# VARIANCE BOUNDS FOR STATIONARY EXPONENTIAL LAST PASSAGE PERCOLATION

MATTHEW LYS

Abstract. In this paper we will review the Exponential Last Passage Percolation model and its stationary counterpart. We will review some of the basic machineries and see how to deduce the Law of Large Numbers for the last passage time from origin to $(N, N)$ in both stationary and non-stationary model. We shall then develop a formula for its variance. Finally, we will show that the variance of the stationary model is of order $N^{\frac{2}{3}}$, as opposed to $N$ that we usually encounter in Central limit theorem.

Contents



## 1. The model and its relatives

The first and last passage percolation models were introduced in the 1960s as models of fluid flow through a random medium. Informally, the model can be described as follows: let us consider a taxi that picks you up at one point and must drop you off at some fixed point. To model Cartesian geometry, we can imagine the road network resembles a rectangular grid. To drop you off, the taxi can choose many different paths, but at each intersection, he will have to wait at a stop light for some random time (see Figure 1). The fastest path the taxi could take would be the first-passage time, and the model is known as the first passage percolation (FPP) model in the literature. We refer to [ADH17] for a review.

In this paper, we will focus on the last passage percolation (LPP) model, which considers the longest path among the set of all up-right paths from pick up to drop off (see Figure 1). Although the geometric properties of LPP and FPP are thought to be similar, the FPP model is notoriously challenging, with many of its key statistical predictions still not fully understood. In contrast, the LPP model, under certain specific conditions, exhibits remarkable identities (as we will see in later sections) that enable us to rigorously determine its scaling exponents.

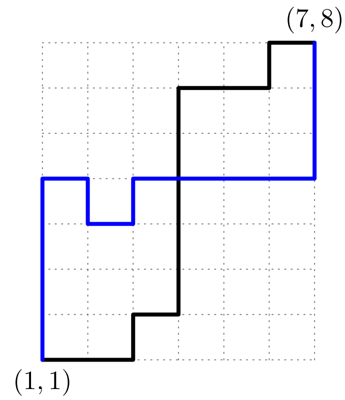


FIGURE 1. Two possible paths from the pickup point (1,1) to the drop off point (7,8). The black one is up-right, while the blue one is not.

To start building this model, consider the non-negative integer lattice $\mathbb{Z}_+^2$, where we place exponentially distributed weights at every point. For this model, let us denote these random variables as $(\omega_{i,j})_{i,j\geq 0}$, where $(i, j)$ refers to the point the weight is attached to. We will consider these weights to be exponential rate 1 variables, so the random variable $\omega \sim \exp(1)$ has the density function $f(x) = e^{-x}$.

We will look at the path from the origin to a point $(m, n)_{m,n\geq 0}$. As mentioned before we restrict ourselves to nearest neighbor up-right paths: $\{\pi \in \Pi_{m,n} | \pi : \pi_0 = (0, 0) \to \pi_1 \to \ldots \to \pi_{m+n} = (m, n)\}$ such that $\pi_{k+1} - \pi_k = (1, 0)$, or $(0, 1)$ for $0 \leq k < m + n$.

The Last passage time is then defined as:

$$\mathcal{G}_{m,n} = \max_{\pi\in\Pi_{m,n}} \sum_{(i,j)\in\pi} \omega_{i,j}. \tag{1.1}$$

This will be the longest path taken, it is known as the **Exponential LPP (ExpLPP)** model. The LPP model is related to various models in probability theory. We mention two of them here and refer the readers to [Sep09] for more details.

**Relation to Random Metric models.** LPP can be interpreted as a random metric. Consider any two points $(k, l), (m, n)$, with $k < m, l < n$. Switching the direction of the inequalities would lead us to consider paths not in the upright direction. Defining the LPP between these two points:

$$\mathcal{G}_{(k,l),(m,n)} = \max_{\pi\in\Pi_{(k,l),(m,n)}} \sum_{(i,j)\in\pi} \omega_{i,j}. \tag{1.2}$$

gives us the general definition of the LPP for any arbitrary points. This acts as a random metric in many ways, since both the path taken, and the value associated with this path are unique for these two points. Consider the last-passage path between $(k, l)$ and $(m, n)$. Picking a point $(p, q)$ in between these two points, such that $k < p < m, l < q < n$, then the sum of the LPP $(k, l)$ to $(p, q)$ and the LPP $(p, q)$ to $(m, n)$ is less than or equal to the original LPP from $(k, l)$ to $(m, n)$. This can be seen since the path through $(p, q)$ is only one of the many paths that can be taken to $(m,n)$, and this may not be the one path that picks up the most weight. This gives the LPP a superadditive structure, such that

$$\mathcal{G}_{(k,l),(m,n)} \geq \mathcal{G}_{(p,q),(m,n)} + \mathcal{G}_{(p,q),(m,n)}. \tag{1.3}$$

Thus the last passage value satisfies the reverse triangle inequality and can be seen as a random distance function (with a negative sign).

**Relations to Random growth model.** The LPP value enjoys the following recursion:

$$\mathcal{G}_{m,n} = \max\{\mathcal{G}_{m-1,n}, \mathcal{G}_{m,n-1}\} + \omega_{i,j}. \tag{1.4}$$

This is because paths in LPP cannot reach a point $(i,j)$ without first reaching its nearest southern and western neighbors. Then a time-dependent growing cluster of points or a random growth model develops from the LPP model.

Let $B(t)$ be called a time-dependent cluster, such that it includes all points that can be reached by an upright path before a time $t$.

$$B(t) = \{(m,n) \in \mathrm{N}^2 | G_{m,n} \leq t\} \tag{1.5}$$

For an example of how a cluster grows, first let $\omega_{2,2} < \omega_{2,1} < \omega_{1,2}$ then

- $t = \omega_{1,1} : B(t) = (1,1)$
- $t = \omega_{1,1} + \omega_{2,1} : B(t) = (1,1), (2,1)$
- $t = \omega_{1,1} + \omega_{1,2} + \omega_{2,2} : B(t) = (1,1), (2,1), (1,2), (2,2)$

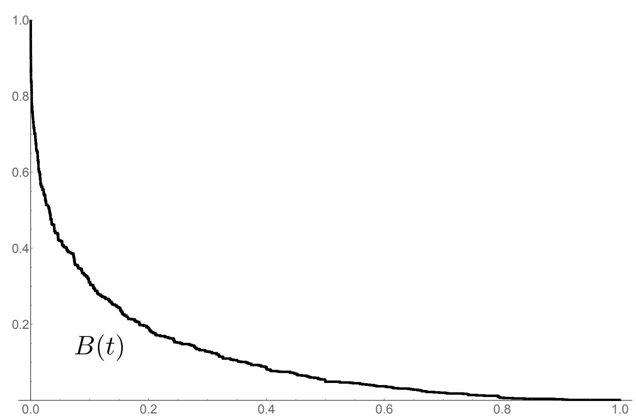


FIGURE 2. Large scale simulation of the growing cluster $B(t)$.

Figure 2 represents a large-scale simulation of the Exponential LPP after appropriate scaling. The growing cluster represents the region in the lower left corner bounded by the curve and the axes. The process $(B(t))_{t\geq 0}$ can be viewed as a random growth model. Indeed, $B(t)$ models a continuous-time Markov chain on a state space:

(1.6)

$$\Gamma = \{\mathcal{U} \subset \mathrm{N}^2 | \text{such that } \mathcal{U} \text{ is finite and } (i,j) \in \mathcal{U} \text{ means } \{1,...,i\} \times \{1,...,j\} \in \mathcal{U}\}.$$

## 2. The Stationary Last Passage Percolation

In this section, we consider the exponential LPP using slightly altered boundary weights

$$\omega_{0,0} = 0 \tag{2.1}$$
$$\omega_{i,0} \sim \exp(1-\varrho) \tag{2.2}$$
$$\omega_{0,j} \sim \exp(\varrho) \tag{2.3}$$
$$\omega_{i,j} \sim \exp(1) \tag{2.4}$$
$$\tag{2.5}$$

for $i, j \geq 1$, and $0 < \varrho < 1$. The Exponential LPP with the above weights enjoys certain stationary properties.

Towards this end, we first define a few variables.

**Definition 2.6.** The *horizontal* and *vertical* increments of last passage times are

$$I_{i,j} = \mathcal{G}_{i,j} - \mathcal{G}_{i-1,j}, \qquad J_{i,j} = \mathcal{G}_{i,j} - \mathcal{G}_{i,j-1}.$$

These are the times it would take to move one increment to the right in a row, or up in a column. These increments also satisfy the equality:

$$I_{i,j} = (I_{i,j-1} - J_{i-1,j})^+ + \omega_{i,j}, \qquad J_{i,j} = (J_{i-1,j} - J_{i,j-1})^+ + \omega_{i,j}.$$

The above equalities follow by algebra and hence their proof is omitted.

Let us also introduce the notation

$$X_{i-1,j-1} = min\{I_{i-1,j}, J_{i,j-1}\}.$$

The next theorem on a property of exponential distributions will help us develop the stationarity of our model.

**Theorem 2.7.** *Fix $0 < \varrho < 1$ and let $I_{i,j-1} \sim exp(1-\varrho), J_{i-1,j} \sim exp(\varrho), \omega_{i,j} \sim exp(1)$ be mutually independent, then $I_{i,j} \sim exp(1-\varrho), J_{i,j} \sim exp(\varrho), X_{i-1,j-1} \sim exp(1)$ and are also mutually independent.*

*Proof.* First, as $X_{i-1,j-1} = min\{I_{i,j-1}, J_{i-1,j}\}$, $P[X_{i-1,j-1} > t] = P[I_{i,j-1} > t, J_{i-1,j} > t] = P[I_{i,j-1} > t]P[J_{i-1,j} > t] = e^{-(1-\varrho)t} \cdot e^{-\varrho t} = e^{-t}$. So $X_{i-1,j-1} \sim \exp(1)$.

The distribution of $I_{i,j}, J_{i,j}$ can be proven with joint moment generating functions. To see the independence, note that $I_{i,j-1} - J_{i-1,j}$ depends on the first step, and the future path after $X_{i-1,j-1}$, so $X_{i-1,j-1}$ and $(I_{i,j-1} - J_{i-1,j})^+, (J_{i-1,j} - I_{i,j-1})^+$ are mutually independent.

Then, $[\omega_{i,j}, ((I_{i,j-1} - J_{i-1,j})^+, (J_{i-1,j} - I_{i,j-1})^+), X_{i-1,j-1}]$ and $[X_{i-1,j-1}, ((I_{i,j-1} - J_{i-1,j})^+, (J_{i-1,j} - I_{i,j-1})^+), \omega_{i,j}]$ are equal in distribution.

Then, $[I_{i,j-1}, J_{i-1,j}, \omega_{i,j}], [I_{i,j}, J_{i,j}, X_{i-1,j-1}]$ are also equal in distribution. □

Let $\mathcal{Y}$ be the set of sequences of doubly-infinite downright paths, indexed $(y_k), k \in$ N such that $(y_k) - (y_{k-1}) = (1, 0)$ or $(0, -1)$. We will restrict allowable sequences to $\mathrm{Z}_+^2$.

These paths will start on the j-axis and move down, maybe to the right, reach the i-axis, and then move to the right for infinity. Therefore this path can be imagined as a boundary line, that will draw an arbitrary separation between an "interior" and "exterior" of the first quadrant. This motivates the definitions

**Definition 2.8.** The Interior of the boundary, or the region to its south-west, will be

$$\mathcal{Y}_- = \{(i,j)|\exists k \in \mathrm{N} \text{ s.t. } (i,j) + (k,K) \in \mathcal{Y}\}, \tag{2.9}$$

and the Exterior of the boundary, or its north-east region will be

$$\mathcal{Y}_+ = \{(i,j)|\exists l \in \mathrm{N} \text{ s.t. } (i,j) - (l,l) \in \mathcal{Y}\}. \tag{2.10}$$

This definition follows the intuition that $\mathcal{Y}_- \cup \mathcal{Y} \cup \mathcal{Y}_+ = \mathrm{Z}_+^2$.

If we introduce this set of sequences on our stationary-expLPP model, we can imagine the terms of this sequence to be equal to horizontal and vertical increments of the last-passage-time.

**Definition 2.11.** We can let $\mathcal{Z}(y_k)$ represent the last-passage-time increments under,

$$\mathcal{Z}(y_k) = \begin{cases} \text{If } (y_k - y_{k-1}) = (1,0) \text{ then we have } \mathcal{G}_{y_k} - \mathcal{G}_{y_{k-1}} = I_{i_{y_k},j} \\ \text{If } (y_k - y_{k-1}) = (0,1) \text{ then we have } \mathcal{G}_{y_k} - \mathcal{G}_{y_{k-1}} = J_{i,j_{y_{k-1}}} \end{cases}. \tag{2.12}$$

Before we prove the stationarity of the expLPP, note with these definitions, the set $\mathcal{Y}$ can be the path that only follows the axes, in this case, $\mathcal{Y}_- = \emptyset$

**Theorem 2.13.** *For any $\mathcal{Y} = (y_k), k \in \mathrm{N}$ the random variables,*

$$\{X_{i,j}|(i,j) \in \mathcal{Y}_-\} \text{ and } \{\mathcal{Z}(y_k)\}, \tag{2.14}$$

*are mutually independent with $X_{i,j} \sim exp(1)$, $I_{i_{y_k},j} \sim exp(1-\varrho)$, $J_{i,j_{y_{k-1}}} \sim exp(\varrho)$.*

*Proof.* .
We will prove this theorem with induction on our down-right path and its random variables.
For the base case, we will consider when $\mathcal{Y}$ consists of the downright path along the axes. This means our $\mathcal{Y}_- = \emptyset$, so there exists no $X_{i,j}$, since $(i,j) \notin \mathcal{Y}_-$. Note also that every vertical and horizontal increment along the axes has the same distribution as the axes weights, $\omega_{0,j}, \omega i, 0$, which are independent with $\exp(\varrho)$ and $\exp(1-\varrho)$, respectively.

We now take any arbitrary down-right path, $\mathcal{Y}$, and assume this stationarity holds. Then we need to prove that this stationarity holds for the next increment of this path sequence. So take our path $(y_k)$, we will create a new path, $(\tilde{y}_k)$ that is the same path at $(y_k)$ except we will turn a southwest corner that appears in the sequence $(y_k)$ into a northeast corner in $(\tilde{y}_k)$ and prove the distribution remains unchanged. So we can write,

$$\{(\tilde{y}_k)_{k\in\mathrm{N}} = (y_k)_{k\in\mathrm{N}} | \forall k \in \mathrm{N}, except k = m, for m \in \mathrm{N}\}$$

Then when we consider $k = m$, the sequence, $(y_k)$ passes through the points, $(y_{m-1}), (y_m), (y_{m+1}) = (i, j+1), (i,j), (i+1, j)$, and the sequence $(\tilde{y}_k)$ passes through the points, $(\tilde{y}_{m-1}), (\tilde{y}_m), (\tilde{y}_{m+1}) = (i, j+1), (i+1, j+1), (i+1, j)$.

This altered path also alters the interior such that, $\tilde{\mathcal{Y}}_{-} = \mathcal{Y}_{-} \cup (i, j)$.

So for the original path $(y_k)$ we had the variables $[I_{i+1,j}, J_{i,j+1}]$ which by assumption are rate $\exp(1-\varrho)$, and $\exp(\varrho)$ and independent from the bulk weights. These variables have been replaced under the path with $[I_{i+1,j+1}, J_{i+1,j+1}]$, and $X_{i,j}$ has been added to the bulk. We need to make sure these weights, are mutually independent and have the specified rates. We know from our previous proofs, that if we show $[I_{i+1,j}, J_{i,j+1}, \omega_{i+1,j+1}]$ are mutually independent with $\exp(1-\varrho)$, $\exp(\varrho)$, $\exp(1)$ distributions, then so is $[I_{i+1,j+1}, J_{i+1,j+1}, X_{i,j}]$. We know that $\omega_{i+1,j+1} \sim \exp(1)$ by construction, and this weight is mutually independent of the increment times since it is not in either the interior or the path. So then the distribution holds for $\tilde{\mathcal{Y}}$ □

Now the corollary follows:

**Corollary 2.15.** *Let the weights,* $\omega_{i,j}, \tilde{\omega}_{i,j}$ *such that* $\omega_{0,0} = \tilde{\omega}_{0,0} = 0$*,* $\omega_{0,j} \geq \tilde{\omega}_{0,j}$*,* $\omega_{i,0} \leq \tilde{\omega}_{i,0}, \omega_{i,j} = \tilde{\omega}_{i,j}$ *then* $I_{i,j} \leq \tilde{I}_{i,j}$*, and* $J_{i,j} \geq \tilde{J}_{i,j}$

*Proof.* .
This proof will follow from induction.
First, consider a base case, the path $(y_k)$ that consists of the union of the two axes. In this case, every $I_{i,j}, \tilde{I}_{i,j}$, $J_{i,j}, \tilde{J}_{i,j}$ are just the sum of their respective axis weight. So considering the inequalities, $\omega_{0,j} \geq \tilde{\omega}_{0,j}$, $\omega_{i,0} \leq \tilde{\omega}_{i,0}$, it holds that $I_{i,j} \leq \tilde{I}_{i,j}$, and $J_{i,j} \geq \tilde{J}_{i,j}$.

Now, assume this holds for any path $(y_k)$. Then consider adding the growth point $(i+1, j+1)$ to this path, analogous to the previous theorem, to construct $\tilde{y}_k$. So then, the inequalities $I_{i,j-1} \leq \tilde{I}_{i,j-1}$, and $J_{i-1,j} \geq \tilde{J}_{i-1,j}$, hold, so $\omega_{i,j} = \tilde{\omega_{i,j}}$.

Finally, note

$$I_{i,j} = (I_{i,j-1} - J_{i-1,j})^+ + \omega_{i,j} = (I_{i,j-1} - J_{i-1,j})^+ + \tilde{\omega}_{i,j} \leq (\tilde{I}_{i,j-1} - \tilde{J}_{i-1,j})^+ + \omega_{i,j} = \tilde{I}_{i,j}$$
.
The proof for $J_{i,j} \geq \tilde{J_{i,j}}$ is analogous.

□

## 3. Law of Large Numbers for the Stationary and non-Stationary exp-LPP

We would now like to tackle the behavior of both the stationary and non-stationary exp-LPP models in terms of how they follow the law of large numbers. We will develop the expectation for the two models, and calculate a function for the limit shape. Our goal for this section is to solve and prove the existence of a function

$$g_N(m, n) = \lim_{N\to\infty} N^{-1} \mathcal{G}_{Nm,Nn}. \tag{3.1}$$

As we have discussed, the LPP model contains a superadditive structure, and so will follow Fekete's superadditive lemma.

**Lemma 3.2.** *For a superadditive sequence* $a_1, a_2, ...$ *the* $\lim_{n\to\infty} \frac{a_n}{n} = sup \frac{a_n}{n}$.

Note that this lemma proves the existence of the function of our limit shape, and shows that this limit will be the supremum of the percolation values to $nM, nN$ as expected by the nature of the LPP. More properties of this limit shape, and the adherence of the LPP to the Law of Large Numbers can be explored in [Sep17].

Now knowing that this limit exists we will solve the expectation and LLN in the stationary case.

**Theorem 3.3.** *Let $\mathcal{G}^{\varrho}_{m,n}$ be the stationary expLPP to point $(m,n)$ with boundary weights, $\omega_{i,0} \sim exp(1-\varrho), \omega_{0,j} \sim exp(\varrho)$ for $0<\varrho<1$, and let $\frac{xN}{N} \to x$, $\frac{yN}{N} \to y$ then both,*

$$\mathbf{E}[\mathcal{G}^{\varrho}_{m,n}] = \frac{m}{1-\varrho} + \frac{n}{\varrho}, \tag{3.4}$$

$$\lim_{N\to\infty} N^{-1}\mathcal{G}^{\varrho}_{Nx,Ny} = \frac{x}{1-\varrho} + \frac{y}{\varrho} \tag{3.5}$$

.

*Proof.* .
Given the stationarity of this model, all horizontal and vertical segments of the path are i.i.d exponential with rate $(1-\varrho), \varrho$ exactly like the boundary segment. Given this and the linearity of expectation, we can easily take this expectation,

$$\mathbf{E}[\mathcal{G}^{\varrho}_{m,n}] = \mathbf{E}[\sum_{i=1}^{m} I_{i,0} + \sum_{j=1}^{n} J_{m,j}] = \mathbf{E}[\sum_{i=1}^{m} I_{i,0}] + \mathbf{E}[\sum_{j=1}^{n} J_{m,j}] = \frac{m}{1-\varrho} + \frac{n}{\varrho}.$$

Then the limit of this process also passes through,

$$\lim_{N\to\infty} \frac{\mathcal{G}^{\varrho}_{m,n}}{N} = \lim_{N\to\infty} \frac{\sum_{i=1}^{xN} I_{i,0} + \sum_{j=1}^{yN} J_{xN,j}}{N} = x\mathbf{E}[I_{i,0}] + y\mathbf{E}[J_{x,j}] = \frac{x}{1-\varrho} + \frac{y}{\varrho}.$$

□

Note that we never claim that the two sums that compose the $\mathcal{G}^{\varrho}$ model are independent, as they are heavily dependent. We note that $J_{m,j}$ changes as $I_{i,0}$ does. Since the path from $(0,0)$ to $(m,n)$ is not a down-right path, like in theorem, we cannot say these increments are independent. In fact, these sums being correlated is essential to the super-concentration we see in the variance of $\mathcal{G}^{\varrho}$ being of order $t^{\frac{2}{3}}$ rather than $t$, which would be observed if they were independent.

Now to take on the non-stationary expLPP, with exp(1) weights on all lattice points, we can rewrite the stationary form in terms of the non-stationary

$$\mathcal{G}^{\varrho}_{m,n} = \max_{0\le a\le x}\{\sum_{i=1}^{a} I_{i,0} + \mathcal{G}_{a,1m,n}\} \vee \max_{0\le b\le y}\{\sum_{j=1}^{b} J_{0,j} + \mathcal{G}_{1,bm,n}\}, \tag{3.6}$$

with $\vee$ indicating the greater of the two.
To see what the Law of Large Numbers will be for the non-stationary expLPP. Let $g_N(x,y) = \lim_{N\to\infty} \frac{\mathcal{G}_{Nx,Ny}}{N}$, and then take the limit of $N\to\infty$ of the rewritten stationary model divided by $N$. We will then consider,the destination point as $(Nx,Ny)$, not $(m,n)$, and use Fekete's lemma and the uniform weight distribution of the non-stationary expLPP. Taking the limit, we get,

$$\frac{x}{1-\varrho} + \frac{y}{\varrho} = \sup_{0\le a\le x}\{\frac{a}{1-\varrho} + g_N(x-a,y)\} \vee \sup_{0\le b\le y}\{\frac{b}{\varrho} + g_N(x,y-b)\}.$$

Then, considering the symmetry of the LPP, let $x = y$, and $0 < \varrho \leq \frac{1}{2}$. Then,

$$\frac{y}{1-\varrho} + \frac{y}{\varrho} = \sup_{0 \leq a \leq y} \{\frac{a}{1-\varrho} + g_N(y-a, y)\} \vee \sup_{0 \leq b \leq y} \{\frac{b}{\varrho} + g_N(y-b, y)\},$$
$$\frac{y}{1-\varrho} + \frac{y}{\varrho} = \sup_{0 \leq b \leq y} \{\frac{b}{\varrho} + g_N(y-b, y)\}$$

We then construct a function, $f$, which we will take the complex dual of to solve for $g_N$. So, let

$$f(b) = \begin{cases} -g_N(y-b, y) & 0 \leq b \leq y \\ \infty & b < 0, b \geq y \end{cases}.$$

Finally, if we let $\alpha - \frac{1}{\varrho}$, then the final equality can be rewritten,

$$y(\alpha + 1 + \frac{1}{\alpha - 1}) = \sup_{b \in \mathrm{R}} (b\alpha - f(b)).$$

From this one can take the complex dual of $f$, and solve for $g_N(b)$, and then plug back in $(x, y)$, and we will find that the LLN for the non-stationary expLPP gives us

$$g_N(x, y) = (\sqrt{x} + \sqrt{y})^2. \tag{3.7}$$

This marks the limit shape of the non-stationary model, which as we have seen the corner growth model similarly represents. As you can see, points further from the axes have higher expectations than points closer to the axes.

In the stationary model, the characteristic direction will be dependent on both the final point and the $\varrho$ chosen for the edge weights, as we have seen in the computation of its expectation. The characteristic direction to a point $(x, y)$ is along the vector $\langle (1-\varrho)^2, \varrho^2 \rangle$, so then $\frac{x}{y} = \frac{(1-\varrho)^2}{\varrho^2}$, simplifying for $\varrho = \frac{\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, then plugging this back into our expectation we get

$$\frac{x}{1-\varrho} + \frac{y}{\varrho} = (\sqrt{x} + \sqrt{y})^2,$$

giving us the non-stationary limit shape.

## 4. Variance of the Stationary Exponential LPP

Now we will take on developing the formula for the variance of the stationary expLPP. A good resource to introduce the structure of the LPP can be [Bal22]. We will need to introduce a few more variables that will serve as ways to split the LPP into the time spent on the boundary, and time spent in the bulk.

Let $-n \leq x \leq m$ and we are adding up the weight on the $i - axis$ to $(x, 0)$ when $x > 0$, and on the $j - axis$ to $(0, -x)$ when $x < 0$

**Definition 4.1.** Let

$$\mathrm{U}_x^{\varrho} = \mathcal{G}_{\{x^+\},\{x^-\}} = \begin{cases} \sum_{i=0}^{x} \omega_{i,0} & \text{if } x \geq 0 \\ \sum_{j=0}^{-x} \omega_{0,j} & \text{if } x \leq 0 \end{cases}, \tag{4.2}$$

be the sum of the boundary weights until some point, x.

After the model has gained some weight along the boundary it will enter the bulk, and so the weight gathered exclusively in the bulk until the point $(m, n)$ will be

**Definition 4.3.** Let $\Pi_x(m, n)$ be the set of nearest neighbor upright paths from $(x \text{ v } 1, -x \text{ v } 1)$ to $(m, n)$.

$$\mathcal{A}_x = \mathcal{A}_x(m, n) = \max_{\pi \in \Pi_x(m,n)} \sum_{(p,q)\in\pi} \omega_{p,q}. \tag{4.4}$$

This then represents the max weight from an exit point of a boundary to $(m, n)$. Note that $\mathcal{A}_{-1} = \mathcal{A}_0 = \mathcal{A}_1$ which describes the scenario that the path enters the bulk at $(1, 1)$, for clarification, $\mathcal{A}_{-2}$ enters the bulk at $(2, 1)$, and $\mathcal{A}_2$ at $(1, 2)$

Since there is a unique maximum path, then there is also a unique exit point, x, that this path goes through. We will label this point $Z^{\varrho}$.

We can also decompose

$$\mathcal{G}^{\varrho}_{m,n} = \mathrm{U}^{\varrho}_{Z^{\varrho}} + \mathcal{A}_{Z^{\varrho}}. \tag{4.5}$$

**Theorem 4.6.** *Let $\mathcal{G}^{\mathbf{W}=0}$ be the last passage time on a model where $\omega_{0,j} = 0$ $\forall j \geq 1$. Then for $v(n) < m_1 < m_2$*

$$\mathcal{A}_0(m_2, n) - \mathcal{A}_0(m_1, n) \leq \mathcal{G}^{\mathbf{W}=0}(m_2, n) - \mathcal{G}^{\mathbf{W}=0}(m_1, n) = \mathcal{G}^{\varrho}_{m,n}(m_2, n) - \mathcal{G}^{\varrho}_{m,n}(m_1, n). \tag{4.7}$$

*Proof.* . The first inequality is true since $\mathcal{A}_0$ is identical to the LPP that doesn't touch the boundaries, or where $\omega_{i,0}, \omega_{0,j} = 0$, so there are less steps in the $\mathcal{A}_0$ process. The second equality comes from understanding the value $v(n)$, this is the first value on the path that takes the shorter of two times at every intersection that hits the horizontal line $j = n$. If $v(n) < m$ as we have in this theorem, then we know that the first LPP step to $(m, n)$ is (1,0). So then note that $v(n) < m_1 < m_2 \leq m$ then the first step in the regular LPP will be to ignore the vertical axis, just as the altered LPP with $\omega_{0,j} = 0$ weights, in order to maximize the time. □

Now let $0 < \varrho < \lambda < 1$ such that we create a modified LPP with $\omega^{\lambda}_{i,0} \sim \exp(1-\lambda)$ weights, and so to actualize the exponential rate, we have the coupled relationship, and following variance:

$$\omega^{\lambda}_{i,0} = \frac{1-\varrho}{1-\lambda}\omega^{\varrho}_{i,0}, \tag{4.8}$$

$$\mathbf{Var}(\omega^{\lambda}_{i,0} - \omega^{\varrho}_{i,0}) = (\frac{1-\varrho}{1-\lambda} - 1)^2 - \frac{1}{(1-\varrho)^2}. \tag{4.9}$$

**Theorem 4.10.** *Fix $m, n \in \mathrm{N}$, and let $Z^{\varrho} > 0$, then*

$$\mathbf{Var}(\mathcal{G}^{\varrho}_{m,n}) = \frac{n}{\varrho^2} - \frac{m}{(1-\varrho)^2} + \frac{2}{1-\varrho}\mathbf{E}[\mathrm{U}^{\varrho}_{Z^{\varrho+}}] \tag{4.11}$$

.

*Proof.* .

To start we will introduce a few new variables, which can be interpreted by cardinal direction.

- $\mathcal{N} = \mathcal{G}^{\varrho}_{m,n} - \mathcal{G}^{\varrho}_{0,n}$
- $\mathcal{E} = \mathcal{G}^{\varrho}_{m,n} - \mathcal{G}^{\varrho}_{m,0}$
- $\mathcal{S} = \mathcal{G}^{\varrho}_{m,0}$
- $\mathcal{W} = \mathcal{G}^{\varrho}_{0,n}$

Note that $\mathcal{N}$ is equal to $\sum_{j=0}^{n} J_{m,j}$ which we know is the sum of $\exp(1-\varrho)$ i.i.d weights.

Also note that our $\mathcal{N}, \mathcal{E}$ act as larger versions of $I, J$ increments and so are mutually independent. Then,

$$\begin{aligned}
\mathbf{Var}(\mathcal{G}^{\varrho}_{m,n}) =& \mathbf{Var}(\mathcal{N} + \mathcal{W}) \\
=& \mathbf{Var}(\mathcal{N}) + \mathbf{Var}(\mathcal{W}) + 2\mathbf{Cov}(\mathcal{N}, \mathcal{W}) \\
=& \mathbf{Var}(\mathcal{N}) + \mathbf{Var}(\mathcal{W}) + 2\mathbf{Cov}(\mathcal{N}, \mathcal{S} + \mathcal{E} - \mathcal{N}) \\
=& \mathbf{Var}(\mathcal{N}) + \mathbf{Var}(\mathcal{W}) + 2\mathbf{Cov}(\mathcal{N}, \mathcal{S}) + 2\mathbf{Cov}(\mathcal{N}, \mathcal{E}) - 2\mathbf{Cov}(\mathcal{N}, \mathcal{N}) \\
=& \mathbf{Var}(\mathcal{N}) + \mathbf{Var}(\mathcal{W}) + 2\mathbf{Cov}(\mathcal{N}, \mathcal{S}) + 0 - 2\mathbf{Var}(\mathcal{N}) \\
=& \mathbf{Var}(\mathcal{W}) - \mathbf{Var}(\mathcal{N}) + 2\mathbf{Cov}(\mathcal{N}, \mathcal{S}) \\
=& \mathbf{Var}(\mathcal{G}^{\varrho}_{0,n}) - \mathbf{Var}(\sum_{j=0}^{n} J_{m,j}) + 2\mathbf{Cov}(\mathcal{N}, \mathcal{S}) \\
=& \frac{n}{\varrho^2} - \frac{m}{(1-\varrho)^2} + 2\mathbf{Cov}(\mathcal{N}, \mathcal{S}).
\end{aligned}$$

Now we will modify the southern boundary, $\mathcal{S}$ so then on the same probability space create a new LPP with all the same conditions, but with new southern boundary weights $\omega^{\lambda}_{i,0} \sim \exp(1-\lambda)$ with the same coupling described. Let $\lambda = \varrho + \epsilon$ and denote this model with a superscript $\epsilon$. Given $\mathcal{S}^{\epsilon}$ is the sum of i.i.d $\exp(1-\varrho-\epsilon)$ weights, it has the following density function, and derivative with respect to epsilon:

$$f_{\mathcal{S}^{\epsilon}}(s) = \frac{(1-\varrho-\epsilon)^m s^{m-1} e^{-(1-\varrho-\epsilon)s}}{(m-1)!},$$

$$\partial_{\epsilon} f_{\mathcal{S}^{\epsilon}}(s) = s f_{\mathcal{S}^{\epsilon}}(s) - \frac{m}{1-\varrho-\epsilon} \cdot f_{\mathcal{S}^{\epsilon}}(s).$$

Note since the sum $\mathcal{S}$ is the sum of i.i.d exponential variables, this sum follows the order statistic property, that $\omega_{i,0}$ weights have identical joint distributions for all parameters. So when fixing the value of $\mathcal{S} = s$, the sum of $\exp(1-\varrho)$ variables, and $\mathcal{S}^{\epsilon} = s$, the LPP models are identical.

So then we have the equality,

$$\mathbf{E}[\mathcal{N}^{\epsilon} | \mathcal{S}^{\epsilon} = s] = \mathbf{E}[\mathcal{N} | \mathcal{S} = s].$$

We can simplify $\mathbf{Cov}(\mathcal{N}, \mathcal{S}) = \mathbf{E}[\mathcal{N}\mathcal{S}] - \mathbf{E}[\mathcal{N}]\mathbf{E}[\mathcal{S}] = \mathbf{E}[\mathcal{N}\mathcal{S}] - \frac{m}{1-\varrho}\mathbf{E}[\mathcal{N}]$. Then,

$$\begin{aligned}
\partial_\epsilon \mathbf{E}[\mathcal{N}^\epsilon]|_{\epsilon=0} =&\partial_\epsilon \int_0^\infty \mathbf{E}[\mathcal{N}^\epsilon|\mathcal{S}^\epsilon = s] f_{\mathcal{S}^\epsilon}(s)ds|_{\epsilon=0}\\
=&\int_0^\infty \mathbf{E}[\mathcal{N}|\mathcal{S} = s]\partial_\epsilon f_{\mathcal{S}^\epsilon}(s)ds|_{\epsilon=0}\\
=&\int_0^\infty \mathbf{E}[\mathcal{N}|\mathcal{S} = s](sf_{\mathcal{S}^\epsilon}(s) - \frac{m}{1-\varrho-\epsilon}\cdot f_{\mathcal{S}^\epsilon}(s))ds|_{\epsilon=0}\\
=&\int_0^\infty \mathbf{E}[\mathcal{N}|\mathcal{S} = s]sf_{\mathcal{S}}(s)ds - \frac{m}{1-\varrho}\int_0^\infty \mathbf{E}[\mathcal{N}|\mathcal{S} = s]f_{\mathcal{S}}(s)ds\\
=&\mathbf{E}[\mathcal{N}\mathcal{S}] - \frac{m}{1-\varrho}\mathbf{E}[\mathcal{N}] = \mathbf{E}[\mathcal{N}\mathcal{S}] - \mathbf{E}[\mathcal{N}]\mathbf{E}[\mathcal{S}] = \mathbf{Cov}(\mathcal{N},\mathcal{S}).
\end{aligned}$$

Now we will tackle this derivative, or how $\mathcal{N}^\epsilon$ instantaneously changes at $\epsilon = 0$ by understanding how $\mathcal{N}^\epsilon$ changes from $\mathcal{N}$. Since the two models differ only in the weights on the southern boundary $\mathcal{N}^\epsilon$ can only vary from $\mathcal{N}$, in the term $\mathcal{G}^\epsilon_{m,n}, \mathcal{G}^\varrho_{m,n}$. Other aspects of the two models that can be different is the maximum lift-off point $Z, Z^\epsilon$ off the southern boundary and the weight accrued on this boundary $\mathrm{U}_x, \mathrm{U}^\epsilon_x$. So $\mathcal{N}, \mathcal{N}^\epsilon$ can change if $Z \neq Z^\epsilon$ and so pick up varying weights, or if $Z = Z^\epsilon$ and they still pick up varying weights. Consider,

$$\begin{aligned}
\mathcal{N}^\epsilon - \mathcal{N} =&(\mathcal{N}^\epsilon - \mathcal{N})\cdot \mathbf{1}\{Z = Z^\epsilon\} + (\mathcal{N}^\epsilon - \mathcal{N})\cdot \mathbf{1}\{Z \neq Z^\epsilon\}\\
=&((\mathcal{A}_{Z^\epsilon} + \mathrm{U}^\epsilon_{Z^\epsilon}) - (\mathcal{A}_Z + \mathrm{U}_Z))\cdot \mathbf{1}\{Z = Z^\epsilon\} + (\mathcal{N}^\epsilon - \mathcal{N})\cdot \mathbf{1}\{Z \neq Z^\epsilon\}\\
=&((\mathcal{A}_Z + \mathrm{U}^\epsilon_Z) - (\mathcal{A}_Z + \mathrm{U}_Z))\cdot \mathbf{1}\{Z = Z^\epsilon\} + (\mathcal{N}^\epsilon - \mathcal{N})\cdot \mathbf{1}\{Z \neq Z^\epsilon\}\\
=&(\mathrm{U}^\epsilon_Z - \mathrm{U}_Z)\cdot \mathbf{1}\{Z = Z^\epsilon\} + (\mathcal{N}^\epsilon - \mathcal{N})\cdot \mathbf{1}\{Z \neq Z^\epsilon\}\\
=&(\mathrm{U}^\epsilon_Z - \mathrm{U}_Z) + (\mathcal{N}^\epsilon - \mathcal{N} - \mathrm{U}^\epsilon_Z + \mathrm{U}_Z)\cdot \mathbf{1}\{Z \neq Z^\epsilon\}.
\end{aligned}$$

We will consider the first part,

$$\mathrm{U}^\epsilon_Z - \mathrm{U}_Z = \mathrm{U}^\epsilon_{Z^+} - \mathrm{U}_{Z^+} = \frac{1-\varrho}{1-\varrho-\epsilon}\mathrm{U}_{Z^+} - \mathrm{U}_{Z^+} = \frac{\epsilon}{1-\varrho-\epsilon}\mathrm{U}_{Z^+}.$$

The right-hand side becomes an error term of the scale $\mathcal{O}(\epsilon^2)$ which we will eventually take the $\epsilon$ derivative when $\epsilon = 0$. The intuition behind this is that when the exit point changes between the two models, then this in itself will be a change on the scale of $\mathcal{O}(\epsilon)$. The next error term is the scale of the expectation of the change of $\mathcal{N}$. We can say $\mathcal{N}^\epsilon - \mathcal{N} \leq \mathcal{S}^\epsilon - \mathcal{S}$ and $\mathrm{U}^\epsilon_x \geq \mathrm{U}_x$, so then $\mathbf{E}[\mathcal{N}^\epsilon - \mathcal{N} - \mathrm{U}^\epsilon_Z + \mathrm{U}_Z] \leq \mathbf{E}[\mathcal{S}^\epsilon - \mathcal{S}]$ which is on the scale $\mathcal{O}(\epsilon)$.

So then we get the equality,

$$\begin{aligned}
\mathcal{N}^\epsilon - \mathcal{N} =&\frac{\epsilon}{1-\varrho-\epsilon}\mathrm{U}_{Z^+} + \mathcal{O}(\epsilon^2)\\
\mathbf{E}[\mathcal{N}^\epsilon] - \mathbf{E}[\mathcal{N}] =&\mathbf{E}[\frac{\epsilon}{1-\varrho-\epsilon}\mathrm{U}_{Z^+}] + \mathcal{O}(\epsilon^2)\\
\lim_{\epsilon\to 0}\frac{\mathbf{E}[\mathcal{N}^\epsilon] - \mathbf{E}[\mathcal{N}]}{\epsilon} =&\lim_{\epsilon\to 0}\frac{\frac{\epsilon}{1-\varrho-\epsilon}\mathbf{E}[\mathrm{U}_{Z^+}] + \mathcal{O}(\epsilon^2)}{\epsilon}\\
\partial_\epsilon \mathbf{E}[\mathcal{N}^\epsilon]|_{\epsilon=0} =&\lim_{\epsilon\to 0}\frac{1}{1-\varrho-\epsilon}\mathbf{E}[\mathrm{U}_{Z^+}] + \mathcal{O}(\epsilon)\\
\mathbf{Cov}(\mathcal{N},\mathcal{S}) =&\frac{1}{1-\varrho}\mathbf{E}[\mathrm{U}_{Z^+}].
\end{aligned}$$

Then,

$$\begin{aligned}\mathbf{Var}(\mathcal{G}^{\varrho}_{m,n}) =& \frac{n}{\varrho^2} - \frac{m}{(1-\varrho)^2} + 2\mathbf{Cov}(\mathcal{N},\mathcal{S}) \\ =& \frac{n}{\varrho^2} - \frac{m}{(1-\varrho)^2} + \frac{2}{1-\varrho}\mathbf{E}[\mathrm{U}^{\varrho}_{Z^{\varrho+}}].\end{aligned}$$

□

The equivalent,

$$\mathbf{Var}(\mathcal{G}^{\varrho}_{m,n}) = \frac{m}{(1-\varrho)^2} - \frac{n}{\varrho^2} + \frac{2}{\varrho}\mathbf{E}[\mathrm{U}^{\varrho}_{-Z^{\varrho-}}], \tag{4.12}$$

can be proven the same way, breaking down $\mathbf{Var}(\mathcal{G}^{\varrho}_{m,n}) = \mathbf{Var}(\mathcal{W}+\mathcal{E})$.

Also, we have the coupling,

**Theorem 4.13.**

$$\mathbf{Var}(\mathcal{G}^{\lambda}_{m,n}) \leq \mathbf{Var}(\mathcal{G}^{\varrho}_{m,n}) + m(\frac{1}{(1-\lambda)^2} - \frac{\varrho^2}{\lambda^2(1-\varrho)^2}). \tag{4.14}$$

*Proof.* .

First note the relations,

- $\omega^{\lambda}_{i,0} = \frac{1-\varrho}{1-\lambda}\omega^{\varrho}_{i,0} \to Z^{\varrho^+} \leq Z^{\lambda^+}$
- $\omega^{\lambda}_{0,j} = \frac{\varrho}{\lambda}\omega^{\varrho}_{0,j} \to -Z^{\lambda^-} \leq -Z^{\varrho^-}$
- $\mathrm{U}^{\lambda}_{-Z^{\lambda-}} = \frac{\varrho}{\lambda}\mathrm{U}^{\varrho}_{-Z^{\lambda-}} \leq \frac{\varrho}{\lambda}\mathrm{U}^{\varrho}_{-Z^{\varrho-}}$.

So then,

$$\begin{aligned}\mathbf{Var}(\mathcal{G}^{\lambda}_{m,n}) =& \frac{m}{(1-\lambda)^2} - \frac{n}{\lambda^2} + \frac{2}{\lambda}\mathbf{E}[\mathrm{U}^{\lambda}_{-Z^{\lambda-}}] \\ \leq& \frac{m}{(1-\lambda)^2} - \frac{n}{\lambda^2} + \frac{2\varrho}{\lambda^2}\mathbf{E}[\mathrm{U}^{\varrho}_{-Z^{\varrho-}}] \\ =& \frac{m}{(1-\lambda)^2} + \frac{m\varrho^2}{\lambda^2(1-\varrho)^2} - \frac{m\varrho^2}{\lambda^2(1-\varrho)^2} - \frac{n\varrho^2}{\lambda^2\varrho^2} + \frac{2\varrho^2}{\lambda^2\varrho}\mathbf{E}[\mathrm{U}^{\varrho}_{-Z^{\varrho-}}] \\ =& \frac{\varrho^2}{\lambda^2}(\frac{m}{(1-\varrho)^2} - \frac{n}{\varrho^2} + \frac{2}{\varrho}\mathbf{E}[\mathrm{U}^{\varrho}_{-Z^{\varrho-}}]) + m(\frac{1}{(1-\lambda)^2} - \frac{\varrho^2}{\lambda^2(1-\varrho)^2}) \\ =& \frac{\varrho^2}{\lambda^2}\mathbf{Var}(\mathcal{G}^{\varrho}_{m,n}) + m(\frac{1}{(1-\lambda)^2} - \frac{\varrho^2}{\lambda^2(1-\varrho)^2}) \\ \leq& \mathbf{Var}(\mathcal{G}^{\varrho}_{m,n}) + m(\frac{1}{(1-\lambda)^2} - \frac{\varrho^2}{\lambda^2(1-\varrho)^2}).\end{aligned}$$

□

## 5. Upper Bound for the Asymptotic Variance of stationary LPP

The goal of this section is to show that the variance of the LPP model scales like $N^{2/3}$ in the characteristic direction. This proof is motivated by methods that can be found in [BCS06]. To do this we will replace the variable point $(m,n)$ with the ray in this direction, so that it can be scaled by t.

**Definition 5.1.** Let $0 < \varrho < \lambda < 1$. Then,

$$m(t) = \lfloor (1-\varrho)^2 t \rfloor \qquad\qquad n(t) = \lfloor \varrho^2 t \rfloor.$$

$\mathcal{A}_x(t), \mathcal{G}^\varrho(t), Z^\varrho(t)$ will also change as a result of this new definition, and

$$\mathcal{G}^\lambda(t) = \mathcal{A}_{Z^\lambda}(t) + \mathrm{U}^\lambda_{Z^\lambda}(t).$$

Will be the stationary LPP on the same probability space, with boundary weights $\omega^\lambda_{i,0}, \omega^\lambda_{0,j}$ with the defined coupling, and bulk weights exp(1).

We need to develop a refined version of $\lambda$ in terms of $\varrho$ that maximizes $\mathbf{E}[\mathrm{U}^\lambda_u - \mathrm{U}^\varrho_u - \mathcal{G}^\lambda(t) + \mathcal{G}^\varrho(t)]$, so we need to maximize

$$\mathbf{E}[\mathrm{U}^\lambda_u - \mathcal{G}^\lambda(t)] = \frac{u}{1-\lambda} - \frac{\lfloor (1-\varrho)^2 t \rfloor}{1-\lambda} - \frac{\lfloor \varrho^2 t \rfloor}{.}\lambda$$

This happens at,

$$\lambda = \frac{\varrho}{\sqrt{(1-\varrho)^2 - \frac{u}{t}} + \varrho} \qquad \frac{1}{\lambda} = 1 + \frac{\sqrt{(1-\varrho)^2 - \frac{u}{t}}}{\varrho} \qquad \frac{1}{1-\lambda} = 1 + \frac{\varrho}{\sqrt{(1-\varrho)^2 - \frac{u}{t}}}.$$

From now on we will use this $\lambda$.

**Theorem 5.2.** *There exists a constant $\mathcal{C}_1 = \mathcal{C}_1(\varrho)$ such that for any $8\varrho^{-2}(1-\varrho)^2 \leq u \leq (1-\varrho)^2 t$ and $t > 0$,*

$$\mathbf{P}\{Z^\varrho(t) > u\} \leq \mathcal{C}_1\left(\frac{t^2}{u^4}\mathbf{E}[\mathrm{U}^\varrho_{Z^\varrho(t)^+}] + \frac{t^2}{u^2}\right). \tag{5.3}$$

*Proof.* .

$$\begin{aligned}
\mathbf{P}\{Z^\varrho(t) > u\} &= \mathbf{P}\{\exists z > u | \mathrm{U}^\varrho_z + \mathcal{A}_z(t) = \mathcal{G}^\varrho(t)\} \\
&= \mathbf{P}\{\exists z > u | \mathrm{U}^\varrho_z - \mathrm{U}^\lambda_z + \mathcal{G}^\lambda(t) \geq \mathcal{G}^\varrho(t)\} && \{\mathrm{U}^\lambda_z + \mathcal{A}_z(t) \leq \mathcal{G}^\lambda(t)\} \\
&\leq \mathbf{P}\{\mathrm{U}^\lambda_u - \mathrm{U}^\varrho_u \geq \mathcal{G}^\lambda(t) - \mathcal{G}^\varrho(t)\} && \{\mathrm{U}^\lambda_z - \mathrm{U}^\varrho_z - (\mathrm{U}^\lambda_u - \mathrm{U}^\varrho_u) \geq 0\} \\
&\leq \mathbf{P}\{\mathrm{U}^\lambda_u - \mathrm{U}^\varrho_u \geq \mathcal{G}^\lambda(t) - \mathcal{G}^\varrho(t) - \mathbf{E}[\mathrm{U}^\lambda_u - \mathrm{U}^\varrho_u - \mathcal{G}^\lambda(t) + \mathcal{G}^\varrho(t)]\} \\
&\leq \mathbf{P}\{\mathrm{U}^\lambda_u - \mathrm{U}^\varrho_u \leq \mathbf{E}[\mathrm{U}^\lambda_u - \mathrm{U}^\varrho_u] - \frac{1}{2}\mathbf{E}[\mathrm{U}^\lambda_u - \mathrm{U}^\varrho_u - \mathcal{G}^\lambda(t) + \mathcal{G}^\varrho(t)]\} \\
&+ \mathbf{P}\{\mathcal{G}^\lambda(t) - \mathcal{G}^\varrho(t) \geq \mathbf{E}[\mathcal{G}^\lambda(t) - \mathcal{G}^\varrho(t)] + \frac{1}{2}\mathbf{E}[\mathrm{U}^\lambda_u - \mathrm{U}^\varrho_u - \mathcal{G}^\lambda(t) + \mathcal{G}^\varrho(t)] \\
&\leq \frac{1}{2}\mathbf{E}[\mathrm{U}^\lambda_u - \mathrm{U}^\varrho_u - \mathcal{G}^\lambda(t) + \mathcal{G}^\varrho(t)]^{-2}(\mathbf{Var}(\mathrm{U}^\lambda_u - \mathrm{U}^\varrho_u) + \mathbf{Var}(\mathcal{G}^\lambda(t) - \mathcal{G}^\varrho(t))). && \text{(By Chebyshev's Inequality)}
\end{aligned}$$

**Lemma 5.4.** *For $u \leq \frac{3}{4}(1-\varrho)^2 t$,*

$$\mathbf{Var}(\mathrm{U}^\lambda_u - \mathrm{U}^\varrho_u) \leq \frac{u\varrho^2}{(1-\varrho)^2}. \tag{5.5}$$

*Proof.* .

$$\mathbf{Var}(\mathrm{U}_u^\lambda - \mathrm{U}_u^\varrho) = u \cdot \mathbf{Var}(\omega_{i,0}^\lambda - \omega_{i,0}^\varrho) = u(\frac{1-\varrho}{1-\lambda} - 1)^2 \cdot \frac{1}{(1-\varrho)^2} = u((\frac{1-\varrho}{1-\lambda} - 1)(\frac{1}{(1-\varrho)}))^2$$

$$\text{Now consider, } (\frac{1-\varrho}{1-\lambda} - 1)(\frac{1}{(1-\varrho)}) = \frac{1}{1-\lambda} - \frac{1}{1-\varrho}$$

$$= \frac{\sqrt{(1-\varrho)^2 - \frac{u}{t}} + \varrho}{\sqrt{(1-\varrho)^2 - \frac{u}{t}}} - \frac{1}{1-\varrho} \leq \frac{1+\varrho}{1-\varrho} - \frac{1}{1-\varrho} = \frac{\varrho}{1-\varrho}.$$

$$\text{So then, } \mathbf{Var}(\mathrm{U}_u^\lambda - \mathrm{U}_u^\varrho) = u(\frac{\varrho}{1-\varrho})^2 = \frac{u\varrho^2}{(1-\varrho)^2}.$$

□

**Lemma 5.6.** *For* $\frac{u}{t} \leq \frac{3}{4}(1-\varrho)^2$,

$$\mathbf{Var}(\mathcal{G}^\lambda(t) - \mathcal{G}^\varrho(t)) \leq \frac{8}{1-\varrho}\mathbf{E}[\mathrm{U}^\varrho_{Z^\varrho(\mathrm{t})+}] + \frac{8(u+1)}{(1-\varrho)^2}. \tag{5.7}$$

*Proof.* .

$$\begin{aligned}
\mathbf{Var}(\mathcal{G}^\lambda(t)) \leq & \mathbf{Var}(\mathcal{G}^\varrho(t)) + t(\frac{(1-\varrho)^2}{(1-\lambda)^2} - \frac{\varrho^2}{\lambda^2}) \\
= & \mathbf{Var}(\mathcal{G}^\varrho(t)) + t((\sqrt{(1-\varrho)^2 - \frac{u}{t}} + \varrho)^2 \cdot \frac{\frac{u}{t}}{(1-\varrho)^2 - \frac{u}{t}}) \\
\leq & \mathbf{Var}(\mathcal{G}^\varrho(t)) + (\frac{1+\varrho}{2} \cdot \frac{\frac{u}{t} \cdot t}{\frac{1}{4}(1-\varrho)^2}) && \text{Note: } \frac{u}{t} \leq \frac{3}{4}(1-\varrho)^2 \\
& \mathbf{Var}(\mathcal{G}^\varrho(t)) + \frac{4u}{(1-\varrho)^2}.
\end{aligned}$$

Now we will use the inequality,

$$\begin{aligned}
\mathbf{Var}(\mathcal{G}^\lambda(t) - \mathcal{G}^\varrho(t)) \leq & 2\mathbf{Var}(\mathcal{G}^\lambda(t)) + 2\mathbf{Var}(\mathcal{G}^\varrho(t)) \\
\leq & 4\mathbf{Var}(\mathcal{G}^\varrho(t)) + \frac{8u}{(1-\varrho)^2} \\
= & \frac{8u}{(1-\varrho)^2} + \frac{4\lfloor \varrho^2 t \rfloor}{\varrho^2} + \frac{4\lfloor (1-\varrho)^2 t \rfloor}{(1-\varrho)^2} + \frac{8}{1-\varrho}\mathbf{E}[\mathrm{U}^\varrho_{Z^\varrho(\mathrm{t})+}] \\
\leq & \frac{8(u+1)}{(1-\varrho)^2} + \frac{8}{1-\varrho}\mathbf{E}[\mathrm{U}^\varrho_{Z^\varrho(\mathrm{t})+}].
\end{aligned}$$

□

**Lemma 5.8.** *For any* $8\varrho^{-2}(1-\varrho)^2 \leq u \leq (1-\varrho)^2 t$,

$$\mathbf{E}[\mathrm{U}_u^\lambda - \mathrm{U}_u^\varrho - \mathcal{G}^\lambda(t) + \mathcal{G}^\varrho(t)] \geq \frac{\varrho}{8(1-\varrho)^3}\frac{u^2}{t}. \tag{5.9}$$

The above lemma follows from elementary real analysis and hence its proof is skipped. Using the lemma we thus get:

$$\begin{aligned}\mathbf{P}\{Z^\varrho(t) > u\} &\leq \frac{1}{2}\mathbf{E}[\mathrm{U}_u^\lambda - \mathrm{U}_u^\varrho - \mathcal{G}^\lambda(t) + \mathcal{G}^\varrho(t)]^{-2}(\mathbf{Var}(\mathrm{U}_u^\lambda - \mathrm{U}_u^\varrho) + \mathbf{Var}(\mathcal{G}^\lambda(t) - \mathcal{G}^\varrho(t)))\\ &\leq \frac{16^2(1-\varrho)^6}{\varrho^2}\frac{t^2}{u^4}(\frac{u\varrho^2}{(1-\varrho)^2} + \frac{8(u+1)}{(1-\varrho)^2} + \frac{8}{1-\varrho}\mathbf{E}[\mathrm{U}^\varrho_{Z^\varrho(t)^+}])\\ &= \mathcal{C}_1(\frac{t^2}{u^4}\mathbf{E}[\mathrm{U}^\varrho_{Z^\varrho(t)^+}] + \frac{t^2}{u^3}).\end{aligned}$$

$\square$

Now we want to see the behavior of $\mathrm{U}^\varrho_{Z^\varrho(t)^+}$. Now we will first notice that the behavior of $\mathrm{U}^\varrho_{Z^\varrho(t)^+}$ is very reliant on $Z^\varrho(t)^+$, given this is the liftoff point. However, removed from this, it is the sum of i.i.d $\exp(1-\varrho)$ random variables, so we can use the Markov inequality, to get the following Chernoff type tail bound.

**Lemma 5.10.** *Let $1-\rho > s > 0$, and fix $s$. Then, let $y = \frac{u}{\alpha(1-\rho)}$ for $0 < \alpha < 1$.*

$$\mathbf{P}\{\mathbf{U}_u^\rho > y\} \leq e^{-(1-\rho)(1-\sqrt{\alpha})^2 y} \tag{5.11}$$

Then using this tail bound, we will combine this with our bound on $Z^\rho(t)^+$,

**Lemma 5.12.** *Let $r \geq \frac{8(1-\rho)}{\alpha\rho^2\mathbf{E}[U^\rho_{Z^\rho(t)^+}]}$, then there exists constants $\mathcal{C}_2, \mathcal{C}_3$ such that,*

$$\mathbf{P}\{U^\rho_{Z^\rho(t)^+} > r\mathbf{E}[U^\rho_{Z^\rho(t)^+}]\} \leq \frac{\mathcal{C}_2 t^2}{\mathbf{E}[U^\rho_{Z^\rho(t)^+}]^3}\cdot(\frac{1}{r^3} + \frac{1}{r^4}) + e^{-\mathcal{C}_3 r\mathbf{E}[U^\rho_{Z^\rho(t)^+}]}. \tag{5.13}$$

*Proof.* .

$$\begin{aligned}\mathbf{P}\{U^\rho_{Z^\rho(t)^+} > y\} &\leq \mathbf{P}\{Z^\rho(t) > u\} + \mathbf{P}\{U_u^\rho > y\}\\ &\leq \mathcal{C}_1(\frac{t^2}{u^4}\mathbf{E}[\mathrm{U}^\varrho_{Z^\varrho(t)^+}] + \frac{t^2}{u^3}) + e^{-(1-\rho)(1-\sqrt{\alpha})^2 y}\\ &\text{Let } y = r\mathbf{E}[U^\rho_{Z^\rho(t)^+}]\\ &= \frac{\mathcal{C}_2 t^2}{\mathbf{E}[U^\rho_{Z^\rho(t)^+}]^3}\cdot(\frac{1}{r^3} + \frac{1}{r^4}) + e^{-\mathcal{C}_3 r\mathbf{E}[U^\rho_{Z^\rho(t)^+}]}\end{aligned}$$

$\square$

**Theorem 5.14.**

$$\limsup_{t\to\infty}\frac{\mathbf{E}[U^\rho_{Z^\rho(t)^+}]}{t^{\frac{2}{3}}} < \infty, \ \textit{and} \ \limsup_{t\to\infty}\frac{\mathbf{Var}(\mathcal{G}^\rho(t))}{t^{\frac{2}{3}}} < \infty \tag{5.15}$$

*Proof.* Note that if the first inequality is true, then by the variance formula, the second also holds. First, let there be a sequence $t_k \to \infty$ such that,

$$\lim_{k\to\infty}\frac{\mathbf{E}[U^\rho_{Z^\rho(t_k)^+}]}{t_k^{\frac{2}{3}}} = \infty.$$

Which means the top of this fraction must dominate as $k\to\infty$, so $\mathbf{E}[U^\rho_{Z^\rho(t_k)^+}] > t_k^{\frac{2}{3}}$.

Now we will use the prior lemma,

$$\mathbf{P}\{U^{\rho}_{Z^{\rho}(t_k)^+} > r\mathbf{E}[U^{\rho}_{Z^{\rho}(t_k)^+}]\} \leq \frac{\mathcal{C}_2 t_k^2}{\mathbf{E}[U^{\rho}_{Z^{\rho}(t_k)^+}]^3} \cdot (\frac{1}{r^3} + \frac{1}{r^4}) + e^{-\mathcal{C}_3 r \mathbf{E}[U^{\rho}_{Z^{\rho}(t_k)^+}]}.$$

Then, if we let $k \to \infty$ we get,

$$\lim_{k\to\infty} \frac{\mathcal{C}_2 t_k^2}{\mathbf{E}[U^{\rho}_{Z^{\rho}(t_k)^+}]^3} \cdot (\frac{1}{r^3} + \frac{1}{r^4}) + e^{-\mathcal{C}_3 r \mathbf{E}[U^{\rho}_{Z^{\rho}(t_k)^+}]} = 0 \cdot (\frac{1}{r^3} + \frac{1}{r^4}) + e^{-\infty} \to 0$$

So,

$$\int_0^\infty \mathbf{P}\{U^{\rho}_{Z^{\rho}(t_k)^+} > r\mathbf{E}[U^{\rho}_{Z^{\rho}(t_k)^+}]\} dr \xrightarrow{k\to\infty} 0.$$

However, we know that $1 = \mathbf{E}[\frac{U^{\rho}_{Z^{\rho}(t_k)^+}}{\mathbf{E}[U^{\rho}_{Z^{\rho}(t_k)^+}]}] \xrightarrow{k\to\infty} 0$, so this sequence cannot exist. Therefore, the expectation has this asymptotic limiting power, and also the Variance of the Stationary Exponential LPP has this power of $\frac{2}{3}$ as its asymptotic limiting power. $\square$

## 6. Acknowledgements

I would like to express my profound gratitude to my mentor, Mr. Sayan Das, for helping me develop my curiosity regarding this topic and probability. I would also like to thank him for his guidance which made this paper possible. Finally, I am very thankful to Peter May for organizing the program.

## References

[ADH17] Antonio Auffinger, Michael Damron, and Jack Hanson. *50 years of first-passage percolation*, volume 68. American Mathematical Soc., 2017.

[Bal22] Márton Balázs. Stationary last passage percolation. 2022.

[BCS06] Márton Balázs, Eric Cator, and Timo Seppalainen. Cube root fluctuations for the corner growth model associated to the exclusion process. 2006.

[Sep09] Timo Seppäläinen. Lecture notes on the corner growth model. *Unpublished notes*, 2009.

[Sep17] Timo Seppäläinen. Variational formulas, busemann functions, and fluctuation exponents for the corner growth model with exponential weights. *arXiv preprint arXiv:1709.05771*, 2017.