1 The Harnack Inequality as a Quantitative Maximum Principle

Caffarelli [1] says that we should interpret the Harnack Inequality as a quantitative version of the maximum principle. Here we describe a simple version of this interpretation, especially for the classic Harnack Inequality.

Let's consider a simple version of the Harnack Inequality for non-negative harmonic functions. That is, let $u \geq 0$ on Ω and $\delta u = 0$ on a bounded domain Ω. In this case, the Harnack Inequality is actually a quantitative version of the *Minimum Principle*.

To see so, let us further assume that $u(x_0) = 0$ for some point on $x_0 \in \partial\Omega$ (so u is in some sense on the extreme version of non-negative harmonic functions on Ω). Recall that the Minimum Principle then says that $u > 0$ in Ω if u is non-constant (i.e. $u \neq 0$ on Ω).

However, the Harnack Inequality tells us more. To see this, recall a simple version of the Harnack Inequality, $\sup_{\Omega'} u \leq C \inf_{\Omega'}$ for any domain $\Omega' \subset\subset \Omega$ and constant C depending on Ω' and Ω. So, while the Minimum Principle tells us that $u > 0$ on Ω', it doesn't give us any information on how large this gap is independent of u. However, the Harnack Inequality tells us that the gap between u and 0 on Ω' can be uniformly bounded below depending only on the relationship between Ω' and Ω; that is, we can't get u arbitralily close to 0 on Ω' by varying the values of u on $\partial\Omega$.

References

[1] Luis A. Caffarelli. Regularity of solutions and level surfaces of elliptic equations. *Proceedings of the AMS Centennial Symposium*, 1988.