Calibration of Credit Default Swaps & Transition Matrices

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Contents

1 CDS Spreads

1.1 Survival Probabilities

Affine Nature of CIR Models: When reviewing the affine formulations of survival probabilities under intensity-based models, it is important to recall that the Feynman-Kac expectation $\mathbb{E}^{\mathbb{Q}}[\cdot]$ is defined with respect to the risk-neutral measure $\mathbb{Q}$.

In term-structure modeling, this expectation is often expressed via zero-coupon bond (ZCB) prices $P(t,T)$, which under affine short-rate models take the form:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_s\,ds}\,\middle|\,\mathcal{F}_t\right] = A(t,T)\cdot\exp\left(-B(t,T)\cdot r_t\right)$$

where r_t is the short rate, and $A(\cdot), B(\cdot)$ are deterministic functions determined by the model.

In intensity-based credit models, this same affine structure is borrowed for modeling the survival probability:

$$P_{\text{surv}}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \lambda_s\,ds}\,\middle|\,\mathcal{F}_t\right] = A(t,T)\cdot\exp\left(-B(t,T)\cdot \lambda_t\right)$$

where λ_t is the hazard (default intensity) rate. Here, the functional analogy holds: the interest rate r_t is replaced by the hazard rate λ_t, and the bond price $P(t,T)$ is reinterpreted as a survival probability.

Non-Affine Nature of GOU and IGOU Models: The Gamma–Ornstein–Uhlenbeck (GOU) and Inverse Gaussian–Ornstein–Uhlenbeck (IGOU) models, see Schoutens & Cariboni (2009), are driven by Lévy subordinators rather than Brownian motion, making their intensity dynamics non-affine. Their survival probabilities are given by:

$$\begin{aligned} P_{surv}(t) &= \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s\,ds\right)\right] \\ &= \exp\big(-f(t;\alpha,a,b,\lambda_0)\big), \end{aligned}$$

where $f(t;\alpha,a,b,\lambda_0)$ is a non-affine function of t, α, a, b, and λ_0.

Unlike affine models (e.g., CIR, JCIR, SCIR), whose survival probabilities can be expressed in the exponential-affine form

$$P_{surv}(t) = A(t)\cdot\exp\left(-\lambda_0 B(t)\right)$$

with $A(t)$, $B(t)$ solving Riccati-type ODEs, the GOU and IGOU models yield closed-form expressions derived from Laplace transforms of integrated Lévy-driven OU processes. This lack of Riccati structure reflects the inherent jump nature of the underlying processes, and precludes the use of standard affine term-structure techniques.

1.2 Intensity Models

The *par spread* of a Credit Default Swap (CDS) is defined as:

$$S = \frac{(1-R)\sum_{i=1}^n D(0,t_i)\left[P_{surv}(t_{i-1}) - P_{surv}(t_i)\right]}{\sum_{i=1}^n D(0,t_i)P_{surv}(t_i)\Delta t_i} \qquad \text{or} \quad S = \frac{(1-R)\left[-\int_0^T D(0,s)\,dP_{surv}(s)\right]}{\int_0^T D(0,s)P_{surv}(s)\,ds}$$

with the discount factor $D(.)$ defined as:$D(0,t) = e^{-rt}$, and the survival probability $P(.)$ selected from the following:

1. **Homogeneous Poisson (HP):**

$$d\lambda_t = 0 \quad \text{(constant)}$$
$$P_{surv}^{HP}(t) = \exp(-\lambda t)$$

2. **Inhomogeneous Poisson (IHP):**

$$\lambda_t = \gamma_i, \quad \text{for } t \in [T_{i-1}, T_i)$$
$$P_{surv}^{IHP}(t) = \exp\left(-\int_0^t \lambda_s\, ds\right) = \exp\left(-\sum_i \gamma_i \Delta T_i\right)$$

3. **Cox-Ingersoll-Ross (CIR):**

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t$$
$$\begin{aligned} P_{surv}^{CIR}(t) &= \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s\, ds\right)\right] \\ &:= \phi^{CIR}(t;\kappa,\theta,\sigma,\lambda_0) = A(t)\cdot\exp(-\lambda_0 B(t)) \end{aligned}$$
$$A(t) = \left(\frac{2\Gamma e^{0.5(\kappa+\Gamma)t}}{(\Gamma+\kappa)(e^{\Gamma t}-1)+2\Gamma}\right)^{\frac{2\kappa\theta}{\sigma^2}}$$
$$B(t) = \frac{2(e^{\Gamma t}-1)}{(\Gamma+\kappa)(e^{\Gamma t}-1)+2\Gamma}, \quad \Gamma = \sqrt{\kappa^2+2\sigma^2}$$

4. **Scaled CIR (SCIR):**

$$\lambda_t = x_t\cdot\beta(t), \quad dx_t = \kappa(\eta - x_t)dt + \sigma\sqrt{x_t}dW_t$$
$$\begin{aligned} P_{surv}^{SCIR}(t) &= \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s\, ds\right)\right] \\ &:= \phi^{SCIR}(t;\kappa,\eta,\sigma,x_0,\beta(t)) = A(t)\cdot\exp(-x_0\cdot\beta(t)\cdot B(t)) \end{aligned}$$
$$A(t) = \left(\frac{2\Gamma e^{0.5(\kappa+\Gamma)t}}{(\Gamma+\kappa)(e^{\Gamma t}-1)+2\Gamma}\right)^{\frac{2\kappa\eta}{\sigma^2}}$$
$$B(t) = \frac{2(e^{\Gamma t}-1)}{(\Gamma+\kappa)(e^{\Gamma t}-1)+2\Gamma}, \quad \Gamma = \sqrt{\kappa^2+2\sigma^2}$$

5. **Jump CIR (JCIR):**

$$d\lambda_t = \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t + dJ_t$$
$$\begin{aligned} P_{surv}^{JCIR}(t) &= \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s\, ds\right)\right] \\ &:= \phi^{JCIR}(t;\cdots) = \exp\left(A(t) + \lambda_0 B(t)\right) \end{aligned}$$
$$\begin{aligned} A(t) = &-\frac{2\kappa\theta}{\sigma^2}\log\left(\frac{c_1+d_1e^{-\alpha t}}{c_1+d_1}\right) + \frac{\kappa\theta t}{c_1} \\ &+ \ell\left(\frac{d_1/c_1 - d_2/c_2}{-\alpha d_2}\right)\log\left(\frac{c_2+d_2e^{-\alpha t}}{c_2+d_2}\right) + \ell\left(\frac{1-c_2}{c_2}\right)t \end{aligned}$$
$$B(t) = \frac{1-e^{-\alpha t}}{c_1+d_1e^{-\alpha t}}$$
$$\alpha = \sqrt{\kappa^2 - 2q\sigma^2}, \quad c_1 = \frac{\kappa+\alpha}{2q}, \quad d_1 = \frac{-\kappa+\alpha}{2q}$$
$$c_2 = 1 - \frac{\mu}{c_1}, \quad d_2 = \frac{d_1+\mu}{c_1}, \quad J_t = \sum_{i=1}^{N_t} Y_i$$

6. **Gamma OU (GOU):**

$$
\begin{aligned}
d\lambda_t &= -\alpha\lambda_t dt + dZ_t, \quad Z_t \sim \text{Gamma Process} \\
P_{surv}^{GOU}(t) &= \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s\,ds\right)\right] \\
&:= \phi^{GOU}(t;\alpha,a,b,y) = \exp\left(-\frac{y}{\alpha}(1-e^{-\alpha t}) - \frac{\alpha a}{1+\alpha b}\left[b\log\left(\frac{b}{b+\frac{1-e^{-\alpha t}}{\alpha}}\right)+t\right]\right)
\end{aligned}
$$

7. **Inverse Gaussian OU (IGOU):**

$$
\begin{aligned}
d\lambda_t &= -\alpha\lambda_t dt + dZ_t, \quad Z_t \sim \text{Inverse Gaussian Process} \\
P_{surv}^{IGOU}(t) &= \mathbb{E}\left[\exp\left(-\int_0^t \lambda_s\,ds\right)\right] \\
&:= \phi^{IGOU}(t;\alpha,a,b,y) = \exp\left(-\frac{y}{\alpha}(1-e^{-\alpha t}) - \frac{2a}{b\alpha}\cdot A(t)\right) \\
A(t) &= \frac{1-\sqrt{1+\kappa(1-e^{-\alpha t})}}{\kappa} + \frac{1}{\sqrt{1+\kappa}}\left[\operatorname{arctanh}\left(\frac{\sqrt{1+\kappa(1-e^{-\alpha t})}}{\sqrt{1+\kappa}}\right) - \operatorname{arctanh}(1)\right] \\
\kappa &= \frac{2}{b^2\alpha}
\end{aligned}
$$

1.3 Calibration

Calibrate each model's parameters $\Theta(.)$ by minimizing the root-mean-square error (RMSE) between model-implied and market spreads:

$$
\text{RMSE}(\Theta) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\Big(S_{\text{model}}(T_i;\Theta) - S_{\text{market}}(T_i)\Big)^2}
$$

The results are given in the next three tables below.

Table 1: Calibrated Global Parameters

Model	θ_1	θ_2	θ_3	θ_4
HP	0.028500	–	–	–
$\Theta^{SCIR} = (\kappa, \eta, \sigma, x_0)$	0.430773	0.109034	1.483678	0.156691
$\Theta^{CIR} = (\kappa, \eta, \sigma, \lambda_0)$	0.137773	0.097110	0.372737	0.015831
$\Theta^{GOU} = (\gamma, a, b, \lambda_0)$	0.430445	0.488751	10.000000	0.014859
$\Theta^{IGOU} = (\gamma, a, b, \lambda_0)$	0.534559	72.921979	1855.001856	0.015394

Table 2: Calibrated Tenor-Specific Parameters

Model	1y	2y	3y	5y	7y	10y
IHP (γ_t)	0.020945	0.027991	0.031578	0.038929	0.037083	0.037272
SCIR ($\beta(t)$)	0.027483	-0.016872	-0.041141	-0.035687	-0.010598	-0.022634

Table 3: Calibration Results Across Models (Market vs Model Spread, Survival Probability)

Model	**Tenor (y)**	**Market**	**Model**	**Survival (%)**	**RMSE**
HP	1.0	126	171	97.2	28.6065
	2.0	147	171	94.5	
	3.0	161	171	91.8	
	5.0	189	171	86.7	
	7.0	198	171	81.9	
	10.0	205	171	75.2	
IHP	1.0	126	126	97.9	0.0035
	2.0	147	147	95.2	
	3.0	161	161	92.3	
	5.0	189	189	85.4	
	7.0	198	198	79.3	
	10.0	205	205	70.9	
SCIR	1.0	126	124	98.0	2.3548
	2.0	147	149	95.2	
	3.0	161	165	92.1	
	5.0	189	187	85.5	
	7.0	198	198	79.3	
	10.0	205	203	71.1	
CIR	1.0	126	125	97.9	2.2521
	2.0	147	148	95.2	
	3.0	161	164	92.1	
	5.0	189	185	85.6	
	7.0	198	197	79.3	
	10.0	205	207	70.6	
GOU	1.0	126	125	97.9	2.4837
	2.0	147	148	95.2	
	3.0	161	164	92.1	
	5.0	189	185	85.7	
	7.0	198	197	79.4	
	10.0	205	207	70.6	
IGOU	1.0	126	125	97.9	2.3362
	2.0	147	148	95.2	
	3.0	161	164	92.1	
	5.0	189	185	85.6	
	7.0	198	197	79.3	
	10.0	205	207	70.6	

We examine the behavior and performance of various credit intensity models using the calibrated results summarized in Tables 1–3. Each model introduces a different structure for the hazard rate process λ_t, influencing both pricing accuracy and risk representation.

1. The HP model assumes a constant hazard rate over time, calibrated at 2.85% (see Table 1). This single parameter governs the entire term structure, resulting in no flexibility to match the observed spread curve. Consequently, the model significantly overestimates short-term spreads and underestimates long-term risk. This mismatch is evident in Table 3, where the model-implied spreads are flat across tenors and yield the highest RMSE (28.6 bps), reflecting a poor fit.

2. The IHP model allows the hazard rate to vary by tenor, using a piecewise-constant structure:

$$\lambda(t) = \gamma_i \quad \text{for } t \in [T_{i-1}, T_i)$$

   Each γ_i is calibrated directly to match the market CDS spread at tenor T_i, with values listed in Table 2. As a result, the model achieves near-perfect accuracy (RMSE ≈ 0), as shown in Table 3.

However, the deterministic and static nature of IHP limits its utility for dynamic risk analysis or pricing of path-dependent products.

3. SCIR extends the CIR framework by combining a stochastic base intensity process x_t with a tenor-specific deterministic scaling $\beta(t)$, such that $\lambda_t = x_t\beta(t)$. The global parameters $(\kappa, \eta, \sigma, x_0)$ are listed in Table 1, while the calibrated $\beta(t)$ values appear in Table 2. This setup balances flexibility and stochastic behavior, allowing for term-structure fitting and credit dynamics. As seen in Table 3, the SCIR model yields a reasonable RMSE (2.35), though negative $\beta(t)$ at some tenors may indicate overfitting or compensatory adjustments.

4. The CIR/ICIR model assumes a stochastic hazard rate evolving according to:

$$d\lambda_t = \kappa(\eta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t$$

   It does not include tenor-specific adjustments, relying purely on the global dynamics (Table 1). Despite this, CIR achieves a reasonable fit (RMSE 2.25), suggesting that a mean-reverting stochastic process alone can partially explain the observed term structure.

5. GOU models the hazard rate as a jump-driven process with rapid mean reversion:

$$d\lambda_t = -\gamma\lambda_t dt + dZ_t, \quad Z_t \sim \text{Gamma process}$$

   The global parameters $(\gamma, a, b, \lambda_0)$ in Table 1 show a moderate jump intensity and fast reversion. The model offers competitive calibration accuracy (RMSE 2.48 in Table 3) and is suitable for capturing short-term credit shocks.

6. IGOU shares the same structure as GOU but with heavier-tailed shocks driven by an *Inverse Gaussian* process. Its parameter set in Table 1, particularly $a = 72.92$ and $b = 1855.00$, indicates rare but extreme jump risk. The model maintains strong calibration performance (RMSE 2.34 in Table 3) and is valuable for stress testing and capturing fat-tailed credit events.

Table 4: Concluding Observations

Model	**Valuation**	**Risk Mgmt**	**Conclusion**
HP	✗	✗	Assumes constant hazard thus fails to capture credit term structure.
IHP	✓	✗	Calibrates via deterministic piecewise hazard with no stochastic dynamics.
SCIR	✓	±	Provides stochastic with tenor scaling thus good fit but negative β's indicate overfit.
CIR	✓	✓	Robust mean-reverting process; interpretable, suitable for pricing and risk scenarios.
GOU	✓	✓	Captures jumps and tail risk; good for stressed valuation and market shocks.
IGOU	±	✓	Extreme fat-tailed shocks; best for stress testing and systemic risk exposure.

1.4 Supporting Note

Survival probabilities in reduced-form credit risk models are often computed as expectations of the form:

$$P(0,T) = \mathbb{E}\left[\exp\left(-\int_0^T \lambda_t dt\right)\right],$$

where the default intensity λ_t follows an affine diffusion process. One commonly used specification is the Cox–Ingersoll–Ross (CIR) model:

$$\begin{aligned} d\lambda_t &= \kappa(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t, \\ \Lambda_t &= \int_0^t \lambda_s ds. \end{aligned}$$

We aim to derive the characteristic function of Λ_t, defined as:

$$\phi_{\Lambda_t}(u) = \mathbb{E}\left[e^{iu\Lambda_t} \mid \lambda_0\right],$$

using the Feynman–Kac theorem. Let $f(t,\lambda) := \mathbb{E}\left[e^{iu\Lambda_T} \mid \lambda_t = \lambda\right]$. Then f satisfies the PDE:

$$\frac{\partial f}{\partial t} + \kappa(\theta - \lambda)\frac{\partial f}{\partial \lambda} + \frac{1}{2}\sigma^2\lambda\frac{\partial^2 f}{\partial \lambda^2} - iu\lambda f = 0,$$

with terminal condition $f(T,\lambda) = 1$.

We postulate an affine solution of the form:

$$f(t,\lambda) = \exp\left(A(t) + B(t)\lambda\right),$$

where functions $A(t)$ and $B(t)$ satisfy the Riccati system:

$$\begin{aligned} \frac{dB}{dt} &= -\kappa B + \frac{1}{2}\sigma^2 B^2 - iu, \quad B(T) = 0, \\ \frac{dA}{dt} &= \kappa\theta B, \quad A(T) = 0. \end{aligned}$$

Solving this system yields the well-known closed-form CIR solution:

$$\begin{aligned} B(\tau) &= \frac{2iu\left(e^{\alpha\tau} - 1\right)}{(\alpha+\kappa)(e^{\alpha\tau} - 1) + 2\alpha}, \\ A(\tau) &= \frac{2\kappa\theta}{\sigma^2}\log\left(\frac{2\alpha e^{(\alpha+\kappa)\tau/2}}{(\alpha+\kappa)(e^{\alpha\tau} - 1) + 2\alpha}\right), \end{aligned}$$

where $\tau = T - t$ and:

$$\alpha = \sqrt{\kappa^2 - 2iu\sigma^2}.$$

Therefore, the characteristic function of Λ_T given λ_0 is:

$$\phi_{\Lambda_T}(u) = \exp\left(A(0) + B(0)\lambda_0\right).$$

In particular, setting $u = i$ recovers the survival probability:

$$P(0,T) = \mathbb{E}\left[\exp\left(-\int_0^T \lambda_t dt\right)\right] = \exp\left(A(0) + B(0)\lambda_0\right).$$

This derivation underpins the affine nature of the integrated CIR framework, mirroring the ZCB pricing structure under affine intensity models, where λ_t follows a stochastic process. It forms the basis for the valuation methodology employed in all models except HP and IHP, which rely on deterministic hazard rates and lack stochastic evolution.

2 Transition Matrices

2.1 Cohort vs. Hazard Rate

A transition matrix depicted as[1]:

$$P = \begin{pmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,k-1} & p_{1,k} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,k-1} & p_{2,k} \\ \vdots & \vdots & \vdots & \vdots & \\ p_{k,1} & p_{k,2} & \cdots & p_{k,k-1} & p_{k,k} \end{pmatrix}$$

with the corresponding cells referring to some observed data L:

$$p_{ij} = \mathbb{P}(L_t = j \mid L_{t-1} = i) \quad \text{subject to:} \quad \sum_{j=1}^{K} \mathbb{P}(L_t = j \mid L_{t-1} = i) = 1$$

The optimization problem is framed as a maximum likelihood:

$$\max_P \mathcal{L}(P \mid L_t, \cdots, L_T) \quad \text{subject to:} \quad \sum_{j=1}^{K} p_{ij} = 1 \quad \forall\, i = 1..K$$

and solved by combining the objective and constraint functions via Lagrange multiplier; hence taking the partials with respect to p_{ij} to yield:

$$\hat{p}_{ij} = \frac{n_{ij}}{\sum_{j=1}^{K} n_{ij}}$$

This is known as the *Cohort* approach. The other method to use is the *hazard rate* approach but only when more granular (intra-year) data is available. In this method the matrix P is generated by $P_t = e^{\Lambda t}$, with Λ containing a matrix of hazard rates λ_{ij}'s, defined as:

$$\lambda_{ij} = \frac{n_{ij}}{\int_t^T N_i(s)ds}$$

with n denoting the number of transitions from state i to state j across the horizon $[t, T]$ and N denoting the number of borrowers in state i at time t. In case of annual observations, the above equation collapses to:

$$\lambda_{ij} = \frac{n_{ij}}{\sum_{t=1}^{T} n_{ij}}$$

Hence, the final matrix produced comprises of:

$$\lambda_{ij} = \frac{n_{ij}}{\sum_{t=1}^{T} n_{ij}} \quad \text{and} \quad \lambda_{ij} = -\sum_{i \neq j} \lambda_{ij}$$

in the off-diagonal and diagonal cells, respectively.

Using the observed transition matrix provided in Table (5), a synthetic loan data is generated for 600 investment and non-investment grade rated borrowers across 5 years. The data is then used to generate new transition matrices using the cohort and hazard rate approaches depicted in Table (6).

[1] The final column represents absorbing (default) state.

Table 5: Transition Matrix (Actuals)

	A	B	C	D
A	0.9000	0.0800	0.0199	0.0001
B	0.0500	0.8500	0.0900	0.0100
C	0.0100	0.0900	0.8000	0.1000
	0.0000	0.0000	0.0000	1.0000

Table 6: Transition Matrix (Cohort vs. Hazard Rate)

	A	B	C	D
A	0.9047	0.0710	0.0243	0.0000
B	0.0529	0.8425	0.0917	0.0129
C	0.0108	0.0723	0.8228	0.0940
	0.0000	0.0000	0.0000	1.0000

	A	B	C	D
A	0.9282	0.0515	0.0193	0.0010
B	0.0397	0.8813	0.0664	0.0127
C	0.0090	0.0514	0.8682	0.0714
	0.0000	0.0000	0.0000	1.0000

To test the accuracy of transition matrices estimated by the cohort and hazard methods, Fisher Information is used to determine how far each method is from the actuals; shown below is the test for cohort only.

Table 7: Distance from the actuals

p_n	$\hat{p}_n$	s/e	c.i	p_n	$\hat{p}_n$	s/e	c.i
p_1	0.9	0.04	[0.79,1.00]	p_7	0.09	0.01	[0.06,0.12]
p_2	0.07	0.01	[0.04,0.10]	p_8	0.01	0.0	[0.00,0.02]
p_3	0.02	0.01	[0.00,0.04]	p_9	0.01	0.0	[0.00,0.02]
p_4	0.0	0.0	[0.00, 0.00]	p_{10}	0.07	0.01	[0.04,0.10]
p_5	0.05	0.01	[0.03,0.07]	p_{11}	0.82	0.04	[0.72,0.93]
p_6	0.84	0.03	[0.76,0.93]	p_{12}	0.09	0.01	[0.06,0.13]

The actual state values appear to fall within the confidence bandwidth.

2.2 Transition Matrix Generator

Using the method introduced in Jarrow et al. (1997) and used in Israel et al. (2001), assume P is an $N \times N$ Markov Transition Matrix (TM) with non-negative entries and each row's sum is 1. The objective is to find a generator matrix Q of dimension $N \times N$ with non-negative off-diagonal entries with each row sum to 1, such that $exp(Q) = P$. Hence for each year t:

$$e^{tQ} = I + tQ + \frac{(tQ)^2}{2!} + \frac{(tQ)^3}{3!} + \cdots$$

where, I is the $N \times N$ identity matrix. Begin by finding the eigenvalues of P, i.e.

$$S = \max\{(a-1)^2 + b^2 \mid a + bi \text{ is an eigenvalue of } P \text{ and } a, b \in \mathcal{R}\}$$

Then:

$$\tilde{Q} = (P - I) - \frac{(P-I)^2}{2} + \frac{(P-I)^3}{3} - \frac{(P-I)^4}{4} + \cdots$$

converges with row-sums 0 such that $exp(\tilde{Q}) = P$ exactly.

In practice, $\tilde{Q}$ is not guaranteed to have non-negative off-diagonal entries. In such cases the resulting P matrix is not a Markov transition and therefore in need of adjustments.

Specifically, each negative off-diagonal entry is replaced by zero with the absolute of negative value added back to the corresponding row. That is once the $\tilde{Q}$ is computed, create an adjusted (new) matrix Q by setting:

$$q_{i,j} = \begin{cases} 0, & \text{if } i \neq j \text{ and } \tilde{q}_{ij} < 0 \\ \tilde{q}_{i,j} - \dfrac{B_i}{G_i} \times |\tilde{q}_{ij}|, & \text{if } G_i > 0 \\ \tilde{q}_{i,j}, & \text{if } G_i = 0 \end{cases}$$

where for each row, the Good (G) and Bad (B) are defined as:

$$G_i = |\tilde{q}_{ij}| + \sum_{j \neq i} \max(\tilde{q}_{ij}, 0)$$
$$B_i = \sum_{j \neq i} \max(-\tilde{q}_{ij}, 0)$$

with $q_{i,j}$ identified as the entries for the new matrix Q.

Exponential of Q should yield the original transition matrix P.

The example used in Jarrow et al. (1997) and Israel et al. (2001) is provided below to demonstrate the method.

$$P = \begin{pmatrix} 0.8910 & 0.0963 & 0.0078 & 0.0019 & 0.0030 & 0.0000 & 0.0000 & 0.0000 \\ 0.0086 & 0.9010 & 0.0747 & 0.0099 & 0.0029 & 0.0029 & 0.0000 & 0.0000 \\ 0.0009 & 0.0291 & 0.8894 & 0.0649 & 0.0101 & 0.0045 & 0.0000 & 0.0000 \\ 0.0006 & 0.0043 & 0.0656 & 0.8427 & 0.0644 & 0.0160 & 0.0018 & 0.0045 \\ 0.0004 & 0.0022 & 0.0079 & 0.0719 & 0.7764 & 0.1043 & 0.0127 & 0.0241 \\ 0.0000 & 0.0019 & 0.0031 & 0.0066 & 0.0517 & 0.8246 & 0.0435 & 0.0685 \\ 0.0000 & 0.0000 & 0.0116 & 0.0116 & 0.0203 & 0.0754 & 0.6493 & 0.2319 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

$$Q = \begin{pmatrix} -0.1156 & 0.1068 & 0.0046 & 0.0013 & 0.0033 & 0.0000 & 0.0000 & 0.0000 \\ 0.0095 & -0.1058 & 0.0826 & 0.0084 & 0.0026 & 0.0029 & 0.0000 & 0.0000 \\ 0.0008 & 0.0321 & -0.1206 & 0.0737 & 0.0093 & 0.0040 & 0.0000 & 0.0000 \\ 0.0006 & 0.0037 & 0.0746 & -0.1756 & 0.0772 & 0.0147 & 0.0014 & 0.0033 \\ 0.0004 & 0.0022 & 0.0062 & 0.0864 & -0.2569 & 0.1263 & 0.0140 & 0.0212 \\ 0.0000 & 0.0021 & 0.0028 & 0.0052 & 0.0624 & -0.1971 & 0.0563 & 0.0683 \\ 0.0000 & 0.0000 & 0.0140 & 0.0134 & 0.0241 & 0.0969 & -0.4224 & 0.2746 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{pmatrix}$$

$$e^Q = \begin{pmatrix} 0.8913 & 0.0957 & 0.0081 & 0.0019 & 0.0030 & 0.0003 & 0.0000 & 0.0000 \\ 0.0086 & 0.9013 & 0.0742 & 0.0101 & 0.0029 & 0.0029 & 0.0001 & 0.0002 \\ 0.0009 & 0.0289 & 0.8900 & 0.0642 & 0.0103 & 0.0045 & 0.0002 & 0.0004 \\ 0.0006 & 0.0044 & 0.0649 & 0.8441 & 0.0631 & 0.0164 & 0.0019 & 0.0045 \\ 0.0004 & 0.0022 & 0.0082 & 0.0704 & 0.7795 & 0.1021 & 0.0127 & 0.0244 \\ 0.0000 & 0.0019 & 0.0031 & 0.0069 & 0.0506 & 0.8265 & 0.0418 & 0.0690 \\ 0.0000 & 0.0003 & 0.0113 & 0.0114 & 0.0200 & 0.0727 & 0.6576 & 0.2271 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix} \approx P$$

Note, using a Generator Matrix we can determine[2]: The survival probabilities of the credit ratings as a function of time, notably if a firm is in a given (Markov chain) rating today, what is the probability of

[2]See Jarrow et al. (1997) for a more comprehensive discussion on Generator Matrix.

surviving within the next t years?

$$e^{0.5\times Q} = \begin{pmatrix} 0.9483 & 0.0422 & 0.0098 & 0.0003 \\ 0.0264 & 0.9209 & 0.0486 & 0.0041 \\ 0.0048 & 0.0486 & 0.8943 & 0.0523 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix} \qquad e^{1.\times Q} = \begin{pmatrix} 00.9004 & 0.0793 & 0.0202 & 0.0013 \\ 0.0495 & 0.8516 & 0.0884 & 0.0105 \\ 0.0102 & 0.0884 & 0.8022 & 0.0992 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

$$e^{1.5\times Q} = \begin{pmatrix} 0.8560 & 0.1120 & 0.0308 & 0.0030 \\ 0.0699 & 0.7906 & 0.1209 & 0.0187 \\ 0.0159 & 0.1208 & 0.7218 & 0.1415 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix} \qquad e^{2.\times Q} = \begin{pmatrix} 0.8149 & 0.1407 & 0.0414 & 0.0053 \\ 0.0877 & 0.7369 & 0.1472 & 0.0283 \\ 0.0218 & 0.1470 & 0.6515 & 0.1797 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}$$

where, in each case, P^t will not produce the desired (accurate) matrix. That is for continuous-time Markov chains, i.e. where the transition from one state to another is a continuous process, the transition rates between states are instantaneous, hence well captured by the generator matrix.

3 Stochastic Generator Matrix for Credit Migration

To construct a dynamic, market-consistent model of credit rating transitions and default, we build on the reduced-form framework of **?**, particularly Sections 6.3.1 and 6.3.2. The method links market-implied default intensities—calibrated from CDS spreads—to a time-inhomogeneous Markov generator matrix $Q(t)$, which governs both rating migration and default risk. This enables simulation of forward-looking credit states, consistent with market data.

Step 1: Calibrate Intensity Model to Market Data We begin by calibrating a stochastic intensity model to observed CDS spreads. A common choice is the CIR process:

$$d\lambda_t = \kappa(\theta - \lambda_t)\,dt + \sigma\sqrt{\lambda_t}\,dW_t, \quad \lambda_0 > 0,$$

with analytical survival probability:

$$P^{\text{CIR}}_{\text{surv}}(t) = A(t) \cdot \exp\left(-\lambda_0 B(t)\right),$$

where the functions $A(t)$ and $B(t)$ are defined by closed-form CIR expressions depending on (κ, θ, σ). This calibration ensures that the model matches observed market spreads across tenors, allowing consistent pricing and simulation.

Step 2: Define a Stochastic Generator Matrix Next, we model credit rating migration using a continuous-time Markov chain over rating states $\mathcal{S} = \{\text{AAA}, \text{AA}, \ldots, \text{D}\}$, where the default state D is absorbing. Instead of a static transition matrix, we specify a *stochastic generator matrix* $Q(t) = [q_{ij}(t)]$, with entries defined as affine functions of the CIR intensity:

$$\begin{aligned} q_{ij}(t) &= \beta_{ij} + \alpha_{ij}\lambda_t, \quad \text{for } i \neq j, \\ q_{ii}(t) &= -\sum_{j\neq i} q_{ij}(t), \end{aligned}$$

ensuring that $Q(t)$ remains a valid generator (rows summing to zero, non-negative off-diagonals). This structure embeds market-observed credit conditions (λ_t) into the evolution of rating states.

This links the CIR intensity λ_t to dynamic migration behavior. Simulated intensity paths over a horizon T yield rating transitions across paths. From this, we construct an empirical migration matrix M via classification of simulated borrower ratings at $t = T$.

Step 3: Simulate Rating Transitions Given the migration matrix M over time Δt, we invert:

$$M = e^{Q\Delta t} \quad \Longrightarrow \quad Q = \log(M),$$

using a matrix logarithm or truncated Taylor series. To preserve generator properties (non-negative off-diagonals, row sums zero), we correct Q as in Israel et al. (2001):

$$q^{\text{adj}}_{ij} = \begin{cases} 0, & \tilde{q}_{ij} < 0,\ i \neq j \\ \tilde{q}_{ij} - \dfrac{B_i}{G_i}|\tilde{q}_{ij}|, & G_i > 0 \\ \tilde{q}_{ij}, & G_i = 0 \end{cases},$$

$$G_i = |\tilde{q}_{ij}| + \sum_{j\neq i} \max(\tilde{q}_{ij}, 0), \quad B_i = \sum_{j\neq i} \max(-\tilde{q}_{ij}, 0).$$

This enables interpolation of transition behavior at arbitrary horizons $t \in [0, T]$ via $P_t = e^{Qt}$.

With the stochastic generator in hand, we simulate credit rating transitions by solving:

$$P(t, T) = \mathbb{E}\left[\exp\left(\int_t^T Q(u)\,du\right)\Big|\mathcal{F}_t\right],$$

which gives the conditional transition probabilities between ratings from time t to T, accounting for the path of λ_u over time.

Therefore, the simulation proceeds as follows:

1. Simulate paths of λ_t using the CIR SDE.
2. At each time step, compute $Q(t)$ based on λ_t.
3. Use the matrix exponential $e^{Q(t)\,\Delta t}$ to transition rating states.
4. For each path, track whether the entity defaults or migrates into other rating categories.

Table (8) depicts the simulated 1-year credit migration matrix based on a stochastic generator framework. This matrix reflects the probability of transitioning from an initial credit rating (rows) to a new rating (columns) after one year, incorporating dynamic, market-implied credit risk.

1. **CDS Calibration (Section 1):** A CIR intensity model calibrated to CDS spreads across tenors, producing market-consistent survival probabilities and intensity dynamics λ_t.
2. **Generator Matrix Construction (Section 3):** Using the Duffie–Singleton framework, each off-diagonal entry of the generator matrix $Q(t)$ is specified as an affine function of λ_t, with coefficients derived from historical transitions (e.g., Jarrow et al.).
3. **Stochastic Simulation:** Thousands of paths of λ_t are simulated using the CIR model. At each step, the corresponding $Q(t)$ is computed, and rating transitions simulated using matrix exponentials $e^{Q(t)\Delta t}$. The result is a forward-looking, market-implied empirical migration matrix.

Table 8: Stochastic Credit Migration Matrix (1-Year Horizon)

From / To	AAA	AA	**A**	BBB	BB	B	CCC	D
AAA	0.855	0.117	0.016	0.005	0.007	0.000	0.000	0.000
AA	0.011	0.882	0.090	0.009	0.005	0.002	0.001	0.000
A	0.000	0.023	0.880	0.062	0.021	0.013	0.000	0.001
BBB	0.002	0.009	0.074	0.824	0.061	0.019	0.002	0.009
BB	0.000	0.006	0.014	0.070	0.758	0.109	0.016	0.027
B	0.000	0.000	0.001	0.011	0.057	0.792	0.042	0.097
CCC	0.000	0.001	0.010	0.020	0.023	0.074	0.624	0.248
D	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Each row entry of M_{ij} in Table (8) represents the probability of transitioning *from* rating i *to* rating j over the horizon. For example, **the row vector** A tells us how borrowers starting in rating A transition over time: (a) 88.0% of A-rated borrowers remain in A, (b) 6.2% are downgraded to BBB, (c) 2.3% are upgraded to AA, and 0.1% default. Therefore, this row is used when simulating the future state of a borrower currently rated A.

The column vector A in the same table shows the composition of borrowers who *end up* in rating A: (a) 88.0% of A-rated entities were already rated A, (b) 9.0% were downgraded from AA, (c) 7.4% were upgraded from BBB, and (d) 1.6% migrated from more distant ratings. Therefore, this column helps understand how the final A-rated cohort was formed, relevant for attribution or population tracking.

References

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