

A dynamic approach to input-output modeling

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Abstract

Previous frameworks for input-output modeling have made the assumption that flows into and out of each economic sector balance, such that there is no accumulation of economic factors or embodied energy within any of the sectors. This may be an adequate assumption for a sector of the economy operating at ‘steady-state’, however the assumption introduces errors for example in analysis of sectors that are growing rapidly, where a non-negligible proportion of input factors may be invested in accumulation within the sector. This paper presents an extension to the traditional input-output framework, wherein accumulation is incorporated explicitly using a dynamic (transient) analysis method. This new approach gives new insight into macro-economics including an alternative metric for social development. It also raises issues for input-output-based methods for net energy analysis. The alternative perspective offered by this new method is used to explore the implications of extraction of declining quality resources from the environment.

Keywords:

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1. Introduction

BLAH BLAH BLAH

1.1. Brief history of input-output (I-O) modeling

Input-output analysis, developed by Wassily Leontief in the 1930's as an extension to the work of Quesnay and Walras ?, is of primary importance in national accounting, allowing determination of the structure of an economy as well as, among other things, calculation of a nation's gross domestic product (GDP), the predominant measure of economic activity.

1.2. Basic I-O method

The basic premise of the I-O method, as outlined in Figure 1A, is that each economic sector takes in factors of production from other sectors (and possibly itself) to produce an economic good at some rate. E.g., the automotive sector takes in steel, rubber, glass, etc. and produces a number of cars per year. In contrast to high-level economic growth models that include only a few factors of production (such as land, capital, and labor), the I-O analysis technique allows many differentiated factors of production and raw material feedstocks. ? In I-O frameworks, each factor of production is considered to be the output from a sector of the economy. As will be discussed later [MAKE SURE TO DISCUSS THIS LATER!], the traditional primary factors of production (land, capital, and labor) are not *flows* into the production processes. Rather, they are *stocks* that, when present, allow factors

22 of production (steel, rubber, and glass) to be transformed into final products
23 (automobiles).

24 In addition to stocks of land, capital, and labor, a flow of energy (or
25 more precisely, the degradation of an exergetic gradient/destruction of ex-
26 ergy) is also required for economic activity. These energy flows originate from
27 the natural environment, recognition of which has provoked researchers from
28 fields of net energy analysis (NEA), material flow analysis (MFA), industrial
29 ecology (IE) and life-cycle assessment (LCA) to extend the traditional (Leon-
30 tief) input-output framework to include important material and energy flows
31 to and from the environment, as depicted in Figure 1B ????????. While the
32 Leontief I-O approach relies exclusively on monetary units to represent value
33 flows among sectors of an economy, the key insight of these extensions of
34 the Leontief I-O framework is to rely upon physical units (especially energy
35 units of joules) to represent some of the value flows among economic sectors.
36 In doing so, energy and material intensities of value flows can be estimated.
37 Their approaches are similar to Figure 1B.

38 Both the original Leontief I-O framework and the extensions cited above
39 assume steady-state conditions in an economy, i.e., flows of value and mate-
40 rial into and out of each economic sector are in balance. Dynamic or transient
41 behavior of the economic system is not considered. Thus, there is no accumu-
42 lation of economic factors or embodied energy within any of the sectors. The
43 analysis techniques provide “snapshots” of economic activity at an instant
44 in time.

45 [MIK’S NEW ADDITION]

46 Assuming no accumulation of materials, within economic sectors or so-

ciety itself, is tantamount to assuming that *all* material flows through the economy are directed toward the production of non-durable goods. However, evidence of the durability of goods and the accumulation of materials surrounds us. Furthermore, energy was required to both fabricate and emplace the durable goods and infrastructure of modern economies. (The energy it took to create the durable goods and infrastructure can be considered “embodied” within the built environment, a point to which we will return in detail later). As Georgescu-Roegen [REFERENCE NEEDED –MKH] notes, “in the everyday world one cannot possibly cross a river only on the flow of main maintenance materials of a non-existent bridge.” [I DON’T HAVE THE SOURCE WITH ME, BUT ”MAIN MAINTENANCE MATERIALS” SOUNDS WEIRD. SHOULD ”MAIN” BE DELETED? SHOULD ”OF A NON-” BE REPLACED WITH ”FOR A NON-”? –MKH]

Analysis methods that neglect the accumulation of materials and embodied energy in the durable goods and infrastructure of the everyday world lack explanatory power. Such models can tell us how at what rates materials and energy are required to *use* our built environment. But, such models cannot tell us *how* the built environment came to be (and how much energy was required to construct it) or *why* flows of non-durable goods are needed. [I DON’T GET THIS SECOND PART: I.E., WHY FLOWS OF NON-DURABLE GOODS ARE NEEDED. LET’S DISCUSS. –MKH] To use Georgescu-Roegen’s imagery, models that neglect accumulation fail to explain why we need any material flows to maintain a non-existent bridge. [IF I UNDERSTOOD THE PREVIOUS SENTENCE, I WOULD PROBABLY UNDERSTAND THE EXTENSION OF THE METAPHOR. BUT,

72 AT THIS POINT, I DON'T. –MKH] The stocks of accumulated materials
73 (capital, appliances, even people) are the drivers of demand. It is to service
74 their needs and wants that we put the economy to work. [THESE LAST
75 TWO SENTENCES ARE EXCELLENT! –MKH]

76 Because economic activity requires energy, we need to understand the way
77 energy flows through economies. The steady-state I-O techniques of Bullard,
78 Herendeen, and others [REFERENCES NEEDED –MKH] offer a means to
79 that end. We contend, however, that these techniques need to be extended
80 and modified to include transient effects that arise when durability of goods
81 and infrastructure (and associated embodied energy) are considered. This
82 paper attempts to address that need.

83 1.3. *Issues of resource quality*

84 [DO WE NEED THIS SECTION? IT FEELS LIKE WE SHOULD MOVE
85 FROM THE PREVIOUS SECTION WHEREIN WE CLEARLY DEMON-
86 STRATE THE NEED FOR OUR WORK TO BEGINNING THE PROCESS
87 OF DEVELOPING THE METHOD. IF WE WANT TO KEEP THIS SEC-
88 TION, WE SHOULD DO A BETTER JOB AT SHOWING WHY IT IS
89 NEEDED. WE SHOULD POINT BACK TO BULLARD AND HEREN-
90 DEEN AND SAY THAT THEIR METHOD DOESN'T ACCOUNT FOR
91 DECREASING RESOURCE QUALITY AND WHY A METHOD IS NEEDED
92 THAT DOES ACCOUNT FOR DECLINING RESOURCE QUALITY. –
93 MKH] Raw material and energy resources must first be extracted from the
94 natural environment before they may be utilized in the economy to pro-
95 vide goods and service to society. Despite increasing levels of technological
96 efficiency, for example in consumer goods such as refrigerators and cars, ev-

97 idence shows that the energy intensity of primary resource extraction, i.e.
 98 the energy required to extract raw materials from the environment, has been
 99 steadily increasing over the last fifty years ????. This increasing energy re-
 100 quirement for primary extraction means that less *net energy* is available for
 101 downstream uses. If this decline in net energy availability outpaces techno-
 102 logical advances in energy efficiency, there may be deleterious impacts on the
 103 economic output of the economy.

104 1.4. An I-O method for dynamic (transient) economic analysis

105 In this paper, we develop a physical input-output, matrix-based method
 106 for modeling multi-sector economies, in the tradition of Georgescu-Roegen’s
 107 “flow-fund” model ??. The method presented in this paper takes a decidedly
 108 engineering approach to extend the techniques of Bullard, Herendeen, and
 109 others to account for durability of goods and embodied energy. This method
 110 allows us to see how energy and materials flow through the economy, where
 111 embodied energy accumulates in the economy, and how declining resource
 112 quality may affect these dynamics. [NEED TO MAKE SURE WE ACHIEVE
 113 THIS LAST POINT]

114 This paper is organized as follows. We first discuss methodology and the
 115 model economy. Thereafter, we present three examples, each with increasing
 116 levels of disaggregation among society, the energy sector, and goods and
 117 services sectors, culminating with a matrix formulation of the new method.
 118 The examples leverage the First Law of Thermodynamics, account for total
 119 energy (T), and develop accounting relationships for embodied energy (B).
 120 Within the examples, we develop a precise definition for embodied energy and
 121 a matrix formulation of the method that can be extended to an arbitrarily

large number of economic sectors. Finally, we draw several implications from the development of the new method.

2. Methodology

2.1. Model economy

The model economy employed herein consists of sectors that produce a single product, either an energy product (energy sectors) or other goods and services (non-energy sectors). Economic sectors receive as inputs direct energy (E) and materials in which energy is embodied (B).¹ Economic sectors emit waste heat (Q).

2.2. Direct energy (E), indirect (embodied) energy (B), and waste heat (Q)

We distinguish between direct energy resources (E), such as coal or oil, and indirect energy (B) “embodied” in outputs from economic sectors. E represents the energetic value of an energy resource (measured as heating value, chemical potential energy, or exergy). In contrast, B represents the energy expended in the production and delivery of goods in the economy, and, as such, measures accumulated upstream energy consumption from the network of economic sectors within the economy. ‘Indirect’ energy and ‘embodied’ energy are synonyms. Both E and B are measured in energy units (joules or BTUs). The flow rates of direct energy (\dot{E}) and indirect energy (\dot{B}) among sectors of the economy, the Earth, and society are in units of power (energy per unit time, J/time or BTU/time).

¹A formal definition for embodied energy (B) is presented in Section 6.4.

143 Waste heat (\dot{Q}) flows from sectors of the economy and society to the
144 Earth and its atmosphere, the necessary result of inefficient consumption of
145 direct energy E . Like \dot{E} and \dot{B} , the units of \dot{Q} are energy per unit time.

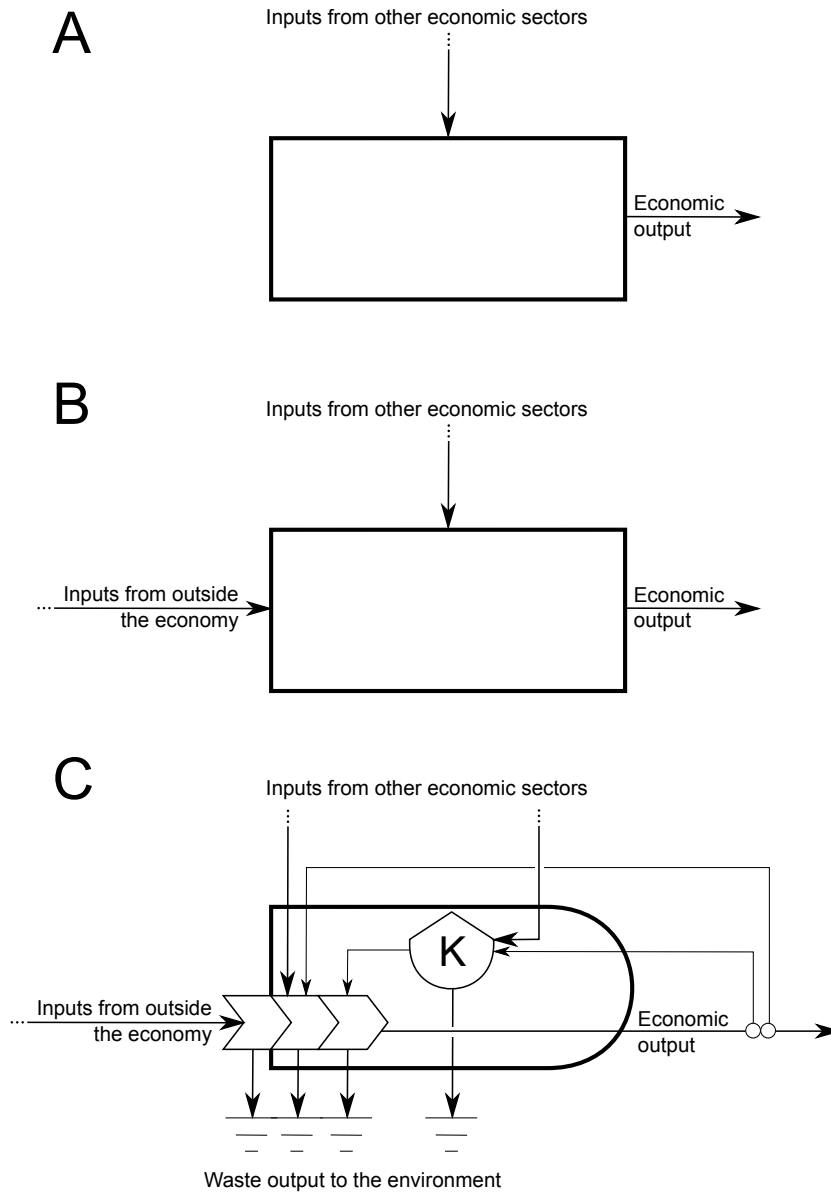


Figure 1: The basic unit of input-output modeling: A) the standard economic approach includes only transactions among sectors of the economy; B) the ecological economics approach models inputs from the natural environment outside the economy as factors of production [MIK: REPLACE 'OUTSIDE THE ECONOMY' WITH 'THE NATURAL ENVIRONMENT' IN THE FIGURE.] and; C) the method presented here accounts also for accumulation, K, of embodied energy within materials in economic sectors.

146 2.3. Total energy (T)

147 Total energy (T) is the sum of the direct and indirect (embodied) energy.

$$T \equiv E + B \quad (1)$$

148 In general, the flow rate of total energy among sectors in the economy,
149 the earth, and society is given by

$$\dot{T} = \dot{E} + \dot{B}. \quad (2)$$

150 In some cases, total energy flows may consist of direct energy (\dot{E}) or
151 embodied energy (\dot{B}) exclusively. For example, the flow of extracted crude
152 oil from the earth consists of direct energy only ($\dot{B} = 0$ and $\dot{T} = \dot{E}$), because,
153 in this method, no embodied energy (B) has been added to the crude oil until
154 it reaches the downstream side of the pump. The flow of goods produced by a
155 non-energy sector of the economy consists of indirect energy only ($\dot{E} = 0$, and
156 therefore $\dot{T} = \dot{B}$), because no direct energy (E) is produced by a non-energy
157 sector in this model economy.

158 In other cases, total energy flows may have both direct *and* indirect com-
159 ponents. For example, the flow of refined petroleum from the energy sector
160 has both a direct energy (\dot{E} , the energy content of the oil product, usually
161 represented by chemical potential energy) and embodied energy (\dot{B} , which
162 accounts for the energy consumed in upstream processes to extract and refine
163 the crude oil).²

²Outputs from agricultural sectors will be similar: both the direct energy component (comprising chemical potential energy) and the embodied energy component will be non-

Single subscripts on T , E , or B can mean one of two things: \dot{T}_i indicates the outflow of total energy from sector i , whereas T_i denotes the total energy content of sector i . Double subscripts on T , E , or B (e.g., \dot{T}_{ij}) indicate a flow from sector i to sector j ,³ in this case for total energy (T).

The I-O literature [REF TO BULLARD AND HERENDEEN, ETC. HERE –MKH] assumes (a) that steady state conditions exist (i.e., no accumulation of total energy in economic sectors) and (b) that flows of total energy (\dot{T}) are *conserved*, where by *conserved*, it is meant that total energy can be neither created nor destroyed. Like the literature, we assume that total energy is conserved. However, we depart from the literature to allow durability of goods as represented by total energy accumulation in economic sectors. Steady state, this approach is not.

Total energy may accumulate within an economic sector as stocks of direct energy materials (piles of coal or tanks of oil) but also as embodied energy in stocks of capital goods (e.g. machinery or buildings). The rate of accumulation of total energy ($\frac{dT}{dt}$) in a sector of the economy, the Earth, or society is given by the time derivative of total energy:

$$\frac{dT}{dt} = \frac{dE}{dt} + \frac{dB}{dt}. \quad (3)$$

We note that the definition of total energy (Equation 1) includes direct energy (E) and embodied energy (B) terms. On the other hand, the First Law of Thermodynamics includes direct energy (E) and waste heat (Q) terms.

zero.

³In the following discussion, the first index always indicates the sector *from* which a quantity flows, and the second index indicates the sector *to* which a quantity flows.

184 The consequence of the foregoing difference is that an interesting relationship
 185 exists between embodied energy (B) and waste heat (Q). We shall see in the
 186 following example that waste heat from an economic sector can be considered
 187 to contribute to energy embodied within the products of that sector.

188 3. Example A: single sector economy

189 In this section, we present an example economic analysis using a single-
 190 sector economy wherein the economy and society are merged together.

191 Figure 2 shows a single-sector Economy (represented by “economy/society,”
 192 2) that extracts direct energy from the earth (\dot{E}_{12}). Direct energy and waste
 193 heat flows are identified by vectors. No direct energy flows from the economy
 194 (2) to the earth (1), only waste heat (\dot{Q}_{21}).

195 3.1. First Law of Thermodynamics

196 Both direct energy (\dot{E} , such as the energy content of coal, oil, and electric-
 197 ity), and waste heat (\dot{Q}) are accounted by the First Law of Thermodynamics.
 198 Accounting for possible accumulation of direct energy in the economy, the
 199 First Law of Thermodynamics indicates that

$$\frac{dE_2}{dt} = \dot{E}_{12} - \dot{Q}_{21}. \quad (4)$$

200 Aside from, for example, the U.S. Strategic Petroleum Reserve, we are not
 201 stockpiling oil and coal at any meaningful rate, i.e. we consume fossil fuels at
 202 a rate equal to the extraction rate. Thus, the world is not accumulating direct

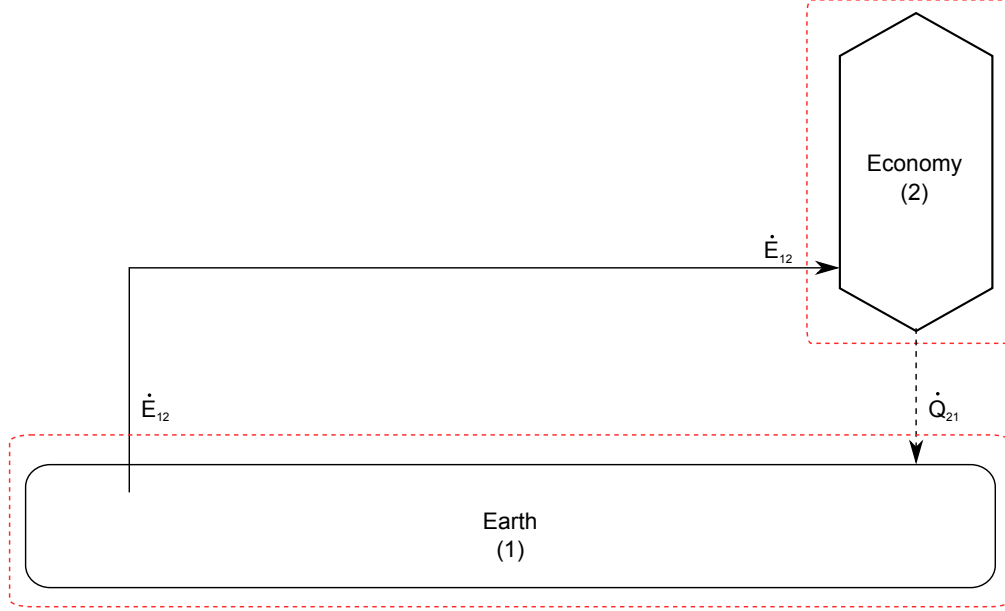


Figure 2: Direct energy (\dot{E}) and waste heat (\dot{Q}) flows for a single-sector economy. [CHANGE "ECONOMY" TO "ECONOMY/SOCIETY" TO MATCH TEXT. OR CHANGE THE TEXT ABOVE. -MKH]

203 energy in the economy.⁴ (The world *is*, however, accumulating embodied
 204 energy in the economy as we shall see shortly.) Thus, the accumulation rate
 205 for direct energy ($\frac{dE_2}{dt}$) in the above equation can be set to zero to obtain

$$0 = \dot{E}_{12} - \dot{Q}_{21}. \quad (5)$$

⁴A counter-example could be made for nuclear fuels where ‘spent’ fuel represents a large exergetic stockpile, however, this reserve is not (presently) economically useful.

206 3.2. Total energy accounting

207 Figure 3 shows the flows of total energy (\dot{T}) through the single-sector
 208 economy.

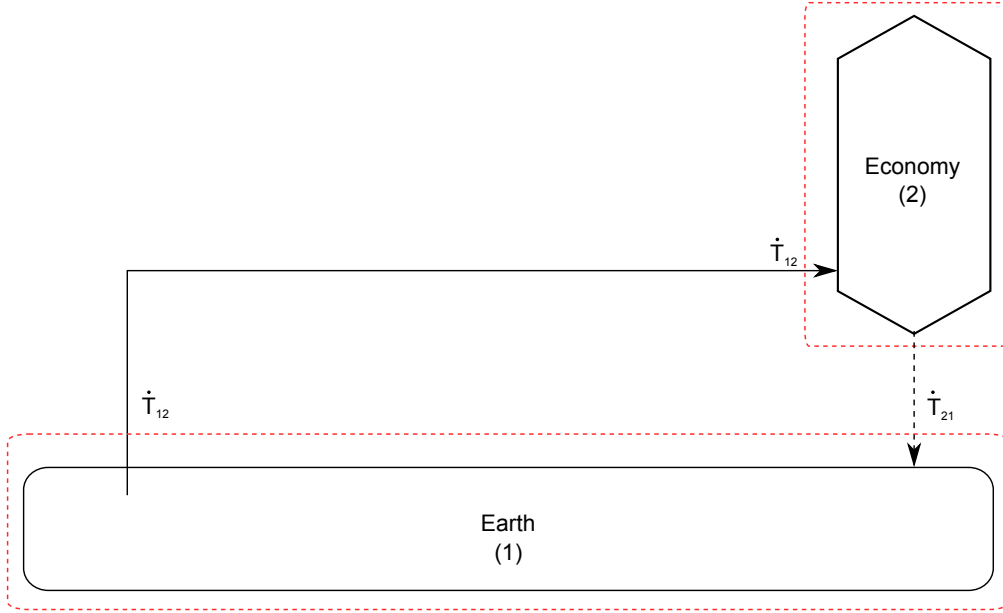


Figure 3: Total Energy Flows (\dot{T}) in a Single-sector Economy.

209 We follow the I-O literature in assuming that total energy (T) is con-
 210 served. The I-O literature assumes steady-state operation of the economy
 211 with no accumulation of embodied energy in the economic sectors. (We will
 212 see later how the assumption in the literature introduces errors into I-O anal-
 213 yses.) We depart from the I-O literature by accounting for both accumulation
 214 and depreciation of energy embodied in sectors of the economy and society.
 215 By doing so, the present analysis does *not* assume a steady-state economy.
 216 A total energy accounting around the single-sector economy (2) gives

$$\frac{dT_2}{dt} = \dot{T}_{12} - \dot{T}_{21}. \quad (6)$$

217 3.3. Embodied energy accounting

218 The First Law of Thermodynamics accounts for both direct energy (E)
 219 and waste heat (Q), whereas total energy (T) accounting tracks direct energy
 220 (E) and embodied energy (B). If we substitute the First Law into the total
 221 energy accounting equation, we can eliminate direct energy (E) to arrive at
 222 an embodied energy accounting equation. We begin by expanding the T
 223 terms in Equation 6 using Equations 1 and 2 to obtain

$$\frac{dE_2}{dt} + \frac{dB_2}{dt} = \dot{E}_{12} + \dot{B}_{12} - \dot{E}_{21} - \dot{B}_{21}. \quad (7)$$

224 Realizing that $\frac{dE_2}{dt} = 0$ (because direct energy does not accumulate in mean-
 225 ingful amounts in the economy) and $\dot{E}_{21} = 0$ (because energy is returned to
 226 the earth as waste heat, see Figure 2) yields

$$\frac{dB_2}{dt} = \dot{E}_{12} + \dot{B}_{12} - \dot{B}_{21}. \quad (8)$$

227 Equation 8 shows that the accumulation rate of embodied energy in the
 228 economy is a function of the inflows of direct and embodied energy less the
 229 outflow of embodied energy.

230 In this example, we substitute⁵ Equation 5 into Equation 8 to obtain an
 231 embodied energy accounting equation:

⁵We shall encounter this move to substitute the First Law of Thermodynamics into the total energy accounting equation repeatedly below.

$$\frac{dB_2}{dt} = \dot{Q}_{21} + \dot{B}_{12} - \dot{B}_{21}. \quad (9)$$

232 An important result of Bullard-Herendeen-style I-O analyses, historically,
 233 has been the quantification of the embodied energy content of economic sector
 234 outputs, in this case \dot{B}_{21} . Equation 8 can be rearranged to give

$$\dot{B}_{21} = \dot{Q}_{21} + \dot{B}_{12} - \frac{dB_2}{dt}. \quad (10)$$

235 Equation 10 indicates that the embodied energy content of the product
 236 of an economic sector (in this case \dot{B}_{21}) can be thought of as the sum of the
 237 embodied energy inputs to the sector (in this case \dot{B}_{12}) and the waste heat
 238 from the sector (in this case \dot{Q}_{21}) less the accumulation rate of embodied
 239 energy in the sector (in this case $\frac{dB_2}{dt}$). This derivation indicates that waste
 240 heat (\dot{Q}) plays an important role⁶ in Bullard-Herendeen-style I-O analyses:
 241 the accumulation of waste heat along a production path leads to energy being
 242 ‘embodied’ in the output of an economic sector.

243 In Equation 10 we also see the first indication that the traditional ap-
 244 proach of neglecting dynamic effects in I-O analyses may lead to errors. If
 245 $\frac{dB_2}{dt}$ is both neglected and nonzero, calculation of the embodied energy out-
 246 flow rate (\dot{B}_{21}) will be in error.

247 3.4. Depreciation

248 It is worthwhile to note that \dot{B}_{21} represents the disposal rate of embodied
 249 energy from the economy back to the earth, akin to depreciation of physical

⁶To our knowledge, there has been no prior identification of the role of waste heat in Bullard-Herendeen-style I-O analyses.

250 assets. This physical depreciation is different from, but related to, financial
 251 depreciation, as financial depreciation is usually faster than physical deprecia-
 252 tion. Embodied energy depreciation (\dot{B}_{21} in this example) can be represented
 253 by a depreciation term such as

$$\dot{B}_{21} = \gamma_2 B_2, \quad (11)$$

254 where γ represents the depreciation rate in units of inverse time (e.g., 1/year)
 255 with $\gamma > 0$. The depreciation rate (γ) indicates that a fraction of the to-
 256 tal stock of embodied energy is disposed over a period of time (e.g, $\gamma =$
 257 0.05/year). In the absence of other inputs or outputs, this depreciation func-
 258 tion provides exponential decay of embodied energy (B). γ is, in general, a
 259 function of time.

260 Equation 11 can be substituted into Equation 9 and rearranged to obtain

$$\frac{dB_2}{dt} = \dot{Q}_{21} + \dot{B}_{12} - \gamma_2 B_2 \quad (12)$$

261 which indicates that the accumulation rate of embodied energy in an eco-
 262 nomic sector (in this case $\frac{dB_2}{dt}$) is equal to the sum of the waste heat rate
 263 from the economic sector (\dot{Q}_{21}) and the inflow rate of embodied energy to
 264 the sector (\dot{B}_{12}) less the embodied energy disposal rate ($\gamma_2 B_2$).

265 **4. Value (X), energy intensity (ε), and the input-output ratio (a)**

266 We now turn to defining flows of value (\dot{X}), energy intensity (ε), and
 267 input-output ratios (a).

268 4.1. Value flows (\dot{X})

269 Among sectors of the economy and society, value (\dot{X}) flows in the same
 270 direction as goods, services, and energy, but in the opposite direction from
 271 currency payments. Typical of the Bullard-Herenden I-O analyses technique
 272 [NEED REFERENCE HERE –MKH], we allow value flows to be in either
 273 monetary units or physical units. For non-energy sectors of the economy,
 274 value outflows are in currency units per time (\$/time). For energy-producing
 275 sectors, value outflows are in units of J/time or BTU/time.

276 4.2. Energy intensity (ε)

277 Energy intensity (ε) is the ratio of total energy and value outflow rates
 278 from an economic sector, such that for the j^{th} economic sector,

$$\varepsilon_j \equiv \frac{\dot{T}_j}{\dot{X}_j}. \quad (13)$$

279 For goods and services sectors of the economy, ε is in units of J/\$, but
 280 for energy-producing sectors of the economy, the units of ε are J/J. For
 281 inter-sector flows, we have

$$\varepsilon_{ij} = \frac{\dot{T}_{ij}}{\dot{X}_{ij}}. \quad (14)$$

282 Furthermore, we note that

$$\varepsilon_i = \varepsilon_{ij} \quad (15)$$

283 for all j , because the energy intensity of a sector's output is the same re-
 284 gardless of its destination. I.e., we assume that all goods produced within a

Table 1: Units for input-output ratios (a).

		Output of	
		Non-energy sector	Energy sector
Inputs from	Non-energy sector	$\frac{\$}{\$}$	$\frac{\$}{J}$
	Energy sector	$\frac{J}{\$}$	$\frac{J}{J}$

sector are produced at the average energy intensity of that sector.⁷

4.3. Input-output ratios (a)

We define a parameter a_{ij} that represents the input of good i required to produce a unit of output from sector j .

$$a_{ij} \equiv \frac{\dot{X}_{ij}}{\dot{X}_j} \quad (16)$$

Input-output ratios are given in mixed units, depending on the purpose of each sector of the economy and the type of input as shown in Table 1.

5. Example B: a one sector economy with external demand

At this point, we move to a second example wherein a single economic sector (3) interacts with Society (2, which provides final demand) and the Earth (1, the destination for waste heat and the source of all resources). In this economy, we assume that the purpose of the goods and services sector

⁷If this approach is unsatisfactory, the sector may be divided into sub-sectors with different energy intensities.

296 is to produce goods and provide services, including the provision of direct
 297 energy available to the economy and society.

298 5.1. First Law of Thermodynamics

299 The First Law of Thermodynamics requires that energy (direct and wast
 300 heat) is conserved around each Sector of the economy (3) as well as around
 301 the Earth (1) and Society (2) as shown in Figure 4.

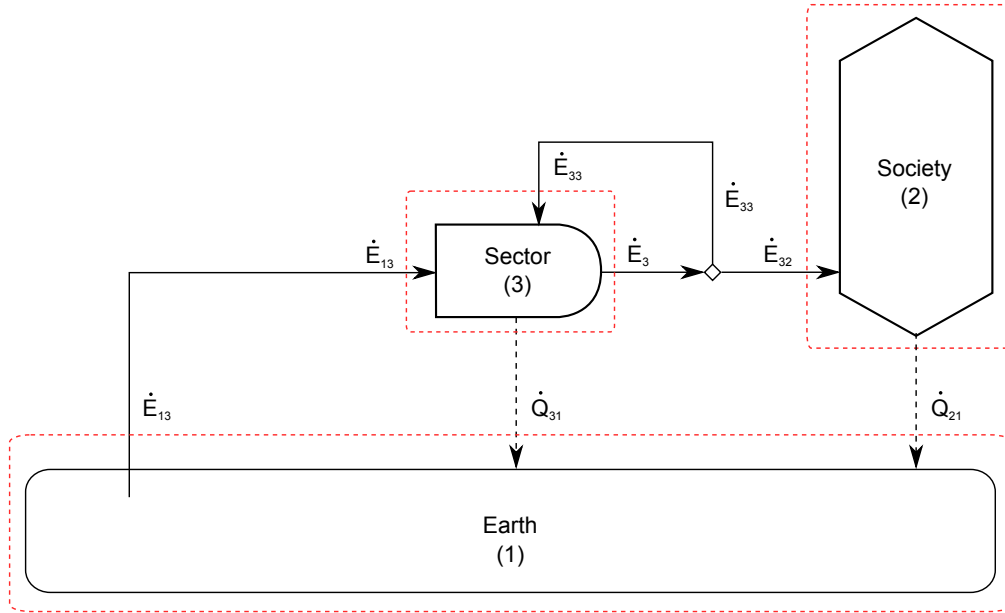


Figure 4: Flows of direct energy (\dot{E}) and waste heat (\dot{Q}) in a one-sector economy with separate demand.

302 The First Law around the economic Sector (3) including the accumulation
 303 rate of direct energy in the sector ($\frac{dE_3}{dt}$) yields

$$\frac{dE_3}{dt} = \dot{E}_{13} + \dot{E}_{33} - \dot{E}_3 - \dot{Q}_{31}. \quad (17)$$

304 It is notable that the economic Sector (3) consumes a portion of its own
 305 energy output (\dot{E}_{33}) as it produces its goods and services: it takes energy to
 306 make energy.

307 First Law energy accounting around the Earth (1) and Society (2) gives

$$\frac{dE_1}{dt} = \dot{Q}_{21} + \dot{Q}_{31} - \dot{E}_{13}, \quad (18)$$

308 and

$$\frac{dE_2}{dt} = \dot{E}_{32} - \dot{Q}_{21}. \quad (19)$$

309 As in Example A, we can set the accumulation of direct energy within
 310 each sector to zero to obtain

$$0 = \dot{E}_{13} + \dot{E}_{33} - \dot{E}_3 - \dot{Q}_{31}, \quad (20)$$

$$0 = \dot{Q}_{21} + \dot{Q}_{31} - \dot{E}_{13}, \quad (21)$$

311 and

$$0 = \dot{E}_{32} - \dot{Q}_{21}, \quad (22)$$

312 5.2. Total energy accounting

313 Again, we follow the I-O literature in assuming that total energy (i.e., the
 314 sum of direct energy and indirect energy) is conserved. Thus, we can draw a
 315 diagram similar to Figure 4 for total energy flows. See Figure 5.

316 Accounting for accumulation of total energy and using the assumption
 317 that total energy is conserved, we can write the following equations.

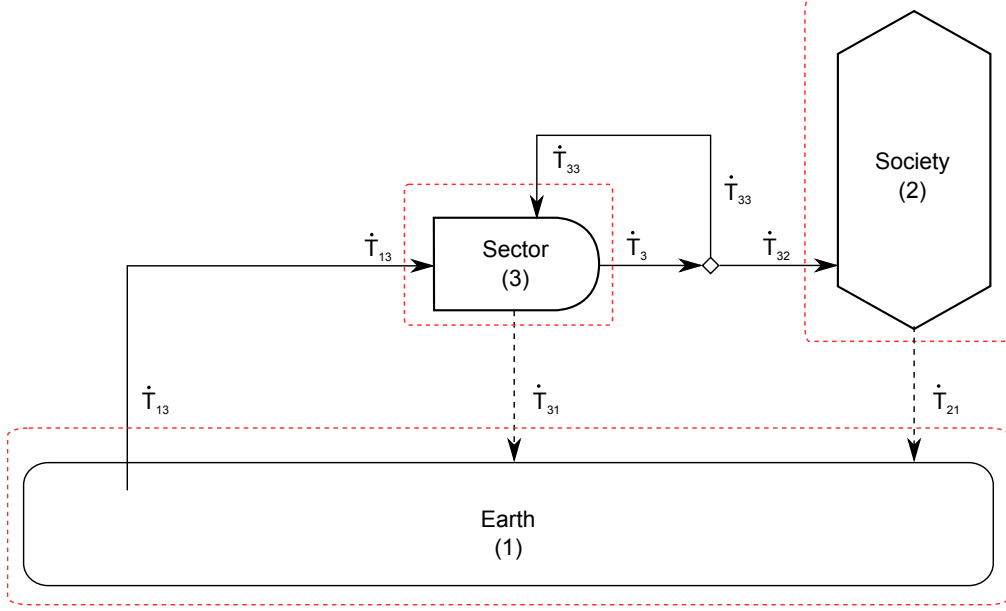


Figure 5: Flows of total energy (\dot{T}) in a one-sector economy with separate demand.

$$\frac{dT_1}{dt} = \dot{T}_{21} + \dot{T}_{31} - \dot{T}_{13}, \quad (23)$$

$$\frac{dT_2}{dt} = \dot{T}_{32} - \dot{T}_{21}, \quad (24)$$

318 and

$$\frac{dT_3}{dt} = \dot{T}_{13} + \dot{T}_{33} - \dot{T}_3 - \dot{T}_{31}. \quad (25)$$

319 5.3. Embodied energy accounting

320 Given that $\frac{dE_i}{dt} = 0$ and $\dot{T} = \dot{E} + \dot{B}$, we note that

$$\frac{dT_i}{dt} = \frac{dB_i}{dt}, \quad (26)$$

321 and we can rewrite the total energy accumulation accounting equations as

$$\frac{dB_1}{dt} = \dot{E}_{21} + \dot{B}_{21} + \dot{E}_{31} + \dot{B}_{31} - \dot{E}_{13} + \dot{B}_{13}, \quad (27)$$

$$\frac{dB_2}{dt} = \dot{E}_{32} + \dot{B}_{32} - \dot{E}_{21} - \dot{B}_{21}, \quad (28)$$

322 and

$$\frac{dB_3}{dt} = \dot{E}_{13} + \dot{B}_{13} + \dot{E}_{33} + \dot{B}_{33} - \dot{E}_3 - \dot{B}_3 - \dot{E}_{31} - \dot{B}_{31}. \quad (29)$$

323 As in Example A, we can substitute the First Law of Thermodynamics for
 324 the economic Sector (Equation 20) into the total energy accounting equation
 325 for the economic Sector (Equation 29). Assuming that $\dot{E}_{31} = 0$ (because
 326 energy is returned to the Earth as waste heat, not direct energy), we obtain

$$\frac{dB_3}{dt} = \dot{Q}_{31} + \dot{B}_{13} + \dot{B}_{33} - \dot{B}_{31} \quad (30)$$

327 Similar to Example A, we observe that the accumulation rate of embodied
 328 energy in the Goods and Services sector (3) is the sum of the rates of waste
 329 heat from the sector (\dot{Q}_{31}) and embodied energy into the sector ($\dot{B}_{13} + \dot{B}_{33}$)
 330 less the rate of embodied energy leaving the sector on its output stream (\dot{B}_{31}).

331 5.4. Depreciation

332 We can substitute a depreciation term for the flow rate of embodied en-
 333 ergy from the economic Sector (3) to the Earth (1) to obtain

$$\frac{dB_3}{dt} = \dot{Q}_{31} + \dot{B}_{13} + \dot{B}_{33} - \gamma_3 B_3. \quad (31)$$

334 5.5. Estimating energy intensity (ε) of the economy

335 The following figure shows value flows (\dot{X}) in the one-sector economy
 336 with separate demand.

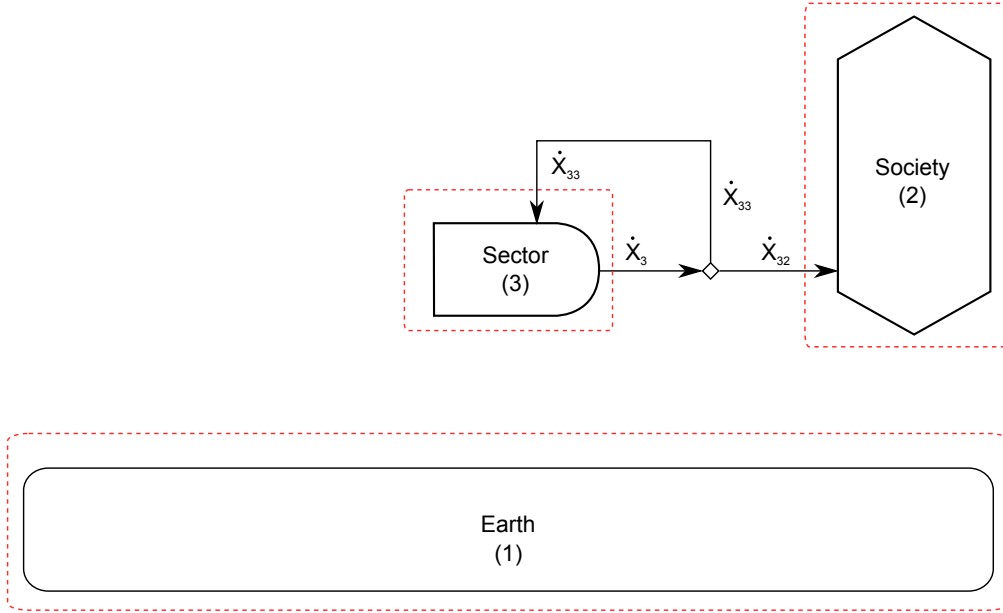


Figure 6: Flows of economic value (\dot{X}) in a one-sector economy with separate demand.

337 The energy intensity (ε) of the economic Sector (3) is given by

$$\varepsilon_3 = \frac{\dot{T}_3}{\dot{X}_3} = \frac{\dot{T}_{33}}{\dot{X}_{33}}. \quad (32)$$

338 The input-output ratio (a) for the economic Sector (3) is

$$a_{33} = \frac{\dot{X}_{33}}{\dot{X}_3}. \quad (33)$$

339 Thus,

$$\dot{T}_3 = \varepsilon_3 \dot{X}_3, \quad (34)$$

340 and

$$\dot{T}_{33} = \varepsilon_3 a_{33} \dot{X}_3. \quad (35)$$

341 Realizing that (a) $\frac{dT_3}{dt} = \frac{dB_3}{dt}$ because $\frac{dE_3}{dt} = 0$, (b) $\dot{T}_{13} = \dot{E}_{13}$ because
 342 $\dot{B}_{13} = 0$ due to processing of raw energy carriers occurring *within* the eco-
 343 nomic Sector (3), and (c) substituting Equations 34 and 35 into Equation 25
 344 gives

$$\frac{dB_3}{dt} = \varepsilon_3 a_{33} \dot{X}_3 + \dot{E}_{13} - \varepsilon_3 \dot{X}_3 - \gamma_3 B_3. \quad (36)$$

345 We can estimate the energy intensity of the economy by solving Equation
 346 36 for ε_3 .

$$\varepsilon_3 = (1 - a_{33})^{-1} \dot{X}_3^{-1} \left[\dot{E}_{13} - \left(\frac{dB_3}{dt} + \gamma_3 B_3 \right) \right] \quad (37)$$

347 Equation 37 is similar to the typical energy intensity equation found in the
 348 I-O literature [REFERENCE BULLARD AND OTHERS HERE. –MKH],
 349 except that Equation 37 applies to a single economic sector and contains
 350 scalar (as opposed to matrix) terms. Using Example C below, we will derive
 351 a matrix representation of Equation 37 that is directly comparable to energy
 352 intensity equations found in the I-O literature.

353 5.6. Derivation of economic sector energy intensity (ε) by a convergent infi-
 354 nite series

355 The single-sector economy of Figures 4 through 6 can be re-drawn as
 356 shown in Figure 7.

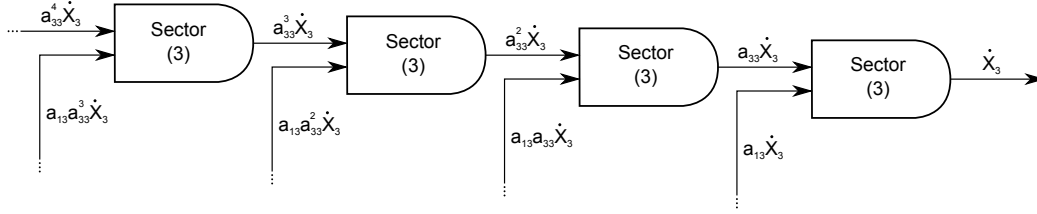


Figure 7: Process flows in a single-sector economy.

357 The economy produces output at a rate of \dot{X}_3 , but it requires energy from
 358 the Earth ($\dot{E}_{13} = a_{13}\dot{X}_3$) to do so. The economy also consumes a fraction of
 359 its own gross output ($\dot{X}_{33} = a_{33}\dot{X}_3$). To produce $a_{33}\dot{X}_3$, the economy requires
 360 an additional $a_{13}a_{33}\dot{X}_3$ of energy from the Earth. The total energy required
 361 for the economy to produce at a rate of \dot{X}_3 is an infinite sum.

$$\dot{E}_{demand} = a_{13}\dot{X}_3 + a_{13}a_{33}\dot{X}_3 + a_{13}a_{33}^2\dot{X}_3 + \dots \quad (38)$$

362 The energy intensity of the economy (ε_3) is

$$\varepsilon_3 = \frac{\dot{E}_{demand}}{\dot{X}_3} = a_{13}(1 + a_{33} + a_{33}^2) + \dots = a_{13} \sum_{n=0}^{\infty} a_{33}^n. \quad (39)$$

363 Realizing that $\sum_{n=0}^{\infty} a_{33}^n = \frac{1}{1-a_{33}}$ and $a_{13} = \frac{\dot{E}_{13}}{\dot{X}_3}$ gives

$$\varepsilon_1 = (1 - a_{33})^{-1} \dot{X}^{-1} \dot{E}_{13}. \quad (40)$$

364 Neglecting accumulation of embodied energy in the economy ($\frac{dB_3}{dt}$) and
 365 depreciation ($\gamma_3 B_3$), Equations 37 and 40 are identical (assuming $\frac{dB_3}{dt} =$
 366 $\gamma_3 = 0$), indicating that the I-O approach accounts for the infinite recursion
 367 of energy demand by the economy.

368 **6. Example C: a two-sector economy**

369 We extend single-sector Example B to derive a matrix representation for
 370 the I-O method that can be generalized to any number of economic sectors.
 371 A two-sector economy consisting of an Energy sector (3) and a Goods and
 372 Services sector (4) is considered. Both the Earth (1) and Society (2) are
 373 also included. Resources are extracted from the Earth (1), and Society (2)
 374 provides the final demand for both the Goods and Services (4) and the Energy
 375 (3) sectors.

376 *6.1. First Law of Thermodynamics*

377 The First Law of Thermodynamics requires that energy is conserved
 378 around each sector of the economy as well as around the Earth (1) and
 379 Society (2) as shown in Figure 8.

380 In this economy, we assume that the purpose of the Goods and Services
 381 sector (4) is to produce goods and provide services, it provides no direct
 382 energy to society. The purpose of the Energy sector (3) is to make direct
 383 energy (\dot{E}) available to the economy and society in a useful form. Both direct
 384 energy (\dot{E}) (such as chemical potential energy in coal, oil, and electricity)
 385 and waste heat (\dot{Q}) are accounted by the First Law of Thermodynamics. The
 386 First Law around the Goods and Services sector (4) including, for now, the
 387 accumulation rate of direct energy in the sector ($\frac{dE_4}{dt}$) yields

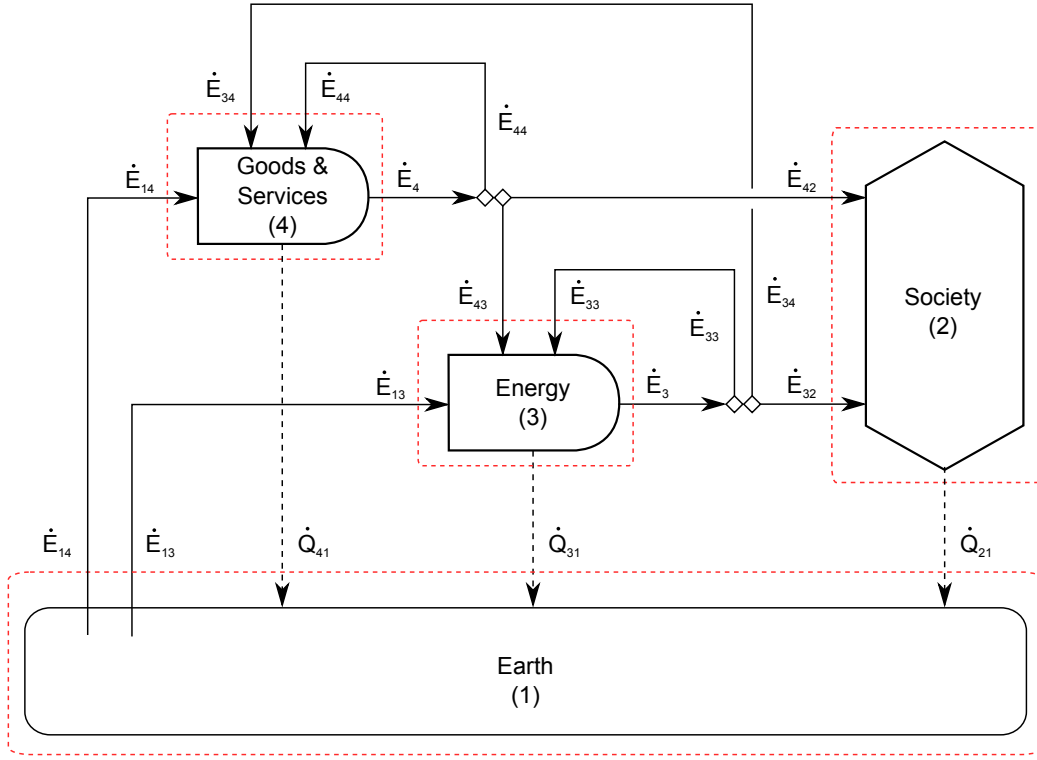


Figure 8: Flows of direct energy (\dot{E}) and waste heat (\dot{Q}) in a two-sector economy.

$$\frac{dE_4}{dt} = \dot{E}_{14} + \dot{E}_{34} + \dot{E}_{44} - \dot{E}_4 - \dot{Q}_{41}. \quad (41)$$

388 Note that we may simplify Equation 41 by realizing that $\dot{E}_4 = \dot{E}_{4i} = 0$,
 389 because the goods and services sector is assumed to produce no flows of
 390 energy, and that $\dot{E}_{14} = 0$, since sector (4) receives no direct energy from the
 391 earth, except via the energy sector (3), hence:

$$\frac{dE_4}{dt} = \dot{E}_{34} - \dot{Q}_{41}. \quad (42)$$

392 The First Law of Thermodynamics around the Earth (1), Society (2),
 393 and the Energy sector (3) gives

$$\frac{dE_1}{dt} = \dot{Q}_{21} + \dot{Q}_{31} + \dot{Q}_{41} - \dot{E}_{13} - \dot{E}_{14}, \quad (43)$$

$$\frac{dE_2}{dt} = \dot{E}_{32} + \dot{E}_{42} - \dot{Q}_{21}, \quad (44)$$

394 and

$$\frac{dE_3}{dt} = \dot{E}_{13} + \dot{E}_{33} + \dot{E}_{43} - \dot{E}_3 - \dot{Q}_{31}. \quad (45)$$

395 As in Examples A and B, we can set the accumulation of direct energy
396 to zero.

$$0 = \dot{Q}_{21} + \dot{Q}_{31} + \dot{Q}_{41} - \dot{E}_{13} - \dot{E}_{14} \quad (46)$$

$$0 = \dot{E}_{32} + \dot{E}_{42} - \dot{Q}_{21} \quad (47)$$

$$0 = \dot{E}_{13} + \dot{E}_{33} + \dot{E}_{43} - \dot{E}_3 - \dot{Q}_{31} \quad (48)$$

397 and

$$0 = \dot{E}_{14} + \dot{E}_{34} + \dot{E}_{44} - \dot{E}_4 - \dot{Q}_{41} \quad (49)$$

398 6.2. Total energy accounting

399 Again, we follow the I-O literature in assuming that total energy (i.e.,
400 the sum of direct energy and embodied energy) is conserved. Thus, we can
401 draw a diagram similar to Figure 8 for total energy flows.

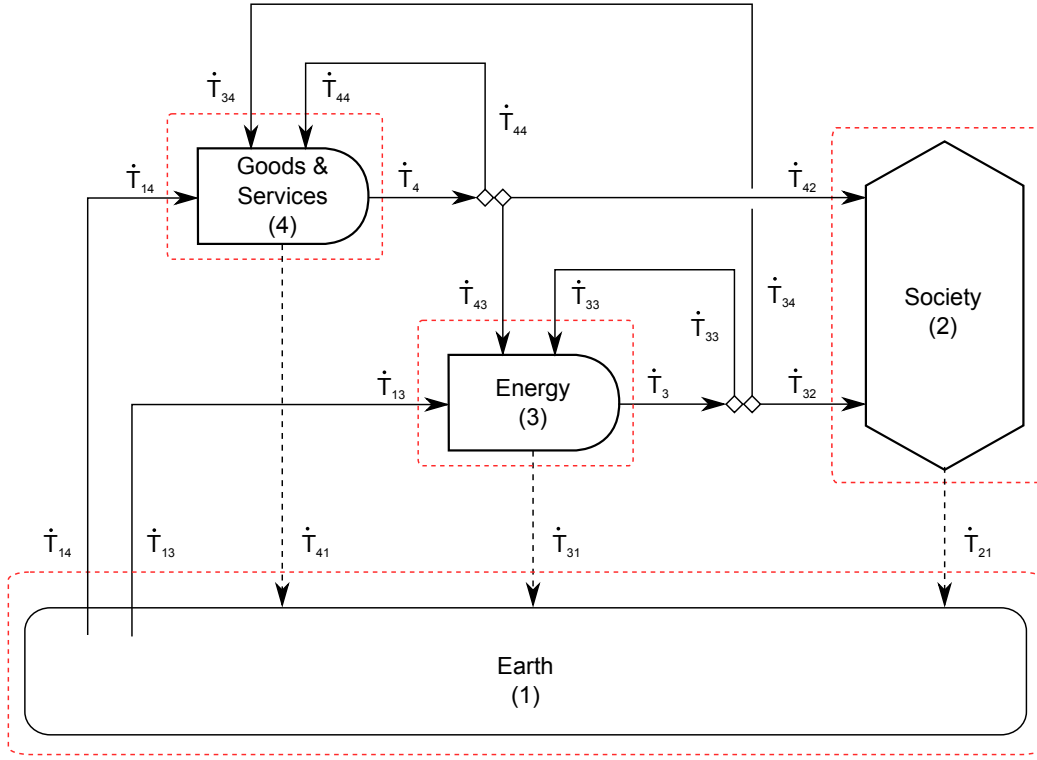


Figure 9: Flows of total energy (\dot{T}) in a two-sector economy.

Accounting for accumulation of total energy and using the assumption that total energy is conserved, we can write the following equations.

$$\frac{dT_1}{dt} = \dot{T}_{21} + \dot{T}_{31} + \dot{T}_{41} - \dot{T}_{13} - \dot{T}_{14}, \quad (50)$$

$$\frac{dT_2}{dt} = \dot{T}_{32} + \dot{T}_{42} - \dot{T}_{21}, \quad (51)$$

$$\frac{dT_3}{dt} = \dot{T}_{13} + \dot{T}_{33} + \dot{T}_{43} - \dot{T}_3 - \dot{T}_{31}, \quad (52)$$

and

$$\frac{dT_4}{dt} = \dot{T}_{14} + \dot{T}_{34} + \dot{T}_{44} - \dot{T}_4 - \dot{T}_{41}. \quad (53)$$

405 *6.3. Embodied energy accounting*

406 Given that $\frac{dE_i}{dt} = 0$, we again note that $\frac{dT_i}{dt} = \frac{dB_i}{dt}$. Substituting $\dot{T} = \dot{E} + \dot{B}$
 407 into the total energy accounting equations gives

$$\frac{dB_1}{dt} = \dot{E}_{21} + \dot{B}_{21} + \dot{E}_{31} + \dot{B}_{31} + \dot{E}_{41} + \dot{B}_{41} - \dot{E}_{13} - \dot{B}_{13} - \dot{E}_{14} - \dot{B}_{14}, \quad (54)$$

$$\frac{dB_2}{dt} = \dot{E}_{32} + \dot{B}_{32} + \dot{E}_{42} + \dot{B}_{42} - \dot{E}_{21} - \dot{B}_{21}, \quad (55)$$

$$\frac{dB_3}{dt} = \dot{E}_{13} + \dot{B}_{13} + \dot{E}_{33} + \dot{B}_{33} + \dot{E}_{43} + \dot{B}_{43} - \dot{E}_3 - \dot{B}_3 - \dot{E}_{31} - \dot{B}_{31}, \quad (56)$$

408 and

$$\frac{dB_4}{dt} = \dot{E}_{14} + \dot{B}_{14} + \dot{E}_{34} + \dot{B}_{34} + \dot{E}_{44} + \dot{B}_{44} - \dot{E}_4 - \dot{B}_4 - \dot{E}_{41} - \dot{B}_{41}. \quad (57)$$

409 Substituting the First Law of Thermodynamics (Equations 46 through
 410 49) into the total energy accounting equations (Equations 54 through 57)
 411 gives embodied energy accounting equations for Example C.

$$\frac{dB_1}{dt} = \dot{B}_{21} + \dot{B}_{31} + \dot{B}_{41} - \dot{B}_{13} - \dot{B}_{14} - \dot{Q}_{21} - \dot{Q}_{31} - \dot{Q}_{41} \quad (58)$$

$$\frac{dB_2}{dt} = \dot{B}_{32} + \dot{B}_{42} + \dot{Q}_{21} - \dot{B}_{21} \quad (59)$$

$$\frac{dB_3}{dt} = \dot{B}_{13} + \dot{B}_{33} + \dot{B}_{43} + \dot{Q}_{31} - \dot{B}_3 - \dot{B}_{31} \quad (60)$$

$$\frac{dB_4}{dt} = \dot{B}_{14} + \dot{B}_{34} + \dot{B}_{44} + \dot{Q}_{41} - \dot{B}_4 - \dot{B}_{41} \quad (61)$$

412 To verify the above derivation, we sum Equations 58 through 61 and use
413 the following identities:

$$\dot{B}_3 = \dot{B}_{32} + \dot{B}_{33} + \dot{B}_{34} \quad (62)$$

414 and

$$\dot{B}_4 = \dot{B}_{42} + \dot{B}_{43} + \dot{B}_{44}; \quad (63)$$

415 to obtain

$$\frac{dB_1}{dt} + \frac{dB_2}{dt} + \frac{dB_3}{dt} + \frac{dB_4}{dt} = 0 \quad (64)$$

416 as expected. The total embodied energy content of the system (Earth (1),
417 Society (2), Energy sector (3), and Goods and Services sector (4)) is constant
418 with respect to time.

419 *6.4. Definition of embodied energy (\dot{B})*

420 [DOES THIS SECTION BELONG HERE?]

421 [I MOVED THIS SECTION TO FOLLOW DIRECTLY AFTER THE
422 EMBODIED ENERGY ACCOUNTING SECTION FOR EXAMPLE C. I
423 THINK IT FITS BETTER HERE. –MKH]

424 At this point we can develop a rigorous definition of embodied energy.
 425 To do so, we use the Goods and Services sector (4) from Example C. Direct
 426 energy accounting around the Goods and Services sector (Figure 8) yields

$$\frac{dE_4}{dt} = \dot{E}_{14} + \dot{E}_{34} + \dot{E}_{44} - \dot{E}_4 - \dot{Q}_{41}, \quad (65)$$

427 Total energy accounting around the Goods and Services sector (Figure 9)
 428 yields

$$\frac{dT_4}{dt} = \dot{T}_{14} + \dot{T}_{34} + \dot{T}_{44} - \dot{T}_4 + \dot{T}_{41}, \quad (66)$$

429 Solving the direct energy equation (Equation 65) for the rate of direct
 430 energy input from the Energy sector (3) to the Goods and Services sector
 431 (4), namely \dot{E}_{34} , substituting into the total energy equation (Equation 66),
 432 solving the result for \dot{B}_4 , and assuming that no direct energy is wasted by
 433 the Goods and Services sector (4) to the Earth (1), i.e. $\dot{E}_{41} = 0$, yields

$$\dot{B}_4 = \dot{B}_{14} + \dot{B}_{34} + \dot{B}_{44} + \dot{Q}_{41} - \frac{dB_4}{dt} - \dot{B}_{41}. \quad (67)$$

434 Written generally, we obtain a formal definition for embodied energy out-
 435 put from an economic sector:

$$\dot{B}_j \equiv \sum_i \dot{B}_{ij} - \frac{dB_j}{dt} - \dot{B}_{j1} + \dot{Q}_{j1}. \quad (68)$$

436 Rearranging, we obtain

$$\frac{dB_j}{dt} = \sum_i \dot{B}_{ij} - \dot{B}_j - \dot{B}_{j1} + \dot{Q}_{j1}. \quad (69)$$

437 In words, the rate of accumulation of embodied energy in a sector of
 438 the economy ($\frac{dB_j}{dt}$) is equal to the sum of the rates of input of embodied
 439 energy into the sector ($\sum_i \dot{B}_{ij}$) less the rate of useful output of embodied
 440 energy from the sector (\dot{B}_j) less the rate of wasting embodied energy by the
 441 sector (\dot{B}_{j1}) *plus* the rate of waste heat from the sector (\dot{Q}_{j1}). The first three
 442 terms on the right side of the equation are expected: accumulation is the
 443 difference between inflow and outflow rates. However, we see that the last
 444 term ($+\dot{Q}_{j1}$) in the above equations indicates that waste heat is *additive* to
 445 both accumulation of embodied energy in a sector of the economy (Equation
 446 69) and outflow of embodied energy from a sector of the economy (Equation
 447 68). Furthermore, because the waste heat appears in the embodied energy
 448 output from a sector, waste heat accumulates along each step of a process
 449 such that the energy embodied in a finished product is the *sum* of waste
 450 heats along a process path.

451 6.5. Depreciation

452 The terms \dot{B}_{21} , \dot{B}_{31} , and \dot{B}_{41} represent material depreciation (i.e., dis-
 453 posal) rates. As before, we can represent the embodied energy content of
 454 material depreciation as $\dot{B}_{i1} = \gamma_i B_i$ to obtain

$$\frac{dB_1}{dt} = \gamma_2 B_2 + \gamma_3 B_3 + \gamma_4 B_4 - \dot{B}_{13} - \dot{B}_{14} - \dot{Q}_{21} - \dot{Q}_{31} - \dot{Q}_{41} \quad (70)$$

$$\frac{dB_2}{dt} = \dot{B}_{32} + \dot{B}_{42} + \dot{Q}_{21} - \gamma_2 B_2 \quad (71)$$

$$\frac{dB_3}{dt} = \dot{B}_{13} + \dot{B}_{33} + \dot{B}_{43} + \dot{Q}_{31} - \dot{B}_3 - \gamma_3 B_3 \quad (72)$$

$$\frac{dB_4}{dt} = \dot{B}_{14} + \dot{B}_{34} + \dot{B}_{44} + \dot{Q}_{41} - \dot{B}_4 - \gamma_4 B_4 \quad (73)$$

455 6.6. Energy ratios

456 [THIS SECTION IS INTERESTING TO US, BUT MAY NOT BE GER-
 457 MANE TO THE PAPER. IF THIS WERE TO BE A BOOK, I THINK
 458 WE SHOULD KEEP THIS SECTION IN. IF WE'RE NOW SHOOTING
 459 FOR A PAPER, WE CAN PROBABLY LEAVE THIS SECTION OUT.
 460 WE SHOULD AT LEAST REFERENCE THE LITERATURE HERE. OR,
 461 WE SHOULD MAKE USE OF THESE RATIOS. FOR EXAMPLE, IF
 462 WE WANT TO TALK ABOUT DECLINING RESOURCE QUALITY IN
 463 TERMS OF *EROI* OR *GER*, WE SHOULD DO SO. WE SHOULD THEN
 464 SUBSTITUTE *GER* INTO THE DERIVED EQUATIONS AND SHOW
 465 HOW A CHANGE IN *GER* AFFECTS ENERGY FLOWS THROUGH
 466 THE ECONOMY. -MKH]

467 Several important energy ratios can be observed in Figure 8. [REFER-
 468 ENCE OTHER PAPERS FOR NOMENCLATURE? -MKH] The Gross En-
 469 ergy Ratio (*GER*) is defined as:

$$GER_\beta \equiv \frac{\dot{E}_3}{\dot{E}_{33}}. \quad (74)$$

470 The Net Energy Ratio (*NER*) is defined as [SHOULD THE NUMERATOR
 471 FOR NER_β BE $\dot{E}_{32} + \dot{E}_{34}$? -MKH]

472 [MIK SHOULD CHECK THE NEXT TWO EQUATIONS. ARE THEY
 473 CORRECT? -MKH]

$$NER_{\beta} \equiv \frac{\dot{E}_{32}}{\dot{E}_{33}} = GER - 1. \quad (75)$$

474 [CAN WE SIMPLIFY THESE? FOR EXAMPLE, $\dot{E}_{43} = 0$, SO THE
475 DENOMINATOR CAN BE SIMPLIFIED TO \dot{B}_{43} -MKH]

$$GER_{\delta} \equiv \frac{\dot{E}_3}{\dot{T}_{33} + \dot{T}_{43}}; \quad (76)$$

476 [THIS EQUATION DOESN'T LOOK RIGHT. IT IS SAME AS THE
477 PREVIOUS EQUATION $GER - 1$, BUT HAS A DIFFERENT DENOMI-
478 NATOR. MIK SHOULD CHECK. -MKH]

$$NER_{\beta} \equiv \frac{\dot{E}_{32}}{\dot{T}_{33} + \dot{T}_{43}} = GER - 1; \quad (77)$$

479 We may also define a Gross External Energy Ratio (GEER) and Net
480 External Energy Ratio (NEER), which account only for inputs from other
481 sectors in the denominator:

$$GEER_{\delta} \equiv \frac{\dot{E}_3}{\dot{T}_{43}}, \quad (78)$$

482 and

$$NEER_{\beta} \equiv \frac{\dot{E}_{32}}{\dot{T}_{43}}. \quad (79)$$

483 6.7. Final demand

484 Society's demand vector for total energy, \dot{T} , can be written as

$$\mathbf{Y}_{\dot{T}} = \left\{ \begin{array}{c} \dot{T}_{32} \\ \dot{T}_{42} \end{array} \right\}. \quad (80)$$

485 In terms of total energy, the ultimate demand ($Y_{\dot{T}}$) is given by

$$Y_{\dot{T}} = \sum_{i=3}^N \dot{T}_{i2} = \dot{T}_{32} + \dot{B}_{42}. \quad (81)$$

486 after realizing that $\dot{E}_{42} = 0$.⁸

487 Using $\dot{T}_{32} = \dot{E}_{32} + \dot{B}_{32}$ and rearranging Equation 81 gives

⁸We acknowledge that there are examples in the real economy which run counter to this model, where output from non-energy sectors are valued for their energetic content, one example being agriculture. [MIK: I SUGGEST THAT THIS SECTION ABOUT LABOR BE MOVED ELSEWHERE, FOR EXAMPLE, NEARER THE SECTIONS WHERE WE DISCUSS FINAL DEMAND VECTORS, Y? –MKH] “Direct” energy inputs also flow in the opposite direction in the form of labor, which we also neglect. This will serve to introduce errors which will be small for industrial economies and larger for less industrial societies. To illustrate this we may compare the United States with India. To feed an adult requires around 2000 kcal/day ≈ 3 GJ/yr. To feed the whole ~ 300 million population of the States requires around 1×10^{18} J (1 EJ) which is around 1% of the roughly 100 EJ of primary energy supply. The US labor force currently stands at around 240 million. Given that a human can supply around 100 W of power and assuming an 8 hour work day, the US labor force will supply 70 TWh/yr ≈ 0.25 EJ. For India, the energy to food to feed 1.25 billion people is nearly 4 EJ which is around 15% of the ~ 25 EJ of primary energy consumed. Assuming that the labor force makes up 500 million people working at 12 hours per day, the energy supplied by labor is around 0.8 EJ or around 3% of the total primary energy. As such, we can see that food energy accounts for around 1% of primary energy in the US and around 15% in India. Similarly, the labor inputs account for around 0.25% in the US and around 3% in India. The implication of including or omitting these flows is different in each case. Our assumptions introduce small errors for industrial societies where most of the world’s energy is consumed.

$$\dot{B}_{32} + \dot{B}_{42} = Y_{\dot{T}} - \dot{E}_{42}. \quad (82)$$

488 Substituting Equation 82 into Equation 71 yields

$$\frac{dB_2}{dt} = Y_{\dot{T}} - \dot{E}_{32} + \dot{Q}_{21} - \gamma_2 B_2. \quad (83)$$

489 Substituting Equation 47 into Equation 83 and realizing that $\dot{E}_{42} = 0$
 490 because direct energy is supplied to society by the energy sector only, we
 491 obtain

$$\frac{dB_2}{dt} = Y_{\dot{T}} - \gamma_2 B_2, \quad (84)$$

492 [IT WOULD BE GOOD TO FIND SOME VERY ROUGH DATA FOR
 493 THIS VALUE, E.G. WHAT IS THE AVERAGE LIFETIME OF MANU-
 494 FACTURED GOODS - INCLUDING PACKAGING AND NON-CONSUMER
 495 GOODS. WHAT IS THE BALANCE OF NON-DURABLE VS. DURABLE
 496 GOODS? WHAT PROPORTION OF GOODS (E.G. FOOD) IS WASTED
 497 BEFORE EVER BEING CONSUMED?]

498 [I AGREE. HOW? –MKH]

499 indicating that the final demand vector for total energy ($Y_{\dot{T}}$) and the ac-
 500 cumulation rate of energy in society ($\frac{dB_2}{dt}$) differ by the rate of disposal
 501 from society ($\gamma_2 B_2$). We note that as total embodied energy in society (B_2)
 502 becomes increasingly large, we need an ever-increasing rate of energy sup-
 503 plied to the society ($Y_{\dot{T}}$) to maintain positive growth ($\frac{dB_2}{dt}$). [MAIN POINT
 504 THAT MUST BE DISCUSSED IN FURTHER DETAIL LATER, PARTIC-
 505 ULARLY IN RELATION TO INCREASING GDP NOT NECESSARILY
 506 SIGNALING INCREASING ACCUMULATION OR GROWTH.]

507 [WE NEVER USE THE FINAL PARAGRAPHS OF THIS SECTION. I
 508 PROPOSE THAT WE DELETE THEM. –MKH]

509 A control volume around the economy gives

$$Y_{\dot{B}} = \sum_{j=1}^N \dot{E}_{1j} - \sum_{i=1}^N \dot{B}_{i1} - \dot{Q}_{21}, \quad (85)$$

510 [I HAVE ADDED \dot{Q}_{21} TERM HERE, AS THERE IS DEFINITELY
 511 WASTE HEAT FROM CONSUMPTION OF FINAL ENERGY $Y_{\dot{B}}$ THAT
 512 IS NOT EMBODIED IN GOOD OR SERVICE, E.G. THE ELECTRICITY
 513 I USE TO COOK MY DINNER, OR WATCH TV]

514 illustrating that final demand to society ($Y_{\dot{B}}$) can be considered the ultimate
 515 energy sink in the absence of depreciation in the economy.

516 It is also interesting to note that

$$\sum_{i=1}^N \dot{T}_i > Y_{\dot{T}}, \quad (86)$$

517 which is a consequence of the self-demand of sectors within the economy.

518 6.8. *Flows of Value (\dot{X})*

519 The following figure shows value flows (\dot{X}) in the two-sector economy.

520 Realizing that the valuable output from energy sectors is direct energy,
 521 $\dot{X}_3 = \dot{E}_3$ and $\dot{X}_{3j} = \dot{E}_{3j}$. Thus, outputs from energy sectors are given in
 522 energy units (joules or BTUs). With reference to Equations ?? and ??, we
 523 can express the Gross Energy Ratio (GER) and Net Energy Ratio (NER)
 524 as

$$GER_{\gamma} = \frac{\dot{E}_3}{\dot{E}_{33}} = \frac{\dot{X}_3}{\dot{X}_{33}} = \frac{1}{a_{33}}, \quad (87)$$

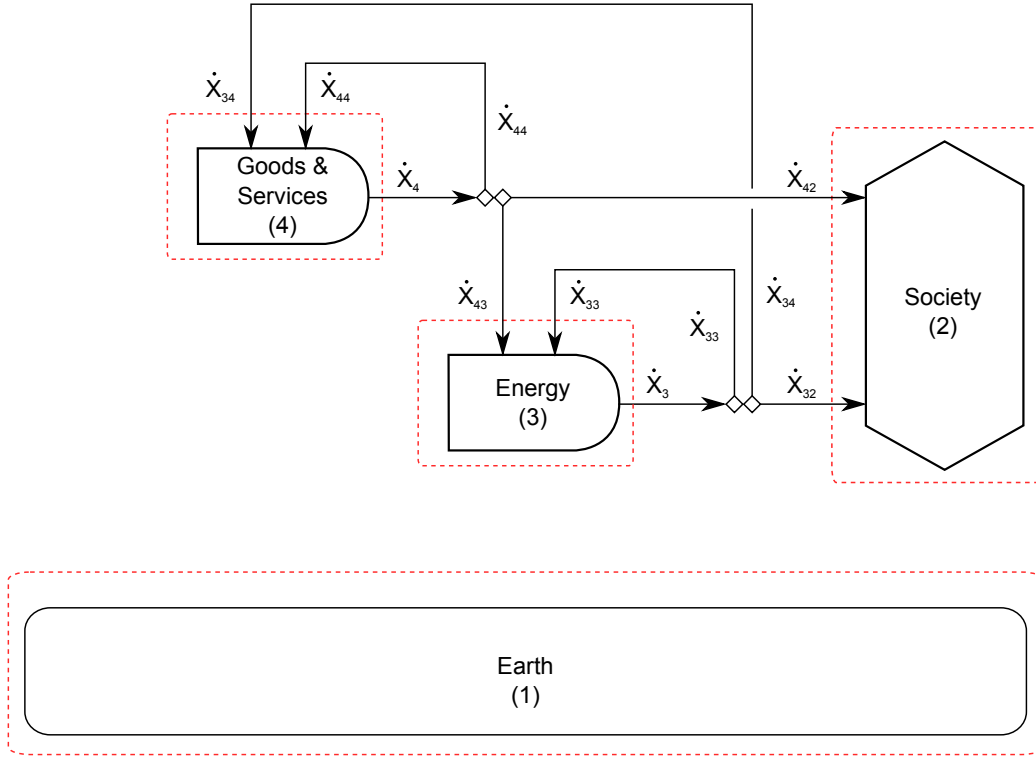


Figure 10: Flows of economic value (\dot{X}) in a two-sector economy.

525 and

$$NER = GER - 1 = \frac{1}{a_{33}} - 1. \quad (88)$$

526 [DO WE NEED THESE NEXT TWO EQUATIONS? DO WE USE
 527 THEM ANYWHERE? THEY MIGHT BE HELPFUL FOR A GDP DIS-
 528 CUSSION, BUT WE HAVEN'T INCLUDED THAT DISCUSSION YET.
 529 –MKH]

530 Written in terms of value flows, the ultimate demand vector (\mathbf{Y}) is given
 531 by

$$\mathbf{Y}_{\dot{X}} = \begin{Bmatrix} \dot{X}_{32} \\ \dot{X}_{42} \end{Bmatrix}, \quad (89)$$

532 and the total value demand from society (Y) is

$$Y_{\dot{X}} = \sum_{i=1}^N \dot{X}_{i2} = \dot{X}_{32} + \dot{X}_{42}. \quad (90)$$

533 6.9. Matrix Formulation

534 We can use Equations 13 through 15 to rewrite Equations ?? and ?? as

$$\dot{X}_{33}\varepsilon_3 + \dot{X}_{43}\varepsilon_4 + \dot{E}_{13} - \frac{dB_3}{dt} - \gamma_3 B_3 = \dot{X}_3 \varepsilon_3 \quad (91)$$

535 and

$$\dot{X}_{34}\varepsilon_3 + \dot{X}_{44}\varepsilon_4 + \dot{E}_{14} - \frac{dB_4}{dt} - \gamma_4 B_4 = \dot{X}_4 \varepsilon_4. \quad (92)$$

536 We can rewrite Equations 91 and 92 in matrix notation with the following
537 definitions:

$$\varepsilon = \begin{Bmatrix} \varepsilon_3 \\ \varepsilon_4 \end{Bmatrix}, \quad (93)$$

$$\mathbf{E} = \begin{Bmatrix} \dot{E}_{13} \\ \dot{E}_{14} \end{Bmatrix}, \quad (94)$$

$$\frac{d\mathbf{B}}{dt} = \begin{Bmatrix} \frac{dB_3}{dt} \\ \frac{dB_4}{dt} \end{Bmatrix}, \quad (95)$$

$$\mathbf{B} = \begin{Bmatrix} B_3 \\ B_4 \end{Bmatrix}, \quad (96)$$

$$\mathbf{A} = \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}, \quad (97)$$

$$\mathbf{X}_t = \begin{bmatrix} \dot{X}_{33} & \dot{X}_{34} \\ \dot{X}_{43} & \dot{X}_{44} \end{bmatrix}, \quad (98)$$

$$\hat{\mathbf{X}} = \delta_{ij} \dot{X}_j, \quad (99)$$

$$\hat{\gamma} = \delta_{ij} \gamma_j, \quad (100)$$

538 and

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \quad (101)$$

539 such that:

$$\mathbf{X}_t^T \varepsilon + \mathbf{E} - \left(\frac{d\mathbf{B}}{dt} + \hat{\gamma} \mathbf{B} \right) = \hat{\mathbf{X}} \varepsilon. \quad (102)$$

540 Additional relationships that will be helpful later include (derived in Ap-
541 pendix):

$$\hat{\mathbf{X}}^{-1} \mathbf{X}_t = \mathbf{A}^T, \quad (103)$$

$$\mathbf{X}_t^T - \hat{\mathbf{X}} = \hat{\mathbf{X}} (\mathbf{A}^T - \mathbf{I}), \quad (104)$$

$$\hat{\mathbf{X}} - \mathbf{X}_t^T = \hat{\mathbf{X}}(\mathbf{I} - \mathbf{A}^T), \quad (105)$$

542 and

$$\left(\hat{\mathbf{X}} - \mathbf{X}_t^T\right)^{-1} = (\mathbf{I} - \mathbf{A}^T)^{-1} \hat{\mathbf{X}}^{-1}. \quad (106)$$

543 *6.10. Estimating ε and $\frac{d\mathbf{B}}{dt}$*

544 With Equation 102, we can solve for either the energy accumulation vector
545 $\left(\frac{d\mathbf{B}}{dt}\right)$ or the energy intensity vector (ε), but not both.

546 Solving for the accumulation vector gives

$$\frac{d\mathbf{B}}{dt} = (\mathbf{X}_t^T - \hat{\mathbf{X}})\varepsilon + \mathbf{E} - \hat{\gamma}\mathbf{B}. \quad (107)$$

547 Finally, we can substitute Equation 104 which gives

$$\frac{d\mathbf{B}}{dt} = \hat{\mathbf{X}}(\mathbf{A}^T - \mathbf{I})\varepsilon + \mathbf{E} - \hat{\gamma}\mathbf{B}, \quad (108)$$

548 which allows estimation of the embodied energy accumulation in economic
549 sectors $\left(\frac{d\mathbf{B}}{dt}\right)$ knowing only sector outputs ($\hat{\mathbf{X}}$), sector input-output ratios (\mathbf{A}),
550 sector energy intensities (ε), energy input to the economy (\mathbf{E}), and sector
551 physical depreciation rates ($\hat{\gamma}\mathbf{b}$). In theory, the transaction matrix (\mathbf{X}_t) is not
552 required if the input-output ratios (\mathbf{A}) are known, though in reality, knowledge
553 of input-output ratios would be derived from the transaction matrix \mathbf{X}_t .

554 Solving for the energy intensity vector gives

$$\varepsilon = (\hat{\mathbf{X}} - \mathbf{X}_t^T)^{-1} \left[\mathbf{E} - \left(\frac{d\mathbf{B}}{dt} + \hat{\gamma}\mathbf{B} \right) \right]. \quad (109)$$

555 Substituting Equation 106 gives

$$\varepsilon = (\mathbf{I} - \mathbf{A}^T)^{-1} \hat{\mathbf{X}}^{-1} \left[\mathbf{E} - \left(\frac{d\mathbf{B}}{dt} + \hat{\gamma}\mathbf{B} \right) \right], \quad (110)$$

which allows estimation of the energy intensity of economic sectors (ε) knowing only sector input-output ratios (\mathbf{A}), sector outputs ($\hat{\mathbf{X}}$), energy input to the economy (\mathbf{E}), sector embodied energy accumulation rates ($\frac{d\mathbf{B}}{dt}$), and sector physical depreciation rates ($\hat{\gamma}\mathbf{B}$).

Comparison of Equations ?? and 110 shows the similarities between the single-sector algebraic formulation and the multi-sector matrix formulation of the I-O analysis method. This newly developed multi-sector matrix formulation can be extended to any desired level of economic and energy sector disaggregation as shown by Bullard (1975, 1978) and others.

***** MATT ENDED HERE *****

7. Implications

Several implications can be drawn from the above detailed development of the I-O method equations in a manner that includes both embodied energy accumulation and depreciation.

7.1. Implications for economic “development”

[IT WOULD BE GOOD TO HAVE A COMPARISON BETWEEN $\frac{d\mathbf{B}}{dt}$ AND STANDARD METRIC OF DEVELOPMENT, I.E. GDP WHICH I GUESS WOULD BE SOMETHING LIKE $\sum_i \dot{X}_i$. WE CAN CERTAINLY ENVISION SITUATIONS WHERE $\sum_i \dot{X}_i$ IS INCREASING AND

One consequence of economic “progress” or “development” is that embodied energy accumulates in economic sectors and society. In fact, accumulation

577 of embodied energy in economic sectors and society could be considered a
 578 *proxy* of development. This proxy for development is overly materialistic,
 579 one-dimensional, and reductionist, but alternatives such as GDP can be sim-
 580 ilarly criticized. In fact, GDP could continue to increase whilst accumulation
 581 of embodied energy or value actually decreased.

582 Figure 9 shows that energy extraction from the Earth is what ultimately
 583 drives development as measured by the accumulation of embodied energy
 584 in the economy and society. Development occurs over time. If embodied
 585 energy is the measure, development can be expressed as the integral of $\frac{d\mathbf{B}}{dt}$
 586 for economic sectors

$$\mathbf{B}(t) = \mathbf{B}(0) + \int_{t=0}^{t=t} \frac{d\mathbf{B}}{dt} dt, \quad (111)$$

587 or, using Equation 84, as the integral of $\frac{dB_2}{dt}$ for society,

$$B_2(t) = B_2(0) + \int_{t=0}^{t=t} \frac{dB_2}{dt} dt = B_2(0) + \int_{t=0}^{t=t} (Y_{\dot{T}} - \gamma_2 B_2 - \dot{Q}_{21}) dt. \quad (112)$$

588 Using embodied energy is obviously an incomplete measure of develop-
 589 ment. We might also use $X(t) = X(0) + \int \frac{dX}{dt} dt$. In fact, B and X are two
 590 complimentary factors to the economic process. For capital, B, to be useful,
 591 we need direct energy, E (to run the capital) and economic value, X (i.e.
 592 money). Therefore each of these factors are necessary, but insufficient.

593 Table 2 describes some of the dynamics that can be observed from Equa-
 594 tion 108. It is quite possible that, especially for regions like the U.S. and
 595 Western Europe, the rate of embodied energy accumulation in the economy
 596 $\left(\frac{d\mathbf{B}}{dt}\right)$ will be small relative to the rate of energy extraction from the Earth

Table 2: Factors from Equation 108 affecting the rate of embodied energy accumulation in the economy.

Right-side term	Implication
$\hat{\mathbf{X}}$	As economic output increases, $\frac{d\mathbf{B}}{dt}$ goes up (as will \mathbf{E})
\mathbf{A}	As input-ouput ratios increase, $\frac{d\mathbf{B}}{dt}$ goes up
ε	As the energy intensity of the economy increases, $\frac{d\mathbf{B}}{dt}$ goes up
\mathbf{E}	As the rate of energy flow from the Earth increases, $\frac{d\mathbf{B}}{dt}$ goes up
$\hat{\gamma}$	As the depreciation rate increases, $\frac{d\mathbf{B}}{dt}$ goes down
\mathbf{B}	As the embodied energy in the economy increases, $\frac{d\mathbf{B}}{dt}$ goes down

597 (\mathbf{E}). On the other hand, in rapidly developing countries, like China or India,
598 the rate of embodied energy accumulation in the economy may be signifi-
599 cantly higher than in a developed economy.

600 The behavior of \mathbf{B} with $\frac{d\mathbf{B}}{dt}$ is vitally important. A developed economy
601 has significantly higher embodied energy (\mathbf{B}) than a developing economy,
602 and, thus, the outflow rate of embodied energy due to depreciation ($\hat{\gamma}\mathbf{B}$)
603 will be higher. As increasingly large amounts of energy are embodied in the
604 economy, increasingly large energy extraction rates (\mathbf{E}) are required to offset
605 depreciation ($\hat{\gamma}\mathbf{B}$) and maintain positive growth ($\frac{d\mathbf{B}}{dt} > 0$) in the sectors of
606 the economy. Depreciation may also be, temporarily, offset by increasing
607 energy efficiency, i.e. by decreasing energy intensity, ε .

608 In a similar manner, Equation 84 indicates that maintaining a positive
609 rate of societal development ($\frac{dB_2}{dt} > 0$) requires ever increasing embodied

610 energy input rates to society ($Y_{\dot{T}}$) as the society “develops.” This mechanism
 611 provides a natural brake to the continued growth of physical economies.

612 7.2. Implications for the I-O method

613 The I-O literature (examples include Bullard (1975) and Cassler (1983))
 614 usually writes Equation 110 as

$$\varepsilon = (\mathbf{I} - \mathbf{A}^T)^{-1} \hat{\mathbf{X}}^{-1} \mathbf{E}. \quad (113)$$

615 It is clear from comparison of Equations 110 and 113 that the literature is
 616 not accounting for accumulation of energy in the economic sectors ($\frac{d\mathbf{B}}{dt}$), nor
 617 does it account for physical depreciation ($\hat{\gamma}\mathbf{B}$). To be precise, the literature
 618 assumes

$$\frac{d\mathbf{B}}{dt} + \hat{\gamma}\mathbf{B} = \mathbf{0}. \quad (114)$$

619 Examining Equation 110, we see that to the extent that $\frac{d\mathbf{B}}{dt} + \hat{\gamma}\mathbf{B} \ll \mathbf{E}$,
 620 estimates of energy intensity (ε) obtained with the assumption of Equation
 621 114 contain little error. However, when the sum of the accumulation and de-
 622 preciation rates ($\frac{d\mathbf{B}}{dt} + \hat{\gamma}\mathbf{B}$) becomes significant relative to the rate of energy
 623 extracted from the Earth (\mathbf{E}), estimates of economic sector energy intensities
 624 (ε) using the assumption of Equation 114 have a high-side bias (assuming
 625 that $\frac{dB}{dt} > 0$ and $\gamma B > 0$). As discussed above, the assumption of Equation
 626 114 can be violated in developing economies because accumulation ($\frac{d\mathbf{B}}{dt}$) is
 627 large or in developed economies because depreciation ($\hat{\gamma}\mathbf{B}$) is large.

628 The assumption of Equation 114 may cause another challenge for energy
 629 analysts. The I-O method is often used to estimate energy intensities for

each sector of the economy (ε) with Equation 114. With ε values in hand, one can estimate changes in energy demand from the Earth (\mathbf{E}) as the output of economic sectors ($\hat{\mathbf{X}}$) increases or decreases by solving Equation 113 for \mathbf{E} .

$$\mathbf{E} = \hat{\mathbf{X}}(\mathbf{I} - \mathbf{A}^T)\varepsilon \quad (115)$$

When accumulation and depreciation terms are included, we see that the energy demands (\mathbf{E}) must be calculated differently. Solving Equation 110 for \mathbf{E} gives

$$\mathbf{E} = \hat{\mathbf{X}}(\mathbf{I} - \mathbf{A}^T)\varepsilon + \left(\frac{d\mathbf{B}}{dt} + \hat{\gamma}\mathbf{B} \right). \quad (116)$$

By comparing Equations 115 and 116, we see that to the extent that accumulation ($\frac{d\mathbf{B}}{dt}$) and depreciation ($\hat{\gamma}\mathbf{B}$) are non-zero, estimates of energy demand are too low. If the sum of accumulation ($\frac{d\mathbf{B}}{dt}$) and depreciation ($\hat{\gamma}\mathbf{B}$) are small relative to total energy demand (\mathbf{E}), then neglecting these effects causes little error. Economies with fast growth rates ($\frac{d\mathbf{B}}{dt}$) or large sizes (\mathbf{B}) are more likely to violate the typical assumptions in the literature.

7.3. Implications for recycling, reuse, and dematerialization

Dematerialization is the idea that economic activity can be unlinked from material or energy demands (UNEP, 2011). One of the primary methods for dematerializing an economy is reuse and recycling of materials. The impact of recycling can be seen in the I-O formulation only when depreciation and accumulation are included.

One effect of recycling is to reduce the magnitude of the disposal rate ($\hat{\gamma}$). Equation 108 indicates that recycling of material in an economy, thereby reducing $\hat{\gamma}$, will slow the effect of depreciation ($\hat{\gamma}\mathbf{B}$) and put upward pressure on growth ($\frac{d\mathbf{B}}{dt}$).

Recycling has a mixed effect on energy demand (\mathbf{E}). Because recycled material displaces newly-produced material in the economy and society, recycling will tend to reduce energy demand (\mathbf{E}). Equation 108 indicates that this displacement effect will put downward pressure on growth ($\frac{dB}{dt}$). However, recycling processes require energy to operate, thereby increasing energy demand (\mathbf{E}). Equation 108 indicates that additional energy demand will put upward pressure on growth ($\frac{dB}{dt}$).

If recycling produces a net reduction in energy demand (\mathbf{E}), that is if the effect of displaced production dominates over the effect of energy consumed in recycling processes, the upward pressure on growth ($\frac{dB}{dt}$) from decrease in $\hat{\gamma}$ and the downward pressure on growth from net reduction of \mathbf{E} offset each other, the growth rate ($\frac{dB}{dt}$) will remain near zero, and total embodied energy (\mathbf{B}) will remain constant. In that scenario, dematerialization can develop: reduced material and energy input (\mathbf{E}) can be accompanied by no change in growth ($\frac{d\mathbf{B}}{dt}$).

7.4. Comparison to a Steady-state Economy

***** Finish this section. In terms of what a SSE would look like in the I-O framework, at first blush, I would think that $dB/dt = 0$ is one aspect. Also, with no growth, inflow rates = depreciation rates. The larger that B is for any society, the larger E must be (to overcome depreciation). To minimize E, hyper-recycling is probably useful. Those are at least a place to

674 start. *****

675 ***** In our discussion, we also addressed the attempts at SSE from
676 point of view of society. In order to achieve this goal *without* recycling,
677 the goods and services sector should have to increase extraction to offset
678 decreasing ore grade, the energy sector should have to increase extraction
679 of energy to allow increasing extraction (unless efficiency could make up the
680 gap - unlikely) in which case the SSE would be violated from these two and
681 from the POV of the earth. *****

682 8. Conceptual and Theoretical Issues

683 8.1. Choice of Energy Input Vector

684 Consistent with traditional I-O methods, the derivation presented above
685 counts energy at the point of inflow to the economy. That is, elements of
686 the energy input vector to the economy (\mathbf{E}) are zero except for those sectors
687 that receive energy directly from the Earth. With the traditional approach,
688 energy input to energy sectors is non-zero, and energy input to non-energy
689 sectors is zero. So, in the two-sector example C above, $\dot{E}_{14} = 0$ and $\dot{E}_{13} \neq 0$.

690 Costanza (1984) suggests an alternative approach, namely to count energy
691 input to the economy at the point of conversion to useful work. Theoretical
692 justification for this direct energy conversion (DEC) approach comes from
693 both thermodynamic and economic considerations. The thermodynamic jus-
694 tification derives from the purpose of energy consumption in an economy,
695 namely to produce useful work. If energy flows *through* a sector, it should
696 not be counted “against” that sector: only energy that is converted to useful
697 work *in* the sector should be counted against that sector.

698 The economic justification derives from the typical treatment of trans-
699 portation sectors of the economy. ***** More here. See Costanza (1984)
700 for the transportation analogy. *****

701 The DEC approach implicitly redefines energy intensity to be the required
702 amount of fossil fuel energy to produce a unit of economic output.

703 ***** Equation redefining ε here.

704 In the DEC approach, electricity consumption is converted to its fossil
705 energy equivalent (coal) before being “applied” to an economic sector. And,
706 refined petroleum is converted to its fossil energy equivalent (crude) before
707 being “applied” to an economic sector.

708 ***** The DEC option is akin to my idea of substituting the 1st Law
709 into the total energy equation. Show this derivation after redefining ε to be
710 embodied energy per dollar, not total energy per dollar. So, there is a second
711 implicit assumption going on with Costanza (1984), namely that we have a
712 re-derivation of energy intensity. *****

713 ***** Show that re-derivation results in only counting the energy burned
714 by each sector (or the waste heat off of each sector). Costanza (1984) shows
715 that distributing energy input at the point of consumption reduces the vari-
716 ance of energy intensity across all sectors of the economy. *****

717 8.2. *What is Endogenous?*

718 Are government and households endogenous? Costanza (1980) was the
719 first to endogenize government and households, because households provide
720 services to the economy (labor) in exchange for wages and government pro-
721 vides services to the economy in exchange for taxes, both of which require en-
722 ergy. Costanza (1980) showed that by including government and households

723 as sectors in the economy, the variation of energy intensity is significantly
724 reduced across all sectors of the economy.

725 8.3. *What About the Sun?*

726 Costanza (1980) includes an option to consider the sun as an input to
727 the economy, thereby significantly increasing the energy intensity of agri-
728 cultural sectors and other sectors that depend upon agricultural outputs,
729 however Costanza (1984) did not include the sun ???. Whether solar input
730 to the economy should be considered is probably dependent upon the ob-
731 jectives of the analysis. In this framework we are primarily interested in
732 the effects of declining energy resource quality in industrial economies, due
733 to depletion of fossil fuels. As such, inclusion of solar flows is unnecessary.
734 However, expanding the framework to include non-industrial or more agrar-
735 ian societies would probably require accounting for these flows. Additionally,
736 similar concerns might be raised in dealing with a society that is largely re-
737 liant on solar or wind energy. [EARLIER FOOTNOTE ON INDUSTRIAL
738 VS. NON-INDUSTRIAL ECONOMIES COULD BE BROUGHT IN HERE
739 - MD]

740 There are a number of means by which solar flows can be accounted.
741 Short-term solar flows could be accounted in the output of agricultural and
742 forestry sectors, as well as some of the renewable energy producers, such
743 as solar thermal and PV, wind, ocean thermal, hydro-power and biomass.
744 This method does not account for longer-term flows of solar energy used to
745 form fossil fuels. The *emergy* accounting method puts all flows in terms of
746 *embodied energy* flows ??. The basic unit of measure is the *emjoule* which
747 is often given in terms of flows of solar energy embodied in the energy (or

material) - the solar emjoules - per unit of resource, abbreviated to seJ/J for
energy resources, or seJ/g for materials. As such, even fossil fuels, e.g. coal,
extracted from the earth have an embodied energy of around 67,000 seJ/J ?.

9. Appendix - Proofs and derivation of useful relationships

9.1. Proof of Equation 104

We begin with a restatement of Equation 104.

$$\mathbf{X}_t^T - \hat{\mathbf{X}} = \hat{\mathbf{X}}(\mathbf{A}^T - \mathbf{I}) \quad (117)$$

We expand the matrices to obtain

$$\begin{bmatrix} \dot{X}_{33} & \dot{X}_{43} \\ \dot{X}_{34} & \dot{X}_{44} \end{bmatrix} - \begin{bmatrix} \dot{X}_3 & 0 \\ 0 & \dot{X}_4 \end{bmatrix} = \begin{bmatrix} \dot{X}_3 & 0 \\ 0 & \dot{X}_4 \end{bmatrix} \begin{bmatrix} a_{33} - 1 & a_{43} \\ a_{34} & a_{44} - 1 \end{bmatrix}. \quad (118)$$

Multiplication of the matrices provides

$$\begin{bmatrix} \dot{X}_{33} - \dot{X}_3 & \dot{X}_{43} \\ \dot{X}_{34} & \dot{X}_{44} - \dot{X}_4 \end{bmatrix} = \begin{bmatrix} \dot{X}_3 a_{33} - \dot{X}_3 & \dot{X}_3 a_{43} \\ \dot{X}_4 a_{34} & \dot{X}_4 a_{44} - \dot{X}_4 \end{bmatrix}. \quad (119)$$

Using $\dot{X}_j a_{ij} = \dot{X}_{ij}$ (see Equation 16) gives

$$\begin{bmatrix} \dot{X}_{33} - \dot{X}_3 & \dot{X}_{43} \\ \dot{X}_{34} & \dot{X}_{44} - \dot{X}_4 \end{bmatrix} = \begin{bmatrix} \dot{X}_{33} - \dot{X}_3 & \dot{X}_{43} \\ \dot{X}_{34} & \dot{X}_{44} - \dot{X}_4 \end{bmatrix} \quad (120)$$

to complete the proof.

758 *9.2. Proof of Equation 105*

759 We begin with a restatement of Equation 105.

$$\hat{\mathbf{X}} - \mathbf{X}_t^T = \hat{\mathbf{X}}(\mathbf{I} - \mathbf{A}^T) \quad (121)$$

760 We expand the matrices to obtain

$$\begin{bmatrix} \dot{X}_3 & 0 \\ 0 & \dot{X}_4 \end{bmatrix} - \begin{bmatrix} \dot{X}_{33} & \dot{X}_{43} \\ \dot{X}_{34} & \dot{X}_{44} \end{bmatrix} = \begin{bmatrix} \dot{X}_3 & 0 \\ 0 & \dot{X}_4 \end{bmatrix} \begin{bmatrix} 1 - a_{33} & -a_{43} \\ -a_{34} & 1 - a_{44} \end{bmatrix}. \quad (122)$$

761 Multiplication of the matrices provides

$$\begin{bmatrix} \dot{X}_3 - \dot{X}_{33} & -\dot{X}_{43} \\ -\dot{X}_{34} & \dot{X}_4 - \dot{X}_{44} \end{bmatrix} = \begin{bmatrix} \dot{X}_3 - \dot{X}_3 a_{33} & -\dot{X}_3 a_{43} \\ -\dot{X}_4 a_{34} & \dot{X}_4 - \dot{X}_4 a_{44} \end{bmatrix}. \quad (123)$$

762 Using $\dot{X}_j a_{ij} = \dot{X}_{ij}$ (see Equation 16) gives

$$\begin{bmatrix} \dot{X}_3 - \dot{X}_{33} & -\dot{X}_{43} \\ -\dot{X}_{34} & \dot{X}_4 - \dot{X}_{44} \end{bmatrix} = \begin{bmatrix} \dot{X}_3 - \dot{X}_{33} & -\dot{X}_{43} \\ -\dot{X}_{34} & \dot{X}_4 - \dot{X}_{44} \end{bmatrix} \quad (124)$$

763 to complete the proof.

764 *9.3. Derivation of Equation 106*

765 We begin with a restatement of Equation 105.

$$\hat{\mathbf{X}} - \mathbf{X}_t^T = \hat{\mathbf{X}}(\mathbf{I} - \mathbf{A}^T) \quad (125)$$

766 We take the inverse of both sides of the equation to obtain

$$\left(\hat{\mathbf{X}} - \mathbf{X}_t^T \right)^{-1} = \left(\hat{\mathbf{X}}(\mathbf{I} - \mathbf{A}^T) \right)^{-1}. \quad (126)$$

767 We now apply the following matrix identity (formula 6.2, pg. 308 from ?).

$$(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \quad (127)$$

768 to the right side of Equation 126 to obtain

$$\left(\hat{\mathbf{X}} - \mathbf{X}_t^{\text{T}}\right)^{-1} = (\mathbf{I} - \mathbf{A}^{\text{T}})^{-1}\hat{\mathbf{X}}^{-1}, \quad (128)$$

769 which is identical to Equation 106.