Ordinary Differential Equations

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Chapter 1

First Order Ordinary Differential Equations

1.1 Antiderivative Theorem

Take $F(t)$ as an antiderivative of $f(t)$. Then all solutions of

$$\frac{dy}{dt} = f(t)$$

are given by

$$y(t) = F(t) + c$$

for some constant c.

1.2 Separable Equations

All solutions of an equation of the form

$$g(y)\frac{dy}{dt} = f(t)$$

are given by solving

$$\int g(y)\ dy = \int f(t)\ dt$$

for y.

1.3 Properties of LDEs

We can write a LDE in standard form:

$$\frac{dy}{dt} + p(t)y(t) = g(t)$$

$g(t)$ is known as the forcing term. The singular points are points where $g(t)$ or $p(t)$ are not continuous. Singular points alter the behaviour of the differential equation in ways that depend on how "bad" the singularity is.

1.4 Existence-Uniqueness Theorem (LDE1)

If $p(t)$ and $g(t)$ are continuous in an open interval I containing $t = t_0$, and $y_0 = y(t_0)$, then there exists a unique function $y(t)$ passing through (t_0, y_0) that satisfies the ODE for all t in that interval. That is, the solution exists and is unique up to a discontinuity in $p(t)$ or $g(t)$.

1.5 Solution of 1st-Order Linear Differential Equations

Multiply the linear differential equation L by some magical function $\mu(t)$ such that we get a perfect derivative.
That is, take $\mu(t)$ such that

$$\frac{d\,(\mu y)}{dt} = \mu(t)L$$

Multiply the LDE by μ:

$$\mu(t)L = \mu\frac{dy}{dt} + \mu(t)p(t)y(t) = \mu(t)g(t)$$

This is a perfect derivative by definition! So we find

$$\mu(t)L = \frac{d\left(\mu y\right)}{dt} = \mu(t)g(t)$$

Integrate and solve for $y(t)$. This gives the full solution:

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)g(t)\ dt + A\frac{1}{\mu(t)}$$

$$\mu(t) = e^{\int p(t)\ dt}$$

The first term is the particular integral, and the last term is the complementary function.

1.6 Full Solution of 1st-Order LDE IVP

$$y(t) = y_0 e^{-\int_{t_0}^{t} p(s)\ ds} + \int_{t_0}^{t} g(\tau)e^{-\int_{\tau}^{t} p(s)\ ds}\ d\tau$$

1.7 Discontinuous Forcing Terms

We can use the Dirac delta function to approximate an impulse forcing term.

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t-\tau)\ dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\ dt = f(\tau)$$

The limits don't have to be ∞ - they just have to encompass τ.

1.8 The One-Sided Green's Function

Finding

This function models the behaviour of a system in response to a forcing term.
Solve

$$\frac{dy}{dt} + p(t)y(t) = \delta(t-\tau)$$

$$y(t_0) = y_0 = 0$$

with $\tau > t_0$.
The solution to this is called the one-sided Green's function.

Finding (the hard way)

Solve the IVP given above for $t_0 \leq t < \tau$ and $t > \tau$, then apply the jump condition $[y(t)]_{\tau^-}^{\tau^+} = 1$.
The solution in the first domain will be found to be 0, and in the second domain it will be non-zero. Apply the jump condition. This gives the constant in the second domain.

Using

The solution to LDE1s can be written in terms of the Green's function. Nobody has done this since 1926.

$$y(t) = y_0 G(t, t_0) + \int_{t_0}^{t} g(\tau)G(t,\tau)\ d\tau$$

$$G(t,\tau) = e^{-\int_{\tau}^{t} p(s)\ ds}$$

1.9 Existence-Uniqueness for LDE1 with Discontinuous Forcing Terms

If $g(t)$ is piecewise continuous, then the solutions are continuous functions $y(t)$ with discontinuous derivatives $y'(t)$ and EU theorem still applies.
This makes intuitive sense - $y(t)$ doesn't respond instantly to $g(t)$, and so it shouldn't jump around.
This doesn't apply if $g(t)$ is not piecewise continuous (i.e. has infinite discontinuities).

1.10 Heaviside Step Function

$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Clearly, $H(t-\tau)$ has the discontinuity at $t = \tau$.

1.11 Existence-Uniqueness for First Order NLDEs

Take a NLDE

$$\frac{dy}{dt} = f(t, y)$$

If f and $\frac{\partial f}{\partial y}$ are defined and continuous throughout the interior of a rectangle R containing (t_0, y_0) in its interior, then the IVP has a unique solution which exists in the interval $t_0 - h < t < t_0 + h$ for some $h > 0$ such that $t_0 - h$ and $t_0 + h$ are both inside R - though finding h is nontrivial.
These conditions are sufficient but not necessary. If $f(t, y)$ is continuous in R then that's sufficient to guarantee at least one solution through (t_0, y_0) (but makes no claims as to uniqueness of that solution).
The solution either exits R or approaches the boundary at infinity. A solution cannot die inside R. We can extend the solution inside R uniquely, but it might exit R.

1.12 Change of Variables

Sometimes you can awkwardly work in a substitution. This is called "change of variables".

1.13 Bernoulli's Equation

$$\frac{dy}{dt} + p(t)y(t) = q(t)y^n$$

Divide through by y^n and set $u = y^{1-n}$. Differentiate to find y^{-n} in terms of u. Substitute, and you get a first order LDE.

1.14 Homogeneous Equations in y/t

$\frac{dy}{dt} = f(y/t)$
Set $y = vt$ and hence $\frac{dy}{dt} = \frac{dv}{dt}t + v$

1.15 First Order Exact Equations

An ODE might be "exact", which means that it is a perfect derivative. We can note that by inspection or do things in an actually logical maths way that makes sense. Since this is MATH2405, we should obviously do this in the former way, but the logical way follows.
Consider an equation

$$M(t, y)\ dt + N(t, y)\ dy = 0$$

This is exact in a region R if $\psi(x, y)$ exists such that

$$\frac{d\psi}{dt} = M(t, y), \quad N(t, y)$$

in R.

The solution is then $\psi(t, y) = c$ for some constant c.
The easy way:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

is sufficient and necessary for the equation to be exact in a region R which is simply connected and M, N, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial t}$ are continuous.

1.16 Integrating Factors

Find some μ such that

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial t} + \mu(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}) = 0$$

This is a PDE for μ which is usually harder to solve than the original ODE, but you might as well try $\mu(t) \neq 0$ or $\mu(y) \neq 0$ seeing as you got this far.

1.17 Autonomous Equations

$$\frac{dy}{dt} = f(y)$$

...that is, the equation is independent of t. The equation is separable.
$y(t) = c$ for some c if $f(c) = 0$, so zeros of $f(y)$ are called critical points. Critical points are called equilibrium solutions of the equation.
If you plot solutions on the $t-y$ plane, the equilibrium solutions are horizontal lines. All other solutions tend toward these solutions - the solutions themselves either attract or repel other solutions. Solutions have a fixed sign inside the bands between equilibrium solutions.
Solutions can be translated along the t axis to get more solutions. Solutions cannot intersect in a region where the solutions and their y partial derivatives are continuous (EU theorem).
To find the behaviour of the solutions, just check the sign of their gradient. Since we already have everything in terms of $\frac{dy}{dt}$, we just have to sub in y and see what we get.

1.18 Bifurcations

Finding Bifurcations

Take some autonomous system with a parameter c.

$$\frac{dy}{dt} = f(y, c)$$

Changes in c may cause equilibrium solutions to split, merge, or vanish, hence changing the behaviour of the entire system. This is called bifurcation.
Because the equilibrium solutions of $y(t)$ are given by the zeros of $f(y, c)$, to find bifurcations you just have to find what values of c allow for solutions, and how many.

Bifurcation Diagrams

Plot y in terms of c.

Types of Bifurcation

- Saddle node - when two equilibriums come from no equilibriums with increasing or decreasing c.
- Pitchfork - when three equilibriums come from one equilibrium.

Stability of Solutions

A stable solution is an attractor. An unstable solution is a repeller. A semi-stable solution attracts from one side and repels from the other.

1.19 Dynamical Systems and Orbits

A dynamical process is one described by some state variables, an independent variable, and some differential equations connecting the state variables. For example, a 1st-order dynamical system with two state variables will have time evolution given by

$$\frac{dx}{dt} = f(x, y, t)$$

$$\frac{dy}{dt} = g(x, y, t)$$

If f and g are independent of t the system is autonomous.
$x(t)$ and $y(t)$ are called component curves.

Chapter 2

Second and Higher Order Ordinary Differential Equations

General form for NLDEs:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

2.1 Existence-Uniqueness for LDEs

Take some LDE

$$b_0(t)\frac{d^n y}{dt^n} + b_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + b_n(t)y(t) = R(t)$$

and define

$$z_1 = y, z_2 = \frac{dy}{dt}, \cdots, z_n = \frac{d^{n-1}y}{dt^{n-1}}$$

The LDE is then equivalent to a system of coupled 1st-order LDEs:

$$\begin{aligned} z_1' &= z_2 \\ z_2' &= z_3 \\ &\cdots \\ z_n' &= -\frac{b_n}{b_0}z_1 - \cdots - \frac{b_1}{b_0}z_n + \frac{R}{b_0} \end{aligned}$$

From this we can derive a parallel EU theorem. There exists a unique solution passing through t_0 satisfying initial conditions

$$\begin{cases} z_1(t_0) & = \alpha_1 \\ z_2(t_0) & = \alpha_2 \\ & \cdots \\ z_n(t_0) & = \alpha_n \end{cases} \Rightarrow \begin{cases} y(t_0) & = \alpha_1 \\ \frac{d^2y}{dt^2}(t_0) & = \alpha_2 \\ & \cdots \\ \frac{d^{n-1}y}{dt^{n-1}}(t_0) & = \alpha_n \end{cases}$$

for continuous $b_j(t), R(t)$ on some interval I containing t_0.

2.2 Self-Adjoint Form of LDE2

Take a 2nd-order LDE in the form

$$\frac{d^2y}{dt^2} + \frac{b_1}{b_0}\frac{dy}{dt} + \frac{b_2}{b_0}y(t) = \frac{R}{b_0}$$

Multiply it all by $p(t) = e^{\int \frac{b_1}{b_0}\ dt}$ and you can then rewrite everything as

$$-\frac{d}{dt}\left(p(t)\frac{dy}{dt}\right) + q(t)y(t) = f(x)$$

This is called self-adjoint form.

2.3 Linind/Lindep of Functions

A set of functions is linearly dependent iff they can be linearly combined to produce 0 with non-zero coefficients, and is linearly independent otherwise.

2.4 Wronskian

$$W(f_1, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ Df_1 & Df_2 & \cdots & Df_n \\ \vdots & \vdots & \ddots & \vdots \\ D^{n-1}f_1 & D^{n-1}f_2 & \cdots & D^{n-1}f_n \end{vmatrix}$$

$$\{f_1, \cdots, f_n\} \text{ lindep} \Rightarrow \forall x. W(f_1, \cdots, f_n)(x) = 0$$
$$\exists x. W(f_1, \cdots, f_n)(x) \neq 0 \Rightarrow \{f_1, \cdots, f_n\} \text{ linind}$$

If all functions in the Wronskian have have continuous partial derivatives of order n, then the first statement becomes an iff statement (i.e. true in both directions).
If the function set is composed of solutions of some homogeneous LDE of order n, and the coefficients of this LDE are continuous, then either

$$\forall x. W(f_1, \cdots, f_n)(x) = 0$$

or

$$\forall x. W(f_1, \cdots, f_n)(x) \neq 0$$

2.5 Abel's Theorem

If u_1, u_2 are solutions of the homogeneous LDE

$$-\frac{d}{dt}\left(p(t)\frac{dy}{dt}\right) + q(t)y(t) = 0$$

with $p(t) \neq 0$ and $q(t)$ continuous, then

$$W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \frac{A}{p(t)}$$

with A some constant depending on the pair of solutions. So iff $A = 0$, the solutions are linearly dependent (note that the solutions are to a homogeneous LDE).

2.6 Basic Solution Sets and Null Space

Take a linearly independent set of n solutions of an nth order homogeneous LDE with continuous coefficients.
We can write any solution of the LDE as a linear combination of the solutions in this set. This means that the solutions form a fundamental set, and are a basis for all solutions to the LDE. The null space has dimension n.
The general solution is a linear combination of all basis solutions with arbitrary coefficients.
If the LDE is not homogeneous, then the general solution is

$$y = y_c + y_p$$

with y_c as the general solution to the homogeneous version of this LDE.

2.7 Sturme Separation Theorem

Take a 2nd-order LDE

$$\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y(t) = 0$$

with continuous $b(t)$ and $c(t)$.
If $y_1(t)$ and $y_2(t)$ are real, independent solutions of the LDE, then y_2 must vanish between any two consecutive zeros of $y_1(t)$ and vice versa. This also implies that the zeros are separate.

2.8 Solving 2nd-order LDEs

We know that $y = y_c + y_p$, so we need to find y_c and y_p.
For a 2nd-order LDE, we know we need two linearly independent solutions to linearly combine to get y_c (we would need n for an nth-order LDE). y_c is called the complementary function, and is also the solution to the corresponding homogeneous LDE.
We can then find y_p using y_c.

2.9 Solving Constant Coefficient LDEs

Finding y_c

We can always find a general solution of a homogeneous LDE with constant coefficients.
One way to do this is by factorising the linear operator, though this is a dumb way to do it. Write the equation in terms of D, then factorise D (because you'll have a polynomial). Let $z(t) = (D-k)y(t)$ (where $D-k$ is one of the factorised components of the LDE), and this gives you a 1st-order LDE in z.
A better way to solve this is to find the characteristic polynomial, which is basically an equation of r in the same form as the equation in D (but with r instead).
If all roots r of the characteristic polynomial are real and distinct, the solutions to the homogeneous LDE will be given by

$$Ae^{rt}$$

If the roots are real and repeated, we get solutions of this form and also

$$Ate^{rt}$$

(one of each for each repeated root).
If we get complex conjugate roots,

$$y_1(t) = e^{(a+ib)t} = e^{at}(\cos bt + i \sin bt)$$

and similarly for the other root. Linearly combine these to get real solutions.
These solutions can now be linearly combined to get y_c. This is the solution of the homogeneous LDE, but if the equation is non-homogeneous we still need y_p.

Method of Undetermined Coefficients

We can use the method of undetermined coefficients or the symbolic D method. These work for particular integrals with the right hand side of the non-homogeneous equation in the form of an exponential, sine, cosine, or polynomial. The symbolic D method is more general.
The method of undetermined coefficients involves judiciously guessing the form of y_p by looking at the forcing term. For example, if the forcing term is a sine or a cosine, then y_p should be a linear combination of sines and cosines.
If $g(t)$ (the forcing term) is a solution of the equation, then our guess will be t multiplied by the form we expect.

Symbolic D Method

If we factorise the D operator out of the LDE we have an equation of the form

$$p(D)y(t) = g(t)$$

for some polynomial p. The solution is then

$$y_p = \frac{1}{p(D)}g(t)$$

We know that $\frac{1}{D}$ is just integration. We can also find

$$\frac{1}{D-a}g(t) = e^{at}\int e^{-at}g(t)$$

$$\frac{1}{(D-a_1)(D-a_2)\cdots(D-a_n)}g(t) = e^{a_n t}\int e^{(a_{n-1}-a_n)t}\cdots\int e^{(a_1-a_2)t}\int e^{-a_1 t}g(t)\,dt\,\cdots\,dt$$

You can find more using Taylor series expansions.

2.10 Harmonic Oscillators

Take a spring with a periodic driving force $F = F_0 \cos(\omega t)$ acting upon it. The tension in the spring is given by Hooke's law and so we find the equation of displacement to be

$$\frac{d^2y}{dt^2} + \frac{\alpha}{m}\frac{dy}{dt} + \frac{k}{m}y(t) = F_0 \cos(\omega t)$$

In general the equation is

$$\frac{d^2y}{dt^2} + 2c\frac{dy}{dt} + k^2 y(t) = F(t)$$

c is the damping constant, which is positive, and k^2 is the normalised spring constant.
For an undamped oscillator, $k = \omega_0$, and period is $T = \frac{2\pi}{\omega_0}$. The solution blows up as $\omega \to \omega_0$.

2.11 Adjoint Equation and Operator

Define

$$\begin{aligned} M[\nu] &= \frac{d^2(p_0\nu)}{dt^2} - \frac{d(p_1\nu)}{dt} + p_2(t)\nu(t) \\ &= p_0\frac{d^2\nu}{dt^2} + \left(2\frac{dp_0}{dt} - p_1(t)\right)\frac{d\nu}{dt} + \left(\frac{d^2p_0}{dt^2} - \frac{dp_1}{dt} + p_2(t)\right)\nu(t) \\ &= 0 \end{aligned}$$

where $\nu(t)$ is the integrating factor.
M is then the adjoint of the original LDE $p_0(t)\frac{d^2y}{dt^2} + p_1(t)\frac{dy}{dt} + p_2(t)y(t)$.
The adjoint of the adjoint is the original equation.
The adjoint of a self-adjoint equation is the original equation.

2.12 Integrating Factors for 2nd-order LDEs

$\nu(t)$ is a function which is an integrating factor iff $M[\nu] = 0$.

2.13 Solution Methods for Non-Constant Coefficient LDEs

Factorising the Linear Operator

If you write the equation in terms of D, sometimes you can factorise the D. For example, you could obtain

$$(D + t)(D - t)y = t$$

Let $z = (D - t)y$ and hence find a 1st-order equation in z, which is then solvable. Do this for $z = (D + t)y$ as well and both solutions will be found.

Reduction of Order

This method requires one solution already.
Take this known solution $y_1(t)$ for a homogeneous LDE. We can use this to reduce the order of the LDE by one, and hence find another solution.
Consider

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0$$

We have $y_1(t)$. We want another solution of the form $y_2 = y_1(t)u(t)$. Substitute this into the LDE to obtain another LDE in $\frac{du}{dt}$. This will be of lower order.

Variation of Parameters

If we known all solutions y_c of a homogeneous LDE with variable coefficients, we still need to find a particular solution, and this method can find those.
Take two solutions y_1 and y_2 of the homogeneous LDE. The solution we are looking for to the non-homogeneous LDE will be of the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

While differentiating this, set $\frac{du_1}{dt}y_1(t) + \frac{du_2}{dt}y_2(t) = 0$ to avoid 2nd-order derivatives.
Substitute y_p into the LDE. You will find that some of the terms make up the original homogeneous LDE, which is equal to 0, so they can be eliminated. This gives us two equations:

$$\frac{du_1}{dt}y_1(t) + \frac{du_2}{dt}y_2(t) = 0 \text{ (from before)}$$

$$\frac{du_1}{dt}\frac{dy_1}{dt} + \frac{du_2}{dt}\frac{dy_2}{dt} = g(t) \text{ (after everything is cancelled out)}$$

This is trivially solvable. The Wronskian will appear (as the determinant of that one matrix) and will be non-zero as y_1, y_2 are a fundamental set of solutions.
The particular solution is then

$$y_p(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(y1,y2)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(y1,y2)}\,dt$$

The constants of integration will cancel out later, as well.

2.14 Green's Function for 2nd-order LDE

We just found a solution for $y_p(t)$:

$$y_p(t) = \int_{t_0}^{t} \frac{g(s)}{W(s)}(y_1(s)y_2(t) - y_2(s)y_1(t))\,ds$$

Rewrite this as

$$y_p(t) = \int_{t_0}^{t} G(t,s)g(s)\,ds$$

with

$$G(t,s) = \begin{cases} \frac{y_1(s)y_2(t)-y_2(s)y_1(t)}{W(s)}, & t \geq s \\ 0, & t < s \end{cases}$$

$G(t,s)$ is Green's function, just like for 1st-order LDEs. It is found in the same way, by solving the IVP

$$L[y] = \delta(t-s), \quad (t \geq t_0)$$

$$y(t_0) = 0, \frac{dy}{dt}(t_0) = 0$$

with jump conditions

$$[y(t)]_{s^-}^{s^+} = 0$$

$$\left[\frac{dy}{dt}(t)\right]_{s^-}^{s^+} = 0$$

Chapter 3

Power Series Solutions

3.1 Power Series

$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x - x_0)^n$
This is a power series expansion about $x = x_0$. This has a radius of convergence ρ for which the series converges when $|x - x_0| < \rho$ and diverges when $|x - x_0| > \rho$ (but may converge or diverge at $|x - x_0| = \rho$). Inside the radius of convergence, convergence is absolute and uniform, and f permits continuous derivatives of all order. Also, $a_n = \frac{f^{(n)}(x_0)}{n!}$.
The ratio test tells us whether a series converges.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x - x_0)^{n+1}}{a_n(x - x_0)^n} \right| < 1$$

From this we find

$$\rho = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

3.2 Ordinary and Singular Points

Consider the homogeneous LDE

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y(x) = 0$$

We want power series solutions around $x = x_0$. The nature of this solution depends on whether x_0 is ordinary or singular.
If the coefficients in standard form, $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$, are analytic at x_0, then x_0 is an ordinary point. "Analytic" here means that there is a convergent Taylor series expansion in some interval about x_0. If this is not the case then x_0 is a singular point.
At any ordinary point, a homogeneous LDE has a power series solution of the form

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 y_1(x) + a_1 y^2(x)$$

where $y_1(x), y_2(x)$ are linind power series solutions with a radius of convergence which is at least equal to the minimum of the radii of convergence of $\frac{Q(x)}{P(x)}$ and $\frac{R(x)}{P(x)}$.
Note that setting some variable $t = x - x_0$ might make algebra easier.

3.3 Airy's Equation

$$\frac{d^2y}{dx^2} - xy(x) = 0$$

No analytic solution - just power series. No singular points. Oscillates for negative x, exponential for positive x.

3.4 Regular Singular Points

Since the solution might not be analytic at a singular point, a Taylor series might not work there.
A singular point x_0 is regular if $(x - x_0)\frac{Q(x)}{P(x)}$ and $(x - x_0)^2\frac{R(x)}{P(x)}$ are analytic.

3.5 Euler Type Equations

Consider expansions at $x = 0$ (just translate x_0 to the origin with $z = x - x_0$ then relabel z as x). If $x = 0$ is a regular singular point then the LDE can be written as

$$x^2\frac{d^2y}{dx^2} + xp(x)\frac{dy}{dx} + q(x)y(x) = 0$$

where $p(x), q(x)$ are analytic.
If we take the Taylor expansion of these two coefficients and find that all terms except the first vanish, then we find

$$x^2\frac{d^2y}{dx^2} + xp_0\frac{dy}{dx} + q_0y(x) = 0$$

This is called a "Euler type" equation and it can be solved analytically.
We want solutions of the form

$$y = Ax^r, \quad x > 0$$

so substitute it in and find a condition for r. This gives the indicial equation

$$F(r) = r^2 + (p_0 - 1)r + q_0 = 0$$

Note that if $y_1 = f(x), \quad x > 0$ is a solution of a Euler type equation, then so is $y_2 = f(-x), \quad x < 0$.
The roots are either both real and distinct, real and equal, or a complex conjugate pair.
If the roots are real and distinct, the solution is $y = C_1x^{r_1} + C_2x^{r_2}, \quad x > 0$.
If the roots are real and equal, the solutions are

$$y_1 = x^r$$

and

$$y_2 = \frac{\partial}{\partial r}x^r = x^r \ln x$$

If the roots are complex conjugates $r = a \pm ib$, the solutions are

$$x^{a+ib} = x^a e^{ib\ln x}$$

and

$$x^{a-ib} = x^a e^{-ib\ln x}$$

The real and imaginary parts of these give lindep real solutions. The solutions oscillate but converge or diverge as x approaches 0, depending on the indicial equation constants.

3.6 Frobenius' Method

For a general equation

$$x^2\frac{d^2y}{dx^2} + xp(x)\frac{dy}{dx} + q(x)y(x) = 0$$

and analytic coefficients, with $x = 0$ as a regular singular point, we look for Euler type solutions multiplied by some other power series.

$$y = x^r\sum_{n=0}^{\infty} a_nx^n$$

Substitute it in, collect like terms, set all the coefficients equal to zero, and find a general solution for the coefficents.
If there are two real roots to the indicial equation, there is always one series solution of the Frobenius form.

3.7 Bessel's Equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y(x) = 0$$

There is at least one solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^{r+k}$$

so sub that in and you find a solution. If ν is not an integer then the second root of the indicial equation ($r = -\nu$) gives another linearly independent solution.
Solutions are known as Bessel functions.

3.8 Bessel Functions of the First Kind of Integer Order n

Assume $\nu = n$, a non-negative integer, and set

$$a_0 = \frac{1}{2^n n!}$$

This gives

$$y(x) = J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left(\frac{x}{2}\right)^{2k}$$

These are known as the Bessel functions of the first kind of integer order.
J_n is a solution to the Bessel equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y(x) = 0$$

if n is a non-negative integer.
These functions represent decaying sinusoids.

3.9 Gamma Function

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x}\, dx$$

- Infinitely differentiable
- $\Gamma(z+1) = z\Gamma(z), \quad z > 0$
- For non-negative integers, $\Gamma(n+1) = n!$
- $\Gamma(1) = 0! = 1$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

For negative z,

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}$$

3.10 Bessel Functions of the First Kind (Any Order)

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+n+1)} \left(\frac{x}{2}\right)^{2k}$$

Worth noting:

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$
$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

Useful identity:

$$J_{-n} = (-1)^n J_n, \quad N \in \mathbb{N}$$

J_n, J_{-n} linearly independent for non-integral n.

3.11 Finding More Solutions

Use reduction of order with a Bessel function. The general solution is then

$$y = AJ_\nu(x) + BJ_\nu(x) \int \frac{dx}{x(J_\nu(x))^2}$$

3.12 Bessel Functions of the Second Kind

For non-integral ν, define

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

This is then used instead of $J_{-\nu}$ as the second linearly independent solution.
To get Bessel functions of the second kind for integral ν, use L'Hôpital's rule to find the limit as $\nu \to n$.
The general solution for Bessel's equation is then

$$y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$$

3.13 Useful Bessel Function Formulae

$$\frac{d}{dx}(x^\nu J_\nu) = x^\nu J_{\nu-1}, \quad \nu \geq 1$$
$$\frac{d}{dx}(x^{-\nu} J_\nu) = -x^{-\nu} J_{\nu+1}, \quad \nu \geq 0$$
$$J_{\nu+1} = \frac{2\nu}{x} J_\nu - J_{\nu-1}, \quad \nu \geq 1$$
$$J_{\nu+1} = -2\frac{d}{dx} J_\nu + J_{\nu-1}, \quad \nu \geq 1$$

3.14 Equal Roots in Indicial Equation

If there are equal roots in the indicial equation then there must be two linind solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} b_n x^{n+r}$$

$$y_2(x) = y_1(x)\ln(x) + \sum_{n=1}^{\infty} c_n x^{n+r}$$

where r is a root.
Use the derivative method to find the coefficients (you can also use reduction of order, but this is messy).

3.15 The Derivative Method

Let $D = \frac{\mathrm{d}}{\mathrm{d}x}$ be a linear operator. Substitute this into your ODE, and then solve for D.

3.16 Roots Differing by an Integer

Take larger root r_1 and smaller root $r_2 = r_1 - 1$. The larger root gives

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+r_1}$$

and the second solution is

$$y_2(x) = \alpha y_1(x) \ln x + \sum_{n=0}^{\infty} c_n x^{n+r_2}$$

Find c_n by substitution. α may be zero. You can, of course, use reduction of order.

3.17 Legendere's Equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dt} + p(p+1)y(x) = 0$$

with constant $p \geq 0$ is called Legendere's equation of order p. $x = \pm 1$ are regular singular points. $x = 0$ is ordinary.

Chapter 4

Systems of Differential Equations

4.1 Systems

$$y^(n) = g(t, y, y', \cdots, y^{(n-1)})$$

Let $x_1 = y, x_2 = y', \cdots, x_n = y^{(n-1)}$, and you get a system of 1st-order ODEs.

$$x_1' = x_2$$
$$\vdots$$
$$x_{n-1}' = x_n$$
$$x_n' = g(\cdots)$$

Other systems are not composed of 1st-order ODEs:

$$x_1' = f_1(\cdots)$$
$$\vdots$$
$$x_n' = f_n(\cdots)$$

We can write this as a vector:

$$\vec{x}' = \vec{f}(t, \vec{x})$$

x_j are called state variables. The space they span is called the state space. Time t is an implicit parameter in solution curves of $\vec{x}$, and these curves are called orbits. Plotting solutions in the $(n+1)$ dimensional space $t - \vec{x}$ gives time-state curves.

4.2 Existence-Uniqueness for Systems

$\vec{x}' = \vec{f}(t, \vec{x}), \quad \vec{x}(t_0) = \vec{x}_0$ with f, f_x continuous in a box B in time-state space, and $(t_0, \vec{x}_0)$ in B.
Solution exists on a t interval containing t_0 here, and there is at most one solution in that interval. Two solutions cannot meet in B.

4.3 Autonomous Systems

A system that does not depend on t is called autonomous. EU for this is a box in state space with continuous f and f_x. Orbits must not meet in the box.
If $\vec{x} = \vec{h}(t)$ is a solution, then so is $\vec{x} = \vec{h}(t - \tau)$.
Equilibrium solutions are found when $\vec{x}(t) = \vec{a}$, when $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) = 0$.
Cycles are periodic solutions of an autonomous system with

$$\vec{x}(t + \tau) = \vec{x}(t)$$

for some τ.

4.4 Stability and Instability

A critical point $\vec{x}_0$ is stable if, when you force a solution though a region close enough to $\vec{x}_0$, you can ensure that it stays within some region around $\vec{x}_0$. Else it is unstable.
Something is asymptotically stable if it approaches $\vec{x}_0$ as $t \to \infty$ and is also stable.

4.5 Separable Systems

$$\frac{dx}{dt} = a, \frac{dy}{dt} = b \Rightarrow \frac{dy}{dx} = \frac{b}{a}$$

4.6 Isolating Nullclines

$$\frac{dx}{dt} = X(x, y), \frac{dy}{dt} = Y(x, y)$$

x nullcline is $X(x, y) = 0$ and y nullcline is $Y(x, y) = 0$ and these intersect at equilibrium points. The sign is fixed on each side of the nullcline, so they divide the plane into regions where orbits rise and fall and move left and move right.

4.7 Predator-Prey Models

Volterra-Lotka Models.

$$\frac{dy}{dt} = my(t) - dy(t)^2$$

First term is birth rate, second term is carrying capacity of environment. Adding a predator term - predator population being $x(t)$ - gives

$$x' = -kx(t) + bx(t)y(t)$$

$$y' = my(t) - cx(t)y(t) - dy(t)^2$$

4.8 Roughly Linearisation

Shift the origin to an equilibrium point and neglect non-linear terms.

4.9 Solution of 1st-order Linear System

Since the system is linear you can write it as a matrix equation, with A as the coefficients and $\vec{F}$ as the constant term. This gives the equation

$$\vec{x}' - A\vec{x} = \vec{F}$$

and you can rewrite this as

$$L[\vec{x}] = \vec{F}$$

where

$$L = \begin{bmatrix} D - a_{11} & -a_{12} & \cdots \\ \vdots & \ddots & \\ -a_{n1} & \cdots & D - a_{nn} \end{bmatrix}$$

Solutions to the equation are linearly combinable to produce more solutions.
There are n dimensions to a 1st-order system of n homogeneous LDEs, so the general solution is a lincom of n linind solutions.
If A is constant, then matrix methods can be used to solve the system. In special cases, the method of undetermined coefficients can find particular integrals.

4.10 Solution for Constant A

Look for solutions $\vec{x}(t) = \vec{u}e^{\lambda t}$. Non-trivial solutions are found when λ is an eigenvalue of A, and $\vec{\mu}$ will be the eigenvector.
If an eigenvalue has a non-unary multiplicity then the second solution will have the form $\vec{k}te^{\lambda t} + \vec{p}e^{\lambda t}$. Solve by substitution.

4.11 Stability of Solutions

This depends on the eigenvalues of A.

- Real, distinct, same-sign eigenvalues: NODE. If positive, unstable equilibrium; if negative, asymptotically stable.
- Real, distinct, opposite-sign eigenvalues: SADDLEPOINT. Unstable.
- Purely imaginary eigenvalues: CENTRE. Stable.
- Complex eigenvalues with real part: SPIRALPOINT. If positive real, unstable; if negative, asymptotically stable spiral.
- Repeated eigenvalue, linind eigenvectors: PROPER/STAR NODE. Either asymptotically stable or unstable depending on sign of eigenvalue (unstable if positive).
- Repeated eigenvalue with only one linind eigenvector: IMPROPER NODE. Asymptotically stable with negative eigenvalue; unstable if positive.

4.12 Stability of Non-linear Systems

If $\vec{x}' = A\vec{x} + \vec{g}(\vec{x})$ and $\vec{g}$ is small around $\vec{x} = 0$ (size of $\vec{g}$ divided by the size of $\vec{x}$ goes to 0 as the latter goes to 0), then the system is locally linear at $\vec{x} = 0$.
If this is the case, then the system is asymptotically stable at $\vec{x} = 0$ if all eigenvalues of A have negative real parts, and unstable otherwise.

4.13 Linearising

Find equilibrium points. Use the Jacobian to change each equilibrium point in the $x - y$ plane to the origin in the $u - v$ plane. Then find the eigenvalues of the resulting system to determine what kind of stability you have.

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} X_x & X_y \\ Y_x & Y_y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

4.14 2nd-order Linear Systems

$$\vec{x}'' = A\vec{x}$$

(constant A)
Look for solutions of the type

$$\vec{x} = \vec{u}e^{rt}$$

with r as the positive and negative roots of the eigenvalues of A and $\vec{u}$ as the eigenvector.
In a system like this, cycles will occur iff the eigenvalues are purely imaginary. A limit cycle is when a non-periodic orbit tends to the periodic orbit (limit cycle) over time.
If a system has a circular cycle, cast it to polar coordinates and find equilibriums.

$$x = r\cos\theta, y = r\sin\theta, r^2 = x^2 + y^2, r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$$

4.15 Hopf Bifurcation

Appearance or disappearance of a periodic orbit through a change in parameters. When two imaginary eigenvalues cross at the axis this will happen.
Unstable spiral points may have a circle around them, so cast to polar coordinates and find that.