Miscellaneous Math Notes

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1 Analysis

Here we record some general notes on analysis.

1.1 Regularity of Eigenvalues and Eigenvectors

Consider a symmetric matrix function $M(x) : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n$. At each point $x \in \mathbb{R}^m$, we have that the eigenvalues λ_i of $M(x)$ are real valued. Therefore, we may uniquely indentify them using an ordering $\lambda_1 \leq ... \leq \lambda_m$. In general, if M is smooth (or even analytic) each λ_i may be only Lipschitz.

The Lipschitz nature of each λ_i may be demonstrated by identifying each λ_i with an inf-sup variational problem. In particular we have that

$$\lambda_i = \inf_{\dim V = i} \sup_{v \in V, \, \|v\|=1} \langle v, Mv \rangle, \tag{1}$$

where V is an i-dimensional sub-space of $\mathbb{R}^n$.

Although the eigenvlaues are at least Lipschitz, there is no guarantee of the regularity of the eigenvectors, not even continuity. Problems occur when eigenvalues have multiplicty greater than one.

Consider the matrix function

$$M(t) = \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix}. \tag{2}$$

We see that the eigenfunctions are solutions to $\lambda^2 - (a+c)\lambda + ac - b^2 = 0$. Therefore,

$$\lambda_i = \frac{a + c \pm \sqrt{(a+c)^2 - 4(ac - b^2)}}{2}, \tag{3}$$

$$= \frac{a + c \pm \sqrt{(a-c)^2 + 4b^2}}{2} \tag{4}$$

We see that λ_i are continuous and their derivatives don't exist only when $(a-c)^2 + 4b^2 = 0$. For the case of $M(t)$ smooth, when this occurs we actually have that $(a-c)^2 + 4b^2 = \mathcal{O}((t - t_0)^2)$. So, we see that λ_i will be Lipschitz.

Example 1.1. Consider the explicit example

$$M(t) = \begin{pmatrix} 1+t & t \\ t & 1-t \end{pmatrix}. \tag{5}$$

Then, we have that

$$\lambda_i = 1 \pm \sqrt{2}|t|. \tag{6}$$

1.2 The Inverse Function Theorem

We will prove a weaker version of the inverse function theorem using differential equations. Consider $f : U \to V$. Let us consider the construction of the inverse of f along a curve $\gamma(t) \in V$ with $\gamma(0) = 0$. We wish to construct a curve $\chi(t)$ with $\chi(0) = 0$ and $f \circ \chi(t) = \gamma(t)$. Differentiating, we see that $\chi(t)$ must necessarily satisfy $Df(\chi(t))\chi'(t) = \gamma'(t)$. Therefore,

$$\chi'(t) = Df^{-1}(\chi(t))\gamma'(t). \tag{7}$$

By choosing a family of curves $\gamma(t)$ exhausting a neighborhood of $y = 0$, we may construct f^{-1} from (7).

Proposition 1.1. Let $U, V \subset \mathbb{R}^n$ be open, and let $0 \in U, V$. Let $f : U \to V$ be such that $f \in C^1(U)$, $f(0) = 0$, Df is invertible at every point of U, Df^{-1} is Lipschitz on U.

Then there exists neighborhoods $0 \in U' \subset U$ and $0 \in V' \subset V$ such that f is a bijection of U' and V'. Furthermore, f^{-1} is continuous on V'.

Proof. Consider the family of curves $\gamma_y(t) = ty$. Then $\gamma'(t) = y$. So, from (7), we seek to solve

$$\chi_y'(t) = Df^{-1}(\chi_y(t))y. \tag{8}$$

We reformulate this as a fixed point problem for an integral equation. For any function $\psi(y, t)$ we define the operator $T\psi(y, t)$ by

$$T\psi(y, t) = \int_0^t Df^{-1}(\psi(y, s))y \, ds. \tag{9}$$

Then, we seek to find $\chi(y,t)$ such that $T\chi(y,t) = \chi(y,t)$. Let L be the Lipschitz constant of Df^{-1} so that $|Df^{-1}(x_1) - Df^{-1}(x_2)| \leq L|x_1 - x_2|$. Then we see that

$$|T\psi(y,t) - T\phi(y,t)| \leq \int_0^t L|\psi(y,s) - \phi(y,s)||y|\, ds, \tag{10}$$

$$\leq L\|\psi - \phi\|_{C^0}|ty|. \tag{11}$$

So, for some neighborhood $V' \subset V$, letting $W = V' \times (-2, 2)$, we get that T takes a convex neighborhood N in $C^0(W)$ of $\psi(y,t) = 0$ to itself. Furthermore,

$$\|T\psi - T\phi\|_{C^0(W)} \leq \frac{1}{2}\|\psi - \phi\|_{C^0(W)}. \tag{12}$$

Therefore, by the contraction mapping principle there is a unique $\chi(y,t) \in N$ such that $T\chi = \chi$. Define $g(y) = \chi(y, 1)$. So, by the definition of $\chi(y,t)$, we have that $f \circ g(y) = y$. Therefore, $g : V' \to U$ is injective.

Now, from the differentiability of f, we have that $f(x) = f(0) + Df(0)x + \mathcal{O}(\|x\|^2)$. Since $Df(0)$ is non-singular, we have $\|Df(0)x\| \geq \|Df^{-1}(0)\|\|x\|$. Hence, for some neighborhood $0 \in U' \subset U$, we have that f is injective on U'. Furthermore, we may take U' such that $f : U' \to V'$. Hence, f and g give continuous bijections between U' and V'. □

Now we extend the regularity.

Proposition 1.2. If $f : U \to V$ is a continuous homeomorphism, $f \in C^1(U)$, and Df non-singular in U. Then, $f^{-1} \in C^1(V)$.

Proof. Let $y = f(x)$ and let us consider restricting h such that $\|Df(x+h) - Df(x)\| < (1/2)\|Df(x)\|$. We have that

$$f(x+h) - f(x) = \int_0^1 Df(x+th)\, dth, \tag{13}$$

$$= \int_0^1 (Df(x+h) - Df(x))\, dth + Df(x)h. \tag{14}$$

Now, $\left\|\int_0^1 (Df(x+h) - Df(x))\, dt\right\| \leq (1/2)\|Df(x)\|$.

We use the first order approximation of f. Since differentiability is local information, we may assume without loss in generality that $1/C \leq \|Df\| \leq C$ on U. Now, we have that for any $x_1, x_2 \in U$ that $f(x_2) = f(x_1) + Df(x_1)(x_2 - x_1) + \mathcal{O}(\|x_2 - x_1\|^2)$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then we have $y_2 = y_1 + Df(f^{-1}(y_1))(f$

At any point $y \in V$ and any direction w consider the curve $\gamma(t) = y + tw \in V$. Let $f(x) = y$ and consider the direction v such that $Df(x)v = w$. Let $\chi(t) = x + tv \in U$.

From the differentiabilty of $f \circ \chi(t)$, we have that $f \circ \chi(t) = y + wt + \mathcal{O}(t^2)$. Therefore, we get that $g(y + tw) - g(y) = g(f \circ \chi(t) + \mathcal{O}(t^2)) - x$. □

2 Viscosity Solutions

Here we record some notes on the discussion of viscosity solutions discussed in the User's Guide [1]. We adopt the notation of the User's Guide [1].

As a reminder, we seek solutions to an elliptic fully non-linear equation

$$F(x, u, Du, D^2u) = 0, x \in \Omega. \tag{15}$$

An upper semi-continuous function $u(x)$ is a **viscosity sub-solution** if for any $x \in \Omega$ and $(p, X) \in J^+u(x)$ we have that

$$F(x, u, p, X) \leq 0. \tag{16}$$

Similarly, a lower semi-continuous function $u(x)$ is a **viscosity super-solution** if for any $x \in \Omega$ and $(p, X) \in J^-u(x)$ we have that

$$F(x, u, p, X) \geq 0. \tag{17}$$

A **viscosity solution** u is both a viscosity sub-solution and viscosity super-solution.

2.1 Motivation

Here we consider an example motivating the need to consider weak viscosity solutions to partial differential equations.

Example 2.1. For $0 < \epsilon < 0$ consider the elliptic operators $L_\epsilon : C^2(\mathbb{R}) \to C(\mathbb{R})$ defined by

$$L_\epsilon u = -u'' + \frac{\epsilon}{(\epsilon + x^2)^2} u. \tag{18}$$

Considering the User's Guide's [1] discussion of Hamilton-Jacobi-Bellman equations, we seek to solve the problem

$$\begin{cases} \sup\limits_{0<\epsilon<1} L_\epsilon u = 0 & x \in (-1, 1), \\ u(-1) = u(1) = 1. \end{cases} \tag{19}$$

Now, note that the solution to each Dirichlet problem

$$\begin{cases} L_\epsilon u_\epsilon = 0 & x \in (-1, 1), \\ u_\epsilon(-1) = u_\epsilon(1) = 1, \end{cases} \tag{20}$$

is the smooth function $u_\epsilon(x) = (\epsilon + 1)^{-1/2}\sqrt{\epsilon + x^2}$.

Now, let us argue by contradiction that there is no C^2 solution to the Dirichlet problem (19). So assume that $u \in C^2(-1, 1)$ is a solution to (19). First, note that at $x = 0$, we have that $L_\epsilon u(0) = -u''(0) + \frac{1}{\epsilon}u(0)$. Therefore, we must have that $u(0) \leq 0$ and

$$-u''(0) + u(0) = 0. \tag{21}$$

Now, consider if $x \neq 0$ and $u \leq 0$. Then $\sup_{0<\epsilon<0} L_\epsilon u(x) = -u''$. Therefore $u''(x) = 0$. Now, if $u(0) < 0$, then from the continuity of u'' and equation (21), we have that $u(0) = 0$ and $u''(0) = 0$. So, for and x such that $u \leq 0$, we have that $u''(0) = 0$. Therefore, since the boundary values are positive, we must have that $u \geq 0$ on all of $(-1, 1)$.

Now consider the case that $u(x) > 0$. The maximum of $\frac{\epsilon}{(\epsilon + x^2)}$ is at $\epsilon = x^2 \in (0, 1)$. Therefore,

$$-u''(x) + \frac{1}{4x^2} u = 0. \tag{22}$$

The general solution to this Euler type equation is readily seen to be $u(x) = c_1 |x|^{(1+\sqrt{2})/2} + c_2 |x|^{(1-\sqrt{2})/2}$, for $x \neq 0$.

Now, let $y = \max\{x \in (-1, 1) : u(x) = 0\}$; we know that $y \geq 0$ and $u'(y) = 0$. Now, if $y > 0$, then we know that (22) has a unique solution to the initival value problem $u(y) = u'(y) = 0$ on the interval $(y, 1)$. However, this solution is clearly $u = 0$, but this is impossible for the given boundary conditions. Therefore, we see that $u > 0$ on $(0, 1)$. Similarly, $u > 0$ on $(-1, 0)$. So $x = 0$ is the only zero of u.

Now we know that on $(0, 1)$, $u(x)$ is of the form $u(x) = c_1 x^{(1+\sqrt{2})/2} + c_2 x^{(1-\sqrt{2})/2}$. However, we have that $u(0) = 0$. Therefore the continuity of u gives us that $c_2 = 0$. The boundary condition $u(1) = 1$ then gives us that $c_1 = 1$. Similar analysis on $(-1, 0)$ then gives us that $u(x) = |x|^{(1+\sqrt{2})/2}$. However, this solution is not C^2.

2.2 The "Closures" of the semi-jets, $\overline{J}^+ u$ and $\overline{J}^- u$

Here we give some worked examples showing that the "closure" of the semi-jets $\overline{J}^+ u$ and $\overline{J}^- u$ are not given by projections of the closures of the graphs of the semi-jets, e.g. the graph $\{(x, J^+u(x))\}$ and the graph $\{(x, u(x), J^+u(x))\}$.

First, a simple example to illustrate the case of continuous functions.

Example 2.2. Consider the function $u : \mathbb{R} \to \mathbb{R}$ defined by $u(x) = |x|$. Note that $J^+u : \mathcal{R} \to \mathcal{P}\mathbb{R}^2$. We see that

$$J^+u(x) = \begin{cases} \{-1\} \times [0, \infty) & x < 0, \\ \emptyset & x = 0, \\ \{1\} \times [0, \infty) & x > 0. \end{cases} \tag{23}$$

We have that

$$\overline{J}^+ u(x) = \begin{cases} \{-1\} \times [0, \infty) & x < 0, \\ \{-1, 1\} \times [0, \infty) & x = 0, \\ \{1\} \times [0, \infty) & x > 0. \end{cases} \tag{24}$$

Now, let us consider the subtle details of the definition of $\overline{J}^+ u$ for upper semi-continuous $u(x)$.

Example 2.3. Let $u : \mathbb{R} \to \mathbb{R}$ be defined by

$$u(x) = \begin{cases} -x & x < 0, \\ 1 + x & x \geq 0. \end{cases} \tag{25}$$

We have that

$$J^+u(x) = \begin{cases} \{-1\} \times [0, \infty) & x < 0, \\ \Big((1, \infty) \times \mathbb{R}\Big) \cup \Big(\{1\} \times [0, \infty)\Big) & x = 0, \\ \{1\} \times [0, \infty) & x > 0. \end{cases} \tag{26}$$

We then have that $\overline{J}^+u(x) = J^+u(x)$. Note that $\overline{J}^+u(0)$ doesn't include $\{-1\} \times [0, \infty)$, because $\limsup_{x \to 0-} u(x) < u(0)$.

References

[1] M. G. Crandall, H. Ishii, and P.-L. Lions. user's guide to viscosity solutions of second order partial differential equations. *ArXiv Mathematics e-prints*, June 1992.