1 The Harnack Inequality as a Quantitative Maximum Principle

Caffarelli [1] says that we should interpret the Harnack Inequality as a quantitative version of the maximum principle. Here we describe a simple version of this interpretation, especially for the classic Harnack Inequality.

Let's consider a simple version of the Harnack Inequality for non-negative harmonic functions. That is, let $u \geq 0$ on Ω and $\delta u = 0$ on a bounded domain Ω. In this case, the Harnack Inequality is actually a quantitative version of the *Minimum Principle*.

To see so, let us further assume that $u(x_0) = 0$ for some point on $x_0 \in \partial\Omega$ (so u is in some sense on the extreme version of non-negative harmonic functions on Ω). Recall that the Minimum Principle then says that $u > 0$ in Ω if u is non-constant (i.e. $u \neq 0$ on Ω).

However, the Harnack Inequality tells us more. To see this, recall a simple version of the Harnack Inequality, $\sup\limits_{\Omega'} u \leq C \inf\limits_{\Omega'}$ for any domain $\Omega' \subset\subset \Omega$ and constant C depending on Ω' and Ω. So, while the Minimum Principle tells us that $u > 0$ on Ω', it doesn't give us any information on how large this gap is independent of u. However, the Harnack Inequality tells us that the gap between u and 0 on Ω' can be uniformly bounded below depending only on the relationship between Ω' and Ω; that is, we can't get u arbitralily close to 0 on Ω' by varying the values of u on $\partial\Omega$.

2 Viscosity Solutions and Comparison Principles

Remark on Sobolev Solutions

When we design weak solutions of a differential equation, we ask that the solutions retain some of the properties of the classical solutions. For example, Sobolev functions provide weak solutions to divergence form elliptic equations where we retain certain integral identities that exist for classical solutions. For Sobolev functions, the function itself and its derivatives are connected enough to still allow the application of integration by parts, when applicable. For example, a classical harmonic function u and a Sobolev harmonic function u both satisfy

$$0 = \int\limits_{\Omega} \langle \nabla u, \nabla \phi \rangle dV = -\int_{\Omega} u \Delta \phi dV, \tag{1}$$

for any $\phi \in C_0^2(\Omega)$.

In particular, we can use the left hand equality applied to $\phi = u\eta^2$ for some $\eta \in C_0^1(\Omega)$. This can be used to establish interior integral estimates for $\|\nabla u\|$;

we see that we get

$$\int_{\Omega} \|\nabla u\|^2 \eta^2 dV = -\int_{\Omega} 2u\eta \langle \nabla u, \nabla \eta \rangle dV. \tag{2}$$

Using Cauchy-Schwarz, we then have that

$$\int_{\Omega} \|\nabla u\|^2 \eta^2 dV \leq 4 \int_{\Omega} u^2 \|\nabla \eta\|^2 dV. \tag{3}$$

With a standard choice of *eta*, this quickly gives us interior gradient estimates

$$\int_{B_\rho(0)} \|\nabla u\|^2 dV \leq \frac{4}{\rho^2} \int_{B_{2\rho}(0)} u^2 dV. \tag{4}$$

Viscosity Solutions

For elliptic equations, one way to interpret *viscosity solutions* is that they allow us to retain a comparison principle. In fact, the comparison principle is baked into the definition; one can also use this viewpoint to keep track of the inequalities and their directions in the definitions of viscosity solutions, sub-solutions, and super-solutions.

So it is no surprise that techniques using viscosity solutions lean heavily on comparison principles.

3 Energy Flux And Estimates for Elliptic Equations

Here I give my best interpretation of statements in [2] that state that the flux particular let us look at solutions to Laplace's equation $\Delta u = 0$.

Integrating $u\Delta u$, we have that $\int_{\partial B} u \nabla_N u dA - \int_B \|\nabla u\|^2 dx = 0$. So we can estimate the energy by the flux.

However, we usually "smooth out" this flux using a cutoff function $\phi \in C_0^1(B)$. That is, we instead integrate $\phi^2 u \Delta u$, and use a Schwartz inequality to obtain energy estimates in the usual fashion.

So we get that

$$\int_B \|\nabla u\| \phi^2 dx = -2 \int_B u\phi \langle \nabla u, \nabla \phi \rangle dx. \tag{5}$$

Note that as $\phi \to \chi_B$ in $C_0^1(B)$, we get that the original flux equality; hence we may justify this as smoothing out the flux.

Now, the main point is that the right hand side is a mixture of the functions values u and its derivatives ∇u. We can then use a Cauchy Schwarz estimate in the usual fashion to obtain an estimate

$$\int_B \|\nabla u\|^2 \phi^2 dx \leq 4 \int_B u^2 \|\nabla \phi\|^2 dx. \tag{6}$$

Then a typical choice of ϕ will provide us with estimates for the energy of u in terms of $\|u\|_{L^2}$.

4 Quasi-conformal Maps and Measures of Non-degenerateness

For a map $f : \mathbb{R}^2 \to \mathbb{R}^2$, we may measure how non-degenerate the map is by using the constant of quasi-conformality. In a sense, this measures how far the map is from "degenerating" into a one-dimensional image; note that the constant of quasi-conformality is scale invariant as such a measure should be.

However, keep in mind that we can still have branch points, e.g. the conformal (and complex) map $g(z) = z^2$ has a degnerate branch point at $z = 0$. Despite this, the image will be missing degenerate behavior such as one-dimensional boundaries (unless it is the image of the boundary of definition).

5 The Suprising Existence and Regularity of the Heat Equation

Here we briefly comment as to why the existence and regularity of the heat equation can be considered suprising. Thinking of the heat equation as an evolution in time, one may be lead to the intuition that the solution exists simply by "evolving" forward at each time t; this is the intuition of short time existence for first order ODE. However, here we show that the impact of the spatial derivative u_{xx} on regularity gives us a competing intuition for the solution not existing even under nice conditions. So not only is it suprising that the heat equation has smooth solutions under nice conditions, but that it has solutions at all.

Let us constider the simple case of the heat equation

$$\begin{cases} u_t = u_{xx} & -1 < x < 1 \text{ and } t > 0, \\ u(-1,t) = -2 & t > 0, \\ u(1,t) = -2 & t > 0, \\ u(x,0) = x^2(|x| - 3) & 0 \le x \le 1. \end{cases} \tag{7}$$

Note that the boundary conditions are continuous. Furthermore, the first order boundardy condition $u_{xx}(\pm 1, 0) = 0 = u_t(\pm 1, t)$ is satisfied as well. Note also that $\phi(x) = u(x, 0)$ is in $C^{2,1}$, in particular we have that $\phi_{xx} = 6|x| - 6$ is Lipschitz.

What is suprising about this case is the heat equations regularity and existence in the interior at $x = 0$ and $t > 0$. Consider the first order approximation

$$u(x, 0 + h) \approx u(x, 0) + u_t(x, 0)h, \tag{8}$$

$$= u(x, 0) + u_x x(x, 0)h, \tag{9}$$

$$= \phi(x) + 6(|x| - 1)h. \tag{10}$$

While $\phi(x) \in C^{2,1}([-1,1])$, the last summand (when freezing h) $6(|x|-1)h$ is only in $C^{0,1}([-1,1])$. Therefore, it seems one iteration of the evolution forward in time destroys two derivatives of the spatial differentiability. It is therefore suprising that the solution $u(x,t)$ should be defined or even C^2 for any $t > 0$.

If instead we had started with an initial condition $u(x,0) = \psi(x)$ with $\psi \in C^k([-1,1]$ for any k, we would still run into the same problem. Look at m first order approximations where $2m > k$, and we run into the same problem (actually we may run into the problem slightly before the mth iteration).

6 Duality and Natural Transformations

Eilenberg and Maclane's classic paper introduces the concept of generalized naturality by discussing the classic example of how an isomorphism between a vector space V and its dual V^* is dependent on the basis you choose to construct the isomorphism. They mention that considering the naturality of this ”...clearly involves a simultaneous consideration of all spaces L and all transformations λ connecting them.”

I would like to comment on two points that I think could use more discussion:

1. Why do we consider transformations when the original concept involves basis?
2. Why must we consider the standard contravariant functor?

For point (1), it is enough to simply observe that a basis $\{\beta_i\}$ for an n-dimensional vector space V is equivalent to a linear isomorphism $C_\beta : \mathbb{R}^n \to V$. Therefore our study of dependence of basis leads us to consider such isomorphisms and to expand our consideration to more general maps.

Point (2) is a little more involved. For this we should consider what it means to be ”independent of basis.” We can think of this as being that the transformation ”looks the same” in every basis. What do we precisely mean by how a transformation ”looks” for a given basis? We can think of this as how the transformation acts on the coordinates of the basis.

For a one-form $\omega \in V^*$, this is given by the pull-back by of the linear transformation $C_\beta : \mathbb{R}^n \to V$ that represents the basis $\{\beta_i\}$. Therefore, we are lead to consider duality as a contravariant functor.

Now, if we don't consider naturality, and we consider any map $T : V \to V^*$, then we are lead to the following diagram: where $S_\beta : \mathbb{R} \to \mathbb{R}^*$ is the map

$$\begin{tikzcd}
\mathbb{R}^* & V^* \arrow[l, "C_\beta^*"'] \\
\mathbb{R} \arrow[u, "S_\beta"] \arrow[r, "C_\beta"'] & V \arrow[u, "T"']
\end{tikzcd}$$

induced by C_β, T, and C^*_β.

Then what it means to be "independent of basis" is that this map T_β is actually indepent of β. So we have that there is a map $S = S_\beta$ for all bases $\{\beta_i\}$; we have the diagram:

$$\begin{tikzcd}
\mathbb{R}^* & V^* \arrow[l, "C^*_\beta"'] \\
\mathbb{R} \arrow[u, "S"] \arrow[r, "C_\beta"'] & V \arrow[u, "T"']
\end{tikzcd}$$

References

[1] Luis A. Caffarelli. Regularity of solutions and level surfaces of elliptic equations. *Proceedings of the AMS Centennial Symposium*, 1988.

[2] Luis A. Caffarelli and Alexis F. Vasseur. The de giorgi method for regularity of solutions of elliptic equations and its applications to fluid dynamics. 2010.