CS 309: Discrete Math (Notes)

Matthew Kosloski

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Chapter 1: Sets and Logic

1.1 Sets

Denoting Sets

A **set** is simply a collection of objects, or elements.
If a set is finite and not large, we can describe it by simply listing the elements:

$$A = \{1, 2, 3, 4\}$$

The above set A is made up of four elements. **The order of the elements in a set does not matter**, therefore, a could also be specified like so:

$$A = \{1, 3, 4, 2\}$$

The elements of a set are assumed to be distinct, so any duplicate occurence of an element can be ignored. Therefore, we could also specify set A like so:

$$A = \{1, 2, 2, 3, 4, 4\}$$

If a set is very large or infinite, we can describe it using a property necessary for membership:

$$B = \{x \mid x \text{ is a positive, even integer}\}$$

The above set B is made up of positive, even integers. The vertical bar "|" is read as "such that" and the text after the bar is the property. Therefore, B can be read as "the set of all x such that x is a positive, even integer." Some sets of numbers occur frequently in mathematics and are given symbols.

Symbol	Set	Example of Members
Z	Integers	-3, 0, 2, 145
Q	Rational numbers	-1/3, 0, 24/15
R	Real numbers	-3, -1.766, 0, 4/15, $\sqrt{2}$, 2.666, $\ldots$, π

Rational numbers are quotients of integers, thus **Q** for *quotient*. The set of real numbers **R** consists of all points on a straight line extending indefinitely in either direction.

We can denote the positive elements in a set using the superscript plus (e.g., $\mathbf{Z}^+$ for positive integers) and the negative elements in a set using the superscript minus (e.g., $\mathbf{Q}^-$ for negative rational numbers).

Set Cardinality

If X is a finite set, we let

$$|X| = \text{number of elements in } X$$

We call $|X|$ the **cardinality** of X.

If we let $A = \{1, 2, 3, 4\}$, then the cardinality of A is 4, or $|A| = 4$. The cardinality of $\{\mathbf{R}, \mathbf{Z}\}$ is 2 since it contains two elements, which just happen to be sets.

Remember: an element in a set can be anything, even a set.

If x is in the set X, we write $x \in X$. If x is NOT in the set X, we write $x \notin X$. For example, both of these are true:

$$3 \in \{1, 2, 3, 4\}$$
$$3 \notin \{x \mid x \text{ is a positive, even integer}\}$$

Empty Set

A set with no elements is called an **empty set** and is denoted by $\varnothing$. In other words, $\varnothing = \{\}$.

Set Equality

Two sets X and Y are **equal** $(X = Y)$ if X and Y have the same elements. To put it differently, for $X = Y$ to be true:

$$\text{For every } x\text{, if } x \in X\text{, then } x \in Y$$
$$\text{For every } x\text{, if } x \in Y\text{, then } x \in X$$

Here are two examples that demonstrate *equality* among sets:

If

$$A = \{1, 3, 2\} \text{ and } B = \{2, 3, 2, 1\},$$

then, by inspection, A and B have the same elements. Therefore $A = B$.

Remember: The elements in a set are unique, so duplicates are removed when evaluating a set.

If

$$A = \{x \mid x^2 + x - 6 = 0\} \text{ and } B = \{2, \text{-}3\},$$

then, $A = B$ in this case, too.

Set Inequality

For a set X to NOT be equal to a set Y $(X \neq Y)$, X and Y must NOT have the same elements. In other words, there must be at least one element in X that is not in Y or at least one element in Y that is not in X (or both).

Here is an example that demonstrates *inequality* among sets:

If

$$A = \{1, 3, 2\} \text{ and } B = \{4, 2\},$$

Then, by inspection, $A \neq B$.

Subsets

Suppose X and Y are sets. If every element of X is an element of Y, we say X is a **subset** of Y and write $X \subseteq Y$. In other words,

If

$$X \text{ and } Y \text{ are sets and, for every x, } x \in X \text{ and } x \in Y.$$

Then, $X \subseteq Y$. Here are some examples demonstrating subsets:

If

$$C = \{1, 3\} \text{ and } A = \{1, 2, 3, 4\},$$

then, every element of C is an element of A. Therefore, $C \subseteq A$.

Let

$$X = \{x \mid x^2 + x - 2 = 0\}$$

We can show that $X \subseteq \mathbf{Z}$:

Remember, $\mathbf{Z}$ is a set of integers, so

$$\mathbf{Z} = \{x \mid x \text{ is an integer}\}.$$

We can solve for the subset X

$$\begin{aligned} x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \end{aligned}$$

which gives $x = -2$ and $x = 1$. So $X = \{\text{-2, 1}\}$. Since every element of set X is an element of set $\mathbf{Z}$, $X \subseteq \mathbf{Z}$.

For a set X to NOT be a subset of a set Y, there must be at least one element of X that is NOT a member of Y.

Let

$$X = \{x \mid 3x^2 - x - 2 = 0\}$$

We can show that X is NOT a subset of $\mathbf{Z}$:

If $x \in X$, then

$$3x^2 - x - 2 = 0.$$

Solving for x, we obtain $x = 1$ and $x = -\frac{2}{3}$, so $X = \{1, -\frac{2}{3}\}$. Since $-\frac{2}{3} \notin \mathbf{Z}$, X is NOT a subset of $\mathbf{Z}$.

Given a set X, $X \subseteq X$, since every element of X is an element of itself.

Proper Subsets

If X is a subset of Y and $X \neq Y$, then X is a **proper subset** of Y and we write $X \subset Y$. If $X \subset Y$, then X is ALWAYS smaller than Y.

Let

$$C = \{1, 3\} \text{ and } A = \{1, 2, 3, 4\},$$

Then $C \subset A$ since $C \neq A$.

Understanding subsets versus proper subsets:

- The symbol for a subset ($\subseteq$) is analogous to $\leq$. In other words, a subset *can* be the same size as the parent set.

- The symbol for a proper subset ($\subset$) is analogous to $<$. In other words, a proper subset is smaller than the parent set.

Power Set

The set of all subsets (proper or not) of a set X, denoted $\mathcal{P}(X)$, is called the **power set** of X.

If A = {a, b, c}, then

$$\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}.$$

All but {a, b, c} are proper subsets of A. $|A| = 3$ and $|\mathcal{P}(A)| = 2^3 = 8$.

In other words, given a set X with n elements, $|\mathcal{P}(X)| = 2^n$.

Given two sets X and Y, there are several operations that can be performed on the sets to produce a new set.

Union, Intersection, and Difference

The **union** of X and Y,

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\},$$

is a set that consists of all elements belonging to X or Y (or both).

The **intersection** of X and Y,

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\},$$

is a set that consists of all elements belonging to X and Y.

The **difference** of X and Y,

$$X - Y = \{x \mid x \in X \text{ and } x \notin Y\},$$

is a set that consists of all elements in X that are not in Y.

If

$$\text{A} = \{1,\ 3,\ 5\} \text{ and } \text{B} = \{4,\ 5,\ 6\}$$

then,

$$
\begin{aligned}
A \cup B &= \{1,3,4,5,6\} \\
A \cap B &= \{5\} \\
A - B &= \{1,3\} \\
B - A &= \{4,6\}
\end{aligned}
$$

In general, A - B $\neq$ B - A.

Union of a Family of Sets

Just like how we took the union of two sets above, we can take the union of a family of sets $\mathcal{S}$.

We define the union of a family $\mathcal{S}$ of sets to be those elements x belonging to at least one set X in the family $\mathcal{S}$. In other words,

$$\cup\, \mathcal{S} = \{x \mid x \in X \text{ for some } X \in \mathcal{S}\}.$$

We can calculate the union of $\mathcal{S}$ like so:

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{n} X_i$$

where X is some set in $\mathcal{S}$ and n is the cardinality of $\mathcal{S}$.

Let

$$
\begin{aligned}
A_1 &= \{1,2,6,7,9\} \\
A_2 &= \{2,5,6,7,8,9,10\} \\
A_3 &= \{1,2,3,4,9\} \\
\mathcal{S} &= \{A_1, A_2, A_3\}
\end{aligned}
$$

Then, the union of $\mathcal{S}$ is

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{3} A_i = A_1 \cup A_2 \cup A_3 = \{1,2,3,\ldots,10\}.$$

Intersection of a Family of Sets

Just like how we took the intersection of two sets above, we can take the intersection of a family of sets $\mathcal{S}$.

We define the intersection of a family $\mathcal{S}$ of sets to be those elements x belonging to at least one set X in the family $\mathcal{S}$. In other words,

$$\cap\, \mathcal{S} = \{x \mid x \in X \text{ for all } X \in \mathcal{S}\}.$$

We can calculate the intersection of $\mathcal{S}$ like so:

$$\bigcap \mathcal{S} = \bigcap_{i=1}^{n} X_i$$

where X is some set in $\mathcal{S}$ and n is the cardinality of $\mathcal{S}$.

Let

$$\begin{aligned} A_1 &= \{1, 2, 6, 7, 9\} \\ A_2 &= \{2, 5, 6, 7, 8, 9, 10\} \\ A_3 &= \{1, 2, 3, 4, 9\} \\ \mathcal{S} &= \{A_1, A_2, A_3\} \end{aligned}$$

Then, the intersection of $\mathcal{S}$ is

$$\bigcap \mathcal{S} = \bigcap_{i=1}^{3} A_i = A_1 \cap A_2 \cap A_3 = \{2, 9\}.$$

Disjoint Sets

Sets X and Y are **disjoint** if $X \cap Y = \varnothing$. In other words, if X and Y share no elements, they are disjoint.

Pairwise Disjoint

A collection of sets $\mathcal{S}$ is said to be **pairwise disjoint** if every pair of sets within the set are disjoint.

Let

$$\mathcal{S} = \{A_1,\, A_2,\, A_3,\, \ldots,\, A_n\}.$$

If

$$\text{For every } i \text{ and } j \text{ in } \mathcal{S},\, A_i \cap A_j = \varnothing, \text{ where } i \neq j.$$

then, $\mathcal{S}$ is a pairwise disjoint set.

For example, If

$$\mathcal{S} = \{\{1,\, 4,\, 5\},\, \{2,\, 6\},\, \{3\},\, \{7,\, 8\}\}.$$

then, by inspection, $\mathcal{S}$ is pairwise disjoint because no set within $\mathcal{S}$ contains common elements.

Universal Set

Every set is a subset of U, which is the universal set. The universal set must be explicitly defined or given from context.

Complement Set

A set $\overline{X} = U - X$ is the **complement** of X. In other words, a *complement* of a set X is the set that contains all elements except those in X.

Let

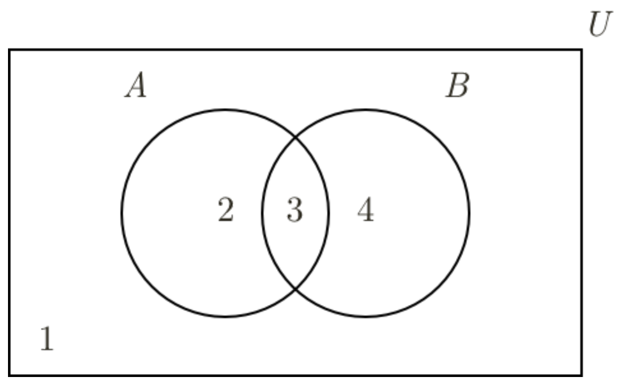
$$\begin{aligned} A &= \{1, 3, 5\} \\ U &= \{1, 2, 3, 4, 5\}. \end{aligned}$$

Then the complement of A is

$$\overline{A} = U - A = \{2, 4\}$$

Venn Diagrams

Venn Diagrams provide pictorial views of a set. In a Venn Diagram, a rectangle depicts a universal set. Subsets of the universal set are drawn as circles, and the members of a set are within the circle.



In the above diagram,

$$
\begin{aligned}
1 &= \overline{(A \cup B)} \\
2 &= A - B \\
3 &= A \cap B \\
4 &= B - A
\end{aligned}
$$

Ordered Pairs

As previously stated, a set is an *unordered* collection of elements. However, sometimes we want to consider the order of elements. An **ordered pair** of elements, written (a, b), is considered distinct from (b, a) so long as $a \neq b$.

Cartesian Product

If X and Y are sets, we let $X \times Y$ denote the set of all ordered pairs (x, y), where $x \in X, y \in Y$. We call this set of ordered pairs a **Cartesian product**.

If $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, then

$$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$
$$Y \times X = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$$

Note, in general, $X \times Y \neq Y \times X$. Also note that $|X \times Y| = |X| \cdot |Y| = 6$. It is always true that $|X \times Y| = |X| \cdot |Y|$.

If $X = \{1, 2\}$ and $Y = \{a, b\}$, and $Z = \{\alpha, \beta\}$, then

$$X \times Y \times Z = \{(1, a, \alpha), (1, a, \beta), (1, b, \alpha), (1, b, \beta), (2, a, \alpha), (2, a, \beta), (2, b, \alpha), (2, b, \beta)\}$$

Set Laws

Let U be a universal set and sets A, B, and C be subsets of U. The following properties hold.

Associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Commutative laws:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cup (A \cup C)$$

Identity laws:

$$A \cup \varnothing = A, A \cap U = A$$

Complement laws:

$$A \cup \overline{A} = U, A \cap \overline{A} = \varnothing$$

Idempotent laws:

$$A \cup A = A, A \cap A = A$$

Bound laws:

$$A \cup U = U, A \cap \varnothing = \varnothing$$

Absorption laws:

$$A \cup (A \cap B) = A, A \cap (A \cup B) = A$$

Involution law:

$$\overline{\overline{A}} = A$$

0/1 laws:

$$\overline{\varnothing} = U$$
$$\overline{U} = \varnothing$$

De Morgan's laws for sets

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$
$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

1.2 Propositions

A sentence that is either true or false, but not both, is called a **proposition.**

The following are examples of propositions:

(a) There are 200 bones in the human body.

(b) Earth is the only planet in the universe that contains life.

(c) The only positive integers that divide 7 are 1 and 7 itself.

The following are *not* propositions:

(i) $x + 4 = 6$.

(ii) Fetch me a stack of papers, please.

(i) is *not* a proposition because the truth value of the equation is predicated on the value of x. (ii) is *not* a proposition because it is neither true nor false, rather a command.

The variables p, q, and r are conventionally used to represent propositions. To define a variable, such as p, to be a proposition, use the following notation:

$$p\colon\ 1 + 1 = 3$$

In everyday language, we combine propositions, such as "It is raining" and "It is cold", with connectives, such as *and* and *or*, to form a single proposition, such as "It is raining and it is cold."

Conjunction

The **conjunction** of p and q, denoted $p \wedge q$, is the proposition of p and q. If

$$p : \text{It is raining,}$$
$$q : \text{It is cold,}$$

then, the conjunction of p and q is

$$p \wedge q\text{: It is raining and it is cold.}$$

The truth values of propositions can be illustrated using **truth tables**. The amount of possible combinations of truth values is 2^n, where n is the amount of propositions.

Here is the truth table of the proposition $p \wedge q$:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

The **disjunction** of p and q, denoted $p \vee q$, is the proposition of p or q. If

$$p : \text{It is spherical,}$$
$$q : \text{It is yellow,}$$

then, the disjunction of p and q is

$$p \vee q\text{: It is spherical or it is yellow.}$$

Here is the truth table of the proposition $p \vee q$, called the *inclusive-or* of p and q:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

In ordinary language, propositions being combined are normally related; but in logic, these propositions are not required to refer to the same subject matter. For example, this proposition is permitted:

$$3 < 5 \text{ or Paris is the capital of England.}$$

Remember: Logic is concerned with the form of propositions and the relation of propositions to each other and not with the subject matter.

Negation

The **negation** of p, denoted $\neg p$, is the proposition not p. If

$$p\text{: Paris is the capital of England,}$$

then, negation of p could be written as one of the following:

$$\neg p : \text{It is not the case that Paris is the capital of England}$$
$$\neg p : \text{Paris is not the capital of England}$$

The truth table of the proposition $\neg p$ is the following:

p	$\neg p$
T	F
F	T

Operator Precedence

In the absence of parentheses, we first evaluate $\neg$, then $\wedge$, and then $\vee$.

For example, consider the following proposition:

$$\neg p \vee q \wedge r$$

We can evaluate the above proposition using the following truth table:

p	q	r	$\neg p$	$q \wedge r$	$\neg p \vee q \wedge r$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	F	T
F	F	F	T	F	T

From this truth table, it is clear that $\neg p \vee q \wedge r$ can be true in 5 cases and false in 3 cases.

1.3 Conditional Propositions and Logical Equivalence

Conditional Proposition

Consider the following proposition:

If it is raining outside, then I will bring an umbrella.

The above proposition is called a **conditional proposition**, and it states that on the condition that it is raining outside, then I will bring an umbrella.

If we let

p : It is raining outside,
q : I will bring an umbrella,

we can denote the conditional proposition as

$$p \to q.$$

The above can be pronounced as "if p then q" or "p implies q." The proposition p is called the **hypothesis** or **sufficient condition**, and the proposition q is called the **conclusion** or **necessary condition**.

How do you determine the truth value of a conditional proposition, such as the one above? Suppose I say,

If I buy a car, then I will let you drive it.

If I end up buying a car and letting you drive it, then the statement is *true*. However, if I do buy the car and do *not* let you drive it, then the statement is *false*. If I do *not* buy a car, the statement is still true (there is no car for you to drive, but there may be one in the future).

The following table illustrates the truth value of $p \to q$:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

From this, it is clear that a conditional proposition is only *false* when the hypothesis is *true* and the conclusion is *false*.

True by Default

To justify how a conditional proposition is always *true* when p is *false*, consider the following proposition:

For all real numbers x, if $x > 0$, then $x^2 > 0$

If we let

$$P(x) : x > 0,$$
$$Q(x) : x^2 > 0$$

Then we can denote the proposition as

if *P(x)* then *Q(x)*.

If we let $x = -2$, then *P(-2)* is *false* and *Q(-2)* is *true*. If we let $x = 0$, then *P(0)* and *Q(0)* are both *false*. This is why we must define $p \to q$ to be *true* no matter what the truth value of p is. This is called **true by default**.

Operator Precedence

In conditional propositions that involve logical operators $\wedge, \vee, \neg, and\ \to$, the conditional operator $\to$ is evaluated last. Therefore, we now have the following order of precedence:

Operator	*Precedence*
$\neg$	1
$\wedge$	2
$\vee$	3
$\to$	4

Let p be *true*, q be *false*, and r be *true*. Evaluate

(a) $p \wedge q \to r$

(b) $p \vee q \to \neg r$

(c) $p \wedge (q \to r)$

(d) $p \to (q \to r)$

(a) We first evaluate $p \wedge q$, which is $false$, and then we evaluate $p \wedge q \to r$, which is $true$.

(b) We first evaluate $\neg r$, which is $false$, then we evaluate $p \vee q$, which is $true$, and finally we evalute the entire proposition $p \vee q \to \neg r$, which is $false$.

(c) We first evaluate $(q \to r)$, which is $true$, and then evaluate $p \wedge (q \to r)$, which is $true$.

(d) We first evaluate $(q \to r)$, which is $true$, and then we evaluate $p \to (q \to r)$, which is $true$.

Rewriting Propositions as Conditional Propositions

For each proposition, rewrite it as a conditional proposition in the form $p \to q$:

(a) Mary will be a good student if she studies hard.

(b) John takes calculus only if he has sophomore, junior, or senior standing.

(c) When you sing, my ears hurt.

(d) A necessary condition for the Cubs to win the World Series is that they sign a right-handed relief pitcher.

(e) A sufficient condition for Maria to visit France is that she goes to the Eiffel Tower.

(a) - (e) can be rewritten as

(a) If Mary studies hard, then she will be a good student.

(b) "p only if q" is the same as "if p then q", therefore, the proposition can be rewritten as:

If John takes calculus, then he has sophomore, junior, or senior standing.

(c) *When* is the same as *if*; thus the proposition is rewritten as

If you sing, then my ears hurt.

(d) A **necessary condition** is a condition that is necessary for an outcome but does not guarantee the outcome, therefore, we can rewrite it as

If the Cubs win the World Series, then they signed a right-handed relief pitcher.

(e) A **sufficient condition** is a condition that, when met, guarantees an outcome; however, if it is *not* met, the outcome is still possible. We can rewrite this proposition as

If Maria goes to the Eiffel tower, then she visits France.

Converse

The **converse** of $p \to q$ is $q \to p$.

Let

$$p : 1 > 2,$$
$$q : 4 < 8.$$

Since p is false and q is true, $p \to q$ is *true*. However, its converse, $q \to p$ is *false*. Thus, a conditional proposition can be *true* while its converse is *false*.

For example, if we have the following conditional proposition $p \to q$, we can write its converse symbolically and in words.

If Jerry receives a scholarship, then he will go to college.

Let

p : Jerry receives a scholarship,

q : Jerry goes to college.

The converse is symbolically expressed as $q \to p$, which can be written in words as

If Jerry goes to college, then he receives a scholarship.

Now, if Jerry does *not* receive a scholarship but goes to college anyway, find the truth values of (a) and (b).

(a) $p \to q$

(b) $q \to p$

(a) Since Jerry did *not* receive a scholarship, p is *false*, but he is still going to college, so q is *true*. Therefore, $p \to q$ is *true*.

(b) Since Jerry is going to college, q is *true*. However, he did *not* receive a scholarship, so p is *false*. Therefore, $q \to p$ is *false*.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Biconditional Proposition

A **biconditional proposition**, expressed as $p \leftrightarrow q$ or "p if and only if q", is true when p and q have the same truth values. Thus, the truth value of $p \leftrightarrow q$ is defined by the following truth table:

In a biconditional proposition, p is both necessary and sufficient for q.

Logical Equivalence

Propositions are said to be **logically equivalent** if they have the same truth value, regardless of the truth values of their constituent propositions $p_1, \ldots, p_n$. If P and Q are made up of the propositions $p_1, \ldots, p_n$, we say P and Q are logically equivalent and write

$$P \equiv Q$$

provided that, given any truth values of $p_1, \ldots, p_n$, P and Q are both *true* or *false*.

For example, we can show that the negation of $p \to q$ is logically equivalent to $p \wedge \neg q$. That is,

$$\neg(p \to q) \equiv p \wedge \neg q.$$

p	q	$p \to q$	$\neg(p \to q)$	$\neg q$	$p \wedge \neg q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

The above demonstrates that $\neg(p \to q)$ is logically equivalent to $p \wedge \neg q$ regardless of the truth values of p and q.

We can use the logical equivalence of $\neg(p \to q)$ and $p \wedge \neg q$ to help us write the negation of conditional propositions. For example, to negate

if Jerry receives a scholarship, then he goes to college,

we let

$$p : \text{Jerry receives a scholarship,}$$
$$q : \text{Jerry goes to college.}$$

The above proposition can be symbolically written as $p \to q$, and its negation is $\neg(p \to q)$. Since $\neg(p \to q)$ is logically equivalent to $p \wedge \neg q$, we can negate the proposition by expressing $p \wedge \neg q$ as words like so:

Jerry receives a scholarship and he does not go to college.

Remember: When evaluating conditional propositions, it is easier to work with the logical operators $\wedge$, $\vee$, and $\neg$ than the conditional operator $\to$.

Additionally, $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$. This is demonstrated by the following truth table:

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	F	T	F	F
F	F	T	T	T	T

De Morgan's Laws for Logic

De Morgan has the following two laws for logic:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q,$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

For the first law, we can demonstrate that $\neg(p \vee q)$ is logically equivalent to $\neg p \wedge \neg q$ using the following truth table:

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Additionally, For the second law, we can demonstrate that $\neg(p \wedge q)$ is logically equivalent to $\neg p \vee \neg q$ using the following truth table:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Contrapositive

The **contrapositive** of the conditional proposition $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.

For example, consider the following proposition (Assume the network is *not* down and Dale can access the Internet):

If the network is down, then Dale cannot access the Internet

Let

$$p : \text{The network is down,}$$
$$q : \text{Dale cannot access the Internet}$$

The given proposition written symbolically is

$$p \rightarrow q.$$

Since the network is *not* down, the hypothesis p is *false*, therefore, the proposition is *true*. The contrapositive can be written symbolically as

$$\neg q \rightarrow \neg p$$

and in words

If Dale can access the Internet, then the network is not down.

Since the hypothesis $\neg q$ is *true* and the conclusion $\neg p$ is *true*, the contrapositive is *true*.

Thus, a conditional proposition and its contrapositive are logically equivalent, as demonstrated in this truth table:

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T