Miscellaneous Math Notes

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Contents

1 Analysis

Here we record some general notes on analysis.

1.1 Different Types of Scalings and Convergence of Limits

Consider a function $f(x,y) : \mathbb{R}^2 \to \mathbb{R}^2$. To determine if the limit $\lim\limits_{\vec{x}\to\vec{0}} f(x,y)$ exists, it is NOT enough to consider the limit of the function along any line approaching $\vec{0}$. To construct such a counter example, it is convenient to consider a function that is homogeneous for parabolic scalings $(x,y) \to (\lambda x, \lambda^2 y)$. In particular we want the function to remain constant under such parabolic scalings.

To construct such a function, start with an ordinary homogeneous function $g(u,v)$ and set $f(x,y) = g(x^2, y)$. We immediately try

$$g(u,v) = \frac{uv}{u^2+v^2}.$$

So,
$$f(x,y) = \frac{x^2 y}{x^4 + y^2}.$$
We see that $f(x,y)$ has the desired properties. In particular, along any line through the origin, the function approaches 0. However, along the parabola $y = mx^2$, we have that $f(x,y) = m/(1+m^2)$, which is non-constant in m.

1.2 Regularity of Eigenvalues and Eigenvectors

Consider a symmetric matrix function $M(x) : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n$. At each point $x \in \mathbb{R}^m$, we have that the eigenvalues λ_i of $M(x)$ are real valued. Therefore, we may uniquely indentify them using an ordering $\lambda_1 \leq ... \leq \lambda_m$. In general, if M is smooth (or even analytic) each λ_i may be only Lipschitz.

The Lipschitz nature of each λ_i may be demonstrated by identifying each λ_i with an inf-sup variational problem. In particular we have that
$$\lambda_i = \inf_{\dim V = i} \sup_{v \in V,\, \|v\|=1} \langle v, Mv \rangle, \tag{1}$$
where V is an i-dimensional sub-space of $\mathbb{R}^n$.

Although the eigenvlaues are at least Lipschitz, there is no guarantee of the regularity of the eigenvectors, not even continuity. Problems occur when eigenvalues have multiplicty greater than one.

Consider the matrix function
$$M(t) = \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix}. \tag{2}$$
We see that the eigenfunctions are solutions to $\lambda^2 - (a+c)\lambda + ac - b^2 = 0$. Therefore,
$$\begin{aligned} \lambda_i &= \frac{a + c \pm \sqrt{(a+c)^2 - 4(ac - b^2)}}{2}, & (3) \\ &= \frac{a + c \pm \sqrt{(a-c)^2 + 4b^2}}{2} & (4) \end{aligned}$$

We see that λ_i are continuous and their derivatives don't exist only when $(a-c)^2 + 4b^2 = 0$. For the case of $M(t)$ smooth, when this occurs we actually have that $(a-c)^2 + 4b^2 = \mathcal{O}((t-t_0)^2)$. So, we see that λ_i will be Lipschitz.

Example 1.1. Consider the explicit example
$$M(t) = \begin{pmatrix} 1+t & t \\ t & 1-t \end{pmatrix}. \tag{5}$$

Then, we have that
$$\lambda_i = 1 \pm \sqrt{2}|t|. \tag{6}$$

For $\lambda_1 = 1 - \sqrt{2}|t|$, we see that the choices of normalized eigenvector are

$$v = \frac{\pm 1}{\sqrt{t^2 + (t + \sqrt{2}|t|)^2}} \begin{pmatrix} -t \\ t + \sqrt{2}|t| \end{pmatrix}. \tag{7}$$

So then we have the one sided limit

$$\lim_{t \to 0+} v = \frac{\pm 1}{\sqrt{1 + (1 + \sqrt{2})^2}} \begin{pmatrix} -1 \\ 1 + \sqrt{2} \end{pmatrix}, \tag{8}$$

and the other one sided limit is

$$\lim_{t \to 0-} v = \frac{\mp 1}{\sqrt{1 + (1 - \sqrt{2})^2}} \begin{pmatrix} -1 \\ 1 - \sqrt{2} \end{pmatrix}. \tag{9}$$

We see that there is no choice of ± 1 in equation (8) and no choice of ± 1 in equation (9) that will make v continuous at $t = 0$. Therefore any choice of normalized eigenvector must have a discontinuity at $t = 0$.

The only regularity of the eigenvalues λ_i that we are guaranteed is Lipschitz; however, symmetric functions f of the eigenvalues have the same regularity as f and the matrix $M(t)$.

Proposition 1.1. Let $M(t) \in C^\infty$ be a $n \times n$ symmetric matrix function, and let $f : \mathbb{R}^n \to \mathbb{R}$ be a symmetric and smooth function. Then $g(t) = f(\lambda_1(t), ..., \lambda_n(t))$ is smooth.

Proof. Consider any fixed time t_0. Let $x_0 = (\lambda_1(t_0), ..., \lambda_n(t_0))$. Since f is a symmetric function, we have that the Nth Taylor Expansion of f around x_0 is of the form:

$$f(x_0 + h) = \sum_{i=0}^{N} + \mathcal{O}(|h|^{N+1}), \tag{10}$$

where $p_i(x)$ is a symmetric homogeneous polynomial of degree i.

□

1.3 Regularity of Systems of ODE

Here we consider how the regularity results for systems of ODE are somewhat surprising. As we discussed in subsection **??**, smooth (and even analytic) matrix functions $M(t)$ have in general at most Lipschitz regularity in their eigenvalues, and their normalized eigenvectors don't need to be continuous.

Now, for a linear system of ODE's such as

$$\vec{x}' = M(t)\vec{x}, \tag{11}$$

the solutions are given by $\vec{x}(t) = \exp\left(\int_0^t M(s)\, ds\right) \vec{x}_0$. From this expression, the regularity of the solutions is immediately clear from the analycity of exp.

However, for the moment forget this fact. Without the use of the exp function, we may be inclined to solve this system by diagonalizing $M(t)$ for each t, and then attempt to derive equations for the coordinates in this time dependent eigenbasis.

Let us consider the case where the normalized eigenvectors are constant on $\{t > 0\}$, constant on $\{t < 0\}$, and have a discontinuity at $t = 0$. For example, we could consider

$$M(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \phi(t) \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + \psi(t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{12}$$

where $\operatorname{suppt} \phi \subset [0, 1]$ and $\operatorname{suppt} \psi \subset [-1, 0]$. For such a situation we get unlinked ODE for the eigen coordinates of $\vec{x}$, and furthermore these are regular on $\mathbb{R}$ (the equations for the coefficients extends smoothly across $t = 0$). However, the discontinuity of the normalized eigenvectors should give us pause as to whether these solutions combined to give a regular $\vec{x}(t)$ at $t = 0$.

From this analysis, the regularity isn't particular clear. Luckily, the closed form solution $u(t) = \exp\left(\int_0^t M(s)\, ds\right) \vec{x}_0$ proves the regularity to us very readily. Compare this to the situation of linear elliptic equations with non-constant coefficients, e.g.

$$\begin{cases} a_{ij}(x)\partial^2_{ij} u = 0 & x \in \Omega \\ u(x) = \phi(x) & x \in \partial\Omega. \end{cases} \tag{13}$$

Here a closed form solution isn't readily available to give us regularity. That must be done using more advanced methods. Until the regularity is proven, it isn't clear that it should even be the case.

1.4 The Inverse Function Theorem

We will prove a weaker version of the inverse function theorem using differential equations. Consider $f : U \to V$. Let us consider the construction of the inverse of f along a curve $\gamma(t) \in V$ with $\gamma(0) = 0$. We wish to construct a curve $\chi(t)$ with $\chi(0) = 0$ and $f \circ \chi(t) = \gamma(t)$. Differentiating, we see that $\chi(t)$ must necessarily satisfy $Df(\chi(t))\chi'(t) = \gamma'(t)$. Therefore,

$$\chi'(t) = Df^{-1}(\chi(t))\gamma'(t). \tag{14}$$

By choosing a family of curves $\gamma(t)$ exhausting a neighborhood of $y = 0$, we may construct f^{-1} from (14).

Proposition 1.2. Let $U, V \subset \mathbb{R}^n$ be open, and let $0 \in U, V$. Let $f : U \to V$ be such that $f \in C^1(U)$, $f(0) = 0$, Df is invertible at every point of U, Df^{-1} is Lipschitz on U.

Then there exists neighborhoods $0 \in U' \subset U$ and $0 \in V' \subset V$ such that f is a bijection of U' and V'. Furthermore, f^{-1} is continuous on V'.

Proof. Consider the family of curves $\gamma_y(t) = ty$. Then $\gamma'(t) = y$. So, from (14), we seek to solve

$$\chi_y'(t) = Df^{-1}(\chi_y(t))y. \tag{15}$$

We reformulate this as a fixed point problem for an integral equation. For any function $\psi(y,t)$ we define the operator $T\psi(y,t)$ by

$$T\psi(y,t) = \int_0^t Df^{-1}(\psi(y,s))y\,ds. \tag{16}$$

Then, we seek to find $\chi(y,t)$ such that $T\chi(y,t) = \chi(y,t)$. Let L be the Lipschitz constant of Df^{-1} so that $|Df^{-1}(x_1) - Df^{-1}(x_2)| \le L|x_1 - x_2|$. Then we see that

$$|T\psi(y,t) - T\phi(y,t)| \le \int_0^t L|\psi(y,s) - \phi(y,s)||y|\,ds, \tag{17}$$

$$\le L\|\psi - \phi\|_{C^0}|ty|. \tag{18}$$

So, for some neighborhood $V' \subset V$, letting $W = V' \times (-2,2)$, we get that T takes a convex neighborhood N in $C^0(W)$ of $\psi(y,t) = 0$ to itself. Furthermore,

$$\|T\psi - T\phi\|_{C^0(W)} \le \frac{1}{2}\|\psi - \phi\|_{C^0(W)}. \tag{19}$$

Therefore, by the contraction mapping principle there is a unique $\chi(y,t) \in N$ such that $T\chi = \chi$. Define $g(y) = \chi(y,1)$. So, by the definition of $\chi(y,t)$, we have that $f \circ g(y) = y$. Therefore, $g : V' \to U$ is injective.

Now, from the differentiability of f, we have that $f(x) = f(0) + Df(0)x + \mathcal{O}(\|x\|^2)$. Since $Df(0)$ is non-singular, we have $\|Df(0)x\| \ge \|Df^{-1}(0)\|\|x\|$. Hence, for some neighborhood $0 \in U' \subset U$, we have that f is injective on U'. Furthermore, we may take U' such that $f : U' \to V'$. Hence, f and g give continuous bijections between U' and V'. □

Now we extend the regularity.

Proposition 1.3. If $f : U \to V$ is a continuous homeomorphism, $f \in C^1(U)$, and Df non-singular in U. Then, $f^{-1} \in C^1(V)$. PROOF IS INCORRECT OR INCOMPLETE. DISREGARD PROPOSITION FOR NOW.

Proof. THIS PROOF IS INCORRECT/INCOMPLETE. Let $y = f(x)$ and let us consider restricting h such that $\|Df(x+h) - Df(x)\| < (1/2)\|Df(x)\|$. We have that

$$f(x+h) - f(x) = \int_0^1 Df(x+th)\,dth, \tag{20}$$

$$= \int_0^1 (Df(x+h) - Df(x))\,dth + Df(x)h. \tag{21}$$

Now, $\left\|\int_0^1 (Df(x+h) - Df(x))\, dt\right\| \leq (1/2)\|Df(x)\|$.

We use the first order approximation of f. Since differentiability is local information, we may assume without loss in generality that $1/C \leq \|Df\| \leq C$ on U. Now, we have that for any $x_1, x_2 \in U$ that $f(x_2) = f(x_1) + Df(x_1)(x_2 - x_1) + \mathcal{O}(\|x_2 - x_1\|^2)$. Let $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then we have $y_2 = y_1 + Df(f^{-1}(y_1))(f$

At any point $y \in V$ and any direction w consider the curve $\gamma(t) = y + tw \in V$. Let $f(x) = y$ and consider the direction v such that $Df(x)v = w$. Let $\chi(t) = x + tv \in U$.

From the differentiabilty of $f \circ \chi(t)$, we have that $f \circ \chi(t) = y + wt + \mathcal{O}(t^2)$. Therefore, we get that $g(y + tw) - g(y) = g(f \circ \chi(t) + \mathcal{O}(t^2)) - x$. $\square$

1.5 Weak Derivatives

Let us motivate the Sobolev notion of weak derivative. For weak derivatives in the Sobolev sense, the weak derivative captures how the function changes in the average. Let us first examine the integration by parts formula for weak derivatives:

$$a \tag{22}$$

2 PDE

Here we record some general notes on PDE.

2.1 Weak Solutions

Let us motivate the use of weak solutions. Let $D = \{\|x\| = 1\| \subset \mathbb{R}^2$ be the unit disk in $\mathbb{R}^2$. Let us consider the case of harmonic functions solving the dirichlet problem on the unit disk $D \subset \mathbb{R}^2$,

$$\begin{cases} \triangle u = 0 & x \in D, \\ u = \phi & x \in \partial D. \end{cases} \tag{23}$$

Let us define the energy of u by

$$E(u) = \int_D |\nabla u|^2 \, dA. \tag{24}$$

Now, for $u \in C^2(D) \cap C(\bar{D})$ and for $\phi \in C^2(D) \cap C(\bar{D})$ with $\phi = 0$ on ∂D one can show that $E(u + \phi) \geq E(u)$, with equality only if $\phi = 0$. That is, u must be a strict minimizer of $E(u)$ for functions $\{w \in C^2(D) \cap C(\bar{D}) | w = \phi \text{ on } \partial D\}$.

However, now note that the energy $E(u)$ is still well-defined for $V = \{u \in C^2(D \setminus \{0\}) \cap C(\bar{D} \setminus \{0\}) : E(u) < \infty\}$. So, we may ask ouselves to search for minimizers in the space of functions V. After all, why should nature be restricted to continuous functions?

For example, the potential function $u(x,y) = \log r$ is harmonic on $D \setminus \{0\}$ and continuous up to the boundary of the disk. Although $E(u) = \infty$ and $u \notin V$, it does add to the motivation to consider the possibility of harmonic functions with singularities. The function u has a singularity at $(x,y) = 0$. We in fact have that $\triangle u = C\delta(x,y)$ as distributions, and so in some sense it is not true that $u = \log r$ is harmonic on $D \setminus \{0\}$. However, for now, let us just consider solutions in the classical sense of derivatives.

Let us first consider the standard example of u bounded and harmonic on $D \setminus \{0\}$.

Proposition 2.1. Let $u \in C^2(D \setminus \{0\}) \cap C(\bar{D} \setminus \{0\})$ be bounded and harmonic in $D \setminus \{0\}$. Then, u is harmonic in D, and so $u \in C^\infty(D)$.

Proof. Consider the function v solving the dirichlet problem

$$\begin{cases} \triangle v = 0 & x \in D, \\ v = u & x \in \partial D. \end{cases} \tag{25}$$

By considering $u - v$, without loss in generality, we may suppose $u = 0$ on ∂D.

Let $|u| \leq M$ on $D \setminus \{0\}$. Let w_δ^+ be the solution to the Dirichlet problem

$$\begin{cases} \triangle w_\delta^+ = 0 & x \in D \setminus D_\delta(0), \\ w_\delta^+ = 0 & x \in \partial D, \\ w_\delta^+ = M & x \in \partial D_\delta(0). \end{cases} \tag{26}$$

A computation gives

$$w_\delta^+ = M \frac{\log r}{\log \delta}. \tag{27}$$

Similary, for the boundary condition $w_\delta^- = -M$ on $\partial D_\delta(0)$, consider the function

$$w_\delta^- = -M \frac{\log r}{\log \delta}. \tag{28}$$

By the maximum principle for harmonic functions, we have that $w_\delta^- \leq u \leq w_\delta^+$ on $D \setminus D_\delta(0)$. So, for some fixed $0 < \delta_0 < 1$, we get that $|u| \leq M \log \delta_0 / -\log \delta$ on $D \setminus D_\delta(0)$. This is true as $\delta \to 0$, and so we get that $u = 0$ on $D \setminus 0$. Therefore, u is trivially harmonic on D. $\square$

Proposition 2.2. Let $u \in C^2(D \setminus \{0\}) \cap C(\bar{D} \setminus \{0\})$ be $\mathcal{O}(|\log r|^\alpha)$ for $0 < \alpha < 1$ and harmonic in $D \setminus \{0\}$. Then, u is harmonic in D, and so $u \in C^\infty(D)$.

Proof. The proof is similar except we now use

$$w_\delta^\pm = \pm K(-\log \delta)^\alpha \frac{\log r}{\log \delta}. \tag{29}$$

$\square$

Note that the result isn't true for one-dimensional singularities. For a certain choice of branch cuts, the function

$$u(x,y) = \operatorname{Im}\left[(z-1)\frac{\log(z-1)}{\log(z+1)}\right], \tag{30}$$

is harmonic and bounded on $D_2 \setminus ([-1,1] \times \{0\})$. Note that as $z_0 \to 0$, the similar harmonic function

$$u(x,y) = \operatorname{Im}\left[(z-1)\frac{\log(z-z_0)}{\log(z+z_0)}\right], \tag{31}$$

converges on compact sets of $D^2 \setminus \{0\}$ to the smooth harmonic function $u(x,y) = y$. So, as the singular set is collapsing from the line segment between $-z_0$ and z_0 to the point $\{0\}$, we see that the harmonic functions converge to a smooth harmonic function. This is sort of expected from the removability of isolated singularities for bounded harmonic functions.

2.2 Estimates

Estimates can be used for showing existence of solutions (see the use of Schauder Estimates and the Continuity Method to prove the existence of solutions to elliptic pde), and they may also be used to show control over how our solutions depend on the coefficients in our equations.

Example 2.1. Consider the boundary value problem

$$\begin{cases} y'' - y = f_\epsilon(x) & 0 < x < 1, \\ y(0) = 0, \\ y(1) = 0, \end{cases} \tag{32}$$

where $f(x)$ is the piecewise function

$$f_\epsilon(x) = \begin{cases} \frac{x}{\epsilon} & 0 \le x < \epsilon, \\ 2 - \frac{x}{\epsilon} & \epsilon \le x < 2\epsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{33}$$

Now, note that $f_\epsilon \to 0$ in a pointwise manner, but $f_\epsilon \not\to 0$ in C^0. So, how much convergence do we need to ensure that the solutions $y \to 0$ as $\epsilon \to 0$? Let us consider this using the Laplace transform.

Now we consider finding a particular solution y_p, i.e. a solution to $y_p'' - y_p = 0$ satisfying $y_p(0) = y_p'(0) = f_\epsilon$. Letting $Y_p(s) = \mathcal{L} y_p$ and $F_\epsilon(s) = \mathcal{L} f$, we see that $(s^2-1)Y_p = F_\epsilon$. Therefore, we have that $Y_p = (s^2-1)^{-1} F_\epsilon$. So, we get that

$$y_p = \int_0^x \sinh(x-\tau) f_\epsilon(\tau)\, d\tau. \tag{34}$$

Therefore,

$$y = c_1 \cosh x + c_2 \sinh x + \int_0^x \sinh(x-\tau) f_\epsilon(\tau)\, d\tau. \tag{35}$$

Using the boundary conditions, we get that $c_1 = 0$ and

$$c_2(\epsilon) = \frac{-1}{\sinh 1} \int_0^1 \sinh(1-\tau) f_\epsilon(\tau)\, d\tau. \tag{36}$$

Since $f_\epsilon \to 0$ in L^1, we see that the solutions $y \to 0$ in C^0. Furthermore,

$$y' = c_2(\epsilon) \cosh x + \int_0^x \cosh(x-\tau) f_\epsilon(\tau)\, d\tau. \tag{37}$$

So, we see that $y \to 0$ in C^1 as $\epsilon \to 0$. However, note that $y \not\to 0$ in C^2. Note that this is to be expected from Schauder estimates since $f \not\to 0$ in C^α for any $0 < \alpha < 1$.

Example 2.2. I THINK THERE IS A MISTAKE IN THIS EXAMPLE. Let us consider the related example of solving the boundary value problem

$$\begin{cases} y'' - y = f_\epsilon(x) & 0 < x < 1, \\ y(0) = 0, \\ y(1) = 0, \end{cases} \tag{38}$$

where f_ϵ is the piecewise function

$$f(x) = \begin{cases} \frac{x}{\epsilon^2} & 0 < x < \epsilon, \\ \frac{2}{\epsilon} - \frac{x}{\epsilon^2} & \epsilon < x < 2\epsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

Note that $f \not\to 0$ in L^1.

Let us solve this in a piecewise manner. We see that

$$y = \begin{cases} -\frac{x}{\epsilon^2} + c_1 \cosh x + c_2 \sinh x & 0 < x < \epsilon, \\ \frac{x}{\epsilon^2} - \frac{2}{\epsilon} + c_3 \cosh x + c_4 \sinh x & \epsilon < x < 2\epsilon, \\ c_5 \cosh x + c_6 \sinh x & 2\epsilon < x < 1. \end{cases} \tag{40}$$

To solve for the constants c_i, we impose the two boundary conditions plus the four conditions from demanding that $y \in C^1$.Let

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & \cosh 1 & \sinh 1 \end{pmatrix}, \tag{41}$$

and

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{42}$$

Then we have that our six conditions give that

$$(M + \epsilon N + \mathcal{O}(\epsilon^2))\vec{c} = \begin{pmatrix} 0 \\ 0 \\ 2/\epsilon^2 \\ 0 \\ -1/\epsilon^2 \\ 0 \end{pmatrix}. \tag{43}$$

Now, note that one can solve

$$M\vec{c}_0 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \\ -1 \\ 0 \end{pmatrix} \tag{44}$$

rather directly (without even using row reduction) to get that

$$\vec{c}_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \tag{45}$$

Therefore, we have that

$$(I + M^{-1}N\epsilon + \mathcal{O}(\epsilon^2))\vec{c} = \frac{1}{\epsilon^2}\vec{c}_0, \tag{46}$$

and so

$$\vec{c} = \frac{1}{\epsilon^2}\vec{c}_0 - \frac{1}{\epsilon}M^{-1}N\vec{c}_0 + \mathcal{O}(1). \tag{47}$$

Now,

$$N\vec{c}_0 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}. \tag{48}$$

Now we solve to get that

$$M^{-1}N\vec{c}_0 = \begin{pmatrix} 0 \\ \cosh 1/\sinh 1 \\ -2 \\ \cosh 1/\sinh 1 \\ -1 \\ \cosh 1/\sinh 1 \end{pmatrix} \tag{49}$$

So, we see that

$$y = \begin{cases} -\frac{x}{\epsilon^2} + \frac{1}{\epsilon^2}\sinh x + \frac{1}{\epsilon}\frac{\cosh 1}{\sinh 1}\sinh x + \mathcal{O}(1) & 0 < x < \epsilon, \\ \frac{x}{\epsilon^2} - \frac{2}{\epsilon} - \frac{1}{\epsilon^2}\sinh x - \frac{2}{\epsilon}\cosh x + \frac{1}{\epsilon}\frac{\cosh 1}{\sinh 1}\sinh x + \mathcal{O}(1) & \epsilon < x < 2\epsilon, \\ -\frac{1}{\epsilon}\cosh x + \frac{1}{\epsilon}\frac{\cosh 1}{\sinh 1}\sinh x + \mathcal{O}(1) & 2\epsilon < x < 1. \end{cases} \tag{50}$$

And we may simplify some of the order to get

$$y = \begin{cases} \mathcal{O}(1) & 0 < x < \epsilon, \\ -\frac{4}{\epsilon} + \mathcal{O}(1) & \epsilon < x < 2\epsilon, \\ -\frac{1}{\epsilon} + \mathcal{O}(1) & 2\epsilon < x < 1. \end{cases} \tag{51}$$

So we see that y doesn't even converge to 0 in a pointwise manner as $\epsilon \to 0$.

2.3 Holder Spaces for Estimates

For different types of PDE's, one obtains estimates in different types of Holder Spaces. For example, the natural Holder spaces are different for Laplace's operator, the heat operator, and parabolic equations with specific degeneracies such as those considered in Krylov[2].

For Poisson's equation $\triangle u = f$, the natural Holder spaces are the standard holder spaces given by the semi-norms

$$\|u\|_\alpha = \frac{|u(x) - u(y)|}{\|x-y\|^\alpha}. \tag{52}$$

For the heat equation $u_t - \triangle u = f$, the natural Holder spaces are the parabolic holder spaces given by the semi-norms

$$\|u\|_\alpha = \frac{|u(x,t) - u(y,s)|}{\|x-y\|^{2\alpha} + |t-s|^\alpha}. \tag{53}$$

Here, we see that the regularity in the spatial x-direction is twice as good as the regularity in the time direction.

Consider the parabolic equation $x^2 u_t = u_{xx} + x^2 u_{yy} + f$, which is of the form considered by Krylov[2]. For this equation the natural Holder spaces are given by the semi-norms

$$\|u\|_\alpha = \frac{|u(x,y,t) - u(z,w,s)|}{|x-z|^{4\alpha} + |y-w|^{2\alpha} + |t-s|^\alpha}. \tag{54}$$

Here, we see that the regularity in the spatial x-direction is twice as good as the regularity in the spatial y-direction, and the regularity in the x-direction is four times as good as the time direction. Note that the x-direction is the "uniformly elliptic direction" and the y-direction is the "elliptic degeneracy direction."

Now, we discuss some heuristic reasons for why these are the appropriate Holder Spaces. Let us consider the equation

$$x^2 u_t = u_{xx} + x^2 u_{yy} + f, \tag{55}$$

as seen in Krylov[2]. Also, consider the case that we have constructed some type of Holder estimate on $B_1(0)$,

$$\|u\|_\alpha \le C(\|f\|_C + \|u\|_C). \tag{56}$$

Note that the choice of semi-norm on (x, y, t) is not important as all semi-norms are comparable on a fixed scale (maybe after changing α for a particular choice in semi-norm). For example,

$$\frac{|x|^\alpha + |y|^\alpha}{|x|^\alpha + |y|^{\alpha/2}}, \tag{57}$$

and

$$\frac{|x|^{2\alpha} + |y|^\alpha}{|x|^\alpha + |y|^\alpha}, \tag{58}$$

are both bounded on $B_1(0)$.

We wish to find a choice in semi-norm on (x, y, t) such that we can use estimates (56) on $B_1(0)$ to create estimates for functions u satisfying (55) on $B_R(0)$. So therefore, let $u(x, y, t)$ satisfy equation (55) on $B_R(0)$. We use a scaling $x = R^a x'$, $y = R^b y'$, and $t = R^c t'$ to get that $u'(x', y', t') = u(R^a x', R^b y', R^c t')$ satisfies

$$R^{2a-c}(x')^2 u'_{t'} = R^{-2a} u'_{x'x'} + R^{2a-2b}(x')^2 u'_{y'y'} + f', \tag{59}$$

where we want all of the powers of R to be the same, and $f'(x', y't') = f(x, y, t)$. We get that $2a - c = -2a = 2a - 2b$. So, we get that $c = 4a$ and $b = 2a$. So on $B_1(0)$, we have

$$(x')^2 u'_{t'} = u'_{x'x'} + (x')^2 u'_{y'y'} + R^{2a} f'. \tag{60}$$

So from our estimates on $B_1(0)$, we have that $\|u'\|_{C^\alpha(B_1)} \le C(R^{2a}\|f\|_{C^0} + \|u\|_{C^0})$. For the appropriate choice of semi-norm on (x', y', t), the Holder norm on u' gives a Holder norm on u:

$$\frac{1}{R^{-4a\alpha}} \frac{|u(x, y, t) - u(z, w, s)|}{|x - z|^{4\alpha} + |y - w|^{2\alpha} + |t - s|^\alpha} \le C(R^{2a}\|f\|_{C^0} + \|u\|_{C^0}). \tag{61}$$

So we get that for the appropriate choice of semi-norm on (x, y, t) that

$$\|u\|_{C^\alpha(W_R)} \le CR^{-4a\alpha}(R^{2a}\|f\|_{C^0} + \|u\|_{C^0}). \tag{62}$$

Taking $a = 1$, we get on $W_R = \{x^4 + y^2 + |t| \le R\}$ that

$$\|u\|_{C^\alpha(W_R)} \le CR^{-4\alpha}(R^2\|f\|_{C^0} + \|u\|_{C^0}). \tag{63}$$

3 Viscosity Solutions

Here we record some notes on the discussion of viscosity solutions discussed in the User's Guide [1]. We adopt the notation of the User's Guide [1].

As a reminder, we seek solutions to an elliptic fully non-linear equation

$$F(x, u, Du, D^2u) = 0, x \in \Omega. \tag{64}$$

An upper semi-continuous function $u(x)$ is a **viscosity sub-solution** if for any $x \in \Omega$ and $(p, X) \in J^+u(x)$ we have that

$$F(x, u, p, X) \leq 0. \tag{65}$$

Similarly, a lower semi-continuous function $u(x)$ is a **viscosity super-solution** if for any $x \in \Omega$ and $(p, X) \in J^-u(x)$ we have that

$$F(x, u, p, X) \geq 0. \tag{66}$$

A **viscosity solution** u is both a viscosity sub-solution and viscosity super-solution.

3.1 Motivation

Here we consider an example motivating the need to consider weak viscosity solutions to partial differential equations.

Example 3.1. For $0 < \epsilon < 0$ consider the elliptic operators $L_\epsilon : C^2(\mathbb{R}) \to C(\mathbb{R})$ defined by

$$L_\epsilon u = -u'' + \frac{\epsilon}{(\epsilon + x^2)^2} u. \tag{67}$$

Considering the User's Guide's [1] discussion of Hamilton-Jacobi-Bellman equations, we seek to solve the problem

$$\begin{cases} \sup\limits_{0<\epsilon<1} L_\epsilon u = 0 & x \in (-1, 1), \\ u(-1) = u(1) = 1. \end{cases} \tag{68}$$

Now, note that the solution to each Dirichlet problem

$$\begin{cases} L_\epsilon u_\epsilon = 0 & x \in (-1, 1), \\ u_\epsilon(-1) = u_\epsilon(1) = 1, \end{cases} \tag{69}$$

is the smooth function $u_\epsilon(x) = (\epsilon + 1)^{-1/2}\sqrt{\epsilon + x^2}$.

Now, let us argue by contradiction that there is no C^2 solution to the Dirichlet problem (68). So assume that $u \in C^2(-1, 1)$ is a solution to (68). First, note that at $x = 0$, we have that $L_\epsilon u(0) = -u''(0) + \frac{1}{\epsilon}u(0)$. Therefore, we must have that $u(0) \leq 0$ and

$$-u''(0) + u(0) = 0. \tag{70}$$

Now, consider if $x \neq 0$ and $u \leq 0$. Then $\sup_{0<\epsilon<0} L_\epsilon u(x) = -u''$. Therefore $u''(x) = 0$. Now, if $u(0) < 0$, then from the continuity of u'' and equation (70), we have that $u(0) = 0$ and $u''(0) = 0$. So, for and x such that $u \leq 0$, we have that $u''(0) = 0$. Therefore, since the boundary values are positive, we must have that $u \geq 0$ on all of $(-1, 1)$.

Now consider the case that $u(x) > 0$. The maximum of $\frac{\epsilon}{(\epsilon+x^2)}$ is at $\epsilon = x^2 \in (0, 1)$. Therefore,

$$-u''(x) + \frac{1}{4x^2}u = 0. \tag{71}$$

The general solution to this Euler type equation is readily seen to be $u(x) = c_1|x|^{(1+\sqrt{2})/2} + c_2|x|^{(1-\sqrt{2})/2}$, for $x \neq 0$.

Now, let $y = \max\{x \in (-1, 1) : u(x) = 0\}$; we know that $y \geq 0$ and $u'(y) = 0$. Now, if $y > 0$, then we know that (71) has a unique solution to the initival value problem $u(y) = u'(y) = 0$ on the interval $(y, 1)$. However, this solution is clearly $u = 0$, but this is impossible for the given boundary conditions. Therefore, we see that $u > 0$ on $(0, 1)$. Similarly, $u > 0$ on $(-1, 0)$. So $x = 0$ is the only zero of u.

Now we know that on $(0, 1)$, $u(x)$ is of the form $u(x) = c_1x^{(1+\sqrt{2})/2} + c_2x^{(1-\sqrt{2})/2}$. However, we have that $u(0) = 0$. Therefore the continuity of u gives us that $c_2 = 0$. The boundary condition $u(1) = 1$ then gives us that $c_1 = 1$. Similar analysis on $(-1, 0)$ then gives us that $u(x) = |x|^{(1+\sqrt{2})/2}$. However, this solution is not C^2.

3.2 The "Closures" of the semi-jets, $\overline{J}^+u$ and $\overline{J}^-u$

Here we give some worked examples showing that the "closure" of the semi-jets $\overline{J}^+u$ and $\overline{J}^-u$ are not given by projections of the closures of the graphs of the semi-jets, e.g. the graph $\{(x, J^+u(x))\}$ and the graph $\{(x, u(x), J^+u(x))\}$.

First, a simple example to illustrate the case of continuous functions.

Example 3.2. Consider the function $u : \mathbb{R} \to \mathbb{R}$ defined by $u(x) = |x|$. Note that $J^+u : \mathcal{R} \to \mathcal{P}\mathbb{R}^2$. We see that

$$J^+u(x) = \begin{cases} \{-1\} \times [0, \infty) & x < 0, \\ \emptyset & x = 0, \\ \{1\} \times [0, \infty) & x > 0. \end{cases} \tag{72}$$

We have that

$$\overline{J}^+u(x) = \begin{cases} \{-1\} \times [0, \infty) & x < 0, \\ \{-1, 1\} \times [0, \infty) & x = 0, \\ \{1\} \times [0, \infty) & x > 0. \end{cases} \tag{73}$$

Now, let us consider the subtle details of the definition of $\overline{J}^+u$ for upper semi-continuous $u(x)$.

Example 3.3. Let $u : \mathbb{R} \to \mathbb{R}$ be defined by

$$u(x) = \begin{cases} -x & x < 0, \\ 1 + x & x \geq 0. \end{cases} \tag{74}$$

We have that

$$J^+u(x) = \begin{cases} \{-1\} \times [0, \infty) & x < 0, \\ \Big((1, \infty) \times \mathbb{R}\Big) \cup \Big(\{1\} \times [0, \infty)\Big) & x = 0, \\ \{1\} \times [0, \infty) & x > 0. \end{cases} \tag{75}$$

We then have that $\overline{J}^+u(x) = J^+u(x)$. Note that $\overline{J}^+u(0)$ doesn't include $\{-1\} \times [0, \infty)$, because $\limsup_{x \to 0-} u(x) < u(0)$.

References

[1] M. G. Crandall, H. Ishii, and P.-L. Lions. user's guide to viscosity solutions of second order partial differential equations. *ArXiv Mathematics e-prints*, June 1992.

[2] N. V. Krylov. Boundedly inhomogeneous elliptic and parabolic equations in a domain. *Izv. Akad. Nauk SSSR Ser. Mat.*, 47(1):75–108, 1983.