Miscellaneous Math Notes

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Contents

1 Analysis

Here we record some general notes on analysis.

1.1 The Inverse Function Theorem

We will prove a weaker version of the inverse function theorem using differential equations. Consider $f : U \to V$. Let us consider the construction of the inverse of f along a curve $\gamma(t) \in V$ with $\gamma(0) = 0$. We wish to construct a curve $\chi(t)$ with $\chi(0) = 0$ and $f \circ \chi(t) = \gamma(t)$. Differentiating, we see that $\chi(t)$ must necessarily satisfy $Df(\chi(t))\chi'(t) = \gamma'(t)$. Therefore,

$$\chi'(t) = Df^{-1}(\chi(t))\gamma'(t). \tag{1}$$

By choosing a family of curves $\gamma(t)$ exhausting a neighborhood of $y = 0$, we may construct f^{-1} from (1).

Proposition 1.1. Let $U, V \subset \mathbb{R}^n$ be open, and let $0 \in U, V$. Let $f : U \to V$ be such that $f \in C^1(U)$, $f(0) = 0$, Df is invertible at every point of U, Df^{-1} is Lipschitz on U.

Then there exists neighborhoods $0 \in U' \subset U$ and $0 \in V' \subset V$ such that f is a bijection of U' and V'. Furthermore, f^{-1} is continuous on V'.

Proof. Consider the family of curves $\gamma_y(t) = ty$. Then $\gamma'(t) = y$. So, from (1), we seek to solve

$$\chi_y'(t) = Df^{-1}(\chi_y(t))y. \tag{2}$$

We reformulate this as a fixed point problem for an integral equation. For any function $\psi(y,t)$ we define the operator $T\psi(y,t)$ by

$$T\psi(y,t) = \int_0^t Df^{-1}(\psi(y,s))y\,ds. \tag{3}$$

Then, we seek to find $\chi(y,t)$ such that $T\chi(y,t) = \chi(y,t)$. Let L be the Lipschitz constant of Df^{-1} so that $|Df^{-1}(x_1) - Df^{-1}(x_2)| \leq L|x_1 - x_2|$. Then we see that

$$|T\psi(y,t) - T\phi(y,t)| \leq \int_0^t L|\psi(y,s) - \phi(y,s)||y|\,ds, \tag{4}$$

$$\leq L\|\psi - \phi\|_{C^0}|ty|. \tag{5}$$

So, for some neighborhood $V' \subset V$, letting $W = V' \times (-2,2)$, we get that T takes a convex neighborhood N in $C^0(W)$ of $\psi(y,t) = 0$ to itself. Furthermore,

$$\|T\psi - T\phi\|_{C^0(W)} \leq \frac{1}{2}\|\psi - \phi\|_{C^0(W)}. \tag{6}$$

Therefore, by the contraction mapping principle there is a unique $\chi(y,t) \in N$ such that $T\chi = \chi$. Define $g(y) = \chi(y,1)$. So, by the definition of $\chi(y,t)$, we have that $f \circ g(y) = y$. Therefore, $g : V' \to U$ is injective.

Now, from the differentiability of f, we have that $f(x) = f(0) + Df(0)x + \mathcal{O}(\|x\|^2)$. Since $Df(0)$ is non-singular, we have $\|Df(0)x\| \geq \|Df^{-1}(0)\|\|x\|$. Hence, for some neighborhood $0 \in U' \subset U$, we have that f is injective on U'. Furthermore, we may take U' such that $f : U' \to V'$. Hence, f and g give continuous bijections between U' and V'. $\square$

Now we extend the regularity.

Proposition 1.2. If $f : U \to V$ is a continuous homeomorphism, $f \in C^1(U)$, and Df non-singular in U. Then, $f^{-1} \in C^1(V)$.

Proof. Let $y = f(x)$ and let us consider restricting h such that $\|Df(x+h) - Df(x)\| < (1/2)\|Df(x)\|$. We have that

$$f(x+h) - f(x) = \int_0^1 Df(x+th)\,dth, \tag{7}$$

$$= \int_0^1 \left(Df(x+h) - Df(x)\right)\,dth + Df(x)h. \tag{8}$$

Now, $\left\|\int_0^1 \left(Df(x+h) - Df(x)\right)\,dt\right\| \leq (1/2)\|Df(x)\|$.

We use the first order approximation of f. Since differentiability is local information, we may assume without loss in generality that $1/C \leq \|Df\| \leq C$ on U. Now, we have that for any $x_1, x_2 \in U$ that $f(x_2) = f(x_1) + Df(x_1)(x_2 -$

$x_1) + \mathcal{O}(\|x_2 - x_1\|^2)$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then we have $y_2 = y_1 + Df(f^{-1}(y_1))(f$

At any point $y \in V$ and any direction w consider the curve $\gamma(t) = y + tw \in V$. Let $f(x) = y$ and consider the direction v such that $Df(x)v = w$. Let $\chi(t) = x + tv \in U$.

From the differentiabilty of $f \circ \chi(t)$, we have that $f \circ \chi(t) = y + wt + \mathcal{O}(t^2)$. Therefore, we get that $g(y + tw) - g(y) = g(f \circ \chi(t) + \mathcal{O}(t^2)) - x$. □

2 Viscosity Solutions

Here we record some notes on the discussion of viscosity solutions discussed in the User's Guide [1]. We adopt the notation of the User's Guide [1].

As a reminder, we seek solutions to an elliptic fully non-linear equation

$$F(x, u, Du, D^2u) = 0, x \in \Omega. \tag{9}$$

An upper semi-continuous function $u(x)$ is a **viscosity sub-solution** if for any $x \in \Omega$ and $(p, X) \in J^+u(x)$ we have that

$$F(x, u, p, X) \leq 0. \tag{10}$$

Similarly, a lower semi-continuous function $u(x)$ is a **viscosity super-solution** if for any $x \in \Omega$ and $(p, X) \in J^-u(x)$ we have that

$$F(x, u, p, X) \geq 0. \tag{11}$$

A **viscosity solution** u is both a viscosity sub-solution and viscosity super-solution.

2.1 Motivation

Here we consider an example motivating the need to consider weak viscosity solutions to partial differential equations.

Example 2.1. For $0 < \epsilon < 0$ consider the elliptic operators $L_\epsilon : C^2(\mathbb{R}) \to C(\mathbb{R})$ defined by

$$L_\epsilon u = -u'' + \frac{\epsilon}{(\epsilon + x^2)^2} u. \tag{12}$$

Considering the User's Guide's [1] discussion of Hamilton-Jacobi-Bellman equations, we seek to solve the problem

$$\begin{cases} \sup\limits_{0<\epsilon<1} L_\epsilon u = 0 & x \in (-1, 1), \\ u(-1) = u(1) = 1. \end{cases} \tag{13}$$

Now, note that the solution to each Dirichlet problem

$$\begin{cases} L_\epsilon u_\epsilon = 0 & x \in (-1, 1), \\ u_\epsilon(-1) = u_\epsilon(1) = 1, \end{cases} \tag{14}$$

is the smooth function $u_\epsilon(x) = (\epsilon+1)^{-1/2}\sqrt{\epsilon+x^2}$.

Now, let us argue by contradiction that there is no C^2 solution to the Dirichlet problem (13). So assume that $u \in C^2(-1,1)$ is a solution to (13). First, note that at $x=0$, we have that $L_\epsilon u(0) = -u''(0) + \frac{1}{\epsilon}u(0)$. Therefore, we must have that $u(0) \leq 0$ and

$$-u''(0) + u(0) = 0. \tag{15}$$

Now, consider if $x \neq 0$ and $u \leq 0$. Then $\sup\limits_{0<\epsilon<0} L_\epsilon u(x) = -u''$. Therefore $u''(x) = 0$. Now, if $u(0) < 0$, then from the continuity of u'' and equation (15), we have that $u(0) = 0$ and $u''(0) = 0$. So, for and x such that $u \leq 0$, we have that $u''(0) = 0$. Therefore, since the boundary values are positive, we must have that $u \geq 0$ on all of $(-1,1)$.

Now consider the case that $u(x) > 0$. The maximum of $\frac{\epsilon}{(\epsilon+x^2)}$ is at $\epsilon = x^2 \in (0,1)$. Therefore,

$$-u''(x) + \frac{1}{4x^2}u = 0. \tag{16}$$

The general solution to this Euler type equation is readily seen to be $u(x) = c_1|x|^{(1+\sqrt{2})/2} + c_2|x|^{(1-\sqrt{2})/2}$, for $x \neq 0$.

Now, let $y = \max\{x \in (-1,1) : u(x) = 0\}$; we know that $y \geq 0$ and $u'(y) = 0$. Now, if $y > 0$, then we know that (16) has a unique solution to the initival value problem $u(y) = u'(y) = 0$ on the interval $(y,1)$. However, this solution is clearly $u = 0$, but this is impossible for the given boundary conditions. Therefore, we see that $u > 0$ on $(0,1)$. Similarly, $u > 0$ on $(-1,0)$. So $x = 0$ is the only zero of u.

Now we know that on $(0,1)$, $u(x)$ is of the form $u(x) = c_1x^{(1+\sqrt{2})/2} + c_2x^{(1-\sqrt{2})/2}$. However, we have that $u(0) = 0$. Therefore the continuity of u gives us that $c_2 = 0$. The boundary condition $u(1) = 1$ then gives us that $c_1 = 1$. Similar analysis on $(-1,0)$ then gives us that $u(x) = |x|^{(1+\sqrt{2})/2}$. However, this solution is not C^2.

2.2 The "Closures" of the semi-jets, $\overline{J}^+u$ and $\overline{J}^-u$

Here we give some worked examples showing that the "closure" of the semi-jets $\overline{J}^+u$ and $\overline{J}^-u$ are not given by projections of the closures of the graphs of the semi-jets, e.g. the graph $\{(x, J^+u(x))\}$ and the graph $\{(x, u(x), J^+u(x))\}$.

First, a simple example to illustrate the case of continuous functions.

Example 2.2. Consider the function $u : \mathbb{R} \to \mathbb{R}$ defined by $u(x) = |x|$. Note that $J^+u : \mathcal{R} \to \mathcal{P}\mathbb{R}^2$. We see that

$$J^+u(x) = \begin{cases} \{-1\} \times [0,\infty) & x < 0, \\ \emptyset & x = 0, \\ \{1\} \times [0,\infty) & x > 0. \end{cases} \tag{17}$$

We have that

$$\overline{J}^{+}u(x) = \begin{cases} \{-1\} \times [0, \infty) & x < 0, \\ \{-1, 1\} \times [0, \infty) & x = 0, \\ \{1\} \times [0, \infty) & x > 0. \end{cases} \tag{18}$$

Now, let us consider the subtle details of the definition of $\overline{J}^{+}u$ for upper semi-continuous $u(x)$.

Example 2.3. Let $u : \mathbb{R} \to \mathbb{R}$ be defined by

$$u(x) = \begin{cases} -x & x < 0, \\ 1 + x & x \geq 0. \end{cases} \tag{19}$$

We have that

$$J^{+}u(x) = \begin{cases} \{-1\} \times [0, \infty) & x < 0, \\ \Big((1, \infty) \times \mathbb{R}\Big) \cup \Big(\{1\} \times [0, \infty)\Big) & x = 0, \\ \{1\} \times [0, \infty) & x > 0. \end{cases} \tag{20}$$

We then have that $\overline{J}^{+}u(x) = J^{+}u(x)$. Note that $\overline{J}^{+}u(0)$ doesn't include $\{-1\} \times [0, \infty)$, because $\limsup_{x \to 0-} u(x) < u(0)$.

References

[1] M. G. Crandall, H. Ishii, and P.-L. Lions. user's guide to viscosity solutions of second order partial differential equations. *ArXiv Mathematics e-prints*, June 1992.