Prove that there exists an animal that weighs more that 500 tons

Proof ...

Not true

The opposite!

All animals weigh less than 500 tons

Proof

$$\forall \underbrace{x} \text{ if } \underbrace{\text{x is an animal}} \text{ Then } \underbrace{\text{x weighs less than}} \text{ 200 tons}$$

Universal statements

The vast majority of mathematical statements to be proved are universal. In discussing how to prove such statements, it is helpful to imagine them in a standard form:

$$\forall x \text{ if } P(x) \text{ then } Q(x)$$

For example,

- If a and b are integers then $6a^2b$ is even.

Proving universal statements: The method of exhaustion

Some theorems can be proved by examining relatively small number of examples.

- Prove that $(n+1)^3 \geq 3^n$ if n is a positive integer with $n \leq 4$.
  - $n = 1$ $\quad 8 \geq 3$
  - $n = 2$
  - $n = 3$
  - $n = 4$

- Prove for every natural number n with $n < 40$ that $n^2 + n + 41$ is prime.

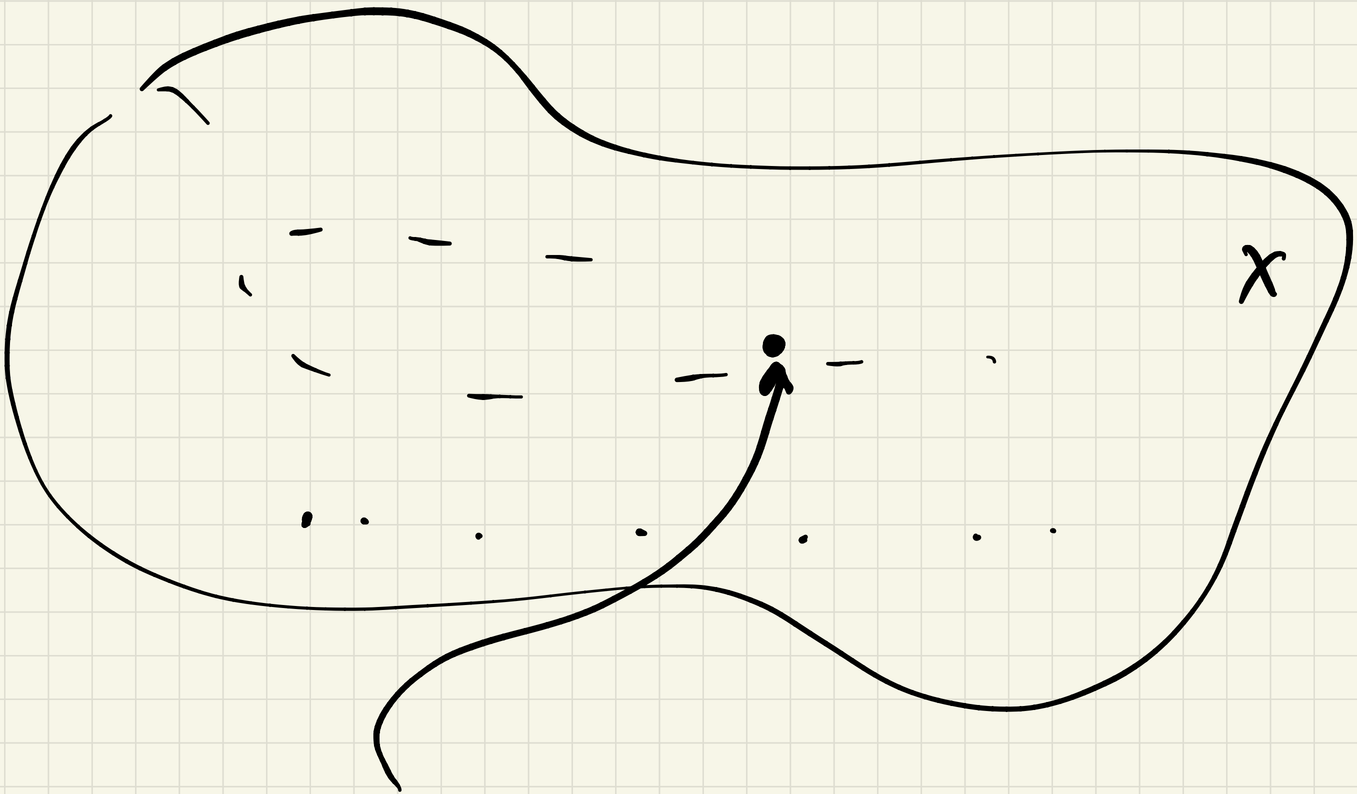
Generalising from the Generic Particular

Motivating example: “Mathematical trick”

Pick any number, add 5, multiply by 4, subtract 6, divide by 2, and subtract twice the original number. The answer is 7.

<table>
<tr><th>Step</th><th>Visual Result</th><th>Algebraic Result</th></tr>
<tr><td>Pick a number.</td><td>☐</td><td>x</td></tr>
<tr><td>Add 5.</td><td>☐|||||</td><td>$x + 5$</td></tr>
<tr><td>Multiply by 4.</td><td>☐|||||
☐|||||
☐|||||
☐|||||</td><td>$(x + 5) \cdot 4 = 4x + 20$</td></tr>
<tr><td>Subtract 6.</td><td>☐||
☐||
☐|||||
☐|||||</td><td>$(4x + 20) - 6 = 4x + 14$</td></tr>
<tr><td>Divide by 2.</td><td>☐||
☐|||||</td><td>$\frac{4x + 14}{2} = 2x + 7$</td></tr>
<tr><td>Subtract twice the original number.</td><td>||
|||||</td><td>$(2x + 7) - 2x = 7$</td></tr>
</table>

x


X

Generalising from the Generic Particular

The most powerful technique for proving a universal statement is one that works regardless of the choice of values for x.

To show that every x satisfies a certain property, suppose x is a particular but arbitrarily chosen and show that x satisfies the property.

Method of direct proof

- Express the statement to be proved in the form
$$\text{“}\forall x,\ \text{if } P(x) \text{ then } Q(x)\text{.”}$$
(This step is often done mentally.)

- Start the proof by supposing x is a particular but arbitrarily chosen element for which the hypothesis $P(x)$ is true.
(This step is often abbreviated “Suppose P(x).”)

- Show that the conclusion $Q(x)$ is true by using definitions, previously established results, and the rules for logical inference.

Prove that the sum of any two even integers is even

Prove that $\forall a, b$ if a and b are even integers that $a+b$ is even

Proof Suppose that a and b are particular but arbitrarily chosen even integers.

By def. of even, $a = 2 \cdot k$, where k is an integer.

$b = 2 \cdot l$, where l is an integer

$$\underline{a+b} = 2k + 2l = \underline{2(k+l)}$$

As k, l are int., so is $k+l$

so by definition of even

$a+b$ is even

QED

$\square$

Prove for all integers n, if n is even then n^2 is even

Proof

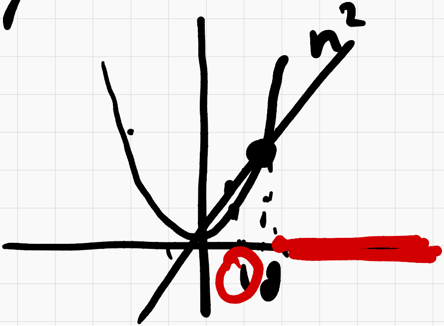
By definition of even, $n = 2k$ where k is an integer

Then $n^2 = (2k)^2 = 2^2 \cdot k^2 = 4 \cdot k^2 = 2(\underbrace{2k^2}_{\text{int}})$

Since $2k^2$ is an int. by def. of even, n^2 is an int.

Is it true that for every positive integer n, $n^2 \geq 2n$?

$n = 1 \Rightarrow \quad 1 \not\geq 2$



$\exists n$: n is a positive integer and $n^2 \not\geq 2n$

Disproving universal statements by counterexample

To disprove a statement means to show that it is false. Consider the question of disproving a statement of the form

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Showing that this statement is false is equivalent to showing that its negation is true. The negation of the statement is existential:

$$\exists x \text{ such that } P(x) \text{ and not } Q(x).$$

Is this true?

Prove for all integers m and n, if $m^2 = n^2$ then $m = n$?

$m = 1$

$n = -1$

Goldbach's conjecture

Every even integer greater than 2 is the sum of two primes.

(Christian Goldbach (1690–1764))

- $4 =$ 2+2
- $6 =$ 3+3
- $8 =$ 5+3
- $10 =$ 7+3
- $12 =$ 5+7
- ...
- up to 10^{17}

$n = 2$

$$a^2 + b^2 = c^2$$

$$3^2 + 4^2 = 5^2$$

$$9 + 16 = 25$$

Fermat's last theorem

No three positive integers a, b, and c satisfy the equation

$$a^n + b^n = c^n$$

for any integer value of n greater than 2.

- Conjectured around 1637 by Pierre de Fermat (1607-1665)
- Proved 1995 by Andrew Wiles