CS 309: Discrete Math (Notes)

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Chapter 1: Sets and Logic

1.1 Sets

Denoting Sets

A **set** is simply a collection of objects, or elements.
If a set is finite and not large, we can describe it by simply listing the elements:

$$A = \{1, 2, 3, 4\}$$

The above set A is made up of four elements. **The order of the elements in a set does not matter**, therefore, a could also be specified like so:

$$A = \{1, 3, 4, 2\}$$

The elements of a set are assumed to be distinct, so any duplicate occurence of an element can be ignored. Therefore, we could also specify set A like so:

$$A = \{1, 2, 2, 3, 4, 4\}$$

If a set is very large or infinite, we can describe it using a property necessary for membership:

$$B = \{x \mid x \text{ is a positive, even integer}\}$$

The above set B is made up of positive, even integers. The vertical bar "|" is read as "such that" and the text after the bar is the property. Therefore, B can be read as "the set of all x such that x is a positive, even integer." Some sets of numbers occur frequently in mathematics and are given symbols.

Symbol	Set	Example of Members
Z	Integers	-3, 0, 2, 145
Q	Rational numbers	-1/3, 0, 24/15
R	Real numbers	-3, -1.766, 0, 4/15, $\sqrt{2}$, 2.666, $\ldots$, π

Rational numbers are quotients of integers, thus **Q** for *quotient*. The set of real numbers **R** consists of all points on a straight line extending indefinitely in either direction.

We can denote the positive elements in a set using the superscript plus (e.g., $\mathbf{Z}^+$ for positive integers) and the negative elements in a set using the superscript minus (e.g., $\mathbf{Q}^-$ for negative rational numbers).

Set Cardinality

If X is a finite set, we let

$$|X| = \text{number of elements in } X$$

We call $|X|$ the **cardinality** of X.

If we let $A = \{1, 2, 3, 4\}$, then the cardinality of A is 4, or $|A| = 4$. The cardinality of $\{\mathbf{R}, \mathbf{Z}\}$ is 2 since it contains two elements, which just happen to be sets.

Remember: an element in a set can be anything, even a set.

If x is in the set X, we write $x \in X$. If x is NOT in the set X, we write $x \notin X$. For example, both of these are true:

$$3 \in \{1, 2, 3, 4\}$$
$$3 \notin \{x \mid x \text{ is a positive, even integer}\}$$

Empty Set

A set with no elements is called an **empty set** and is denoted by $\varnothing$. In other words, $\varnothing = \{\}$.

Set Equality

Two sets X and Y are **equal** $(X = Y)$ if X and Y have the same elements. To put it differently, for $X = Y$ to be true:

For every x, if $x \in X$, then $x \in Y$
For every x, if $x \in Y$, then $x \in X$

Here are two examples that demonstrate *equality* among sets:

If

$$A = \{1, 3, 2\} \text{ and } B = \{2, 3, 2, 1\},$$

then, by inspection, A and B have the same elements. Therefore $A = B$.

Remember: The elements in a set are unique, so duplicates are removed when evaluating a set.

If

$$A = \{x \mid x^2 + x - 6 = 0\} \text{ and } B = \{2, \text{-}3\},$$

then, $A = B$ in this case, too.

Set Inequality

For a set X to NOT be equal to a set Y $(X \neq Y)$, X and Y must NOT have the same elements. In other words, there must be at least one element in X that is not in Y or at least one element in Y that is not in X (or both).

Here is an example that demonstrates *inequality* among sets:

If

$$A = \{1, 3, 2\} \text{ and } B = \{4, 2\},$$

Then, by inspection, $A \neq B$.

Subsets

Suppose X and Y are sets. If every element of X is an element of Y, we say X is a **subset** of Y and write $X \subseteq Y$. In other words,

If

$$X \text{ and } Y \text{ are sets and, for every x, } x \in X \text{ and } x \in Y.$$

Then, $X \subseteq Y$. Here are some examples demonstrating subsets:

If

$$C = \{1, 3\} \text{ and } A = \{1, 2, 3, 4\},$$

then, every element of C is an element of A. Therefore, $C \subseteq A$.

Let

$$X = \{x \mid x^2 + x - 2 = 0\}$$

We can show that $X \subseteq \mathbf{Z}$:

Remember, $\mathbf{Z}$ is a set of integers, so

$$\mathbf{Z} = \{x \mid x \text{ is an integer}\}.$$

We can solve for the subset X

$$\begin{aligned} x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \end{aligned}$$

which gives $x = -2$ and $x = 1$. So X = {-2, 1}. Since every element of set X is an element of set $\mathbf{Z}$, $X \subseteq \mathbf{Z}$.

For a set X to NOT be a subset of a set Y, there must be at least one element of X that is NOT a member of Y.

Let

$$X = \{x \mid 3x^2 - x - 2 = 0\}$$

We can show that X is NOT a subset of $\mathbf{Z}$:

If $x \in X$, then

$$3x^2 - x - 2 = 0.$$

Solving for x, we obtain $x = 1$ and $x = -\frac{2}{3}$, so $X = \{1, -\frac{2}{3}\}$. Since $-\frac{2}{3} \notin \mathbf{Z}$, X is NOT a subset of $\mathbf{Z}$.

Given a set X, $X \subseteq X$, since every element of X is an element of itself.

Proper Subsets

If X is a subset of Y and $X \neq Y$, then X is a **proper subset** of Y and we write $X \subset Y$. If $X \subset Y$, then X is ALWAYS smaller than Y.

Let

$$C = \{1,\ 3\} \text{ and } A = \{1,\ 2,\ 3,\ 4\},$$

Then $C \subset A$ since $C \neq A$.

Understanding subsets versus proper subsets:

- The symbol for a subset ($\subseteq$) is analogous to $\leq$. In other words, a subset *can* be the same size as the parent set.
- The symbol for a proper subset ($\subset$) is analogous to $<$. In other words, a proper subset is smaller than the parent set.

Power Set

The set of all subsets (proper or not) of a set X, denoted $\mathcal{P}(X)$, is called the **power set** of X.

If A = {a, b, c}, then

$$\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

All but {a, b, c} are proper subsets of A. $|A| = 3$ and $|\mathcal{P}(A)| = 2^3 = 8$.

In other words, given a set X with n elements, $|\mathcal{P}(X)| = 2^n$.

Given two sets X and Y, there are several operations that can be performed on the sets to produce a new set.

Union, Intersection, and Difference

The **union** of X and Y,

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\},$$

is a set that consists of all elements belonging to X or Y (or both).

The **intersection** of X and Y,

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\},$$

is a set that consists of all elements belonging to X and Y.

The **difference** of X and Y,

$$X - Y = \{x \mid x \in X \text{ and } x \notin Y\},$$

is a set that consists of all elements in X that are not in Y.

If

$$\text{A} = \{1, 3, 5\} \text{ and } \text{B} = \{4, 5, 6\}$$

then,

$$
\begin{aligned}
A \cup B &= \{1, 3, 4, 5, 6\} \\
A \cap B &= \{5\} \\
A - B &= \{1, 3\} \\
B - A &= \{4, 6\}
\end{aligned}
$$

In general, A - B $\neq$ B - A.

Union of a Family of Sets

Just like how we took the union of two sets above, we can take the union of a family of sets $\mathcal{S}$.

We define the union of a family $\mathcal{S}$ of sets to be those elements x belonging to at least one set X in the family $\mathcal{S}$. In other words,

$$\cup\, \mathcal{S} = \{x \mid x \in X \text{ for some } X \in \mathcal{S}\}.$$

We can calculate the union of $\mathcal{S}$ like so:

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{n} X_i$$

where X is some set in $\mathcal{S}$ and n is the cardinality of $\mathcal{S}$.

Let

$$
\begin{aligned}
A_1 &= \{1, 2, 6, 7, 9\} \\
A_2 &= \{2, 5, 6, 7, 8, 9, 10\} \\
A_3 &= \{1, 2, 3, 4, 9\} \\
\mathcal{S} &= \{A_1, A_2, A_3\}
\end{aligned}
$$

Then, the union of $\mathcal{S}$ is

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{3} A_i = A_1 \cup A_2 \cup A_3 = \{1, 2, 3, \ldots, 10\}.$$

Intersection of a Family of Sets

Just like how we took the intersection of two sets above, we can take the intersection of a family of sets $\mathcal{S}$.

We define the intersection of a family $\mathcal{S}$ of sets to be those elements x belonging to at least one set X in the family $\mathcal{S}$. In other words,

$$\cap\, \mathcal{S} = \{x \mid x \in X \text{ for all } X \in \mathcal{S}\}.$$

We can calculate the intersection of $\mathcal{S}$ like so:

$$\bigcap \mathcal{S} = \bigcap_{i=1}^{n} X_i$$

where X is some set in $\mathcal{S}$ and n is the cardinality of $\mathcal{S}$.

Let

$$\begin{aligned} A_1 &= \{1, 2, 6, 7, 9\} \\ A_2 &= \{2, 5, 6, 7, 8, 9, 10\} \\ A_3 &= \{1, 2, 3, 4, 9\} \\ \mathcal{S} &= \{A_1, A_2, A_3\} \end{aligned}$$

Then, the intersection of $\mathcal{S}$ is

$$\bigcap \mathcal{S} = \bigcap_{i=1}^{3} A_i = A_1 \cap A_2 \cap A_3 = \{2, 9\}.$$

Disjoint Sets

Sets X and Y are **disjoint** if $X \cap Y = \varnothing$. In other words, if X and Y share no elements, they are disjoint.

Pairwise Disjoint

A collection of sets $\mathcal{S}$ is said to be **pairwise disjoint** if every pair of sets within the set are disjoint.

Let

$$\mathcal{S} = \{A_1,\ A_2,\ A_3,\ \ldots,\ A_n\}.$$

If

$$\text{For every } i \text{ and } j \text{ in } \mathcal{S},\ A_i \cap A_j = \varnothing, \text{ where } i \neq j.$$

then, $\mathcal{S}$ is a pairwise disjoint set.

For example, If

$$\mathcal{S} = \{\{1,\ 4,\ 5\},\ \{2,\ 6\},\ \{3\},\ \{7,\ 8\}\}.$$

then, by inspection, $\mathcal{S}$ is pairwise disjoint because no set within $\mathcal{S}$ contains common elements.

Universal Set

Every set is a subset of U, which is the universal set. The universal set must be explicitly defined or given from context.

Complement Set

A set $\overline{X} = U - X$ is the **complement** of X. In other words, a *complement* of a set X is the set that contains all elements except those in X.

Let

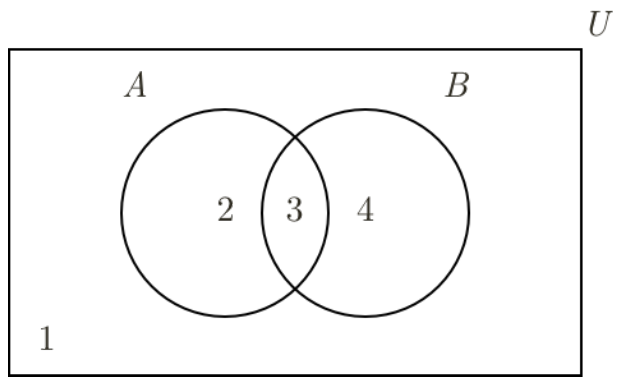
$$A = \{1, 3, 5\}$$
$$U = \{1, 2, 3, 4, 5\}.$$

Then the complement of A is

$$\overline{A} = U - A = \{2, 4\}$$

Venn Diagrams

Venn Diagrams provide pictorial views of a set. In a Venn Diagram, a rectangle depicts a universal set. Subsets of the universal set are drawn as circles, and the members of a set are within the circle.



In the above diagram,

$$
\begin{aligned}
1 &= \overline{(A \cup B)} \\
2 &= A - B \\
3 &= A \cap B \\
4 &= B - A
\end{aligned}
$$

Ordered Pairs

As previously stated, a set is an *unordered* collection of elements. However, sometimes we want to consider the order of elements. An **ordered pair** of elements, written (a, b), is considered distinct from (b, a) so long as $a \neq b$.

Cartesian Product

If X and Y are sets, we let $X \times Y$ denote the set of all ordered pairs (x, y), where $x \in X, y \in Y$. We call this set of ordered pairs a **Cartesian product**.

If $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, then

$$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$
$$Y \times X = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$$

Note, in general, $X \times Y \neq Y \times X$. Also note that $|X \times Y| = |X| \cdot |Y| = 6$. It is always true that $|X \times Y| = |X| \cdot |Y|$.

If $X = \{1, 2\}$ and $Y = \{a, b\}$, and $Z = \{\alpha, \beta\}$, then

$$X \times Y \times Z = \{(1, a, \alpha), (1, a, \beta), (1, b, \alpha), (1, b, \beta), (2, a, \alpha), (2, a, \beta), (2, b, \alpha), (2, b, \beta)\}$$

Set Laws

Let U be a universal set and sets A, B, and C be subsets of U. The following properties hold.

Associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

Commutative laws:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cup (A \cup C)$$

Identity laws:

$$A \cup \varnothing = A, A \cap U = A$$

Complement laws:

$$A \cup \overline{A} = U, A \cap \overline{A} = \varnothing$$

Idempotent laws:

$$A \cup A = A, A \cap A = A$$

Bound laws:

$$A \cup U = U, A \cap \varnothing = \varnothing$$

Absorption laws:

$$A \cup (A \cap B) = A, A \cap (A \cup B) = A$$

Involution law:

$$\overline{\overline{A}} = A$$

0/1 laws:

$$\overline{\varnothing} = U$$
$$\overline{U} = \varnothing$$

De Morgan's laws for sets

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$$
$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

1.2 Propositions

A sentence that is either true or false, but not both, is called a **proposition.**

The following are examples of propositions:

(a) There are 200 bones in the human body.

(b) Earth is the only planet in the universe that contains life.

(c) The only positive integers that divide 7 are 1 and 7 itself.

The following are *not* propositions:

(i) $x + 4 = 6$.

(ii) Fetch me a stack of papers, please.

(i) is *not* a proposition because the truth value of the equation is predicated on the value of x. (ii) is *not* a proposition because it is neither true nor false, rather a command.

The variables p, q, and r are conventionally used to represent propositions. To define a variable, such as p, to be a proposition, use the following notation:

$$p\colon\ 1 + 1 = 3$$

In everyday language, we combine propositions, such as "It is raining" and "It is cold", with connectives, such as *and* and *or*, to form a single proposition, such as "It is raining and it is cold."

Conjunction

The **conjunction** of p and q, denoted $p \wedge q$, is the proposition of p and q. If

$$\begin{aligned} p &: \text{It is raining,} \\ q &: \text{It is cold,} \end{aligned}$$

then, the conjunction of p and q is

$$p \wedge q\text{: It is raining and it is cold.}$$

The truth values of propositions can be illustrated using **truth tables**. The amount of possible combinations of truth values is 2^n, where n is the amount of propositions.

Here is the truth table of the proposition $p \wedge q$:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

The **disjunction** of p and q, denoted $p \vee q$, is the proposition of p or q. If

$$p : \text{It is spherical,}$$
$$q : \text{It is yellow,}$$

then, the disjunction of p and q is

$$p \vee q\text{: It is spherical or it is yellow.}$$

Here is the truth table of the proposition $p \vee q$, called the *inclusive-or* of p and q:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

In ordinary language, propositions being combined are normally related; but in logic, these propositions are not required to refer to the same subject matter. For example, this proposition is permitted:

$$3 < 5 \text{ or Paris is the capital of England.}$$

Remember: Logic is concerned with the form of propositions and the relation of propositions to each other and not with the subject matter.

Negation

The **negation** of p, denoted $\neg p$, is the proposition not p. If

p: Paris is the capital of England,

then, negation of p could be written as one of the following:

$\neg p$: It is not the case that Paris is the capital of England

$\neg p$: Paris is not the capital of England

The truth table of the proposition $\neg p$ is the following:

p	$\neg p$
T	F
F	T

Operator Precedence

In the absence of parentheses, we first evaluate $\neg$, then $\wedge$, and then $\vee$.

For example, consider the following proposition:

$$\neg p \vee q \wedge r$$

We can evaluate the above proposition using the following truth table:

p	q	r	$\neg p$	$q \wedge r$	$\neg p \vee q \wedge r$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	F	F	F
T	F	F	F	F	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	F	T
F	F	F	T	F	T

From this truth table, it is clear that $\neg p \vee q \wedge r$ can be true in 5 cases and false in 3 cases.

1.3 Conditional Propositions and Logical Equivalence