Rupture Dynamics and Memory Persistence in Discontinuous Systems

Abstract

We develop a rigorous mathematical framework for analyzing systems exhibiting discontinuous transitions while maintaining information persistence. By introducing the notion of memory trace subspaces and resonance mappings, we establish conditions under which global coherence emerges from local discontinuities. We prove the existence of persistent subspaces and demonstrate applications to quantum mechanics, neural networks, and dynamical systems.

1 Introduction

Discontinuous transitions in physical and mathematical systems often present challenges for traditional analytical methods. This work introduces a formal framework for studying systems where information persists across such discontinuities, which we term "rupture points." Our approach unifies treatments across multiple domains, from quantum measurement to neural computation.

1.1 Motivation

Consider a dynamical system that undergoes discrete transitions. Classical theory typically struggles with such discontinuities, yet many natural systems exhibit robust information preservation across similar transitions. Examples include:

- Quantum systems under measurement
- Neural networks during learning events
- Phase transitions in physical systems
- Cognitive state transitions in biological systems

Our framework provides a unified mathematical treatment of such phenomena.

2 Mathematical Framework

2.1 Preliminary Definitions

Definition 1 (Strong Continuity with Uniform Boundedness)**.** Let X, Y be Banach spaces. A family of operators $\{T(t)\}_{t\in\mathbb{R}}$ from X to Y is said to be strongly continuous and uniformly bounded if:

1. For each $x \in X$, the map $t \mapsto T(t)x$ is continuous in the norm topology of Y.
2. There exists a constant $C > 0$ such that $\|T(t)\| \leq C$ for all $t \in \mathbb{R}$.

Definition 2 (Sectional Limits in the Strong Operator Topology)**.** For an operator-valued function $M(t) : \mathbb{R} \to \mathcal{L}(V)$, where $\mathcal{L}(V)$ denotes the space of bounded linear operators on V, the sectional limits at t_0 are defined as:

$$M(t_0^-) = \lim_{t\to t_0^-} M(t), \quad M(t_0^+) = \lim_{t\to t_0^+} M(t),$$

where the limits are taken in the strong operator topology. That is, for all $x \in V$:

$$\lim_{t\to t_0^\pm} \|M(t)x - M(t_0^\pm)x\| = 0.$$

2.2 Rupture Systems

Definition 3 (Rupture System)**.** A rupture system is a tuple $(V, M, \mathcal{T}, S, \mathcal{F})$, where:

- V is a Banach space.
- $\mathcal{T} = \{t_1, \ldots, t_k\} \subset \mathbb{R}$ is a finite set of rupture points.
- $M : \mathbb{R} \setminus \mathcal{T} \to \mathcal{L}(V)$ is a strongly continuous operator-valued function, where $\mathcal{L}(V)$ is the space of bounded linear operators on V.
- $S \subseteq V$ is a closed subspace (the memory trace subspace) such that S is invariant under $M(t)$ for all $t \notin \mathcal{T}$.
- $\mathcal{F} = \{f_{ij}\}_{i<j}$ is a collection of resonance functions.

The following axioms must be satisfied:

2.3 Axioms

Axiom 1 (Local Invertibility with Uniform Boundedness): For each $t \notin \mathcal{T}$, there exists $\varepsilon > 0$ and a bounded operator-valued function $N(s) : (t-\varepsilon, t+\varepsilon) \to \mathcal{L}(V)$ such that:

1. $M(s)N(s) = N(s)M(s) = I_V$ for all $s \in (t-\varepsilon, t+\varepsilon)$.
2. The map $s \mapsto N(s)$ is strongly continuous.
3. $\|N(s)\| \leq C$, where $C > 0$ is uniform across $(t-\varepsilon, t+\varepsilon)$.
4. At the boundaries $t-\varepsilon$ and $t+\varepsilon$, the operator $N(s)$ remains well-defined and satisfies the above properties.

Axiom 2 (Memory Persistence with Continuity of Projections): There exists a projection $P_S : V \to S$ satisfying:

1. P_S is bounded with $\|P_S\| = 1$.
2. P_S is strongly continuous: for any $x \in V$, the map $x \mapsto P_S(x)$ is continuous in the norm topology.
3. For all $v \in S$ and $t_i \in \mathcal{T}$, both sectional limits $M(t_i^-)v$ and $M(t_i^+)v$ exist in the strong operator topology.
4. $P_S(M(t_i^-)v) = P_S(M(t_i^+)v)$ for all $v \in S$.

Axiom 3 (Resonance): For each pair $t_i < t_j \in \mathcal{T}$, the resonance function $f_{ij} : V \to V$ satisfies:

1. f_{ij} is bounded and strongly continuous.
2. $f_{ij}(M(t_i^+)v) = M(t_j^-)v$ for all $v \in S$.
3. $f_{ij} \circ f_{jk} = f_{ik}$ for all $t_i < t_j < t_k$.

3 Applications

3.1 Quantum Mechanics

In quantum measurement theory, rupture points correspond to state collapse events:

$$M(t) = U(t)PU^\dagger(t).$$

3.2 Neural Networks

Neural networks exhibit rupture points during learning events.

3.3 Dynamical Systems

For a dynamical system (X, ϕ_t), rupture systems characterize bifurcation points.

4 Conclusions and Future Directions

This framework opens several promising directions:

1. Classification of rupture systems by persistent subspace structure.
2. Quantitative measures of information preservation.
3. Applications to machine learning architecture design.
4. Connections to category theory via functor properties.