

# Calculation for Finding Intersection of Geodesic Ray in $S^3$ with Euclidean Ball

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We are given  $p \in S^3$  with  $v \in T_p S^3$  a unit vector. Furthermore, we are given a ball  $B_r(q) \subset S^3$  such that  $B_r(q) = \{\|x - q\| < r\}$  where  $\|x\|$  is the ordinary Euclidean vector norm. We wish to find if the forward geodesic ray in  $S^3$ , starting at  $p$  and in the direction of  $v$ , will intersect  $B_r(q)$ .

Note that the geodesic ray is parameterized by  $x(t) = \cos(t)p + \sin(t)v$  where  $t \geq 0$ . So, finding the intersection points of  $x(t)$  with  $B_r(q)$  is equivalent to finding the non-negative solutions  $t \geq 0$  to

$$\|\cos(t)p + \sin(t)v - q\|^2 = r^2.$$

Using that  $\{p, v\}$  is an orthonormal set of vectors in Euclidean space and that  $q$  is a unit vector, we obtain that

$$2 - 2\langle p, q \rangle \cos(t) - 2\langle v, q \rangle \sin(t) = r^2.$$

So we have intersection points if and only if there are solutions to

$$\langle p, q \rangle \cos(t) + \langle v, q \rangle \sin(t) = 1 - \frac{r^2}{2}.$$

This is more informatively expressed as a dot product of vectors in  $\mathbb{R}^2$ :

$$\begin{pmatrix} \langle p, q \rangle \\ \langle v, q \rangle \end{pmatrix} \cdot \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} = |1 - \frac{r^2}{2}|.$$

Since the minimum and maximum of the dot product on the left hand side is  $\pm \|(\langle p, q \rangle, \langle v, q \rangle)\|$ , we see that there are intersection points if and only if  $\sqrt{\langle p, q \rangle^2 + \langle v, q \rangle^2} \geq 1 - r^2/2$ .

Now let  $(a, b)$  be the normalization of  $(\langle p, q \rangle, \langle v, q \rangle)$ , and let  $D = (1 - r^2/2)(\langle p, q \rangle^2 + \langle v, q \rangle^2)^{-1/2}$ . So our equation becomes

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} = D.$$

Note that solutions exist if and only if  $|D| \leq 1$ .

We now get two sets of equations. We must compare the solutions of each set to find the smallest solution  $t > 0$ . The first set of equations is

$$\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} = D \begin{pmatrix} a \\ b \end{pmatrix} + \sqrt{1 - D^2} \begin{pmatrix} -b \\ a \end{pmatrix}.$$

The second set of equations is

$$\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix} = D \begin{pmatrix} a \\ b \end{pmatrix} - \sqrt{1 - D^2} \begin{pmatrix} -b \\ a \end{pmatrix}.$$

The first set of equations may be solved by finding the solutions  $t > 0$  of

$$\tan(t) = \frac{Db + \sqrt{1 - D^2}a}{Da - \sqrt{1 - D^2}b},$$

that are in the correct quadrant. For this, one must check the signs of  $Db + \sqrt{1 - D^2}a$  and  $Da - \sqrt{1 - D^2}b$ .

The second set of equations may be solved by finding the solutions  $t > 0$  of

$$\tan(t) = \frac{Db - \sqrt{1 - D^2}a}{Da + \sqrt{1 - D^2}b},$$

that are in the correct quadrant. Thus one must check the signs of  $Db - \sqrt{1 - D^2}a$  and  $Da + \sqrt{1 - D^2}b$ .

Finally, one must compare the solutions of both equations to find the smallest possible solutions  $t > 0$ . This gives the true intersection time  $t > 0$  of the light ray.