

CS 309: Discrete Math (Notes)

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Chapter 1: Sets and Logic

Sets

Denoting Sets

A **set** is simply a collection of objects, or elements.

If a set is finite and not large, we can describe it by simply listing the elements:

$$A = \{1, 2, 3, 4\}$$

The above set A is made up of four elements. **The order of the elements in a set does not matter**, therefore, a could also be specified like so:

$$A = \{1, 3, 4, 2\}$$

The elements of a set are assumed to be distinct, so any duplicate occurrence of an element can be ignored. Therefore, we could also specify set A like so:

$$A = \{1, 2, 2, 3, 4, 4\}$$

If a set is very large or infinite, we can describe it using a property necessary for membership:

$$B = \{x \mid x \text{ is a positive, even integer}\}$$

The above set B is made up of positive, even integers. The vertical bar “ \mid ” is read as “such that” and the text after the bar is the property. Therefore, B can be read as “the set of all x such that x is a positive, even integer.” Some sets of numbers occur frequently in mathematics and are given symbols.

Symbol	Set	Example of Members
Z	Integers	-3, 0, 2, 145
Q	Rational numbers	-1/3, 0, 24/15
R	Real numbers	-3, -1.766, 0, 4/15, $\sqrt{2}$, 2.666, ..., π

Rational numbers are quotients of integers, thus **Q** for *quotient*. The set of real numbers **R** consists of all points on a straight line extending indefinitely in either direction.

We can denote the positive elements in a set using the superscript plus (e.g., **Z**⁺ for positive integers) and the negative elements in a set using the superscript minus (e.g., **Q**⁻ for negative rational numbers).

Set Cardinality

If X is a finite set, we let

$$|X| = \text{number of elements in } X$$

We call $|X|$ the **cardinality** of X .

If we let $A = \{1, 2, 3, 4\}$, then the cardinality of A is 4, or $|A| = 4$. The cardinality of $\{\mathbf{R}, \mathbf{Z}\}$ is 2 since it contains two elements, which just happen to be sets.

Remember: an element in a set can be anything, even a set.

If x is in the set X , we write $x \in X$. If x is NOT in the set X , we write $x \notin X$. For example, both of these are true:

$$\begin{aligned} 3 &\in \{1, 2, 3, 4\} \\ 3 &\notin \{x \mid x \text{ is a positive, even integer}\} \end{aligned}$$

Empty Set

A set with no elements is called an **empty set** and is denoted by \emptyset . In other words, $\emptyset = \{\}$.

Set Equality

Two sets X and Y are **equal** ($X = Y$) if X and Y have the same elements. To put it differently, for $X = Y$ to be true:

For every x , if $x \in X$, then $x \in Y$
For every x , if $x \in Y$, then $x \in X$

Here are two examples that demonstrate *equality* among sets:

If

$$A = \{1, 3, 2\} \text{ and } B = \{2, 3, 2, 1\},$$

then, by inspection, A and B have the same elements. Therefore $A = B$.

Remember: The elements in a set are unique, so duplicates are removed when evaluating a set.

If

$$A = \{x \mid x^2 + x - 6 = 0\} \text{ and } B = \{2, -3\},$$

then, $A = B$ in this case, too.

Set Inequality

For a set X to NOT be equal to a set Y ($X \neq Y$), X and Y must NOT have the same elements. In other words, there must be at least one element in X that is not in Y or at least one element in Y that is not in X (or both).

Here is an example that demonstrates *inequality* among sets:

If

$$A = \{1, 3, 2\} \text{ and } B = \{4, 2\},$$

Then, by inspection, $A \neq B$.

Subsets

Suppose X and Y are sets. If every element of X is an element of Y , we say X is a **subset** of Y and write $X \subseteq Y$. In other words,

If

X and Y are sets and, for every x , $x \in X$ and $x \in Y$.

Then, $X \subseteq Y$. Here are some examples demonstrating subsets:

If

$$C = \{1, 3\} \text{ and } A = \{1, 2, 3, 4\},$$

then, every element of C is an element of A . Therefore, $C \subseteq A$.

Let

$$X = \{x \mid x^2 + x - 2 = 0\}$$

We can show that $X \subseteq \mathbf{Z}$:

Remember, \mathbf{Z} is a set of integers, so

$$\mathbf{Z} = \{x \mid x \text{ is an integer}\}.$$

We can solve for the subset X

$$\begin{aligned}x^2 + x - 2 &= 0 \\(x + 2)(x - 1) &= 0\end{aligned}$$

which gives $x = -2$ and $x = 1$. So $X = \{-2, 1\}$. Since every element of set X is an element of set \mathbf{Z} , $X \subseteq \mathbf{Z}$.

For a set X to NOT be a subset of a set Y , there must be at least one element of X that is NOT a member of Y .

Let

$$X = \{x \mid 3x^2 - x - 2 = 0\}$$

We can show that X is NOT a subset of \mathbf{Z} :

If $x \in X$, then

$$3x^2 - x - 2 = 0.$$

Solving for x , we obtain $x = 1$ and $x = -\frac{2}{3}$, so $X = \{1, -\frac{2}{3}\}$. Since $-\frac{2}{3} \notin \mathbf{Z}$, X is NOT a subset of \mathbf{Z} .

Given a set X , $X \subseteq X$, since every element of X is an element of itself.

Proper Subsets

If X is a subset of Y and $X \neq Y$, then X is a **proper subset** of Y and we write $X \subset Y$. If $X \subset Y$, then X is ALWAYS smaller than Y .

Let

$$C = \{1, 3\} \text{ and } A = \{1, 2, 3, 4\},$$

Then $C \subset A$ since $C \neq A$.

Understanding subsets versus proper subsets:

- The symbol for a subset (\subseteq) is analogous to \leq . In other words, a subset *can* be the same size as the parent set.
- The symbol for a proper subset (\subset) is analogous to $<$. In other words, a proper subset is smaller than the parent set.

Power Set

The set of all subsets (proper or not) of a set X , denoted $\mathcal{P}(X)$, is called the **power set** of X .

If $A = \{a, b, c\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

All but $\{a, b, c\}$ are proper subsets of A . $|A| = 3$ and $|\mathcal{P}(A)| = 2^3 = 8$.

In other words, given a set X with n elements, $|\mathcal{P}(X)| = 2^n$.

Given two sets X and Y , there are several operations that can be performed on the sets to produce a new set.

Union, Intersection, and Difference

The **union** of X and Y ,

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\},$$

is a set that consists of all elements belonging to X or Y (or both).

The **intersection** of X and Y ,

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\},$$

is a set that consists of all elements belonging to X and Y .

The **difference** of X and Y ,

$$X - Y = \{x \mid x \in X \text{ and } x \notin Y\},$$

is a set that consists of all elements in X that are not in Y .

If

$$A = \{1, 3, 5\} \text{ and } B = \{4, 5, 6\}$$

then,

$$A \cup B = \{1, 3, 4, 5, 6\}$$

$$A \cap B = \{5\}$$

$$A - B = \{1, 3\}$$

$$B - A = \{4, 6\}$$

In general, $A - B \neq B - A$.

Union of a Family of Sets

Just like how we took the union of two sets above, we can take the union of a family of sets \mathcal{S} .

We define the union of a family \mathcal{S} of sets to be those elements x belonging to at least one set X in the family \mathcal{S} . In other words,

$$\bigcup \mathcal{S} = \{x \mid x \in X \text{ for some } X \in \mathcal{S}\}.$$

We can calculate the union of \mathcal{S} like so:

$$\bigcup \mathcal{S} = \bigcup_{i=1}^n X_i$$

where X is some set in \mathcal{S} and n is the cardinality of \mathcal{S} .

Let

$$A_1 = \{1, 2, 6, 7, 9\}$$

$$A_2 = \{2, 5, 6, 7, 8, 9, 10\}$$

$$A_3 = \{1, 2, 3, 4, 9\}$$

$$\mathcal{S} = \{A_1, A_2, A_3\}$$

Then, the union of \mathcal{S} is

$$\bigcup \mathcal{S} = \bigcup_{i=1}^3 A_i = A_1 \cup A_2 \cup A_3 = \{1, 2, 3, \dots, 10\}.$$

Intersection of a Family of Sets

Just like how we took the intersection of two sets above, we can take the intersection of a family of sets \mathcal{S} .

We define the intersection of a family \mathcal{S} of sets to be those elements x belonging to at least one set X in the family \mathcal{S} . In other words,

$$\cap \mathcal{S} = \{x \mid x \in X \text{ for all } X \in \mathcal{S}\}.$$

We can calculate the intersection of \mathcal{S} like so:

$$\cap \mathcal{S} = \bigcap_{i=1}^n X_i$$

where X is some set in \mathcal{S} and n is the cardinality of \mathcal{S} .

Let

$$A_1 = \{1, 2, 6, 7, 9\}$$

$$A_2 = \{2, 5, 6, 7, 8, 9, 10\}$$

$$A_3 = \{1, 2, 3, 4, 9\}$$

$$\mathcal{S} = \{A_1, A_2, A_3\}$$

Then, the intersection of \mathcal{S} is

$$\cap \mathcal{S} = \bigcap_{i=1}^3 A_i = A_1 \cap A_2 \cap A_3 = \{2, 9\}.$$

Disjoint Sets

Sets X and Y are **disjoint** if $X \cap Y = \emptyset$. In other words, if X and Y share no elements, they are disjoint.

Pairwise Disjoint

A collection of sets \mathcal{S} is said to be **pairwise disjoint** if every pair of sets within the set are disjoint.

Let

$$\mathcal{S} = \{A_1, A_2, A_3, \dots, A_n\}.$$

If

$$\text{For every } i \text{ and } j \text{ in } \mathcal{S}, A_i \cap A_j = \emptyset, \text{ where } i \neq j.$$

then, \mathcal{S} is a pairwise disjoint set.

For example, If

$$\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}.$$

then, by inspection, \mathcal{S} is pairwise disjoint because no set within \mathcal{S} contains common elements.

Universal Set

Every set is a subset of U , which is the universal set. The universal set must be explicitly defined or given from context.

Complement Set

A set $\overline{X} = U - X$ is the **complement** of X . In other words, a *complement* of a set X is the set that contains all elements except those in X .

Let

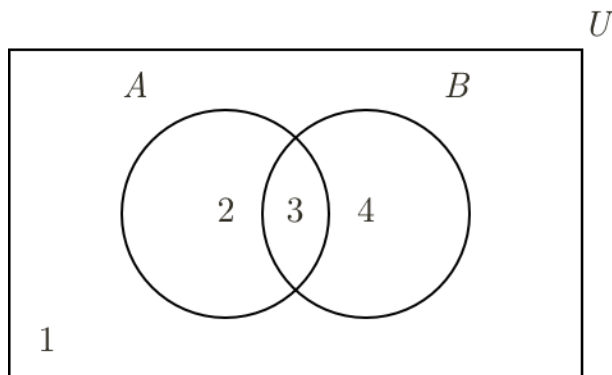
$$\begin{aligned} A &= \{1, 3, 5\} \\ U &= \{1, 2, 3, 4, 5\}. \end{aligned}$$

Then the complement of A is

$$\overline{A} = U - A = \{2, 4\}$$

Venn Diagrams

Venn Diagrams provide pictorial views of a set. In a Venn Diagram, a rectangle depicts a universal set. Subsets of the universal set are drawn as circles, and the members of a set are within the circle.



In the above diagram,

$$1 = \overline{(A \cup B)}$$

$$2 = A - B$$

$$3 = A \cap B$$

$$4 = B - A$$

Ordered Pairs

As previously stated, a set is an *unordered* collection of elements. However, sometimes we want to consider the order of elements. An **ordered pair** of elements, written (a, b) , is considered distinct from (b, a) so long as $a \neq b$.

Cartesian Product

If X and Y are sets, we let $X \times Y$ denote the set of all ordered pairs (x, y) , where $x \in X, y \in Y$. We call this set of ordered pairs a **Cartesian product**.

If $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, then

$$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$$Y \times X = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$$

Note, in general, $X \times Y \neq Y \times X$. Also note that $|X \times Y| = |X| \cdot |Y| = 6$. It is always true that $|X \times Y| = |X| \cdot |Y|$.

If $X = \{1, 2\}$ and $Y = \{a, b\}$, and $Z = \{\alpha, \beta\}$, then

$$X \times Y \times Z = \{(1, a, \alpha), (1, a, \beta), (1, b, \alpha), (1, b, \beta), (2, a, \alpha), (2, a, \beta), (2, b, \alpha), (2, b, \beta)\}$$

Set Laws

Let U be a universal set and sets A , B , and C be subsets of U . The following properties hold.

Associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Commutative laws:

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Identity laws:

$$A \cup \emptyset = A, A \cap U = A$$

Complement laws:

$$A \cup \overline{A} = U, A \cap \overline{A} = \emptyset$$

Idempotent laws:

$$A \cup A = A, A \cap A = A$$

Bound laws:

$$A \cup U = U, A \cap \emptyset = \emptyset$$

Absorption laws:

$$A \cup (A \cap B) = A, A \cap (A \cup B) = A$$

Involution law:

$$\overline{\overline{A}} = A$$

0/1 laws:

$$\overline{\emptyset} = U$$

$$\overline{U} = \emptyset$$

De Morgan's laws for sets

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$