Calculation for Finding Intersection of Geodesic Ray in S^3 with Euclidean Ball

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We are given $p \in S^3$ with $v \in T_pS^3$ a unit vector. Furthermore, we are given a ball $B_r(q) \subset S^3$ such that $B_r(q) = \{\|x - q\| < r\}$ where $\|x\|$ is the ordinary Euclidean vector norm. We wish to find if the forward geodesic ray in S^3, starting at p and in the direction of v, will intersect $B_r(q)$.

Note that the geodesic ray is parameterized by $x(t) = \cos(t)p + \sin(t)v$ where $t \geq 0$. So, finding the intersection points of $x(t)$ with $B_r(q)$ is equivalent to finding the non-negative solutions $t \geq 0$ to

$$\|\cos(t)p + \sin(t)v - q\|^2 = r^2.$$

Using that $\{p, v\}$ is an orthonormal set of vectors in Euclidean space and that q is a unit vector, we obtain that

$$2 - 2\langle p, q\rangle \cos(t) - 2\langle v, q\rangle \sin(t) = r^2.$$

So we have intersection points if and only if there are solutions to

$$\langle p, q\rangle \cos(t) + \langle v, q\rangle \sin(t) = 1 - \frac{r^2}{2}.$$

This is more informatively expressed as a dot product of vectors in $\mathbb{R}^2$:

$$\begin{pmatrix} \langle p, q\rangle \\ \langle v, q\rangle \end{pmatrix} \cdot \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} = 1 - \frac{r^2}{2}.$$

Since the minimum and maximum of the dot product on the left hand side is $\pm\|(\langle p, q\rangle, \langle v, q\rangle)\|$, we see that there is are intersection points if and only if $\sqrt{\langle p, q\rangle^2 + \langle v, q\rangle^2} \geq |1 - r^2/2|$.

Now let (a, b) be the normalization of $(\langle p, q\rangle, \langle v, q\rangle)$, and let $D = (1 - r^2/2)(\langle p, q\rangle^2 + \langle v, q\rangle^2)^{-1/2}$. So our equation becomes

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} cos(t) \\ sin(t) \end{pmatrix} = D.$$

Note that solutions exist if and only if $|D| \leq 1$.

We now get two sets of equations. We must compare the solutions of each set to find the smallest solution $t > 0$. The first set of equations is

$$\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} = D \begin{pmatrix} a \\ b \end{pmatrix} + \sqrt{1 - D^2} \begin{pmatrix} -b \\ a \end{pmatrix}.$$

The second set of equations is

$$\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} = D \begin{pmatrix} a \\ b \end{pmatrix} - \sqrt{1 - D^2} \begin{pmatrix} -b \\ a \end{pmatrix}.$$

The first set of equations may be solved by finding the solutions $t > 0$ of

$$\tan(t) = \frac{Db + \sqrt{1 - D^2}a}{Da - \sqrt{1 - D^2}b},$$

that are in the correct quadrant. For this, one must check the signs of $Db + \sqrt{1 - D^2}a$ and $Da - \sqrt{1 - D^2}b$.

The second set of equations may be solved by finding the solutions $t > 0$ of

$$\tan(t) = \frac{Db - \sqrt{1 - D^2}a}{Da + \sqrt{1 - D^2}b},$$

that are in the correct quadrant. Thus one must check the signs of $Db - \sqrt{1 - D^2}a$ and $Da + \sqrt{1 - D^2}b$.

Finally, one must compare the solutions of both equations to find the smallest possible solutions $t > 0$. This gives the true intersection time $t > 0$ of the light ray.