

1 The Harnack Inequality as a Quantitative Maximum Principle

Caffarelli [1] says that we should interpret the Harnack Inequality as a quantitative version of the maximum principle. Here we describe a simple version of this interpretation, especially for the classic Harnack Inequality.

Let's consider a simple version of the Harnack Inequality for non-negative harmonic functions. That is, let $u \geq 0$ on Ω and $\delta u = 0$ on a bounded domain Ω . In this case, the Harnack Inequality is actually a quantitative version of the *Minimum Principle*.

To see so, let us further assume that $u(x_0) = 0$ for some point on $x_0 \in \partial\Omega$ (so u is in some sense on the extreme version of non-negative harmonic functions on Ω). Recall that the Minimum Principle then says that $u > 0$ in Ω if u is non-constant (i.e. $u \neq 0$ on Ω).

However, the Harnack Inequality tells us more. To see this, recall a simple version of the Harnack Inequality, $\sup_{\Omega'} u \leq C \inf_{\Omega'} u$ for any domain $\Omega' \subset\subset \Omega$ and constant C depending on Ω' and Ω . So, while the Minimum Principle tells us that $u > 0$ on Ω' , it doesn't give us any information on how large this gap is independent of u . However, the Harnack Inequality tells us that the gap between u and 0 on Ω' can be uniformly bounded below depending only on the relationship between Ω' and Ω ; that is, we can't get u arbitrarily close to 0 on Ω' by varying the values of u on $\partial\Omega$.

2 Viscosity Solutions and Comparison Principles

Remark on Sobolev Solutions

When we design weak solutions of a differential equation, we ask that the solutions retain some of the properties of the classical solutions. For example, Sobolev functions provide weak solutions to divergence form elliptic equations where we retain certain integral identities that exist for classical solutions. For Sobolev functions, the function itself and its derivatives are connected enough to still allow the application of integration by parts, when applicable. For example, a classical harmonic function u and a Sobolev harmonic function u both satisfy

$$0 = \int_{\Omega} \langle \nabla u, \nabla \phi \rangle dV = - \int_{\Omega} u \Delta \phi dV, \quad (1)$$

for any $\phi \in C_0^2(\Omega)$.

In particular, we can use the left hand equality applied to $\phi = u\eta^2$ for some $\eta \in C_0^1(\Omega)$. This can be used to establish interior integral estimates for $\|\nabla u\|$;

we see that we get

$$\int_{\Omega} \|\nabla u\|^2 \eta^2 dV = - \int_{\Omega} 2u\eta \langle \nabla u, \nabla \eta \rangle dV. \quad (2)$$

Using Cauchy-Schwarz, we then have that

$$\int_{\Omega} \|\nabla u\|^2 \eta^2 dV \leq 4 \int_{\Omega} u^2 \|\nabla \eta\|^2 dV. \quad (3)$$

With a standard choice of η , this quickly gives us interior gradient estimates

$$\int_{B_{\rho}(0)} \|\nabla u\|^2 dV \leq \frac{4}{\rho^2} \int_{B_{2\rho}(0)} u^2 dV. \quad (4)$$

Viscosity Solutions

For elliptic equations, one way to interpret *viscosity solutions* is that they allow us to retain a comparison principle. In fact, the comparison principle is baked into the definition; one can also use this viewpoint to keep track of the inequalities and their directions in the definitions of viscosity solutions, sub-solutions, and super-solutions.

So it is no surprise that techniques using viscosity solutions lean heavily on comparison principles.

3 Energy Flux And Estimates for Elliptic Equations

Here I give my best interpretation of statements in [2] that state that the flux particular let us look at solutions to Laplace's equation $\Delta u = 0$.

Integrating $u\Delta u$, we have that $\int_{\partial B} u \nabla_N u dA - \int_B \|\nabla u\|^2 dx = 0$. So we can estimate the energy by the flux.

However, we usually "smooth out" this flux using a cutoff function $\phi \in C_0^1(B)$. That is, we instead integrate $\phi^2 u \Delta u$, and use a Schwartz inequality to obtain energy estimates in the usual fashion.

So we get that

$$\int_B \|\nabla u\|^2 \phi^2 dx = -2 \int_B u \phi \langle \nabla u, \nabla \phi \rangle dx. \quad (5)$$

Note that as $\phi \rightarrow \chi_B$ in $C_0^1(B)$, we get that the original flux equality; hence we may justify this as smoothing out the flux.

Now, the main point is that the right hand side is a mixture of the functions values u and its derivatives ∇u . We can then use a Cauchy Schwarz estimate in the usual fashion to obtain an estimate

$$\int_B \|\nabla u\|^2 \phi^2 dx \leq 4 \int_B u^2 \|\nabla \phi\|^2 dx. \quad (6)$$

Then a typical choice of ϕ will provide us with estimates for the energy of u in terms of $\|u\|_{L^2}$.

References

- [1] Luis A. Caffarelli. Regularity of solutions and level surfaces of elliptic equations. *Proceedings of the AMS Centennial Symposium*, 1988.
- [2] Luis A. Caffarelli and Alexis F. Vasseur. The de giorgi method for regularity of solutions of elliptic equations and its applications to fluid dynamics. 2010.