SDEs

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August 2025

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Abstract

This note examines derivative pricing under the Black–Scholes framework and its extensions to models with stochastic volatility, stochastic interest rates, and jump-diffusions. Starting from geometric Brownian motion (GBM), the Black–Scholes PDE is derived and interpreted via option sensitivities. The Feynman–Kac Theorem links SDEs to PDEs through expected discounted payoffs under the risk-neutral measure. By relaxing constant assumptions and incorporating models like Heston, Hull–White, and Merton, the pricing framework generalizes to partial integro-differential equations (PIDEs) that capture market phenomena such as skew, smile, and fat tails.

1 Feynman-Kac

Consider a generalized SDE:

$$dX_t = \mu(X_t, t)\, dt + \sigma(X_t, t)\, dW_t$$

and a terminal payoff $\psi(X_T)$. The Feynman–Kac provides the discounted price as:

$$f(x,t) = \mathbb{E}\left[e^{-\int_t^T r(u)\, du}\, \psi(X_T) \,\middle|\, X_t = x \right]$$

which can be written as:

$$f(x,t) = e^{-\int_t^T r(u)\, du} \int \psi(y)\, p(x,t;y,T)\, dy$$

assuming a transition density $p(x,t;y,T)$, solved by:

(1) Differentiating $f(x,t)$ with respect to time and space

Time derivative:

$$\frac{\partial f}{\partial t}(x,t) = r(t) f(x,t) + e^{-\int_t^T r(u)\, du} \int \psi(y)\, \frac{\partial}{\partial t} p(x,t;y,T)\, dy$$

First spatial derivative:

$$\frac{\partial f}{\partial x}(x,t) = e^{-\int_t^T r(u)\, du} \int \psi(y)\, \frac{\partial}{\partial x} p(x,t;y,T)\, dy$$

Second spatial derivative:

$$\frac{\partial^2 f}{\partial x^2}(x,t) = e^{-\int_t^T r(u)\, du} \int \psi(y)\, \frac{\partial^2}{\partial x^2} p(x,t;y,T)\, dy$$

(2) Substituting back into the PDE $\frac{\partial f}{\partial t} + \mu(x,t)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 f}{\partial x^2} - r(t) f = 0$ to yield:

$$r(t) f + e^{-\int_t^T r(u)\, du} \int \psi(y)\, \frac{\partial p}{\partial t}\, dy + \mu(x,t) e^{-\int_t^T r(u)\, du} \int \psi(y)\, \frac{\partial p}{\partial x}\, dy$$

$$+ \frac{1}{2}\sigma^2(x,t) e^{-\int_t^T r(u)\, du} \int \psi(y)\, \frac{\partial^2 p}{\partial x^2}\, dy - r(t) f = 0.$$

(3) Tidying up to obtain

$$e^{-\int_t^T r(u)\,du} \int \psi(y) \left[\frac{\partial p}{\partial t} + \mu(x,t)\frac{\partial p}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 p}{\partial x^2} \right] dy = 0.$$

Since $\psi(y)$ is arbitrary, the integrand satisfies:

$$\frac{\partial p}{\partial t} + \mu(x,t)\frac{\partial p}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 p}{\partial x^2} = 0.$$

This derivation shows that the Feynman–Kac representation implies that $p(x,t;y,T)$ satisfies the Kolmogorov backward PDE. The proof establishes equivalence between the Feynman–Kac probabilistic representation and the PDE—but does *not* specify what the actual density p is.

So, how to determine the transition density $p(x,t;y,T)$? In option pricing, the key quantity is the expected payoff under the risk-neutral measure $\mathbb{Q}$. For a European call option with strike K and maturity T, this is given by the Feynman–Kac representation:

$$C(S_0,0) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(S_T - K)^+\right] = e^{-rT}\int_K^\infty (s-K)\,p_{S_T}(s)\,ds$$

where $p_{S_T}(s)$ is the risk-neutral density (a probability) of the terminal asset price S_T. This density may be derived using one of the approaches below:

(i) **Implied**, recovered from option prices via Breeden–Litzenberger,

(ii) **Analytical**, solving the underlying asset process to obtain its moments (say mean and variance) and thereby its probability distribution.

(iii) **Simulation**, using Monte Carlo sampling.

(iv) **Characteristic-Functions**, via Fourier methods (COS/FFT).

Although we cannot say deterministically whether the option will finish in-the-money (ITM) or out-of-the-money (OTM), the distribution tells us the *probability weight* associated with each possible terminal price s.

2 Volatility Modeling

In the previous section, the focus was on how to price derivatives using Feynman-Kac and their resulting PDEs. Now, we need to address distributions and parameters, starting with the recovery of the risk-neutral distribution and the implied local volatility surface. We do so by providing a framework that connects the **probabilistic**, **differential**, and **market-based** views of option pricing.

The framework allows for the extraction of the market-implied risk-neutral density $p(S_T)$ from observed option prices via Breeden–Litzenberger, and the local volatility surface $\sigma_{\text{LV}}(K,T)$ via Dupire's formula. These quantities serve as inputs for forward simulation models, numerical PDE solvers, or risk analytics engines.

Most importantly, this framework captures the observed volatility smile and skew in the market by calibrating a non-constant, strike- and time-dependent local volatility function. Despite being deterministic, the local volatility model is calibrated to match market option prices and hence reproduces the implied smile observed across maturities.

This results in more accurate and market-consistent option pricing than constant-volatility models, as it naturally adapts to the observed volatility surface, ensuring arbitrage-free interpolation across strikes and maturities.

2.1 Recovering Risk-Neutral Density via Breeden–Litzenberger

Let $C(K,T)$ denote the t_0 price of a European call option with strike K and maturity T. Under the risk-neutral measure $\mathbb{Q}$, discounted expectation pricing as defined by Feynman-Kac is:

$$C(K,T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] = e^{-rT}\int_K^\infty (S-K)\,p(S,T)\,dS$$

where $p(S,T)$ is the risk-neutral probability density function (pdf) of the terminal stock price S_T.

Differentiate once with respect to strike K:

$$\frac{\partial C}{\partial K}(K,T) = -e^{-rT}\int_K^\infty p(S,T)\,dS$$

Differentiate again:

$$\frac{\partial^2 C}{\partial K^2}(K,T) = e^{-rT}p(K,T)$$

Thus, the second partial derivative of the call option price with respect to strike recovers the **pointwise** risk-neutral density of the underlying asset:

$$p(K,T) = e^{rT}\frac{\partial^2 C}{\partial K^2}(K,T)$$

This is the **Breeden–Litzenberger identity**, which provides a direct, model-free method for extracting the implied risk-neutral probability distribution from observed option prices.

2.2 Recovering Local Volatility via Dupire

Using Feynman-Kac, at t_0 the price of a European call option with strike K and maturity T is:

$$C(K,T) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T - K)^+\right] = e^{-rT}\int_K^\infty (S-K)\,p(S,T)\,dS$$

where $p(S,T)$ is the risk-neutral density of S_T.

As before, differentiate $C(K,T)$ only this time with respect to maturity T:

$$\frac{\partial C}{\partial T} = -re^{-rT}\int_K^\infty (S-K)p(S,T)\,dS + e^{-rT}\int_K^\infty (S-K)\frac{\partial p}{\partial T}(S,T)\,dS$$

Apply the Fokker–Planck equation for the local volatility process:

$$\frac{\partial p}{\partial T} = -\frac{\partial}{\partial S}(rSp) + \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma_{\mathrm{LV}}^2(S,T)S^2p\right)$$

Substitute this into the integral:

$$\frac{\partial C}{\partial T} = -rC(K,T) + e^{-rT}\int_K^\infty (S-K)\left[-\frac{\partial}{\partial S}(rSp) + \frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma_{\mathrm{LV}}^2 S^2 p\right)\right]dS$$

Apply integration by parts to both terms:

Term 1:

$$\int_K^\infty (S-K)\left(-\frac{\partial}{\partial S}(rSp)\right)dS = rKp(K,T) + r\int_K^\infty p(S,T)\,dS = rK\frac{\partial C}{\partial K} + rC$$

Term 2:

$$\int_K^\infty (S-K)\cdot\frac{1}{2}\frac{\partial^2}{\partial S^2}\left(\sigma_{\mathrm{LV}}^2 S^2 p\right)dS = \frac{1}{2}\sigma_{\mathrm{LV}}^2(K,T)K^2p(K,T) = \frac{1}{2}\sigma_{\mathrm{LV}}^2(K,T)K^2\frac{\partial^2 C}{\partial K^2}$$

Putting all together:

$$\frac{\partial C}{\partial T} = -rC + rK\frac{\partial C}{\partial K} + \frac{1}{2}\sigma_{\mathrm{LV}}^2(K,T)K^2\frac{\partial^2 C}{\partial K^2}$$

Solving for $\sigma_{\mathrm{LV}}^2(K,T)$, we obtain the **Dupire formula**:

$$\sigma_{\mathrm{LV}}^2(K,T) = \frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K} - rC}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$$

Note that the resulting local volatility surface $\sigma_{\mathrm{LV}}^2(K,T)$:

(a) is evaluated at the option's strike K and maturity T, not at the spot price S,

(b) is expressed entirely in terms of partial derivatives of observed call option prices $C(K,T)$.

Also note if we were to take the first partial of

2.3 Constructing the Local Volatility Surface in Practice

Dupire's formula translates a continuum of call prices across strikes and maturities into a strike- and time-dependent local volatility function, which can be used for forward simulation, option pricing, and risk management.

In theory, Dupire's formula requires a smooth continuum of call prices $C(K,T)$ or implied volatilities $\sigma_{\mathrm{imp}}(K,T)$. However, market data is available only at discrete strikes and maturities, and may contain noise or inconsistencies. To apply model-free formulas like Dupire's in practice, one must first construct a smooth, arbitrage-free surface from raw quotes:

- **Grid Construction:** Organize market-implied volatilities into a grid over strike K and maturity T.
- **No-Arbitrage Constraints:**
  - *Calendar (vertical) arbitrage:* $\frac{\partial C}{\partial T} \geq 0$
  - *Butterfly (horizontal) arbitrage:* $\frac{\partial^2 C}{\partial K^2} \geq 0$
- Once a smooth surface is constructed, Dupire's formula is applied:

$$\sigma_{\text{LV}}^2(K,T) = \frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial K} - rC}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}$$

Smile Dynamics and Factor Decomposition

While the local volatility model $\sigma_{\text{LV}}(K,T)$ is calibrated to match observed option prices, it generates a deterministic volatility surface that does not evolve in a way consistent with observed market smile dynamics, i.e. Dupire gives you a static snapshot of the surface today.

A key distinction arises when studying how the implied volatility surface responds to changes in the underlying asset price S:

- **Sticky Strike Smile:** Implied volatilities $\sigma_{\text{imp}}(K,T)$ remain unchanged when S changes — observed in local volatility models.
- **Sticky Delta (Floating Smile):** Implied volatilities shift with S, preserving the moneyness or delta structure — observed in stochastic volatility markets (e.g., FX, equity).

Mathematically, taking the partial derivative $\partial\sigma_{\text{imp}}/\partial S$ distinguishes the two behaviors:

- If $\partial\sigma_{\text{imp}}/\partial S \approx 0$: sticky strike
- If $\partial\sigma_{\text{imp}}/\partial S \neq 0$: floating smile

These dynamics are particularly important for exotic options (e.g., barriers, cliquets) where payoff sensitivity depends on the evolution of the volatility smile.

Smile Decomposition via Principal Components. Implied volatility smiles for each maturity can be summarized by three main features:

- **Level (Volatility)**: $\sigma_0 = \sigma_{\text{imp}}(K_0,T)$ — ATM volatility
- **Skew (Slope)**:

$$X(T) = \frac{\sigma_+ - \sigma_-}{\sigma_0}$$

  where σ_+, σ_- are implied vols at symmetric OTM call/put strikes.

- **Convexity (Curvature)**:
$$Y(T) = \frac{\sigma_+ + \sigma_- - 2\sigma_0}{\sigma_0}$$

By tracking (Level, Skew, Convexity) across maturities and time, one can apply **Principal Component Analysis (PCA)** to identify dominant modes of smile variation:

PCA Factor	Interpretation
PC1	Parallel shift in volatility (level)
PC2	Rotation of the smile (skew)
PC3	Flattening or steepening (convexity)

Such factor-based analysis is widely used in risk management, scenario testing, and model calibration for exotic derivatives where the smile's term structure and dynamics materially affect pricing.

Therefore despite its theoretical appeal, the local volatility model is rarely the practitioner's first choice for surface construction or exotic pricing. Key limitations include:

1. Poor Fit to Path-Dependent Exotics Local volatility models assume a deterministic volatility function $\sigma^2_{\mathrm{LV}}(K, T)$, calibrated to match market prices of vanillas. However, because it lacks randomness in future volatility evolution, it fails to capture the volatility-of-volatility effects required to price path-dependent options accurately.

2. Overfitting and Instability Dupire's formula involves second derivatives of call prices w.r.t. strike, and a time derivative — all of which amplify noise in observed option quotes. This leads to an unstable and often jagged surface unless extremely careful smoothing is applied.

3. Static Surface; No Dynamics Local volatility surfaces are calibrated statically. They do not evolve naturally over time or respond to changes in market conditions. This makes them unsuitable for dynamic hedging or for pricing long-dated or forward-starting options.

4. No Smile Dynamics The local volatility model, once calibrated, produces deterministic forward smiles. In reality, implied volatilities move stochastically.

Due to the above shortcomings, practitioners opt to fit option prices directly using parametric models, namely:

- **SVI (Stochastic Volatility Inspired):**
$$w(k) = a + b\left[\rho(k - m) + \sqrt{(k - m)^2 + \sigma^2}\right]$$
where $w(k) = \sigma^2_{\mathrm{imp}} T$ is the total implied variance and $k = \log(K/F)$ is log-moneyness.
Note: This function ensures convexity in implied variance and is built to model the smile observed in equity and FX markets.

- **Heston Model:**
$$dS_t = \mu S_t\, dt + \sqrt{v_t}\, S_t\, dW^1_t$$
$$dv_t = \kappa(\theta - v_t)\, dt + \xi\sqrt{v_t}\, dW^2_t$$
with $\mathrm{corr}(dW^1, dW^2) = \rho$

- **SABR Model:** Common in interest rate derivatives:

$$\begin{aligned} dF_t &= \sigma_t F_t^{\beta}\, dW_t^1 \\ d\sigma_t &= \nu \sigma_t\, dW_t^2 \end{aligned}$$

with $\mathrm{corr}(dW^1, dW^2) = \rho$.

3 Parameter Estimation

For a risk-neutral diffusion process X_t, pricing a contingent claim with payoff $\psi(X_T)$ reduces to computing the conditional expectation:

$$V(x,t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\psi(X_T) \mid X_t = x\right]$$

There are multiple strategies to compute this:

- **PDE Method (FDM)**: Solve the backward Kolmogorov equation.
- **Characteristic Function (CF)**: Use Fourier inversion if CF is known.
- **Monte Carlo Simulation (MCS)**: Simulate paths to compute expectations numerically.
- **Filtering**: Estimate hidden states and parameters from partial observations.

Consider the Four Models: PDE, CF, MCS, Filtering

3.1 Geometric Brownian Motion (GBM)

SDE:

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

PDE (FDM):

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

CF: Lognormal distribution:

$$\phi(u) = \exp\left(iu\log S_0 + iu\left(r - \frac{1}{2}\sigma^2\right)T - \frac{1}{2}u^2\sigma^2 T\right)$$

MCS (Euler scheme):

$$S_{t+\Delta t} = S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}Z_t\right)$$

Filtering: Not required (fully observable).

MLE: Use closed-form log-likelihood of log-returns.

3.2 Cox–Ingersoll–Ross (CIR)

SDE:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

PDE (FDM):

$$\frac{\partial V}{\partial t} + \kappa(\theta - r)\frac{\partial V}{\partial r} + \frac{1}{2}\sigma^2 r\frac{\partial^2 V}{\partial r^2} = rV$$

CF: Bond prices are known in closed form; PDF is non-central χ^2.

MCS: Use full truncation or Milstein schemes:

$$r_{t+\Delta t} = r_t + \kappa(\theta - r_t)\Delta t + \sigma\sqrt{r_t^+}\sqrt{\Delta t}Z_t$$

Filtering: Use:

- **Extended Kalman Filter (EKF)** or
- **Unscented Kalman Filter (UKF)** — for better nonlinear approximation.

MLE: Leverage the known χ^2 density.

3.3 Hull–White (HW)

SDE:

$$dr_t = [\theta(t) - \alpha r_t]dt + \sigma dW_t$$

PDE (FDM):

$$\frac{\partial V}{\partial t} + [\theta(t) - \alpha r]\frac{\partial V}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2} = rV$$

CF: Transition density is Gaussian; closed-form solution for bond prices.

MCS: Exact simulation via:

$$r_{t+\Delta t} = r_t e^{-\alpha\Delta t} + \int_t^{t+\Delta t} e^{-\alpha(t+\Delta t-s)}\theta(s)\,ds + \sigma\sqrt{\frac{1-e^{-2\alpha\Delta t}}{2\alpha}}Z$$

Filtering: Kalman Filter (KF)

Setup:

- **State:** $x_t = r_t$
- **Observation:** $y_t = r_t + \varepsilon_t$, with $\varepsilon_t \sim \mathcal{N}(0, \sigma^2_{\text{obs}})$

State Transition Equation (discretized):

$$r_{t+\Delta t} = \phi r_t + m_t + \eta_t, \quad \eta_t \sim \mathcal{N}(0, Q_t)$$

where:

$$\phi = e^{-\alpha\Delta t}, \quad m_t = \int_t^{t+\Delta t} e^{-\alpha(t+\Delta t-s)}\theta(s)ds, \quad Q_t = \frac{\sigma^2}{2\alpha}(1-e^{-2\alpha\Delta t})$$

Algorithm:

At each time step t:

1. **Prediction Step:**

$$\hat{r}_{t|t-1} = \phi \hat{r}_{t-1|t-1} + m_{t-1}, \quad P_{t|t-1} = \phi^2 P_{t-1|t-1} + Q_{t-1}$$

2. **Update Step:**

$$K_t = \frac{P_{t|t-1}}{P_{t|t-1} + \sigma^2_{\text{obs}}}$$
$$\hat{r}_{t|t} = \hat{r}_{t|t-1} + K_t(y_t - \hat{r}_{t|t-1})$$
$$P_{t|t} = (1 - K_t)P_{t|t-1}$$

The log-likelihood can be accumulated via:

$$\log p(y_t) = -\frac{1}{2}\left(\log(2\pi S_t) + \frac{(y_t - \hat{r}_{t|t-1})^2}{S_t}\right), \quad S_t = P_{t|t-1} + \sigma^2_{\text{obs}}$$

3.4 Heston Model

SDE:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW^1_t$$
$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t} dW^2_t, \quad dW^1_t dW^2_t = \rho dt$$

PDE (FDM):

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \kappa(\theta - v)\frac{\partial V}{\partial v} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\xi^2 v\frac{\partial^2 V}{\partial v^2} + \rho\xi vS\frac{\partial^2 V}{\partial S \partial v} = rV$$

CF: Known in closed form for $\log S_T$; Carr–Madan or COS method applies.

MCS: Use **QE scheme**:

- Moment-match the conditional distribution of $v_{t+\Delta t}$.
- Sample $S_{t+\Delta t}$ conditionally using log-normal or exponential mixtures.

Filtering: Particle Filter (PF)

Setup:

- **Latent state:** $x_t = v_t$
- **Observed:** $y_t = \log S_t$
- **State evolution:**

$$v_{t+\Delta t} = v_t + \kappa(\theta - v_t)\Delta t + \xi\sqrt{v_t}\sqrt{\Delta t} Z_t$$

- **Observation model (discretized):**

$$\log S_{t+\Delta t} = \log S_t + \left(r - \frac{1}{2}v_t\right)\Delta t + \sqrt{v_t}\sqrt{\Delta t} Z'_t$$

Bootstrap Particle Filter (BPF) Algorithm:

1. **Initialize:** N particles $\{v_0^{(i)}\}_{i=1}^N \sim p(v_0)$
2. For each time t:
   (a) **Propagate:** For each particle, sample
   $$v_t^{(i)} \sim p(v_t \mid v_{t-1}^{(i)})$$
   (b) **Weight:**
   $$w_t^{(i)} \propto p(y_t \mid v_t^{(i)})$$
   This uses the likelihood from the log-price model, e.g.,
   $$y_t \mid v_t^{(i)} \sim \mathcal{N}(\mu_t^{(i)}, \sigma_t^{2(i)}), \quad \text{based on discretized SDE}$$
   (c) **Normalize:** $\tilde{w}_t^{(i)} = \frac{w_t^{(i)}}{\sum_j w_t^{(j)}}$
   (d) **Resample:** Draw new particles $\{v_t^{(i)}\}$ from the weighted set
3. **Estimate:** The filtered mean:
   $$\hat{v}_t = \sum_i \tilde{w}_t^{(i)} v_t^{(i)}$$

Likelihood: Approximate at each step:

$$p(y_t \mid y_{1:t-1}) \approx \sum_{i=1}^N w_t^{(i)}$$

Total log-likelihood is accumulated over time:

$$\log \mathcal{L}(\theta) = \sum_{t=1}^T \log \left(\sum_{i=1}^N w_t^{(i)} \right)$$

4 Market Price of Risk and Measure Change

Traders may hold different beliefs about the real-world expected return μ, but pricing derivatives requires agreement on a risk-neutral measure $\mathbb{Q}$. Girsanov's theorem explains how the transition from a subjective real-world measure $\mathbb{P}^{(i)}$ to the common pricing measure $\mathbb{Q}$ is governed by the market price of risk $\lambda^{(i)}$, where:

$$\lambda^{(i)} = \frac{\mu^{(i)} - r}{\sigma}$$

with each trader i applying their own Girsanov's change of measure:

$$dW_t^{\mathbb{Q}} = dW_t^{(i)} + \lambda^{(i)} dt$$

However, the financial market ensures **arbitrage-free pricing** through a unique, observable $\mathbb{Q}$, determined from traded asset and option prices. Thus, traders must price derivatives using this shared $\mathbb{Q}$, even if their beliefs under $\mathbb{P}^{(i)}$ differ.

Disagreement over μ leads to differing real-world dynamics, but pricing is done under the same risk-neutral measure $\mathbb{Q}$, ensuring all market participants agree on the derivative's fair value. Girsanov's theorem enables this shift by removing the subjective drift.

Formally, if asset price S_t follow GBM under the physical measure $\mathbb{P}$:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t^{\mathbb{P}}$$

with the market price of risk defined as:

$$\lambda = \frac{\mu - r}{\sigma}$$

Girsanov's transformation guarantees the existence of an equivalent measure $\mathbb{Q}$, under which:

$$dW_t^{\mathbb{Q}} = dW_t^{\mathbb{P}} + \lambda \, dt$$

Under this new measure $\mathbb{Q}$, the process for S_t becomes:

$$dS_t = r S_t \, dt + \sigma S_t \, dW_t^{\mathbb{Q}}$$

This removes the subjective drift μ and replaces it with the objective risk-free rate r, enabling risk-neutral pricing.

The change of measure from $\mathbb{P}$ to $\mathbb{Q}$ is characterized by the Radon–Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \lambda \, dW_t^{\mathbb{P}} - \frac{1}{2} \int_0^T \lambda^2 \, dt \right)$$

This reweights probability paths to reflect a world where investors are indifferent to risk, thus making the discounted asset price a $\mathbb{Q}$-martingale:

$$\tilde{S}_t = e^{-rt} S_t \quad \text{is a martingale under } \mathbb{Q}$$

Going back to Feynman-Kac, Girsanov's theorem formalizes the transition to a risk-neutral world, enabling arbitrage-free pricing of derivatives using the expectation under $\mathbb{Q}$:

$$V(S_t, t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-t)} \cdot \psi(S_T) \,\middle|\, \mathcal{F}_t \right]$$

4.1 Constrained Stochastic Model

Now that we know about basic pricing setup and how to solve them numerically, let's proceed and investigate the Black-Scholes model. This model assumes the underlying asset follows GBM under $\mathbb{Q}$:

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}}$$

where both volatility σ and risk-free rate r are assumed constant.
Assuming $V(S_t, t)$ is the option price, Feynman–Kac prices this as:

$$V(X_t, t) = \mathbb{E}\left[e^{-\int_t^T r(u)\,du}\,\psi(X_T)\,\middle|\,\mathcal{F}_t\right]$$

Or equivalently:

$$V(x, t) = e^{-\int_t^T r(u)\,du}\int_{-\infty}^{\infty}\psi(y)\,p(x, t; y, T)\,dy$$

by conditioning X_t on x or $X_t = x$. This way, the random variable X_T has a transition density $p(x, t; y, T)$ in the y-variable. The variable y is the dummy integration variable that represents a possible future state $X_T = y$. In other words:

$$\begin{cases} x: & \text{current position at } t \\ y: & \text{future position at } T \\ p(x, t; y, T) & \text{transition density from } x \text{ at } t \text{ to } y \text{ at } T \end{cases}$$

The transition density satisfies two parabolic PDEs — the forward and backward Kolmogorov equations defined as:
Forward Kolmogorov Equation

$$\frac{\partial}{\partial T}p(x, t; y, T) = -\frac{\partial}{\partial y}\left[\mu(y, T)\,p(x, t; y, T)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[\sigma^2(y, T)\,p(x, t; y, T)\right]$$

Backward Kolmogorov Equation

$$\frac{\partial}{\partial t}p(x, t; y, T) = -\mu(x, t)\frac{\partial}{\partial x}p(x, t; y, T) - \frac{1}{2}\sigma^2(x, t)\frac{\partial^2}{\partial x^2}p(x, t; y, T)$$

Although the transition density $p(x, t; y, T)$ satisfies the Kolmogorov PDEs, the value function $V(x, t)$ defined by the expectation

$$V(x, t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\psi(X_T) \mid X_t = x\right]$$

also satisfies a parabolic PDE — derived via the Feynman–Kac theorem — of the form:

$$\frac{\partial V}{\partial t} + rx\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 V}{\partial x^2} = rV(x, t)$$

Therefore, we obtain:

$$\frac{\partial}{\partial t}V(S_t, t) + rS\frac{\partial}{\partial S}V(S_t, t) + \frac{1}{2}\sigma^2 S^2\frac{\partial^2}{\partial S^2}V(S_t, t) = rV(S_t, t)$$

or equivalently in terms of Greeks:

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV(S_t, t)$$

$\Theta = \frac{\partial V}{\partial t}$	Time decay	$\Delta = \frac{\partial V}{\partial S}$	Price sensitivity
$\Gamma = \frac{\partial^2 V}{\partial S^2}$	Price convexity		

4.2 Unconstrained Stochastic Model

Now, let's remove the constant constraints (constant short rate and constant volatility) by considering a 3-dimensional process:

$$\begin{aligned} dS_t &= r_t S_t\, dt + \sigma_t S_t\, dW_t^{(1)} \\ d\sigma_t &= \mu_\sigma(\sigma_t, t)\, dt + \eta(\sigma_t, t)\, dW_t^{(2)} \\ dr_t &= \mu_r(r_t, t)\, dt + \xi(r_t, t)\, dW_t^{(3)} \end{aligned}$$

with correlations amongst Brownian processes assumed to exist and defined as $\rho_{ij} = dW_t^{(i)} dW_t^{(j)}/dt$. In this case, the resulting PDE is:

$$\begin{aligned} &\frac{\partial V}{\partial t} + r_t S_t \frac{\partial V}{\partial S} + \mu_\sigma \frac{\partial V}{\partial \sigma} + \mu_r \frac{\partial V}{\partial r} && \cdots \text{first partials} \\ &+ \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\eta^2 \frac{\partial^2 V}{\partial \sigma^2} + \frac{1}{2}\xi^2 \frac{\partial^2 V}{\partial r^2} && \cdots \text{second partials} \\ &+ \rho_{12}\sigma_t S_t \eta \frac{\partial^2 V}{\partial S \partial \sigma} + \rho_{13}\sigma_t S_t \xi \frac{\partial^2 V}{\partial S \partial r} + \rho_{23}\eta\xi \frac{\partial^2 V}{\partial \sigma \partial r} && \cdots \text{cross-term partials} \\ &= r_t V \end{aligned}$$

Expressed in terms of Greeks:

$$\begin{aligned} &\Theta + rS\Delta + \mu_\sigma \mathcal{V} + \mu_r \mathcal{R} && \cdots \text{sensitivities} \\ &+ \frac{1}{2}\sigma^2 S^2 \Gamma + \frac{1}{2}\eta^2 \mathcal{C}_{\sigma\sigma} + \frac{1}{2}\xi^2 \mathcal{C}_{rr} && \cdots \text{convexities} \\ &+ \rho_{12}\sigma S \eta \mathcal{C}_{S\sigma} + \rho_{13}\sigma S \xi \mathcal{C}_{Sr} + \rho_{23}\eta\xi \mathcal{C}_{\sigma r} && \cdots \text{cross terms} \\ &= rV \end{aligned}$$

$\Theta = \frac{\partial V}{\partial t}$	Time decay	$\Delta = \frac{\partial V}{\partial S}$	Delta (price sensitivity)
$\mathcal{V} = \frac{\partial V}{\partial \sigma}$	Vega (vol sensitivity)	$\mathcal{R} = \frac{\partial V}{\partial r}$	Rho (rate sensitivity)
$\Gamma = \frac{\partial^2 V}{\partial S^2}$	Gamma (price convexity)	$\mathcal{C}_{\sigma\sigma} = \frac{\partial^2 V}{\partial \sigma^2}$	Volatility convexity
$\mathcal{C}_{rr} = \frac{\partial^2 V}{\partial r^2}$	Rate convexity	$\mathcal{C}_{S\sigma} = \frac{\partial^2 V}{\partial S \partial \sigma}$	Price–vol cross
$\mathcal{C}_{Sr} = \frac{\partial^2 V}{\partial S \partial r}$	Price–rate cross	$\mathcal{C}_{\sigma r} = \frac{\partial^2 V}{\partial \sigma \partial r}$	Vol–rate cross

The Feynman-Kac is:

$$V(s, \sigma, r, t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(u)\, du}\, \psi(S_T, \sigma_T, r_T) \,\middle|\, S_t = s,\ \sigma_t = \sigma,\ r_t = r \right]$$

4.3 Challenging the Unconstrained Model

At this point, we must scrutinize the suitability of the underlying processes assumed in the 3-dimensional setting[1], namely asset price S_t, stochastic volatility σ_t, and stochastic interest rate r_t.

1. **Asset Price Process:** Most financial asset returns deviate significantly from Gaussianity. Empirical evidence shows that returns exhibit *negative skewness* and *excess kurtosis*, features that cannot be captured by geometric Brownian motion. A more flexible alternative is to use a jump-diffusion process, such as Merton's model:

$$dS_t = r_t S_t \, dt + \sigma_t S_t \, dW_t^{(1)} + S_{t^-} (J-1) \, dN_t$$

where N_t is a Poisson process with intensity λ, and J represents the random jump size, typically $\log J \sim \mathcal{N}(\mu_J, \delta_J^2)$. This formulation captures fat tails and discontinuities.

2. **Volatility Process:** The volatility smile and empirical persistence in volatility suggest a mean-reverting process that also allows for stochastic behavior. A widely accepted model is the **Heston stochastic volatility** model:

$$d\sigma_t = \kappa(\theta - \sigma_t) \, dt + \eta\sqrt{\sigma_t} \, dW_t^{(2)}$$

where κ is the rate of mean reversion, θ is the long-term volatility level, and η controls volatility of volatility.

3. **Interest Rate Process:** Interest rates often exhibit mean reversion and cannot be appropriately modeled as Brownian motions. A standard alternative is the **Hull–White model**:

$$dr_t = \alpha(t)(\theta(t) - r_t) \, dt + \xi(t) \, dW_t^{(3)}$$

where $\alpha(t)$ controls the speed of reversion and $\theta(t)$ is the time-dependent long-term mean.

Therefore, the revised processes are:

$$\begin{aligned}
dS_t &= r_t S_t \, dt + \sigma_t S_t \, dW_t^{(1)} + S_{t^-} (J-1) \, dN_t \\
d\sigma_t &= \kappa(\theta - \sigma_t) \, dt + \eta\sqrt{\sigma_t} \, dW_t^{(2)} \\
dr_t &= \alpha(t)(\theta(t) - r_t) \, dt + \xi(t) \, dW_t^{(3)}
\end{aligned}$$

With the three Brownian motions allowed to be correlated as before:

$$\mathbb{E}[dW_t^{(i)} dW_t^{(j)}] = \rho_{ij} \, dt, \quad i, j \in \{1, 2, 3\}.$$

Then the option price $V(S, \sigma, r, t)$ satisfies the following partial integro-differential equation (PIDE):

$$\begin{aligned}
&\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \kappa(\theta - \sigma)\frac{\partial V}{\partial \sigma} + \alpha(t)(\theta(t) - r)\frac{\partial V}{\partial r} \\
&\quad + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\eta^2 \sigma \frac{\partial^2 V}{\partial \sigma^2} + \frac{1}{2}\xi^2(t)\frac{\partial^2 V}{\partial r^2} \\
&\quad + \rho_{12}\sigma S \eta\sqrt{\sigma}\frac{\partial^2 V}{\partial S \partial \sigma} + \rho_{13}\sigma S \xi(t)\frac{\partial^2 V}{\partial S \partial r} + \rho_{23}\eta\sqrt{\sigma}\xi(t)\frac{\partial^2 V}{\partial \sigma \partial r} \\
&\quad + \lambda \int_{\mathbb{R}_+} [V(jS, \sigma, r, t) - V(S, \sigma, r, t)] \, f_J(j) \, dj = rV
\end{aligned}$$

[1] The 1D case was presented for pedagogical reasons and serves no practical purpose.

The Feynman-Kac for the above SDEs is:

$$V(s, \sigma, r, t) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_t^T r(u)\, du}\, \psi(S_T, \sigma_T, r_T) \,\middle|\, S_t = s,\ \sigma_t = \sigma,\ r_t = r \right]$$

This revised model now captures skew, smile, fat tails, and interest rate curvature — all within a consistent risk-neutral valuation framework.