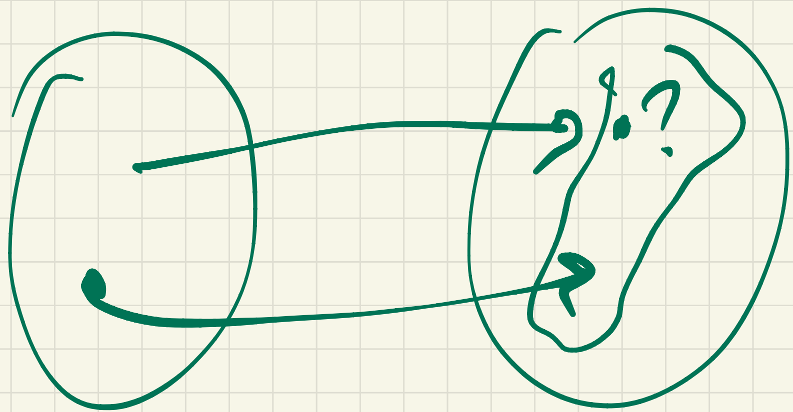
$f(x) = x^2$

$f: \mathbb{R} \to \mathbb{R}$

range$(f) = \mathbb{R}_{\geq 0}$



Codomain vs range

range(f) f(A)

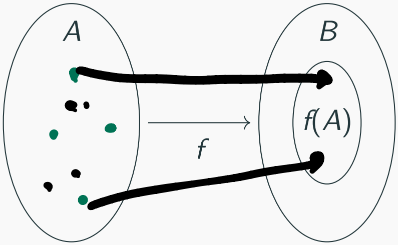
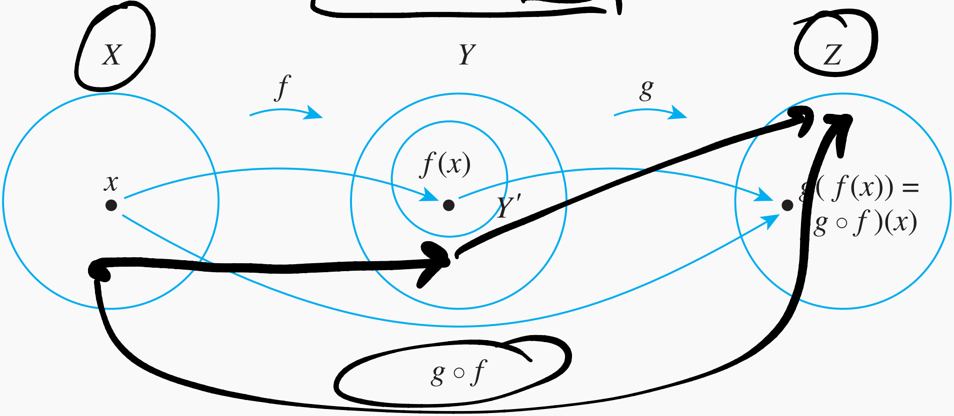


Figure 4: the range of f

Composition of functions

If $f : X \to Y$ and $g : Y \to Z$ are functions, then their **composition** $g \circ f$ is a function from X to Z given by

$$\boxed{(g \circ f)(x) = g(f(x)).}$$



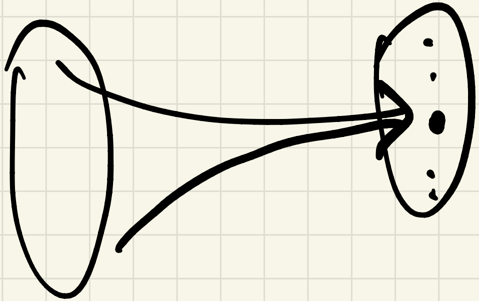
f, g, h

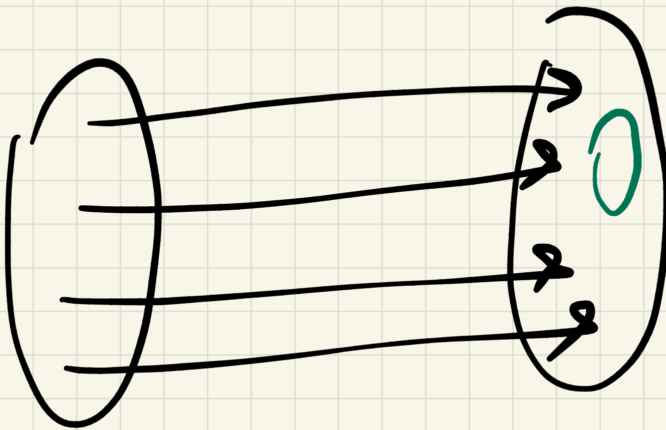
$$(f \circ g)(x) = f(g(x))$$

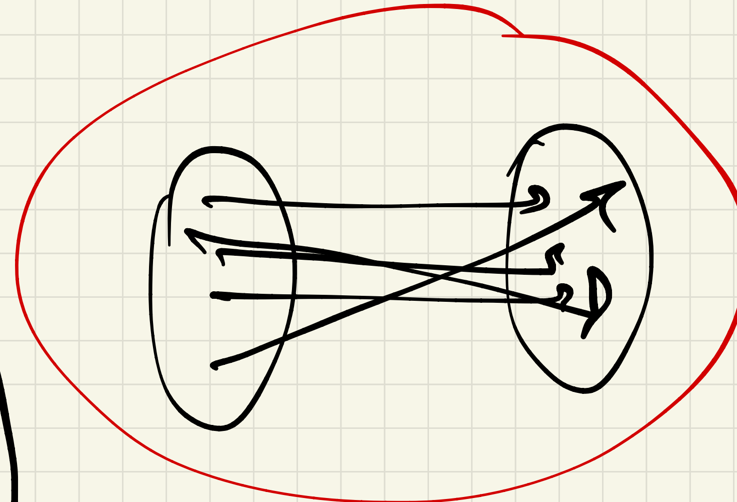
Example

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = 4x + 3$.

- $g \circ f(x) =$ $g(f(x)) = g(x^2) = 4x^2 + 3$
- $f \circ g(x) =$ $f(g(x)) = f(4x+3) = (4x+3)^2$
- $f \circ f(x) =$ x^4
- $g \circ g(x) =$ $16x + 15$







Injective (one-to-one) functions

Definition Let $f : A \to B$ be a function. We call f an *injective* (or *one-to-one*) function if

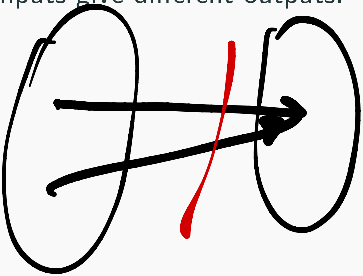
$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \text{ for all } a_1, a_2 \in A.$$

This is logically equivalent to $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ and so injective functions never repeat values. In other words, different inputs give different outputs.

Examples

$f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not injective.

$h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is injective.

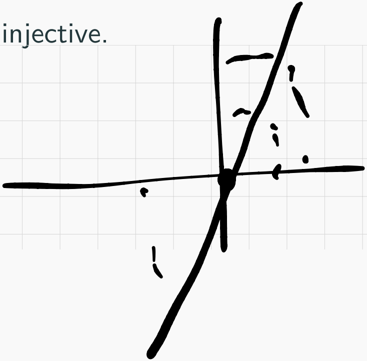


Prove that $f(x)$ is not injective and $h(x) = 2x$ is injective

- $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not injective.

  $f(2) = f(-2)$

- $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is injective.

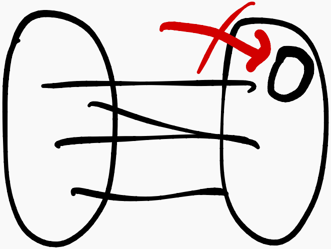


More examples

- $\mathit{first\_letter} : \mathit{People} \to \mathit{Char}$
- $\mathit{ID} : \mathit{People} \to \mathbb{N}$

Surjective (or onto) functions

Definition $f : A \to B$ is *surjective* (or onto) if the range of f coincides with the codomain of f. This means that for every $b \in B$ there exists $a \in A$ with $b = f(a)$.



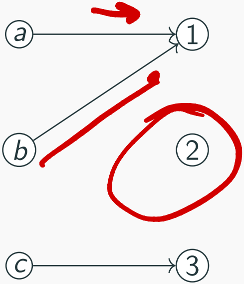
Examples

$f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not surjective.

$h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is not surjective.

$h' : \mathbb{Q} \to \mathbb{Q}$ given by $h'(x) = 2x$ is surjective.

Classify $f\colon \{a, b, c\} \to \{1, 2, 3\}$ given by

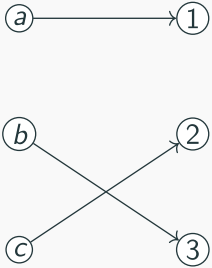


Function

Non-injective

Non-surjective

Classify $g : \{a, b, c\} \to \{1, 2, 3\}$ given by

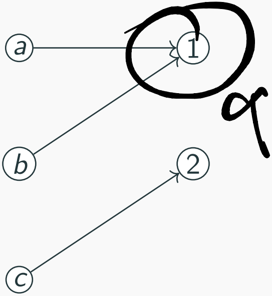


Function

Injective

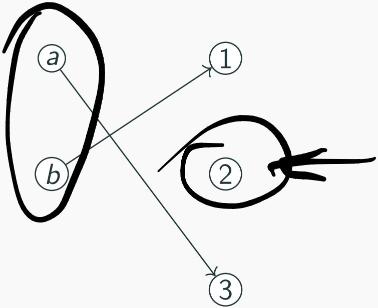
Surjective

Classify $h : \{a, b, c\} \to \{1, 2\}$ given by



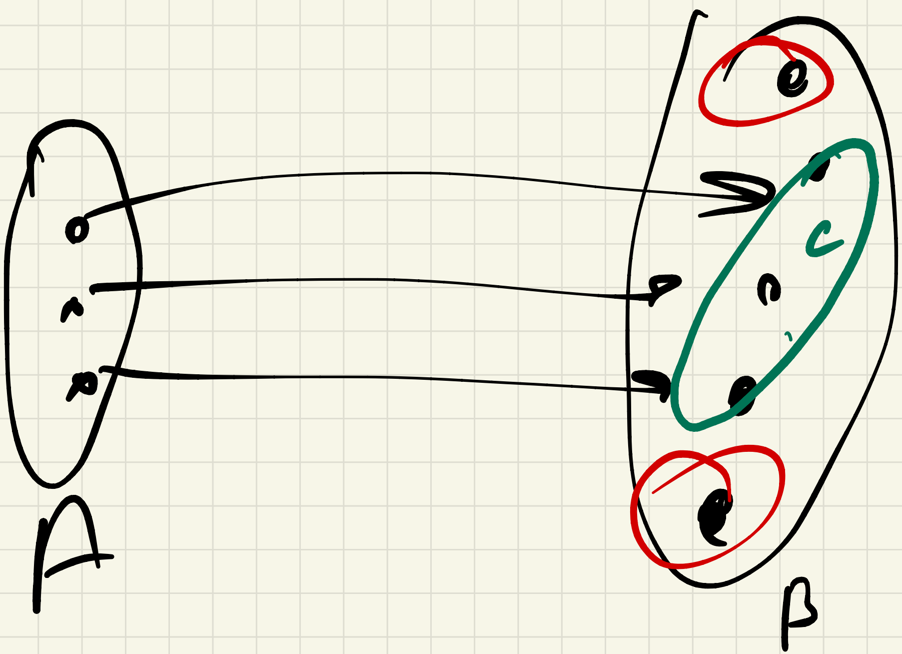
Function
Not injective
Surjective

Classify $h' : \{a, b, c\} \to \{1, 2, 3\}$ given by



$f: A \to B$

$f: A \to C$



$f(A)$

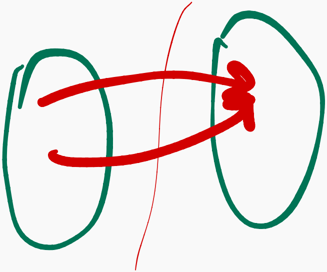
$\mathrm{ran}(A)$

Bijections

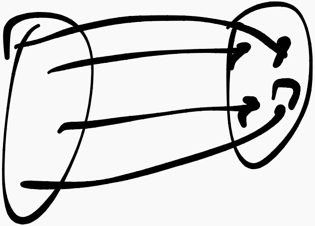
We call f *bijective* if f is both injective and surjective.

Examples

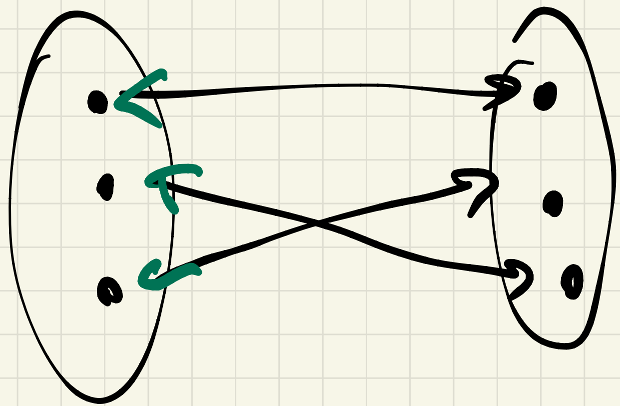
$f : \mathbb{Q} \to \mathbb{Q}$ given by $f(x) = 2x$ is bijective.



Injective



ran(f) = co-domain of f

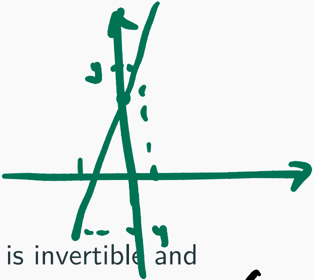


Inverse functions

If f is a bijection from a set X to a set Y, then there is a function f^{-1} from Y to X that "undoes" the action of f; that is, it sends each element of Y back to the element of X that it came from. This function is called the inverse function for f.

Then $f(a) = b$ if, and only if, $f^{-1}(b) = a$.

Example

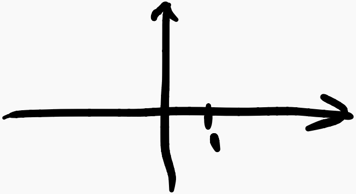


$k : \mathbb{R} \to \mathbb{R}$ given by $k(x) = 4x + 3$ is invertible and

$$k^{-1}(y) = \frac{1}{4}(y - 3).$$

$$y = 4x + 3 \Rightarrow 4x = y - 3 \Rightarrow x = \frac{1}{4}(y-3)$$

Example



Let $A = \{x \mid x \in \mathbb{R}, x \neq 1\}$ and $f : A \to A$ be given by

$$f(x) = \frac{x}{x-1}.$$

Show that f is bijective and determine the inverse function.

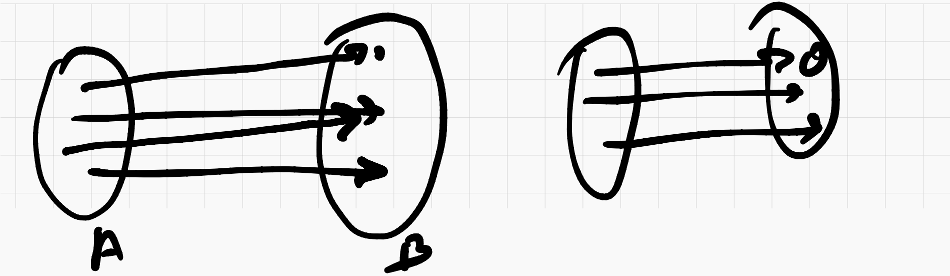
Cardinality of finite sets and functions

Recall: *The cardinality of a* finite *set* S *is the number of elements in* S

A bijection $f\colon S \to \{1, \ldots, n\}$.

For finite sets A and B

- $|A| \geq |B|$ iff there is a surjective function from A to B.
- $|A| \leq |B|$ iff there is a injective function from A to B.
- $|A| = |B|$ iff there is a bijection from A to B.



The pigeonhole principle

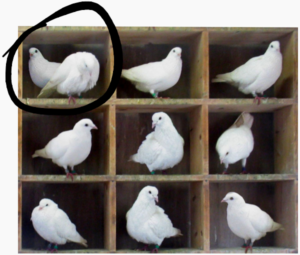
Let $f : A \to B$ be a function where A and B are finite sets.

The *pigeonhole principle* states that if $|A| > |B|$ then at least one value of f occurs more than once.

In other words, we have $f(a) = f(b)$ for some distinct elements a, b of A.

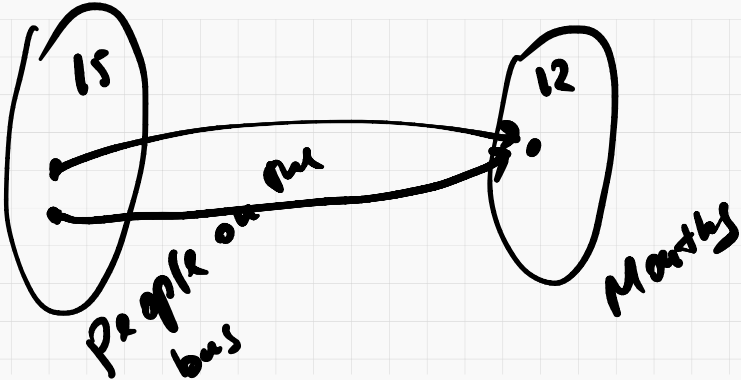
Pigeons and pigeonholes

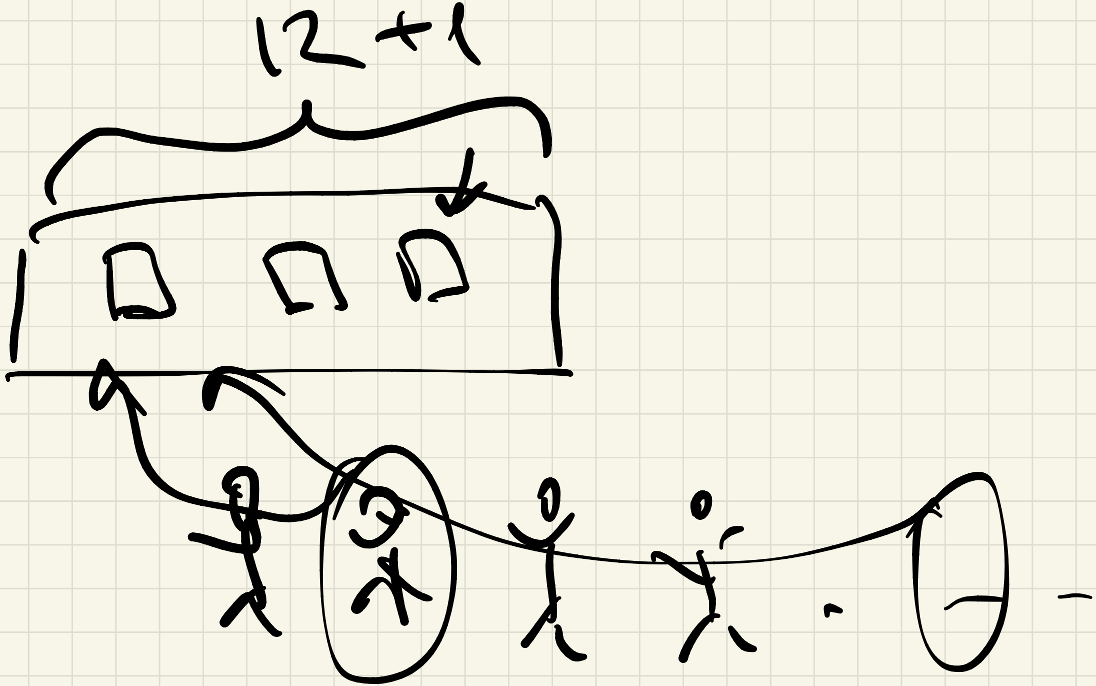
If (N+1) pigeons occupy N holes, then some hole must have at least 2 pigeons.

 Image by McKay from en.wikipedia

Example

Problem. There are 15 people on a bus. Show that at least two of them have a birthday in the same month of the year.




12+1

Example

Problem. How many different surnames must appear in a telephone directory to guarantee that at least two of the surnames begin with the same letter of the alphabet and end with the same letter of the alphabet?

kone ✓

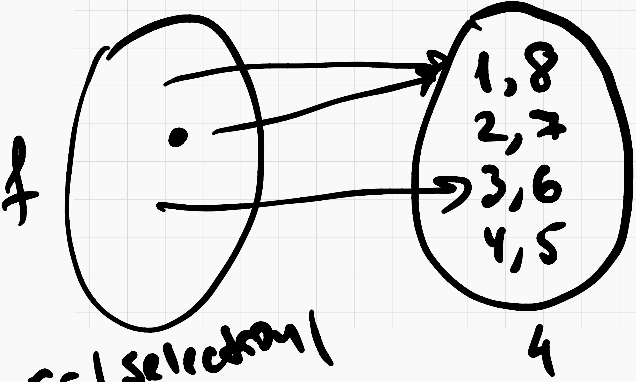
$$\underbrace{26 \times 26}_{} + 1$$

Example

Problem. Five numbers are selected from the numbers 1, 2, 3, 4, 5, 6, 7 and 8. Show that there will always be two of the numbers that sum to 9.

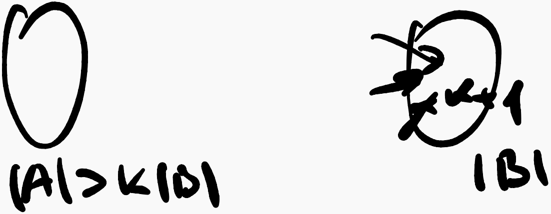
$4+5=9$

$1, 2, 3, 7, 8$

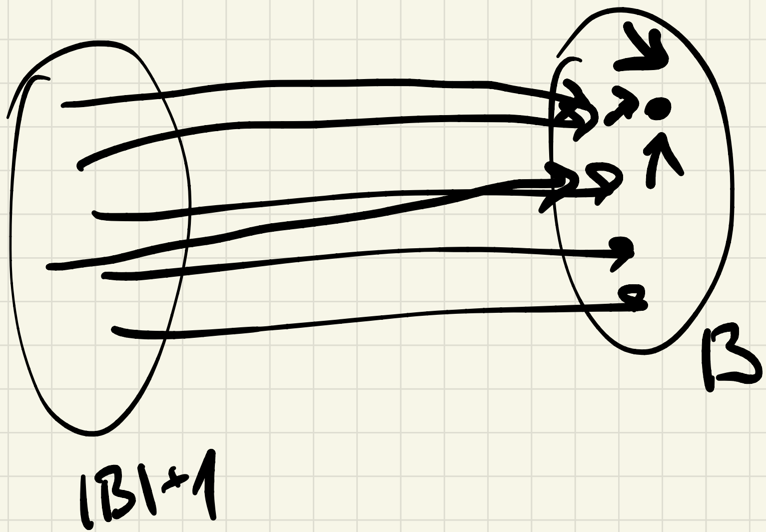


$5 = |\text{selection}|$

Extended pigeonhole principle



Consider a function $f\colon A \to B$ where A and B are finite sets and $|A| > k|B|$ for some natural number k. Then, there is a value of f which occurs at least $k+1$ times.



$k=2$

$2 * |B| + 1$

Example

Problem. How many different surnames must appear in a telephone directory to guarantee that at least five of the surnames begin with the same letter of the alphabet and end with the same letter of the alphabet?

$4 \times 26 \times 26 + 1$

$\cancel{k=5}$

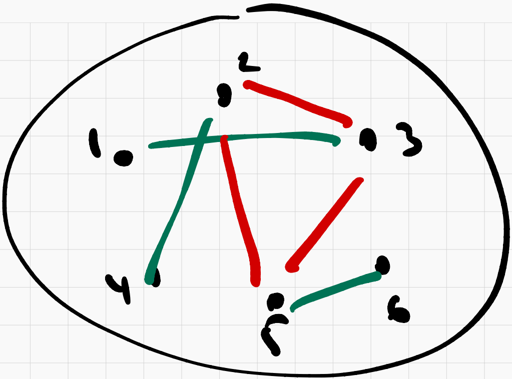
$k=4$

$\cancel{k+1=6}$

$k+1=5$

Example

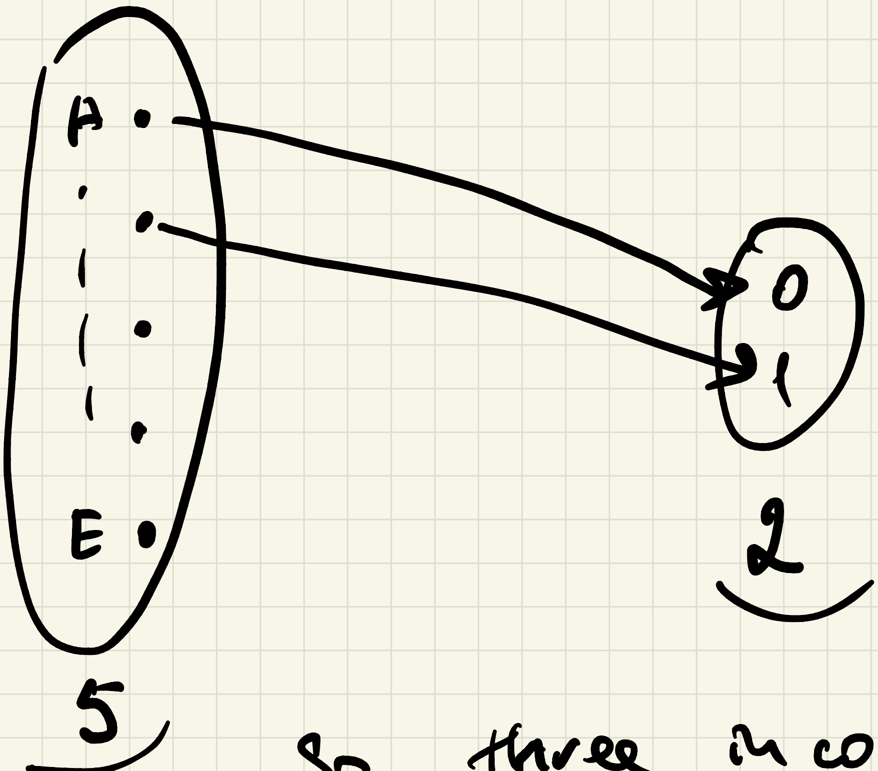
Problem. Show that in any group of six people there are either three who all know each other or three complete strangers.



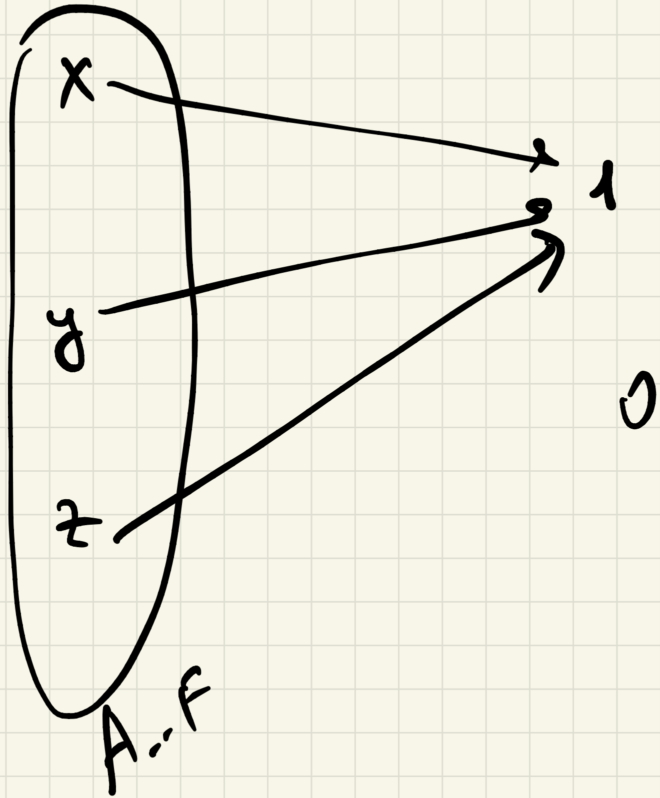
$\underbrace{A, B, C, D, E}, \underbrace{F}$

F

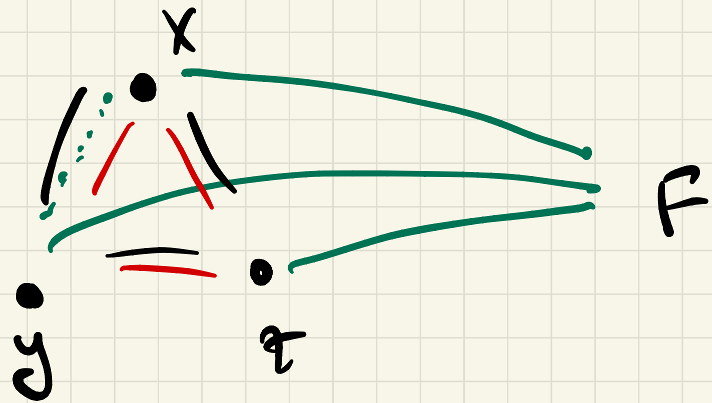
$5 > 2 \times 2$



So, three incoming arrows to either 0 or 1



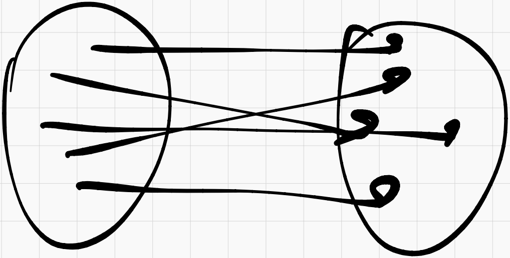
x, y, z know F



Bijections and cardinality

Recall that the cardinality of a finite set is the number of elements in the set.

Sets A and B have the same cardinality iff there is a bijection from A to B.



$f: A \to B$

$|A| = |B|$

Recall: the powerset and bit vectors

Let $S = \{1, 2, \ldots, n\}$ and let B^n be the set of bit strings of length n. The function

$$f \colon Pow(S) \to B^n$$

which assigns each subset A of S to its characteristic vector is a bijection.

We used this to compute the cardinality of the powerset.