$}$	$\frac{J}{J}$

sector are produced at the average energy intensity of that sector.[7]

4.3. Input-output ratios (a)

We define a parameter a_{ij} that represents the input of good i required to produce a unit of output from sector j.

$$a_{ij} \equiv \frac{\dot{X}_{ij}}{\dot{X}_j} \tag{16}$$

Input-output ratios are given in mixed units, depending on the purpose of each sector of the economy and the type of input as shown in Table 1.

5. Example B: a one sector economy with external demand

At this point, we move to a second example wherein a single economic sector (3) interacts with Society (2, which provides final demand) and the Earth (1, the destination for waste heat and the source of all resources). In this economy, we assume that the purpose of the goods and services sector

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[7]If this approach is unsatisfactory, the sector may be divided into sub-sectors with different energy intensities.

296 is to produce goods and provide services, including the provision of direct
297 energy available to the economy and society.

5.1. First Law of Thermodynamics

299 The First Law of Thermodynamics requires that energy (direct and wast
300 heat) is conserved around each Sector of the economy (3) as well as around
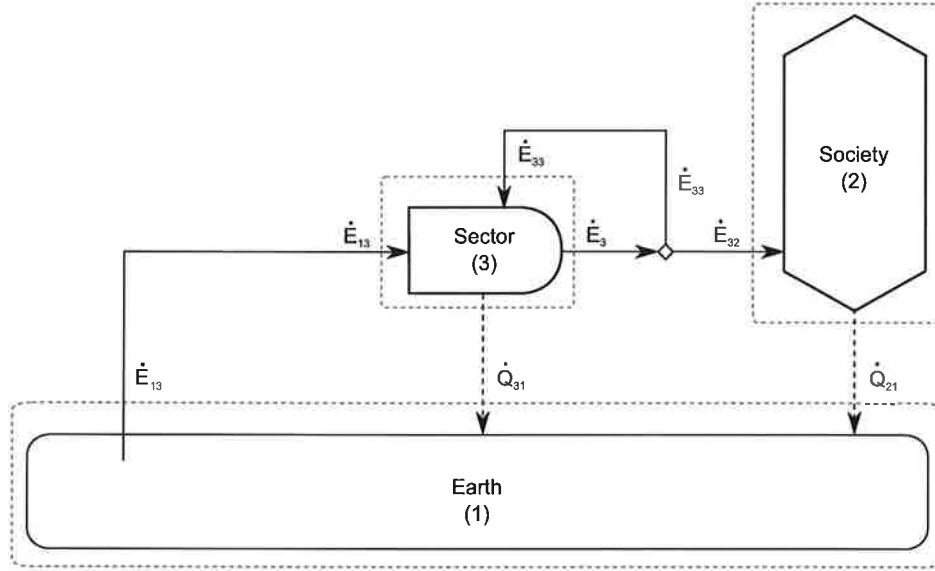301 the Earth (1) and Society (2) as shown in Figure 4.

Figure 4: Flows of direct energy ($\dot{E}$) and waste heat ($\dot{Q}$) in a one-sector economy with separate demand.

302 The First Law around the economic Sector (3) including the accumulation
303 rate of direct energy in the sector $\left(\frac{dE_3}{dt}\right)$ yields

$$\frac{dE_3}{dt} = \dot{E}_{13} + \dot{E}_{33} - \dot{E}_3 - \dot{Q}_{31}. \tag{17}$$

20

It is notable that the economic Sector (3) consumes a portion of its own energy output ($\dot{E}_{33}$) as it produces its goods and services: it takes energy to make energy.

First Law energy accounting around the Earth (1) and Society (2) gives

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} = \dot{Q}_{21} + \dot{Q}_{31} - \dot{E}_{13}, \tag{18}$$

and

$$\frac{\mathrm{d}E_2}{\mathrm{d}t} = \dot{E}_{32} - \dot{Q}_{21}. \tag{19}$$

As in Example A, we can set the accumulation of direct energy within each sector to zero to obtain

$$0 = \dot{E}_{13} + \dot{E}_{33} - \dot{E}_3 - \dot{Q}_{31}, \tag{20}$$

$$0 = \dot{Q}_{21} + \dot{Q}_{31} - \dot{E}_{13}, \tag{21}$$

and

$$0 = \dot{E}_{32} - \dot{Q}_{21}, \tag{22}$$

5.2. Total energy accounting

Again, we follow the I-O literature in assuming that total energy (i.e., the sum of direct energy and indirect energy) is conserved. Thus, we can draw a diagram similar to Figure 4 for total energy flows. See Figure 5.

Accounting for accumulation of total energy and using the assumption that total energy is conserved, we can write the following equations.
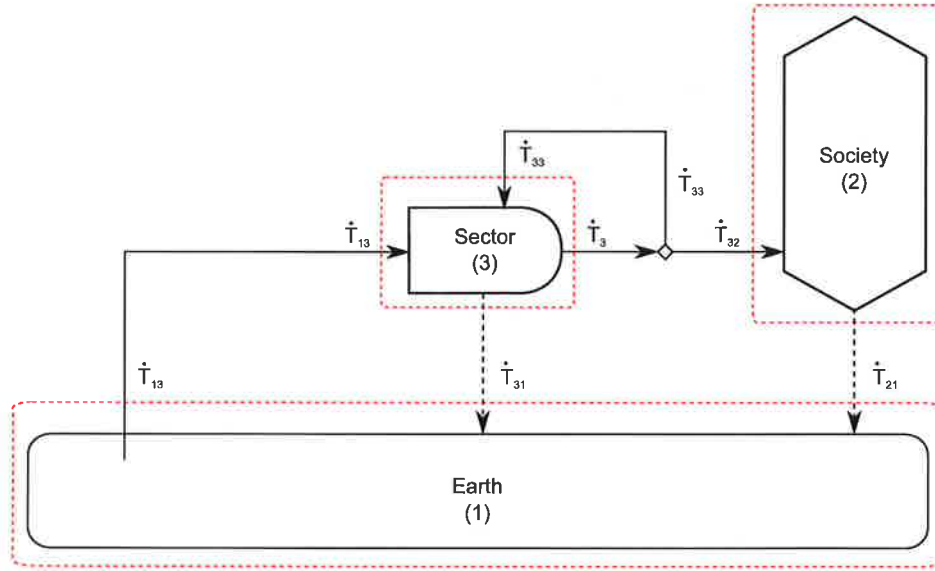
Figure 5: Flows of total energy $(\dot{T})$ in a one-sector economy with separate demand.

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = \dot{T}_{21} + \dot{T}_{31} - \dot{T}_{13}, \tag{23}$$

$$\frac{\mathrm{d}T_2}{\mathrm{d}t} = \dot{T}_{32} - \dot{T}_{21}, \tag{24}$$

and

$$\frac{\mathrm{d}T_3}{\mathrm{d}t} = \dot{T}_{13} + \dot{T}_{33} - \dot{T}_3 - \dot{T}_{31}. \tag{25}$$

5.3. Embodied energy accounting

Given that $\frac{\mathrm{d}E_i}{\mathrm{d}t} = 0$ and $\dot{T} = \dot{E} + \dot{B}$, we note that

$$\frac{\mathrm{d}T_i}{\mathrm{d}t} = \frac{\mathrm{d}B_i}{\mathrm{d}t}, \quad = \frac{dL_i}{dt} \tag{26}$$

22

and we can rewrite the total energy accumulation accounting equations as

$$\frac{dL_1}{dt} = \dot{E}_{21} + \dot{S}_{21} + \dot{L}_{21} + \dot{E}_{31} + \dot{S}_{31} + \dot{L}_{31} - \dot{E}_{13} + \dot{S}_{13} + \dot{L}_{13}$$

$$\frac{dB_1}{dt} = \dot{E}_{21} + \dot{B}_{21} + \dot{E}_{31} + \dot{B}_{31} - \dot{E}_{13} + \dot{B}_{13}, \qquad (27)$$

$$\frac{dL_2}{dt} = \dot{E}_{32} + \dot{S}_{32} + \dot{L}_{32} - \dot{E}_{21} - \dot{S}_{21} - \dot{L}_{21} \qquad \dot{S}_{32} = \dot{S}_{21}$$

$$\frac{dB_2}{dt} = \dot{E}_{32} + \dot{B}_{32} - \dot{E}_{21} - \dot{B}_{21}, \qquad (28)$$

and

$$\frac{dL_3}{dt} = \dot{E}_{13} + \dot{S}_{13} + \dot{L}_{13} + \dot{E}_{33} + \dot{S}_{33} + \dot{L}_{33} - \dot{E}_3 - \dot{S}_3 - \dot{L}_3 - \dot{E}_{31} - \dot{S}_{31} - \dot{L}_{31}$$

$$\dot{S}_{31} = \dot{S}_{33} + \dot{S}_{13}$$

$$\frac{dB_3}{dt} = \dot{E}_{13} + \dot{B}_{13} + \dot{E}_{33} + \dot{B}_{33} - \dot{E}_3 - \dot{B}_3 - \dot{E}_{31} - \dot{B}_{31}. \qquad (29)$$

As in Example A, we can substitute the First Law of Thermodynamics for the economic Sector (Equation 20) into the total energy accounting equation for the economic Sector (Equation 29). Assuming that $\dot{E}_{31} = 0$ (because energy is returned to the Earth as waste heat, not direct energy), we obtain

$$\frac{dL_3}{dt} = \dot{Q}_{31} + \dot{S}_{13} + \dot{L}_{13} + \dot{S}_{33} + \dot{L}_{33} - \dot{S}_{31} - \dot{L}_{31}$$

$$\frac{dB_3}{dt} = \dot{Q}_{31} + \dot{B}_{13} + \dot{B}_{33} - \dot{B}_{31} \qquad (30)$$

Similar to Example A, we observe that the accumulation rate of embodied energy in the Goods and Services sector (3) is the sum of the rates of waste heat from the sector ($\dot{Q}_{31}$) and embodied energy into the sector ($\dot{B}_{13} + \dot{B}_{33}$) less the rate of embodied energy leaving the sector on its output stream ($\dot{B}_{31}$).

5.4. Depreciation

We can substitute a depreciation term for the flow rate of embodied energy from the economic Sector (3) to the Earth (1) to obtain

$$\frac{dB_3}{dt} = \dot{Q}_{31} + \dot{B}_{13} + \dot{B}_{33} - \gamma_3 B_3. \qquad (31)$$

$$\frac{dL_3}{dt} = \dot{Q}_{31} + \dot{S}_{13} + \dot{L}_{13} + \dot{S}_{33} + \dot{L}_{33} - \dot{S}_{31} - \gamma_3 L_3$$

$$\Rightarrow \quad \frac{dL_3}{dt} = \dot{Q}_{31} + \dot{L}_{13} + \dot{L}_{33} - \gamma_3 L_3$$

5.5. Estimating energy intensity (ε) of the economy

335   The following figure shows value flows $(\dot{X})$ in the one-sector economy

336   with separate demand.

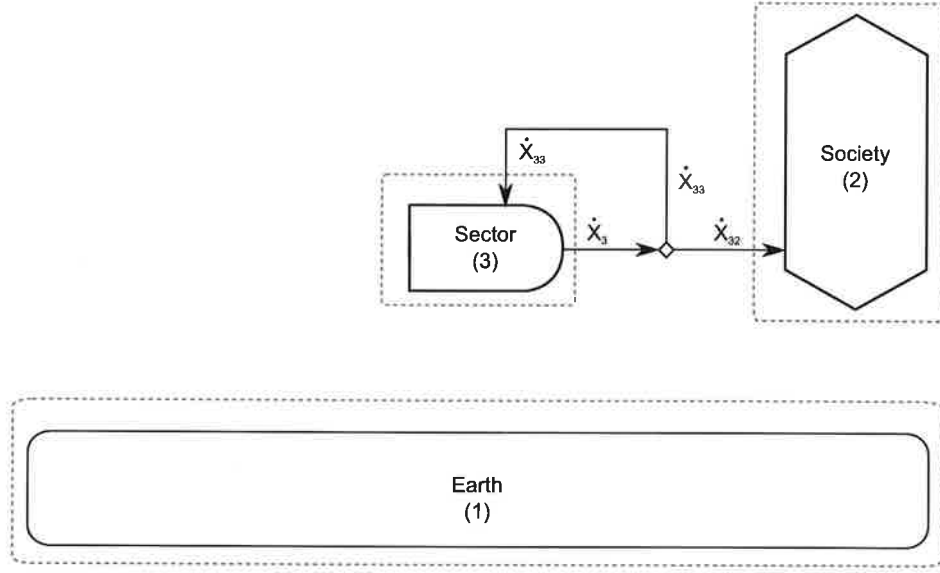Figure 6: Flows of economic value $(\dot{X})$ in a one-sector economy with separate demand.

337   The energy intensity (ε) of the economic Sector (3) is given by

$$\varepsilon_3 = \frac{\dot{T}_3}{\dot{X}_3} = \frac{\dot{T}_{33}}{\dot{X}_{33}}. \tag{32}$$

338   The input-output ratio (a) for the economic Sector (3) is

$$a_{33} = \frac{\dot{X}_{33}}{\dot{X}_3}. \tag{33}$$

339   Thus,

24

$$\dot{T}_3 = \varepsilon_3 \dot{X}_3, \tag{34}$$

340 and

$$\dot{T}_{33} = \varepsilon_3 a_{33}\dot{X}_3. \tag{35}$$

341 Realizing that (a) $\frac{dT_3}{dt} = \frac{dB_3}{dt}$ because $\frac{dE_3}{dt} = 0$, (b) $\dot{T}_{13} = \dot{E}_{13}$ because
342 $\dot{B}_{13} = 0$ due to processing of raw energy carriers occurring *within* the eco-
343 nomic Sector (3), and (c) substituting Equations 34 and 35 into Equation 25
344 gives

$$\frac{dB_3}{dt} = \varepsilon_3 a_{33}\dot{X}_3 + \dot{E}_{13} - \varepsilon_3 \dot{X}_3 - \gamma_3 B_3. \tag{36}$$

345 We can estimate the energy intensity of the economy by solving Equation
346 36 for ε_3.

$$\varepsilon_3 = (1 - a_{33})^{-1}\dot{X}_3^{-1}\left[\dot{E}_{13} - \left(\frac{dB_3}{dt} + \gamma_3 B_3\right)\right] \tag{37}$$

347 Equation 37 is similar to the typical energy intensity equation found in the
348 I-O literature [REFERENCE BULLARD AND OTHERS HERE. –MKH],
349 except that Equation 37 applies to a single economic sector and contains
350 scalar (as opposed to matrix) terms. Using Example C below, we will derive
351 a matrix representation of Equation 37 that is directly comparable to energy
352 intensity equations found in the I-O literature.

25

5.6. Derivation of economic sector energy intensity (ε) by a convergent infi-
     nite series

355   The single-sector economy of Figures 4 through 6 can be re-drawn as
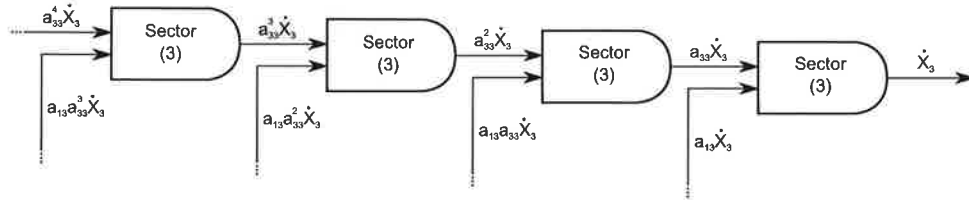356 shown in Figure 7.

Figure 7: Process flows in a single-sector economy.

[handwritten margin note: Is this true?]

[handwritten margin note: $a_{13} > 1$, $a_{33} < 1$]

357   The economy produces output at a rate of $\dot{X}_3$, but it requires energy from
358 the Earth ($\dot{E}_{13} = a_{13}\dot{X}_3$) to do so. The economy also consumes a fraction of
359 its own gross output ($\dot{X}_{33} = a_{33}\dot{X}_3$). To produce $a_{33}\dot{X}_3$, the economy requires
360 an additional $a_{13}a_{33}\dot{X}_3$ of energy from the Earth. The total energy required
361 for the economy to produce at a rate of $\dot{X}_3$ is an infinite sum.

[handwritten margin note: does this sequence always converge?]

$$\dot{E}_{demand} = a_{13}\dot{X}_3 + a_{13}a_{33}\dot{X}_3 + a_{13}a_{33}^2\dot{X}_3 + \cdots \tag{38}$$

362   The energy intensity of the economy (ε_3) is

$$\varepsilon_3 = \frac{\dot{E}_{demand}}{\dot{X}_3} = a_{13}(1 + a_{33} + a_{33}^2 + \cdots) = a_{13}\sum_{n=0}^{\infty} a_{33}^n. \tag{39}$$

[handwritten annotation: a_{33}^n)]

363   Realizing that $\sum_{n=0}^{\infty} a_{33}^n = \frac{1}{1-a_{33}}$ and $a_{13} = \frac{\dot{E}_{13}}{\dot{X}_3}$ gives

$$\varepsilon_1 = (1 - a_{33})^{-1}\dot{X}^{-1}\dot{E}_{13}. \tag{40}$$

26

364   Neglecting accumulation of embodied energy in the economy $\left(\frac{dB_3}{dt}\right)$ and

365   depreciation ($\gamma_3 B_3$), Equations 37 and 40 are identical (assuming $\frac{dB_3}{dt} =$

366    $\gamma_3 = 0$), indicating that the I-O approach accounts for the infinite recursion

367   of energy demand by the economy.

6. Example C: a two-sector economy

369   We extend single-sector Example B to derive a matrix representation for

370   the I-O method that can be generalized to any number of economic sectors.

371   A two-sector economy consisting of an Energy sector (3) and a Goods and

372   Services sector (4) is considered. Both the Earth (1) and Society (2) are

373   also included. Resources are extracted from the Earth (1), and Society (2)

374   provides the final demand for both the Goods and Services (4) and the Energy

375   (3) sectors.

6.1. First Law of Thermodynamics

377   The First Law of Thermodynamics requires that energy is conserved

378   around each sector of the economy as well as around the Earth (1) and

379   Society (2) as shown in Figure 8.

380   In this economy, we assume that the purpose of the Goods and Services

381   sector (4) is to produce goods and provide services, it provides no direct

382   energy to society. The purpose of the Energy sector (3) is to make direct

383   energy ($\dot{E}$) available to the economy and society in a useful form. Both direct

384   energy ($\dot{E}$) (such as chemical potential energy in coal, oil, and electricity)

385   and waste heat ($\dot{Q}$) are accounted by the First Law of Thermodynamics. The

386   First Law around the Goods and Services sector (4) including, for now, the

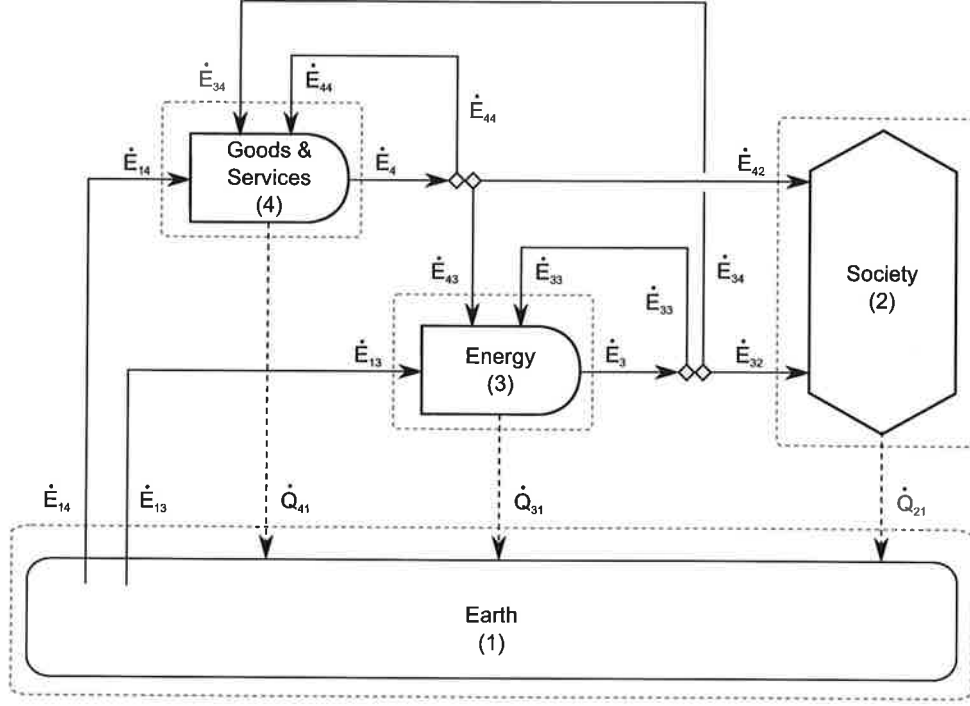387   accumulation rate of direct energy in the sector $\left(\frac{dE_4}{dt}\right)$ yields

Figure 8: Flows of direct energy ($\dot{E}$) and waste heat ($\dot{Q}$) in a two-sector economy.

$$\frac{\mathrm{d}E_4}{\mathrm{d}t} = \dot{E}_{14} + \dot{E}_{34} + \dot{E}_{44} - \dot{E}_4 - \dot{Q}_{41}. \tag{41}$$

388   Note that we may simplify Equation 41 by realizing that $\dot{E}_4 = \dot{E}_{4i} = 0$,
389 because the goods and services sector is assumed to produce no flows of
390 energy, and that $\dot{E}_{14} = 0$, since sector (4) receives no direct energy from the
391 earth, except via the energy sector (3), hence:

$$\frac{\mathrm{d}E_4}{\mathrm{d}t} = \dot{E}_{34} - \dot{Q}_{41}. \tag{42}$$

392   The First Law of Thermodynamics around the Earth (1), Society (2),
393 and the Energy sector (3) gives

28

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} = \dot{Q}_{21} + \dot{Q}_{31} + \dot{Q}_{41} - \dot{E}_{13} - \dot{E}_{14}, \tag{43}$$

$$\frac{\mathrm{d}E_2}{\mathrm{d}t} = \dot{E}_{32} + \dot{E}_{42} - \dot{Q}_{21}, \tag{44}$$

394 and

$$\frac{\mathrm{d}E_3}{\mathrm{d}t} = \dot{E}_{13} + \dot{E}_{33} + \dot{E}_{43} - \dot{E}_3 - \dot{Q}_{31}. \tag{45}$$

395 As in Examples A and B, we can set the accumulation of direct energy
396 to zero.

$$0 = \dot{Q}_{21} + \dot{Q}_{31} + \dot{Q}_{41} - \dot{E}_{13} - \dot{E}_{14} \tag{46}$$

$$0 = \dot{E}_{32} + \dot{E}_{42} - \dot{Q}_{21} \tag{47}$$

$$0 = \dot{E}_{13} + \dot{E}_{33} + \dot{E}_{43} - \dot{E}_3 - \dot{Q}_{31} \tag{48}$$

397 and

$$0 = \dot{E}_{14} + \dot{E}_{34} + \dot{E}_{44} - \dot{E}_4 - \dot{Q}_{41} \tag{49}$$

398 *6.2. Total energy accounting*

399 Again, we follow the I-O literature in assuming that total energy (i.e.,
400 the sum of direct energy and embodied energy) is conserved. Thus, we can
401 draw a diagram similar to Figure 8 for total energy flows.

29
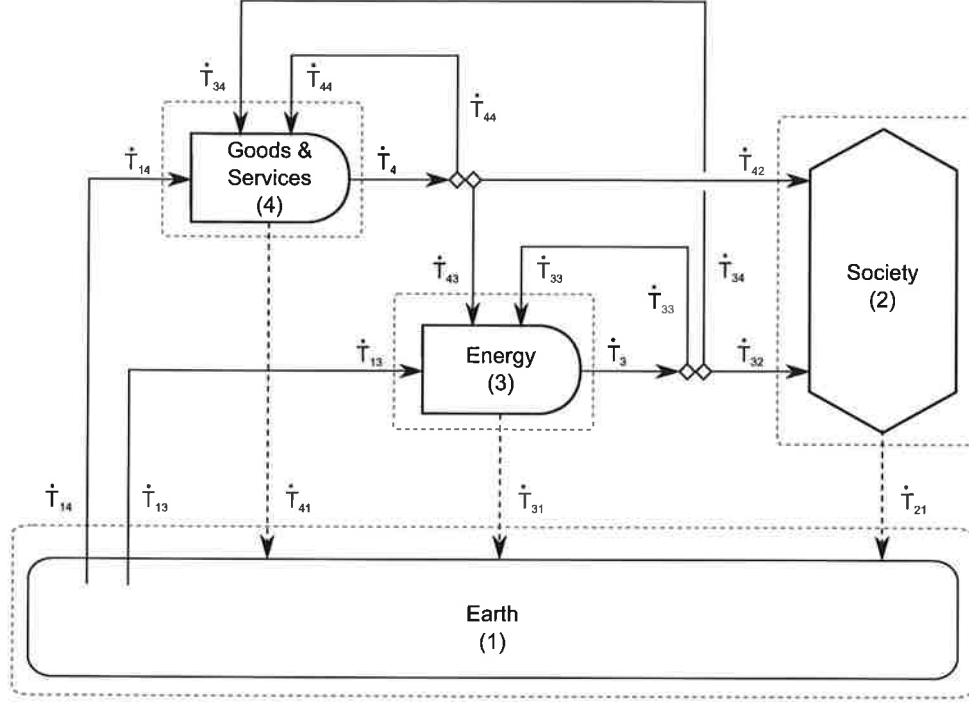
Figure 9: Flows of total energy ($\dot{T}$) in a two-sector economy.

Accounting for accumulation of total energy and using the assumption that total energy is conserved, we can write the following equations.

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = \dot{T}_{21} + \dot{T}_{31} + \dot{T}_{41} - \dot{T}_{13} - \dot{T}_{14}, \tag{50}$$

$$\frac{\mathrm{d}T_2}{\mathrm{d}t} = \dot{T}_{32} + \dot{T}_{42} - \dot{T}_{21}, \tag{51}$$

$$\frac{\mathrm{d}T_3}{\mathrm{d}t} = \dot{T}_{13} + \dot{T}_{33} + \dot{T}_{43} - \dot{T}_3 - \dot{T}_{31}, \tag{52}$$

and

30

$$\frac{dT_4}{dt} = \dot{T}_{14} + \dot{T}_{34} + \dot{T}_{44} - \dot{T}_4 - \dot{T}_{41}. \tag{53}$$

6.3. Embodied energy accounting

Given that $\frac{dE_i}{dt} = 0$, we again note that $\frac{dT_i}{dt} = \frac{dB_i}{dt}$. Substituting $\dot{T} = \dot{E} + \dot{B}$ into the total energy accounting equations gives

(handwritten, left margin: (54))
$$\frac{dS_1}{dt} + \frac{dL_1}{dt} = \dot{E}_{21} + \dot{S}_{21} + \dot{L}_{21} + \dot{E}_{31} + \dot{S}_{31} + \dot{L}_{31} + \dot{E}_{41} + \dot{S}_{41} + \dot{L}_{41} - \dot{E}_{13}\left(-\dot{S}_{13} - \dot{L}_{13}\right) - \dot{L}_{14}\left(-\dot{S}_{14} - \dot{L}_{14}\right)$$

$$\frac{dB_1}{dt} = \dot{E}_{21} + \dot{B}_{21} + \dot{E}_{31} + \dot{B}_{31} + \dot{E}_{41} + \dot{B}_{41} - \dot{E}_{13} - \dot{B}_{13} - \dot{E}_{14} - \dot{B}_{14}, \tag{54}$$

(handwritten: (55))
$$\frac{dL_2}{dt} = \dot{E}_{32} + \dot{L}_{32} + \dot{E}_{42} + \dot{L}_{42} - \dot{E}_{21} + \left(\dot{S}_{32} + \dot{S}_{42} - \dot{S}_{21}\right) - \dot{L}_{21}$$

$$\frac{dB_2}{dt} = \dot{E}_{32} + \dot{B}_{32} + \dot{E}_{42} + \dot{B}_{42} - \dot{E}_{21} - \dot{B}_{21}, \tag{55}$$

(handwritten: (56))
$$\frac{dL_3}{dt} = \dot{E}_{13} + \dot{L}_{13} + \dot{E}_{33} + \dot{L}_{33} + \dot{E}_{43} + \dot{L}_{43} - \dot{E}_3 - \dot{S}_3 - \dot{L}_3 - \dot{E}_{31} - \dot{L}_{31} + \left(\dot{S}_{13} + \dot{S}_{33} + \dot{S}_{43} - \dot{S}_{31}\right)$$

$$\frac{dB_3}{dt} = \dot{E}_{13} + \dot{B}_{13} + \dot{E}_{33} + \dot{B}_{33} + \dot{E}_{43} + \dot{B}_{43} - \dot{E}_3 - \dot{B}_3 - \dot{E}_{31} - \dot{B}_{31}, \tag{56}$$

and

(handwritten: (57))
$$\frac{dL_4}{dt} = \dot{E}_{14} + \dot{L}_{14} + \dot{E}_{34} + \dot{L}_{34} + \dot{E}_{44} + \dot{L}_{44} - \dot{E}_4 - \dot{S}_4 - \dot{L}_4 - \dot{E}_{41} - \dot{L}_{41} + \left(\dot{S}_{14} + \dot{S}_{44} + \dot{S}_{34} - \dot{S}_{41}\right)$$

$$\frac{dB_4}{dt} = \dot{E}_{14} + \dot{B}_{14} + \dot{E}_{34} + \dot{B}_{34} + \dot{E}_{44} + \dot{B}_{44} - \dot{E}_4 - \dot{B}_4 - \dot{E}_{41} - \dot{B}_{41}. \tag{57}$$

Substituting the First Law of Thermodynamics (Equations 46 through 49) into the total energy accounting equations (Equations 54 through 57) gives embodied energy accounting equations for Example C.

(handwritten: (58))
$$\frac{dS_1}{dt} + \frac{dL_1}{dt} = \dot{S}_{21} + \dot{L}_{21} + \dot{S}_{31} + \dot{L}_{31} + \dot{S}_{41} + \dot{L}_{41}\left(-\dot{S}_{13} + \dot{L}_{13} - \dot{S}_{14} - \dot{L}_{14}\right) - \dot{Q}_{21} - \dot{Q}_{31} - \dot{Q}_{41}$$

$$\frac{dB_1}{dt} = \dot{B}_{21} + \dot{B}_{31} + \dot{B}_{41} - \dot{B}_{13} - \dot{B}_{14} - \dot{Q}_{21} - \dot{Q}_{31} - \dot{Q}_{41} \tag{58}$$

$$\frac{dB_2}{dt} = \dot{B}_{32} + \dot{B}_{42} + \dot{Q}_{21} - \dot{B}_{21} \tag{59}$$

31

(handwritten: (59))
$$\frac{dL_2}{dt} = \dot{L}_{32} + \dot{L}_{42} + \dot{Q}_{21} - \dot{L}_{21} + \left(\dot{S}_{32} + \dot{S}_{42} - \dot{S}_{21}\right)$$

$$(60) \quad \frac{dL_3}{dt} = \dot{L}_{13} + \dot{L}_{33} + \dot{L}_{43} + \dot{Q}_{31} - \dot{S}_3 - \dot{L}_3 \overset{-\dot{L}_{31}}{\left(+ \dot{S}_{13} + \dot{S}_{33} + \dot{S}_{43} - \dot{S}_{31} \right)} = 0$$

$$\frac{dB_3}{dt} = \dot{B}_{13} + \dot{B}_{33} + \dot{B}_{43} + \dot{Q}_{31} - \dot{B}_3 - \dot{B}_{31} \qquad (60)$$

$$(61) \quad \frac{dL_4}{dt} = \dot{L}_{14} + \dot{L}_{34} + \dot{L}_{44} + \dot{Q}_{41} - \dot{S}_4 - \dot{L}_4 - \dot{L}_{41} \left(+ \dot{S}_{14} + \dot{S}_{34} + \dot{S}_{44} - \dot{S}_{41} \right) = 0$$

$$\frac{dB_4}{dt} = \dot{B}_{14} + \dot{B}_{34} + \dot{B}_{44} + \dot{Q}_{41} - \dot{B}_4 - \dot{B}_{41} \qquad (61)$$

412   To verify the above derivation, we sum Equations 58 through 61 and use

413   the following identities:

$$\dot{S}_3 + \dot{L}_3 = \dot{L}_{32} + \dot{L}_{33} + \dot{L}_{34} + \dot{S}_{32} + \dot{S}_{33} + \dot{S}_{34} \qquad (62)$$

$$\dot{B}_3 = \dot{B}_{32} + \dot{B}_{33} + \dot{B}_{34} \qquad (62)$$

414   and

$$\dot{S}_4 + \dot{L}_4 = \dot{L}_{41} \dot{S}_{42} + \dot{L}_{42} + \dot{S}_{43} + \dot{L}_{43} + \dot{S}_{44} + \dot{L}_{44} \qquad (63)$$

$$\dot{B}_4 = \dot{B}_{42} + \dot{B}_{43} + \dot{B}_{44}; \qquad (63)$$

415   to obtain

$$\frac{dB_1}{dt} + \frac{dB_2}{dt} + \frac{dB_3}{dt} + \frac{dB_4}{dt} = 0 \qquad (64)$$

416   as expected. The total embodied energy content of the system (Earth (1),

417   Society (2), Energy sector (3), and Goods and Services sector (4)) is constant

418   with respect to time.

419   *6.4. Definition of embodied energy ($\dot{B}$)*

420   [DOES THIS SECTION BELONG HERE?]

421   [I MOVED THIS SECTION TO FOLLOW DIRECTLY AFTER THE

422   EMBODIED ENERGY ACCOUNTING SECTION FOR EXAMPLE C. I

423   THINK IT FITS BETTER HERE. –MKH]

$$\frac{dS_1}{dt} + \frac{dL_1}{dt} + \frac{dL_2}{dt} + \frac{dL_3}{dt} + \frac{dL_4}{dt} = \left(\dot{S}_{21} + \dot{L}_{21} + \dot{S}_{31} + \dot{L}_{31} + \dot{S}_{41} + \dot{L}_{41} - \dot{Q}_{21} - \dot{Q}_{31} - \dot{Q}_{41} \right) + \left(\dot{L}_{32} + \dot{L}_{42} + \dot{Q}_{21} - \dot{L}_{21} \right)$$

32

$$+ \left(\dot{L}_{13} + \dot{L}_{33} + \dot{L}_{43} + \dot{Q}_{31} - \dot{S}_3 - \dot{L}_3 \right) + \left(\dot{S}_{13} + \dot{S}_{33} + \dot{S}_{43} - \dot{S}_{31} \right) - \dot{L}_{31}$$

$$+ \left(\dot{L}_{14} + \dot{L}_{34} + \dot{L}_{44} + \dot{Q}_{41} - \dot{S}_4 - \dot{L}_4 \right) + \left(\dot{S}_{14} + \dot{S}_{34} + \dot{S}_{44} - \dot{S}_{41} \right) - \dot{L}_{41}$$

(64) check

$$+ \left(\dot{S}_{32} + \dot{S}_{42} - \dot{S}_{21} \right) = \varnothing \quad \checkmark$$

At this point we can develop a rigorous definition of embodied energy. To do so, we use the Goods and Services sector (4) from Example C. Direct energy accounting around the Goods and Services sector (Figure 8) yields

$$\frac{\mathrm{d}E_4}{\mathrm{d}t} = \dot{E}_{14} + \dot{E}_{34} + \dot{E}_{44} - \dot{E}_4 - \dot{Q}_{41}, \tag{65}$$

Total energy accounting around the Goods and Services sector (Figure 9) yields d

$$\frac{\mathrm{d}T_4}{\mathrm{d}t} = \dot{T}_{14} + \dot{T}_{34} + \dot{T}_{44} - \dot{T}_4 + \dot{T}_{41}, \tag{66}$$

Solving the direct energy equation (Equation 65) for the rate of direct energy input from the Energy sector (3) to the Goods and Services sector (4), namely $\dot{E}_{34}$, substituting into the total energy equation (Equation 66), solving the result for $\dot{B}_4$, and assuming that no direct energy is wasted by the Goods and Services sector (4) to the Earth (1), i.e. $\dot{E}_{41} = 0$, yields

$$\dot{S}_4 + \dot{L}_4 = \dot{S}_{14} + \dot{L}_{14} + \dot{L}_{34} + \dot{L}_{44} + \dot{Q}_{41} - \frac{d L_4}{d t_4} - \dot{L}_{41} + \left(\dot{S}_{34} + \dot{S}_{44} - \dot{S}_{41} \right)^{=0} \quad \left(67\right)$$

$$\dot{B}_4 = \dot{B}_{14} + \dot{B}_{34} + \dot{B}_{44} + \dot{Q}_{41} - \frac{\mathrm{d}B_4}{\mathrm{d}t} - \dot{B}_{41}. \tag{67}$$

Written generally, we obtain a formal definition for embodied energy output from an economic sector:

$$\dot{S}_j + \dot{L}_j \equiv \sum_i \dot{L}_{ij} - \frac{d L_j}{d t} - \dot{L}_{ji} + \dot{Q}_{j1} \quad \left(68\right)$$

$$\dot{B}_j \equiv \sum_i \dot{B}_{ij} - \frac{\mathrm{d}B_j}{\mathrm{d}t} - \dot{B}_{j1} + \dot{Q}_{j1}. \tag{68}$$

Rearranging, we obtain

$$\frac{\mathrm{d}B_j}{\mathrm{d}t} = \sum_i \dot{B}_{ij} - \dot{B}_j - \dot{B}_{j1} + \dot{Q}_{j1}. \tag{69}$$

$$\frac{d L_j}{d t} = \sum_i \dot{L}_{ij} - \dot{L}_j \overset{33}{-} \dot{S}_j - \dot{L}_{ji} + \dot{Q}_{j1} \quad \left(69\right)$$

In words, the rate of accumulation of embodied energy in a sector of the economy $\left(\frac{\mathrm{d}B_j}{\mathrm{d}t}\right)$ is equal to the sum of the rates of input of embodied energy into the sector $\left(\sum_i \dot{B}_{ij}\right)$ less the rate of useful output of embodied energy from the sector $(\dot{B}_j)$ less the rate of wasting embodied energy by the sector $(\dot{B}_{j1})$ *plus* the rate of waste heat from the sector $(\dot{Q}_{j1})$. The first three terms on the right side of the equation are expected: accumulation is the difference between inflow and outflow rates. However, we see that the last term $(+\,\dot{Q}_{j1})$ in the above equations indicates that waste heat is *additive* to both accumulation of embodied energy in a sector of the economy (Equation 69) and outflow of embodied energy from a sector of the economy (Equation 68). Furthermore, because the waste heat appears in the embodied energy output from a sector, waste heat accumulates along each step of a process such that the energy embodied in a finished product is the *sum* of waste heats along a process path.

6.5. Depreciation $\dot{L}_{21},\ \dot{L}_{31}\quad \dot{L}_{41}$

The terms $\dot{B}_{21}$, $\dot{B}_{31}$, and $\dot{B}_{41}$ represent material depreciation (i.e., disposal) rates. As before, we can represent the embodied energy content of material depreciation as $\dot{B}_{i1} = \gamma_i B_i$ to obtain

$$\ddot{L}_{i1} = \gamma_i \dot{L}_i$$

$$\frac{\mathrm{d}B_1}{\mathrm{d}t} = \gamma_2 B_2 + \gamma_3 B_3 + \gamma_4 B_4 - \dot{B}_{13} - \dot{B}_{14} - \dot{Q}_{21} - \dot{Q}_{31} - \dot{Q}_{41} \qquad (70)$$

$$\frac{d\dot{S}_i}{dt} + \frac{dL_1}{dt} = \gamma_2 L_2 + \gamma_3 L_3 + \gamma_4 L_4 + \dot{S}_{21} + \dot{S}_{31} + \dot{S}_{41} - \dot{S}_{13} - \dot{L}_{13} - \dot{S}_{14} - \dot{L}_{14} - \dot{Q}_{21} - \dot{Q}_{31} - \dot{Q}_{41}$$

$$\frac{\mathrm{d}B_2}{\mathrm{d}t} = \dot{B}_{32} + \dot{B}_{42} + \dot{Q}_{21} - \gamma_2 B_2 \qquad (71)$$

$$= 0$$

$$\frac{dL_2}{dt} = \dot{L}_{32} + \dot{L}_{42} + \dot{Q}_{21} - \gamma_2 L_2 + \left(\dot{S}_{32} + \dot{S}_{42} - \dot{S}_{21}\right)$$

$$\frac{\mathrm{d}B_3}{\mathrm{d}t} = \dot{B}_{13} + \dot{B}_{33} + \dot{B}_{43} + \dot{Q}_{31} - \dot{B}_3 - \gamma_3 B_3 \qquad (72)$$

34

$$\frac{dL_3}{dt} = \dot{L}_{13} + \dot{S}_{13} + \dot{L}_{33} + \dot{L}_{43} + \dot{Q}_{31} - \dot{S}_3 - \dot{L}_3 - \gamma_3 \dot{L}_3 + \left(\dot{S}_{33} + \dot{S}_{43} - \dot{S}_{31}\right)$$

$$= 0$$

$$\frac{dL_4}{dt} = \mathcal{B}\,\dot{S}_{14} + \dot{L}_{14} + \dot{L}_{34} + \dot{L}_{44} + \dot{Q}_{41} - \dot{S}_4 - \dot{L}_4 - \gamma_4 L_4 \left(+ \dot{S}_{34} + \dot{S}_{44} \overset{=\ 0}{-} \dot{S}_{41} \right)$$

$$\frac{\mathrm{d}B_4}{\mathrm{d}t} = \dot{B}_{14} + \dot{B}_{34} + \dot{B}_{44} + \dot{Q}_{41} - \dot{B}_4 - \gamma_4 B_4 \tag{73}$$

6.6. Energy ratios

[THIS SECTION IS INTERESTING TO US, BUT MAY NOT BE GERMANE TO THE PAPER. IF THIS WERE TO BE A BOOK, I THINK WE SHOULD KEEP THIS SECTION IN. IF WE'RE NOW SHOOTING FOR A PAPER, WE CAN PROBABLY LEAVE THIS SECTION OUT. WE SHOULD AT LEAST REFERENCE THE LITERATURE HERE. OR, WE SHOULD MAKE USE OF THESE RATIOS. FOR EXAMPLE, IF WE WANT TO TALK ABOUT DECLINING RESOURCE QUALITY IN TERMS OF *EROI* OR *GER*, WE SHOULD DO SO. WE SHOULD THEN SUBSTITUTE *GER* INTO THE DERIVED EQUATIONS AND SHOW HOW A CHANGE IN *GER* AFFECTS ENERGY FLOWS THROUGH THE ECONOMY. –MKH]

Several important energy ratios can be observed in Figure 8. [REFERENCE OTHER PAPERS FOR NOMENCLATURE? –MKH] The Gross Energy Ratio (*GER*) is defined as:

$$GER_\beta \equiv \frac{\dot{E}_3}{\dot{E}_{33}}. \tag{74}$$

The Net Energy Ratio (*NER*) is defined as [SHOULD THE NUMERATOR FOR NER_β BE $\dot{E}_{32} + \dot{E}_{34}$? –MKH] — No β only includes own pathway flows.

[MIK SHOULD CHECK THE NEXT TWO EQUATIONS. ARE THEY CORRECT? –MKH]

35

$$NER_\beta \equiv \frac{\dot{E}_{32}}{\dot{E}_{33}} = GER - 1. \tag{75}$$

[handwritten: IS THIS CORRECT? ✓]

474 [CAN WE SIMPLIFY THESE? FOR EXAMPLE, $\dot{E}_{43} = 0$, SO THE
475 DENOMINATOR CAN BE SIMPLIFIED TO $\dot{B}_{43}$ –MKH]

$$GER_\delta \equiv \frac{\dot{E}_3}{\dot{T}_{33} + \dot{T}_{43}}; \tag{76}$$

[handwritten: $= \dfrac{\dot{E}_3}{\dot{T}_{33} + \dot{B}_{43}}$] [handwritten: $= \dfrac{\dot{E}_3}{\dot{T}_{33} + \dot{S}_{43} + \dot{L}_{43}}$]

476 [THIS EQUATION DOESN'T LOOK RIGHT. IT IS SAME AS THE
477 PREVIOUS EQUATION $GER - 1$, BUT HAS A DIFFERENT DENOMI-
478 NATOR. MIK SHOULD CHECK. –MKH]

[handwritten: $= \dfrac{\dot{E}_3}{\dot{S}_{33} + \dot{L}_{33} + \dot{E}_{33} + \dot{S}_{43} + \dot{L}_{43}}$]

$$NER_\beta \equiv \frac{\dot{E}_{32}}{\dot{T}_{33} + \dot{T}_{43}} = GER - 1; \tag{77}$$

[handwritten: $= \dfrac{\dot{E}_{32}}{\dot{S}_{33} + \dot{L}_{33} + \dot{E}_{33} + \dot{S}_{43} + \dot{L}_{43}}$]

479 We may also define a Gross External Energy Ratio (GEER) and Net
480 External Energy Ratio (NEER), which account only for inputs from other
481 sectors in the denominator:

$$GEER_\delta \equiv \frac{\dot{E}_3}{\dot{T}_{43}}, \tag{78}$$

[handwritten: $= \dfrac{\dot{E}_3}{\dot{S}_{43} + \dot{L}_{43}}$]

482 and

$$NEER_\beta \equiv \frac{\dot{E}_{32}}{\dot{T}_{43}}. \tag{79}$$

[handwritten: $= \dfrac{\dot{E}_{32}}{\dot{S}_{43} + \dot{L}_{43}}$]

483 *6.7. Final demand*

484 Society's demand vector for total energy, $\dot{T}$, can be written as

$$\mathbf{Y}_{\dot{T}} = \begin{Bmatrix} \dot{T}_{32} \\ \dot{T}_{42} \end{Bmatrix} = \begin{Bmatrix} \dot{E}_{32} + \dot{S}_{32} + \dot{L}_{32} \\ \dot{E}_{42} + \dot{S}_{42} + \dot{L}_{42} \end{Bmatrix} \tag{80}$$

In terms of total energy, the ultimate demand $(Y_{\dot{T}})$ is given by

$$Y_{\dot{T}} = \sum_{i=3}^{N} \dot{T}_{i2} = \dot{T}_{32} + \dot{B}_{42}. \quad = \dot{E}_{32} + \dot{S}_{32} + \dot{L}_{32} \quad (81)$$
$$+ \dot{S}_{42} + \dot{L}_{42}$$

after realizing that $\dot{E}_{42} = 0$.[8]

Using $\dot{T}_{32} = \dot{E}_{32} + \dot{B}_{32}$ and rearranging Equation 81 gives

---

[8]We acknowledge that there are examples in the real economy which run counter to this model, where output from non-energy sectors are valued for their energetic content, one example being agriculture. [MIK: I SUGGEST THAT THIS SECTION ABOUT LABOR BE MOVED ELSEWHERE, FOR EXAMPLE, NEARER THE SECTIONS WHERE WE DISCUSS FINAL DEMAND VECTORS, Y? –MKH] "Direct" energy inputs also flow in the opposite direction in the form of labor, which we also neglect. This will serve to introduce errors which will be small for industrial economies and larger for less industrial societies. To illustrate this we may compare the United States with India. To feed an adult requires around 2000 kcal/day $\approx$ 3 GJ/yr. To feed the whole $\sim$300 million population of the States requires around 1×10^{18}J (1 EJ) which is around 1% of the roughly 100 EJ of primary energy supply. The US labor force currently stands at around 240 million. Given that a human can supply around 100 W of power and assuming an 8 hour work day, the US labor force will supply 70 TWh/yr $\approx$ 0.25 EJ. For India, the energy to food to feed 1.25 billion people is nearly 4 EJ which is around 15% of the $\sim$25 EJ of primary energy consumed. Assuming that the labor force makes up 500 million people working at 12 hours per day, the energy supplied by labor is around 0.8 EJ or around 3% of the total primary energy. As such, we can see that food energy accounts for around 1% of primary energy in the US and around 15% in India. Similarly, the labor inputs account for around 0.25% in the US and around 3% in India. The implication of including or omitting these flows is different in each case. Our assumptions introduce small errors for industrial societies where most of the world's energy is consumed.

37

$$\dot{B}_{32} + \dot{B}_{42} = Y_{\dot{T}} - \dot{E}_{42}. \tag{82}$$

488 Substituting Equation 82 into Equation 71 yields

$$\frac{\mathrm{d}B_2}{\mathrm{d}t} = Y_{\dot{T}} - \dot{E}_{32} + \dot{Q}_{21} - \gamma_2 B_2. \tag{83}$$

489 Substituting Equation 47 into Eqaution 83 and realizing that $\dot{E}_{42} = 0$
490 because direct energy is supplied to society by the energy sector only, we
491 obtain

$$\frac{\mathrm{d}B_2}{\mathrm{d}t} = Y_{\dot{T}} - \gamma_2 B_2, \tag{84}$$

492 [IT WOULD BE GOOD TO FIND SOME VERY ROUGH DATA FOR
493 THIS VALUE, E.G. WHAT IS THE AVERAGE LIFETIME OF MANU-
494 FACTURED GOODS - INCLUDING PACKAGING AND NON-CONSUMER
495 GOODS. WHAT IS THE BALANCE OF NON-DURABLE VS. DURABLE
496 GOODS? WHAT PROPORTION OF GOODS (E.G. FOOD) IS WASTED
497 BEFORE EVER BEING CONSUMED?]

498 [I AGREE. HOW? –MKH]

499 indicating that the final demand vector for total energy $(Y_{\dot{T}})$ and the ac-
500 cumulation rate of energy in society $\left(\frac{\mathrm{d}B_2}{\mathrm{d}t}\right)$ differ by the rate of disposal
501 from society $(\gamma_2 B_2)$. We note that as total embodied energy in society (B_2)
502 becomes increasingly large, we need an ever-increasing rate of energy sup-
503 plied to the society $(Y_{\dot{T}})$ to maintain positive growth $\left(\frac{\mathrm{d}B_2}{\mathrm{d}t}\right)$. [MAIN POINT
504 THAT MUST BE DISCUSSED IN FURTHER DETAIL LATER, PARTIC-
505 ULARLY IN RELATION TO INCREASING GDP NOT NECESSARILY
506 SIGNALING INCREASING ACCUMULATION OR GROWTH.]

507 [WE NEVER USE THE FINAL PARAGRAPHS OF THIS SECTION. I

508 PROPOSE THAT WE DELETE THEM. –MKH]

[handwritten: if we keep this it would be worth drawing CV on Figure]

509 A control volume around the economy gives

[handwritten: SHOULD BE $Y_{\dot{T}}$ ←]

$$Y_{\dot{B}} = \sum_{j=1}^{N} \dot{E}_{1j} - \sum_{i=1}^{N} \dot{B}_{i1} - \dot{Q}_{21}, \qquad (85)$$

[handwritten: $\Rightarrow \sum_i^N \dot{B}_{i1} = \sum_i^N \dot{S}_{i1} + \sum_i^N \dot{L}_{i1}$]

510 [I HAVE ADDED $\dot{Q}_{21}$ TERM HERE, AS THERE IS DEFINITELY

511 WASTE HEAT FROM CONSUMPTION OF FINAL ENERGY $Y_{\dot{B}}$ THAT

512 IS NOT EMBODIED IN GOOD OR SERVICE, E.G. THE ELECTRICITY

513 I USE TO COOK MY DINNER, OR WATCH TV]

514 illustrating that final demand to society ($Y_{\dot{B}}$) can be considered the ultimate

515 energy sink in the absence of depreciation in the economy.

516 It is also interesting to note that

$$\sum_{i=1}^{N} \dot{T}_i > Y_{\dot{T}}, \qquad (86)$$

517 which is a consequence of the self-demand of sectors within the economy.

518 *6.8. Flows of Value $(\dot{X})$*

519 The following figure shows value flows $(\dot{X})$ in the two-sector economy.

520 Realizing that the valuable output from energy sectors is direct energy,

521 $\dot{X}_3 = \dot{E}_3$ and $\dot{X}_{3j} = \dot{E}_{3j}$. Thus, outputs from energy sectors are given in

522 energy units (joules or BTUs). With reference to Equations ?? and ??, we

523 can express the Gross Energy Ratio (GER) and Net Energy Ratio (NER)

524 as

$$GER_\gamma = \frac{\dot{E}_3}{\dot{E}_{33}} = \frac{\dot{X}_3}{\dot{X}_{33}} = \frac{1}{a_{33}}, \qquad (87)$$

[handwritten: β under GER; $\frac{\dot{E}_3}{\dot{E}_{33}} \alpha_{33}$]

39
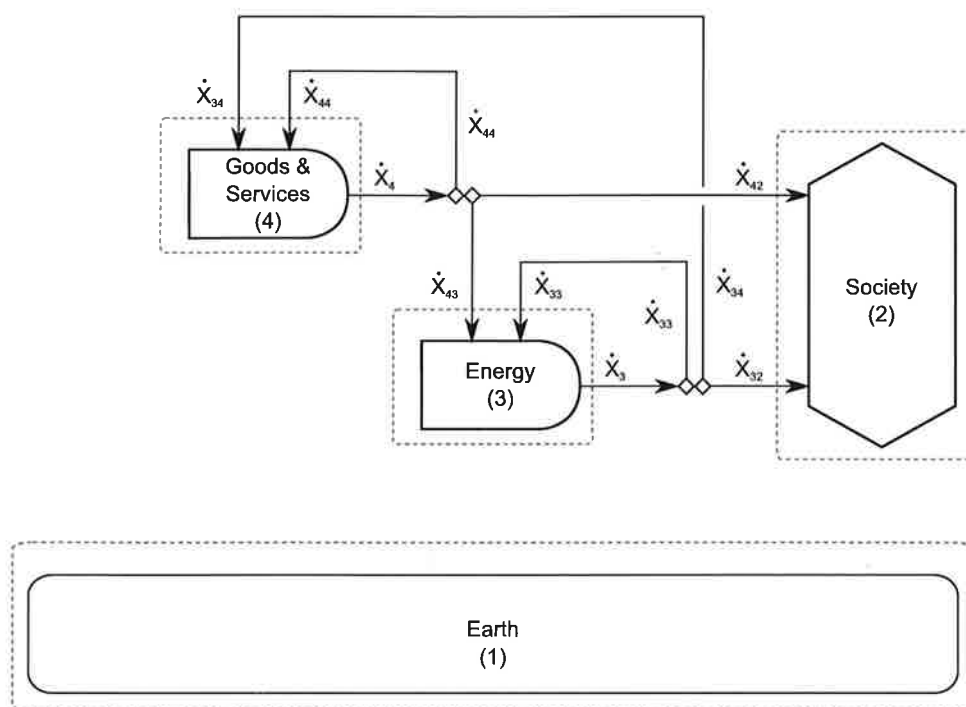
Figure 10: Flows of economic value ($\dot{X}$) in a two-sector economy.

and

$$NER_{\beta} = GER_{\beta} - 1 = \frac{1}{a_{33}} - 1. \tag{88}$$

[DO WE NEED THESE NEXT TWO EQUATIONS? DO WE USE THEM ANYWHERE? THEY MIGHT BE HELPFUL FOR A GDP DISCUSSION, BUT WE HAVEN'T INCLUDED THAT DISCUSSION YET. –MKH]

Written in terms of value flows, the ultimate demand vector ($\mathbf{Y}$) is given by

40

$$\dot{T}_{33} + \dot{T}_{43} + \ddot{E}_{13} - \frac{dB_3}{dt} - \gamma_3 \beta_3 = \dot{X} \dot{T}_3 \quad \left| \quad \dot{E}_{13} + \dot{E}_{33} + \overset{=0}{\dot{E}_{43}} = \dot{E}_3 + \dot{Q}_{31} \right.$$

$$(91) \quad \dot{E}_{33} + \dot{S}_{33} + \dot{L}_{33} + \dot{E}_{43} + \dot{S}_{43} + \dot{L}_{43} + \dot{E}_{13} - \frac{dL_3}{dt} - \gamma_3 B_3 - \dot{S}_{31} = \dot{S}_3 + \dot{L}_3 \cdot \dot{E}_3$$

$$(72) \quad \frac{dL_3}{dt} = \overset{=0}{\dot{L}_{13}} + \overset{=0}{\dot{S}_{13}} + \dot{S}_{33} + \dot{L}_{33} + \dot{S}_{43} + \dot{L}_{43} + \dot{Q}_{31} - \dot{S}_3 - \dot{L}_3 - \gamma_3 B_3 - \dot{S}_{31}$$

$$\mathbf{Y}_{\dot{X}} = \left\{ \begin{array}{c} \dot{X}_{32} \\ \dot{X}_{42} \end{array} \right\}, \tag{89}$$

₅₃₂ and the total value demand from society (Y) is

$$Y_{\dot{X}} = \sum_{i=1}^{N} \dot{X}_{i2} = \dot{X}_{32} + \dot{X}_{42}. \tag{90}$$

$(13) \quad \varepsilon_i = \dfrac{\dot{T}_i}{\dot{X}_i}$

₅₃₃ *6.9. Matrix Formulation*

$(72) \quad (73)$

₅₃₄ We can use Equations 13 through 15 to rewrite Equations ?? and ?? as

$(14) \quad \varepsilon_{ij} = \dfrac{\dot{T}_{ij}}{\dot{X}_{ij}}$

$$\overset{\dot{T}_{33}}{\dot{X}_{33}\varepsilon_3} + \overset{\dot{T}_{43}}{\dot{X}_{43}\varepsilon_4} + \overset{+\dot{E}_{13}}{\dot{E}_{13}} - \frac{dB_3}{dt} - \gamma_3 B_3 = \overset{\dot{T}_3}{\dot{X}_3\varepsilon_3} \tag{91}$$

I don't get this → see above.

$(15) \quad \varepsilon_i = \varepsilon_{ij}$ ₅₃₅ and

$$\dot{X}_{33}\varepsilon_3 + \dot{X}_{43}\varepsilon_4 + \dot{E}_{13} - \frac{dL_3}{dt} - \gamma_3 B_3 - \dot{S}_{31} = \dot{X}_3 \varepsilon_3 \qquad (91)$$

$$\dot{X}_{34}\varepsilon_3 + \dot{X}_{44}\varepsilon_4 + \dot{E}_{14} - \frac{dB_4}{dt} - \gamma_4 B_4 = \dot{X}_4\varepsilon_4. \tag{92}$$

₅₃₆ We can rewrite Equations 91 and 92 in matrix notation with the following
₅₃₇ definitions:

$$\dot{X}_{34}\varepsilon_3 + \dot{X}_{44}\varepsilon_4 + \dot{E}_{14} - \frac{dL_4}{dt} - \gamma_4 B_4 - \dot{S}_{44} = \dot{X}_{e4} \varepsilon_4 \qquad (92)$$

$$\varepsilon = \left\{ \begin{array}{c} \varepsilon_3 \\ \varepsilon_4 \end{array} \right\}, \tag{93}$$

$$\mathbf{E} = \left\{ \begin{array}{c} \dot{E}_{13} \\ \dot{E}_{14} \end{array} \right\}, \tag{94}$$

$$\frac{d\mathbf{B}}{dt} = \left\{ \begin{array}{c} \frac{dB_3}{dt} \\ \frac{dB_4}{dt} \end{array} \right\}, \quad = \frac{dL}{dt} = \left\{ \begin{array}{c} \frac{dL_3}{dt} \\ \frac{dL_4}{dt} \end{array} \right\} \tag{95}$$

41

$$\bar{S} = \left\{ \begin{matrix} S_{31} \\ S_{41} \end{matrix} \right\}$$

$$\mathbf{B} = \left\{ \begin{matrix} B_3 \\ B_4 \end{matrix} \right\}, \quad \Rightarrow \quad L = \left\{ \begin{matrix} L_3 \\ L_4 \end{matrix} \right\} \tag{96}$$

$$\mathbf{A} = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}, \tag{97}$$

$$\mathbf{X}_t = \begin{bmatrix} \dot{X}_{33} & \dot{X}_{34} \\ \dot{X}_{43} & \dot{X}_{44} \end{bmatrix}, \tag{98}$$

Is this right?

$$\hat{\mathbf{X}} = \delta_{ij}\dot{X}_j, \tag{99}$$

$$\hat{\gamma} = \delta_{ij}\gamma_j, \tag{100}$$

538 and

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \tag{101}$$

I'm not sure these are defined quite right should it be:

$$\hat{\dot{X}} = \delta_{ij}\dot{X}_i \quad ?$$

$$\hat{\gamma} = \delta_{ij}\gamma_i$$

539 such that:

(handwritten) $X_t^T \varepsilon + E - \left(\dfrac{d\bar{L}}{dt} + \hat{\gamma}\bar{b} + \bar{S} \right) = \hat{X}\varepsilon$

$$\mathbf{X}_t^{\mathrm{T}}\varepsilon + \mathbf{E} - \left(\frac{d\mathbf{B}}{dt} + \hat{\gamma}\mathbf{B} \right) = \hat{\mathbf{X}}\varepsilon. \tag{102}$$

540   Additional relationships that will be helpful later include (derived in Appendix):

541   pendix):

$$\hat{\mathbf{X}}^{-1}\mathbf{X}_t = \mathbf{A}^{\mathrm{T}}, \tag{103}$$

$$\mathbf{X}_t^{\mathrm{T}} - \hat{\mathbf{X}} = \hat{\mathbf{X}}(\mathbf{A}^{\mathrm{T}} - \mathbf{I}), \tag{104}$$

42

$$\hat{\mathbf{X}} - \mathbf{X}_t^{\mathrm{T}} = \hat{\mathbf{X}}(\mathbf{I} - \mathbf{A}^{\mathrm{T}}), \tag{105}$$

542 and

$$\left(\hat{\mathbf{X}} - \mathbf{X}_t^{\mathrm{T}}\right)^{-1} = (\mathbf{I} - \mathbf{A}^{\mathrm{T}})^{-1}\hat{\mathbf{X}}^{-1}. \tag{106}$$

543 *6.10. Estimating ε and $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}$*

544   With Equation 102, we can solve for either the energy accumulation vector

545 $\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)$ or the energy intensity vector (ε), but not both. *[handwritten: → IMPORTANT POINT TO BE RE-ITERATED]*

546   Solving for the accumulation vector gives

[handwritten left margin: $\frac{dL}{dt} = \left(X_t^T - \hat{X}\right)\varepsilon + E - \hat{\gamma}B - S$]

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = (\mathbf{X}_t^{\mathrm{T}} - \hat{\mathbf{X}})\varepsilon + \mathbf{E} - \hat{\gamma}\mathbf{B}. \tag{107}$$

547 Finally, we can substutute Equation 104 which gives

[handwritten left margin: $\frac{dL}{dt} = \hat{X}\left(A^T - I\right)\varepsilon + E - \hat{\gamma}L - S$]

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} = \hat{\mathbf{X}}(\mathbf{A}^{\mathrm{T}} - \mathbf{I})\varepsilon + \mathbf{E} - \hat{\gamma}\mathbf{B}, \tag{108}$$

548 which allows estimation of the embodied energy accumulation in economic

549 sectors $\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)$ knowing only sector outputs $(\hat{\mathbf{X}})$, sector input-output ratios $(\mathbf{A})$,

550 sector energy intensities (ε), energy input to the economy $(\mathbf{E})$, and sector

551 physical depreciation rates $(\hat{\gamma}\mathbf{b})$. In theory, the transaction matrix $(\mathbf{X}_t)$ is not

552 required if the input-ouput ratios $(\mathbf{A})$ are known, though in reality, knowledge

553 of input-output ratios would be derived from the transaction matrix $\mathbf{X}_t$.

554   Solving for the energy intensity vector gives

[handwritten left margin: $\varepsilon = \left(\hat{X} - X_t^T\right)^{-1}\left[E - \left(\frac{dL}{dt} + \hat{\gamma}L - S\right)\right]$]

$$\varepsilon = (\hat{\mathbf{X}} - \mathbf{X}_t^{\mathrm{T}})^{-1}\left[\mathbf{E} - \left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} + \hat{\gamma}\mathbf{B}\right)\right]. \tag{109}$$

555 Substituting Equation 106 gives

43

$$\varepsilon = \left(I - A^T\right)^{-1} \hat{X}^{-1} \left[E - \left(\frac{dL}{dt} + \hat{\gamma}L + \beta\right)\right]$$

$$\varepsilon = (\mathbf{I} - \mathbf{A}^{\mathrm{T}})^{-1}\hat{\mathbf{X}}^{-1}\left[\mathbf{E} - \left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} + \hat{\gamma}\mathbf{B}\right)\right], \tag{110}$$

which allows estimation of the energy intensity of economic sectors (ε) knowing only sector input-output ratios ($\mathbf{A}$), sector outputs ($\hat{\mathbf{X}}$), energy input to the economy ($\mathbf{E}$), sector embodied energy accumulation rates ($\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}$), and sector physical depreciation rates ($\hat{\gamma}\mathbf{B}$). (91) 9 (92) ?

Comparison of Equations **??** and 110 shows the similarities between the single-sector algebraic formulation and the multi-sector matrix formulation of the I-O analysis method. This newly developed multi-sector matrix formulation can be extended to any desired level of economic and energy sector disaggregation as shown by Bullard (1975, 1978) and others.

*************************** MATT ENDED HERE ************************

7. Implications

Several implications can be drawn from the above detailed development of the I-O method equations in a manner that includes both embodied energy accumulation and depreciation.

7.1. Implications for economic "development"

[IT WOULD BE GOOD TO HAVE A COMPARISON BETWEEN $\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}$ AND STANDARD METRIC OF DEVELOPMENT, I.E. GDP WHICH I GUESS WOULD BE SOMETHING LIKE $\sum_i \dot{X}_i$. WE CAN CERTAINLY ENVISION SITUATIONS WHERE $\sum_i \dot{X}_i$ IS INCREASING AND

One consequence of economic "progress" or "development" is that embodied energy accumulates in economic sectors and society. In fact, accumulation

44

of embodied energy in economic sectors and society could be considered a *proxy* of development. This proxy for development is overly materialistic, one-dimensional, and reductionist, but alternatives such as GDP can be similarly criticized. In fact, GDP could continue to increase whilst accumulation of embodied energy or value actually decreased.

Figure 9 shows that energy extraction from the Earth is what ultimately drives development as measured by the accumulation of embodied energy in the economy and society. Development occurs over time. If embodied energy is the measure, development can be expressed as the integral of $\frac{dB}{dt}$ for economic sectors

$$\mathbf{B}(t) = \mathbf{B}(0) + \int_{t=0}^{t=t} \frac{d\mathbf{B}}{dt}dt, \tag{111}$$

[handwritten: $L(t) = L(0) + \int_{t=0}^{t} \frac{dL}{dt} dt$]

[handwritten margin note: Eqn 84 does not have Q_{21} term!]

or, using Equation 84, as the integral of $\frac{dB_2}{dt}$ for society,

$$B_2(t) = B_2(0) + \int_{t=0}^{t=t} \frac{dB_2}{dt}dt = B_2(0) + \int_{t=0}^{t=t} (Y_{\dot{T}} - \gamma_2 B_2 - \dot{Q}_{21})dt. \tag{112}$$

[handwritten: $L_2(t) = L_2(0) + \int_0^t \frac{dL_2}{dt} = L_2(0) + \int_0^t \left(Y_{\dot{T}} - \gamma_2 L_2 - \frac{dL_2}{dt} S_{21} - \dot{Q}_{21} \right) dt$]

Using embodied energy is obviously an incomplete measure of development. We might also use $X(t) = X(0) + \int \frac{dX}{dt}dt$. In fact, B and X are two complimentary factors to the economic process. For capital, B, to be useful, we need direct energy, E (to run the capital) and economic value, X (i.e. money). Therefore each of these factors are necessary, but insufficient.

Table 2 describes some of the dynamics that can be observed from Equation 108. It is quite possible that, especially for regions like the U.S. and Western Europe, the rate of embodied energy accumulation in the economy $\left(\frac{dB}{dt}\right)$ will be small relative to the rate of energy extraction from the Earth

45

Table 2: Factors from Equation 108 affecting the rate of embodied energy accumulation in the economy.

Right-side term	Implication
$\hat{\mathbf{X}}$	As economic output increases, $\frac{d\mathbf{B}}{dt}$ goes up (as will $\mathbf{E}$)
$\mathbf{A}$	As input-ouput ratios increase, $\frac{d\mathbf{B}}{dt}$ goes up
ε	As the energy intensity of the economy increases, $\frac{d\mathbf{B}}{dt}$ goes up
$\mathbf{E}$	As the rate of energy flow from the Earth increases, $\frac{d\mathbf{B}}{dt}$ goes up
$\hat{\gamma}$	As the depreciation rate increases, $\frac{d\mathbf{B}}{dt}$ goes down
$\mathbf{B}$	As the embodied energy in the economy increases, $\frac{d\mathbf{B}}{dt}$ goes down

($\mathbf{E}$). On the other hand, in rapidly developing countries, like China or India, the rate of embodied energy accumulation in the economy may be significantly higher than in a developed economy.

The behavior of $\mathbf{B}$ with $\frac{d\mathbf{B}}{dt}$ is vitally important. A developed economy has significantly higher embodied energy ($\mathbf{B}$) than a developing economy, and, thus, the outflow rate of embodied energy due to depreciation ($\hat{\gamma}\mathbf{B}$) will be higher. As increasingly large amounts of energy are embodied in the economy, increasingly large energy extraction rates ($\mathbf{E}$) are required to offset depreciation ($\hat{\gamma}\mathbf{B}$) and maintain positive growth ($\frac{d\mathbf{B}}{dt} > 0$) in the sectors of the economy. Depreciation may also be, temporarily, offset by increasing energy efficiency, i.e. by decreasing energy intensity, ε.

In a similar manner, Equation 84 indicates that maintaining a positive rate of societal development ($\frac{dB_2}{dt} > 0$) requires ever increasing embodied

46

610 energy input rates to society ($Y_{\dot{T}}$) as the society "develops." This mechanism

611 provides a natural brake to the continued growth of physical economies.

612 *7.2. Implications for the I-O method*

613   The I-O literature (examples include Bullard (1975) and Cassler (1983))

614 usually writes Equation 110 as

$$\varepsilon = (\mathbf{I} - \mathbf{A}^T)^{-1}\hat{\mathbf{X}}^{-1}\mathbf{E}. \tag{113}$$

615 It is clear from comparison of Equations 110 and 113 that the literature is

616 not accounting for accumulation of energy in the economic sectors $\left(\frac{d\mathbf{B}}{dt}\right)$, nor

617 does it account for physical depreciation ($\hat{\gamma}\mathbf{B}$). To be precise, the literature

618 assumes

$$\frac{dL}{dt} + \hat{\gamma}L \cdot = 0$$

Not sure if this should include $\bar{S}$ term; probably.

$$\frac{d\mathbf{B}}{dt} + \hat{\gamma}\mathbf{B} = \mathbf{0}. \qquad \frac{dL}{dt} + \hat{\gamma}L \tag{114}$$

619   Examining Equation 110, we see that to the extent that $\frac{d\mathbf{B}}{dt} + \hat{\gamma}\mathbf{B} \ll \mathbf{E}$,

620 estimates of energy intensity (ε) obtained with the assumption of Equation

621 114 contain little error. However, when the sum of the accumulation and de-

622 preciation rates $\left(\frac{d\mathbf{B}}{dt} + \hat{\gamma}\mathbf{B}\right)$ becomes significant relative to the rate of energy

623 extracted from the Earth ($\mathbf{E}$), estimates of economic sector energy intensities

624 (ε) using the assumption of Equation 114 have a high-side bias (assuming

625 that $\frac{d\mathbf{B}}{dt} > 0$ and $\gamma\mathbf{B} > 0$). As discussed above, the assumption of Equation

626 114 can be violated in developing economies because accumulation $\left(\frac{d\mathbf{B}}{dt}\right)$ is

627 large or in developed economies because depreciation ($\hat{\gamma}\mathbf{B}$) is large.

628   The assumption of Equation 114 may cause another challenge for energy

629 analysts. The I-O method is often used to estimate energy intensities for

47

630 each sector of the economy (ε) with Equation 114. With ε values in hand,
631 one can estimate changes in energy demand from the Earth ($\mathbf{E}$) as the output
632 of economic sectors ($\hat{\mathbf{X}}$) increases or decreases by solving Equation 113 for
633 $\mathbf{E}$.

$$\mathbf{E} = \hat{\mathbf{X}}(\mathbf{I} - \mathbf{A}^\mathrm{T})\varepsilon \tag{115}$$

634 When accumulation and depreciation terms are included, we see that the
635 energy demands ($\mathbf{E}$) must be calculated differently. Solving Equation 110 for
636 $\mathbf{E}$ gives

$$E = \hat{X}\left(I - A^\mathsf{T}\right)\varepsilon +_* \left(\frac{dL}{dt} + \hat{\gamma}L + S\right)$$

$$\mathbf{E} = \hat{\mathbf{X}}(\mathbf{I} - \mathbf{A}^\mathrm{T})\varepsilon + \left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t} + \hat{\gamma}\mathbf{B}\right). \tag{116}$$

637 By comparing Equations 115 and 116, we see that to the extent that ac-
638 cumulation $\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)$ and depreciation $(\hat{\gamma}\mathbf{B})$ are non-zero, estimates of energy
639 demand are too low. If the sum of accumulation $\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)$ and depreciation $(\hat{\gamma}\mathbf{B})$
640 are small relative to total energy demand ($\mathbf{E}$), then neglecting these effects
641 causes little error. Economies with fast growth rates $\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\right)$ or large sizes $(\mathbf{B})$
642 are more likely to violate the typical assumptions in the literature.

643 *7.3. Implications for recycling, reuse, and dematerialization*

644 Dematerialization is the idea that economic activity can be unlinked from
645 material or energy demands (UNEP, 2011). One of the primary methods for
646 dematerializing an economy is reuse and recycling of materials. The impact
647 of recycling can be seen in the I-O formulation only when depreciation and
648 accumulation are included.

48

One effect of recycling is to reduce the magnitude of the disposal rate ($\hat{\gamma}$). Equation 108 indicates that recycling of material in an economy, thereby reducing $\hat{\gamma}$, will slow the effect of depreciation ($\hat{\gamma}\mathbf{B}$) and put upward pressure on growth $\left(\frac{dB}{dt}\right)$.

Recycling has a mixed effect on energy demand ($\mathbf{E}$). Because recycled material displaces newly-produced material in the economy and society, recycling will tend to reduce energy demand ($\mathbf{E}$). Equation 108 indicates that this displacement effect will put downward pressure on growth $\left(\frac{dB}{dt}\right)$. However, recycling processes require energy to operate, thereby increasing energy demand ($\mathbf{E}$). Equation 108 indicates that additional energy demand will put upward pressure on growth $\left(\frac{dB}{dt}\right)$.

If recycling produces a net reduction in energy demand ($\mathbf{E}$), that is if the effect of displaced production dominates over the effect of energy consumed in recycling processes, the upward pressure on growth $\left(\frac{dB}{dt}\right)$ from decrease in $\hat{\gamma}$ and the downward pressure on growth from net reduction of $\mathbf{E}$ offset each other, the growth rate $\left(\frac{dB}{dt}\right)$ will remain near zero, and total embodied energy ($\mathbf{B}$) will remain constant. In that scenario, dematerialization can develop: reduced material and energy input ($\mathbf{E}$) can be accompanied by no change in growth $\left(\frac{dB}{dt}\right)$.

7.4. Comparison to a Steady-state Economy

****** Finish this section. In terms of what a SSE would look like in the I-O framework, at first blush, I would think that $dB/dt = 0$ is one aspect. Also, with no growth, inflow rates = depreciation rates. The larger that B is for any society, the larger E must be (to overcome depreciation). To minimize E, hyper-recycling is probably useful. Those are at least a place to

[margin note: THIS WILL NEED TO BE RE-THOUGHT IN TERMS OF S & L]

674 start. ******

675 ****** In our discussion, we also addressed the attempts at SSE from
676 point of view of society. In order to achieve this goal *without* recycling,
677 the goods and services sector should have to increase extraction to offset
678 decreasing ore grade, the energy sector should have to increase extraction
679 of energy to allow increasing extraction (unless efficiency could make up the
680 gap - unlikely) in which case the SSE would be violated from these two and
681 from the POV of the earth. ******

682 ## 8. Conceptual and Theoretical Issues

683 *8.1. Choice of Energy Input Vector*

684 Consistent with traditional I-O methods, the derivation presented above
685 counts energy at the point of inflow to the economy. That is, elements of
686 the energy input vector to the economy ($\mathbf{E}$) are zero except for those sectors
687 that receive energy directly from the Earth. With the traditional approach,
688 energy input to energy sectors is non-zero, and energy input to non-energy
689 sectors is zero. So, in the two-sector example C above, $\dot{E}_{14} = 0$ and $\dot{E}_{13} \neq 0$.

690 Costanza (1984) suggests an alternative approach, namely to count energy
691 input to the economy at the point of conversion to useful work. Theoretical
692 justification for this direct energy conversion (DEC) approach comes from
693 both thermodynamic and economic considerations. The thermodynamic jus-
694 tification derives from the purpose of energy consumption in an economy,
695 namely to produce useful work. If energy flows *through* a sector, it should
696 not be counted "against" that sector: only energy that is converted to useful
697 work *in* the sector should be counted against that sector.

50

698 The economic justification derives from the typical treatment of trans-
699 portation sectors of the economy. ****** More here. See Costanza (1984)
700 for the transportation analogy. ******

701 The DEC approach implicitly redefines energy intensity to be the required
702 amount of fossil fuel energy to produce a unit of economic output.

703 ****** Equation redefining ε here.

704 In the DEC approach, electricity consumption is converted to its fossil
705 energy equivalent (coal) before being "applied" to an economic sector. And,
706 refined petroleum is converted to its fossil energy equivalent (crude) before
707 being "applied" to an economic sector.

708 ****** The DEC option is akin to my idea of substituting the 1st Law
709 into the total energy equation. Show this derivation after redefining ε to be
710 embodied energy per dollar, not total energy per dollar. So, there is a second
711 implicit assumption going on with Costanza (1984), namely that we have a
712 re-derivation of energy intensity. ******

713 ****** Show that re-derivation results in only counting the energy burned
714 by each sector (or the waste heat off of each sector). Costanza (1984) shows
715 that distributing energy input at the point of consumption reduces the vari-
716 ance of energy intensity across all sectors of the economy. ******

717 *8.2. What is Endogenous?*

718 Are government and households endogenous? Costanza (1980) was the
719 first to endogenize government and households, because households provide
720 services to the economy (labor) in exchange for wages and government pro-
721 vides services to the economy in exchange for taxes, both of which require en-
722 ergy. Costanza (1980) showed that by including government and households

723 as sectors in the economy, the variation of energy intensity is significantly
724 reduced across all sectors of the economy.

725 *8.3. What About the Sun?*

726 Costanza (1980) includes an option to consider the sun as an input to
727 the economy, thereby significantly increasing the energy intensity of agri-
728 cultural sectors and other sectors that depend upon agricultural outputs,
729 however Costanza (1984) did not include the sun ??. Whether solar input
730 to the economy should be considered is probably dependent upon the ob-
731 jectives of the analysis. In this framework we are primarily interested in
732 the effects of declining energy resource quality in industrial economies, due
733 to depletion of fossil fuels. As such, inclusion of solar flows is unnecessary.
734 However, expanding the framework to include non-industrial or more agrar-
735 ian societies would probably require accounting for these flows. Additionally,
736 similar concerns might be raised in dealing with a society that is largely re-
737 liant on solar or wind energy. [EARLIER FOOTNOTE ON INDUSTRIAL
738 VS. NON-INDUSTRIAL ECONOMIES COULD BE BROUGHT IN HERE
739 - MD]

740 There are a number of means by which solar flows can be accounted.
741 Short-term solar flows could be accounted in the output of agricultural and
742 forestry sectors, as well as some of the renewable energy producers, such
743 as solar thermal and PV, wind, ocean thermal, hydro-power and biomass.
744 This method does not account for longer-term flows of solar energy used to
745 form fossil fuels. The *emergy* accounting method puts all flows in terms of
746 *em*bodied en*ergy* flows ??. The basic unit of measure is the *em*joule which
747 is often given in terms of flows of solar energy embodied in the energy (or

52

748 material) - the solar emjoules - per unit of resource, abbreviated to seJ/J for
749 energy resources, or seJ/g for materials. As such, even fossil fuels, e.g. coal,
750 extracted from the earth have an embodied energy of around 67,000 seJ/J ?.

9. Appendix - Proofs and derivation of useful relationships

9.1. Proof of Equation 104

753 We begin with a restatement of Equation 104.

$$\mathbf{X}_t^T - \hat{\mathbf{X}} = \hat{\mathbf{X}}(\mathbf{A}^T - \mathbf{I}) \tag{117}$$

754 We expand the matrices to obtain

$$\begin{bmatrix} \dot{X}_{33} & \dot{X}_{43} \\ \dot{X}_{34} & \dot{X}_{44} \end{bmatrix} - \begin{bmatrix} \dot{X}_3 & 0 \\ 0 & \dot{X}_4 \end{bmatrix} = \begin{bmatrix} \dot{X}_3 & 0 \\ 0 & \dot{X}_4 \end{bmatrix} \begin{bmatrix} a_{33} - 1 & a_{43} \\ a_{34} & a_{44} - 1 \end{bmatrix}. \tag{118}$$

755 Multiplication of the matrices provides

$$\begin{bmatrix} \dot{X}_{33} - \dot{X}_3 & \dot{X}_{43} \\ \dot{X}_{34} & \dot{X}_{44} - \dot{X}_4 \end{bmatrix} = \begin{bmatrix} \dot{X}_3 a_{33} - \dot{X}_3 & \dot{X}_3 a_{43} \\ \dot{X}_4 a_{34} & \dot{X}_4 a_{44} - \dot{X}_4 \end{bmatrix}. \tag{119}$$

756 Using $\dot{X}_j a_{ij} = \dot{X}_{ij}$ (see Equation 16) gives

$$\begin{bmatrix} \dot{X}_{33} - \dot{X}_3 & \dot{X}_{43} \\ \dot{X}_{34} & \dot{X}_{44} - \dot{X}_4 \end{bmatrix} = \begin{bmatrix} \dot{X}_{33} - \dot{X}_3 & \dot{X}_{43} \\ \dot{X}_{34} & \dot{X}_{44} - \dot{X}_4 \end{bmatrix} \tag{120}$$

757 to complete the proof.

9.2. Proof of Equation 105

759   We begin with a restatement of Equation 105.

$$\hat{\mathbf{X}} - \mathbf{X}_t^{\mathrm{T}} = \hat{\mathbf{X}}(\mathbf{I} - \mathbf{A}^{\mathrm{T}}) \qquad (121)$$

760 We expand the matrices to obtain

$$\begin{bmatrix} \dot{X}_3 & 0 \\ 0 & \dot{X}_4 \end{bmatrix} - \begin{bmatrix} \dot{X}_{33} & \dot{X}_{43} \\ \dot{X}_{34} & \dot{X}_{44} \end{bmatrix} = \begin{bmatrix} \dot{X}_3 & 0 \\ 0 & \dot{X}_4 \end{bmatrix} \begin{bmatrix} 1 - a_{33} & -a_{43} \\ -a_{34} & 1 - a_{44} \end{bmatrix}. \qquad (122)$$

761 Multiplication of the matrices provides

$$\begin{bmatrix} \dot{X}_3 - \dot{X}_{33} & -\dot{X}_{43} \\ -\dot{X}_{34} & \dot{X}_4 - \dot{X}_{44} \end{bmatrix} = \begin{bmatrix} \dot{X}_3 - \dot{X}_3 a_{33} & -\dot{X}_3 a_{43} \\ -\dot{X}_4 a_{34} & \dot{X}_4 - \dot{X}_4 a_{44} \end{bmatrix}. \qquad (123)$$

762 Using $\dot{X}_j a_{ij} = \dot{X}_{ij}$ (see Equation 16) gives

$$\begin{bmatrix} \dot{X}_3 - \dot{X}_{33} & -\dot{X}_{43} \\ -\dot{X}_{34} & \dot{X}_4 - \dot{X}_{44} \end{bmatrix} = \begin{bmatrix} \dot{X}_3 - \dot{X}_{33} & -\dot{X}_{43} \\ -\dot{X}_{34} & \dot{X}_4 - \dot{X}_{44} \end{bmatrix} \qquad (124)$$

763 to complete the proof.

764 *9.3. Derivation of Equation 106*

765   We begin with a restatement of Equation 105.

$$\hat{\mathbf{X}} - \mathbf{X}_t^{\mathrm{T}} = \hat{\mathbf{X}}(\mathbf{I} - \mathbf{A}^{\mathrm{T}}) \qquad (125)$$

766 We take the inverse of both sides of the equation to obtain

$$\left(\hat{\mathbf{X}} - \mathbf{X}_t^{\mathrm{T}}\right)^{-1} = \left(\hat{\mathbf{X}}(\mathbf{I} - \mathbf{A}^{\mathrm{T}})\right)^{-1}. \qquad (126)$$

767 We now apply the following matrix identity (formula 6.2, pg. 308 from ?).

$$(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \tag{127}$$

768 to the right side of Equation 126 to obtain

$$\left(\hat{\mathbf{X}} - \mathbf{X}_t^{\mathrm{T}}\right)^{-1} = (\mathbf{I} - \mathbf{A}^{\mathrm{T}})^{-1}\hat{\mathbf{X}}^{-1}, \tag{128}$$

769 which is identical to Equation 106.

Questions → Does the embodied energy that flows into a sector / ever leave as flow $\dot{B}_{ji}$?
$$\overset{\wedge}{\dot{B}_{ij}}$$

Not in the case of $\dot{S}$ flows.

Should we stipulate that:

$$\dot{S}_{j1} = \sum_i \dot{S}_{ij} + (x)$$

One S passive $\Rightarrow S_{i1} = 0$ $\dfrac{dS_1}{dt} \neq 0$ $\left| \dfrac{dS_i}{dt} \begin{cases} = 0, & i \neq 1 \\ \neq 0, & i = 1 \end{cases} \right.$

We could define $\dot{S}_i$ as some fraction, say ϕ_i of $\dot{L}_i$, such that

$$\dot{S}_i = \phi_i \dot{L}_i$$

$$\Rightarrow \dot{b}_i = \dot{S}_i + \dot{L}_i = \dot{L}_i (1 + \phi_i)$$