Social Network Analysis

Graph Theory preliminaries

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Graphs

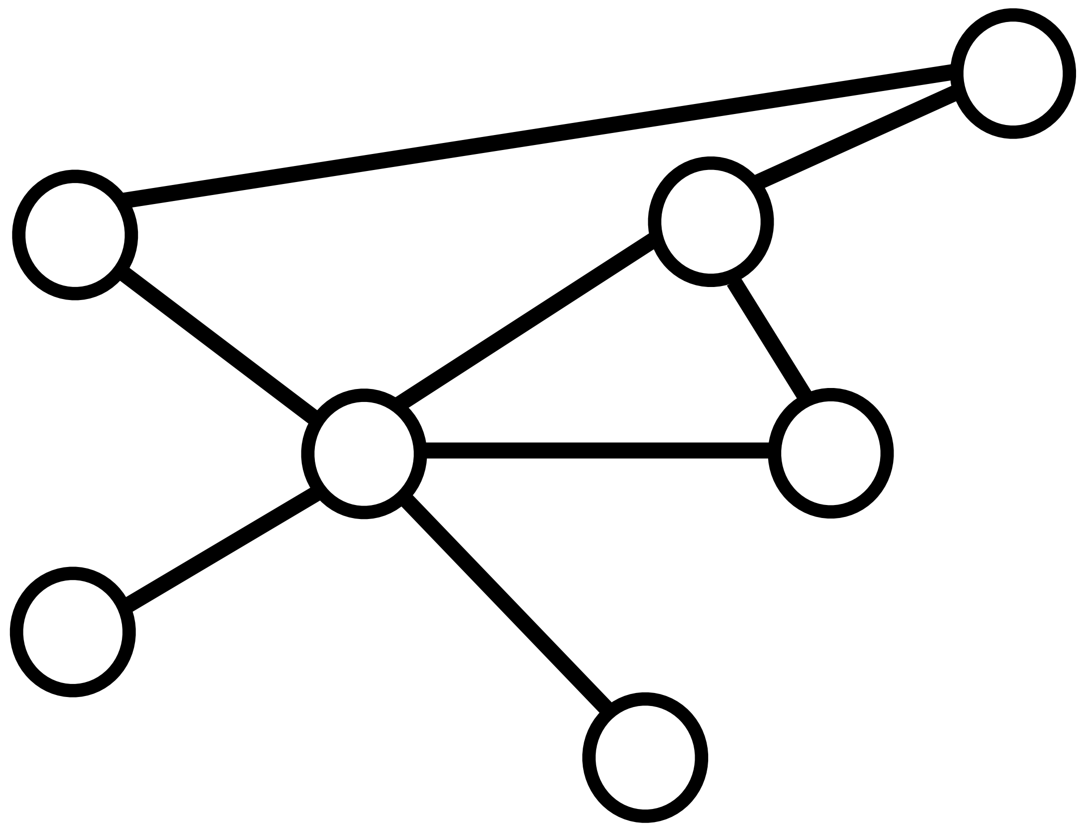
A graph $G = (V, E)$ consists of a set of **nodes** (**vertices**) V and a set of pairs of nodes E.

The elements of E are called **edges** or **links**.

If there is an edge between u and v, then we say that u and v are **adjacent**.

For an edge $e = (u, v)$, the vertices u and v are called the endpoints of e.

We say that the edge e is **incident with** each of its endpoints.

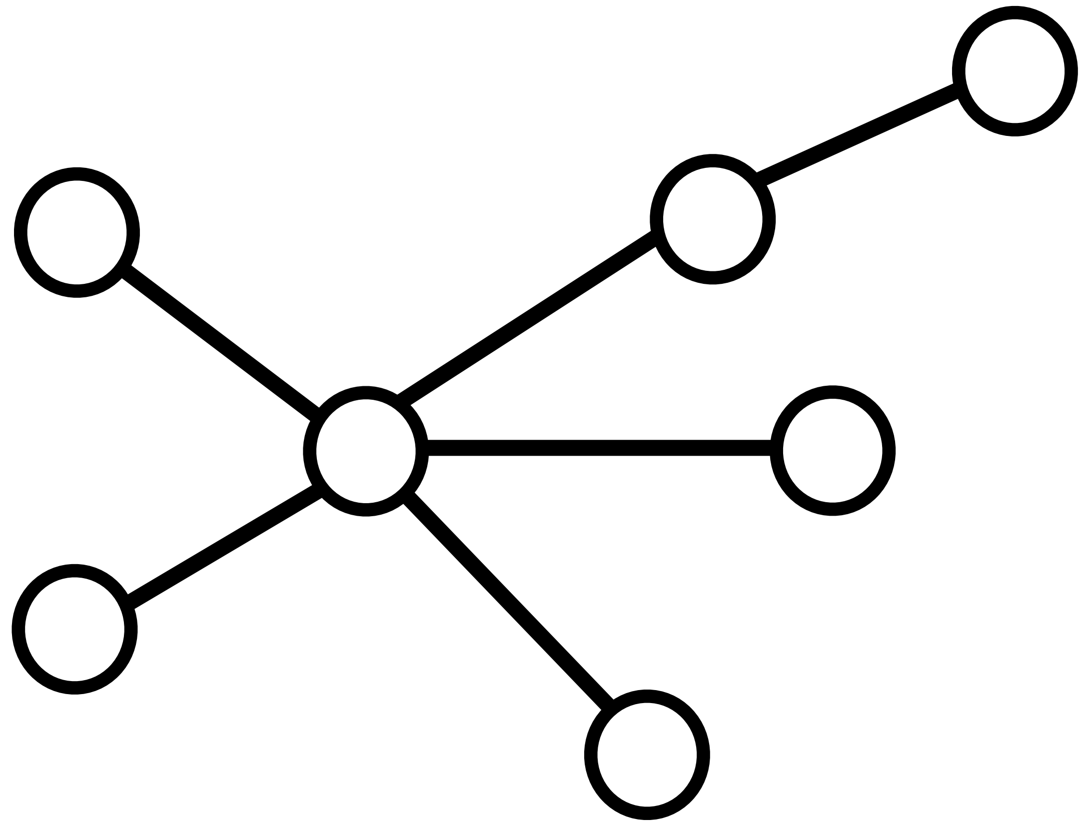


Graphs can be represented visually:

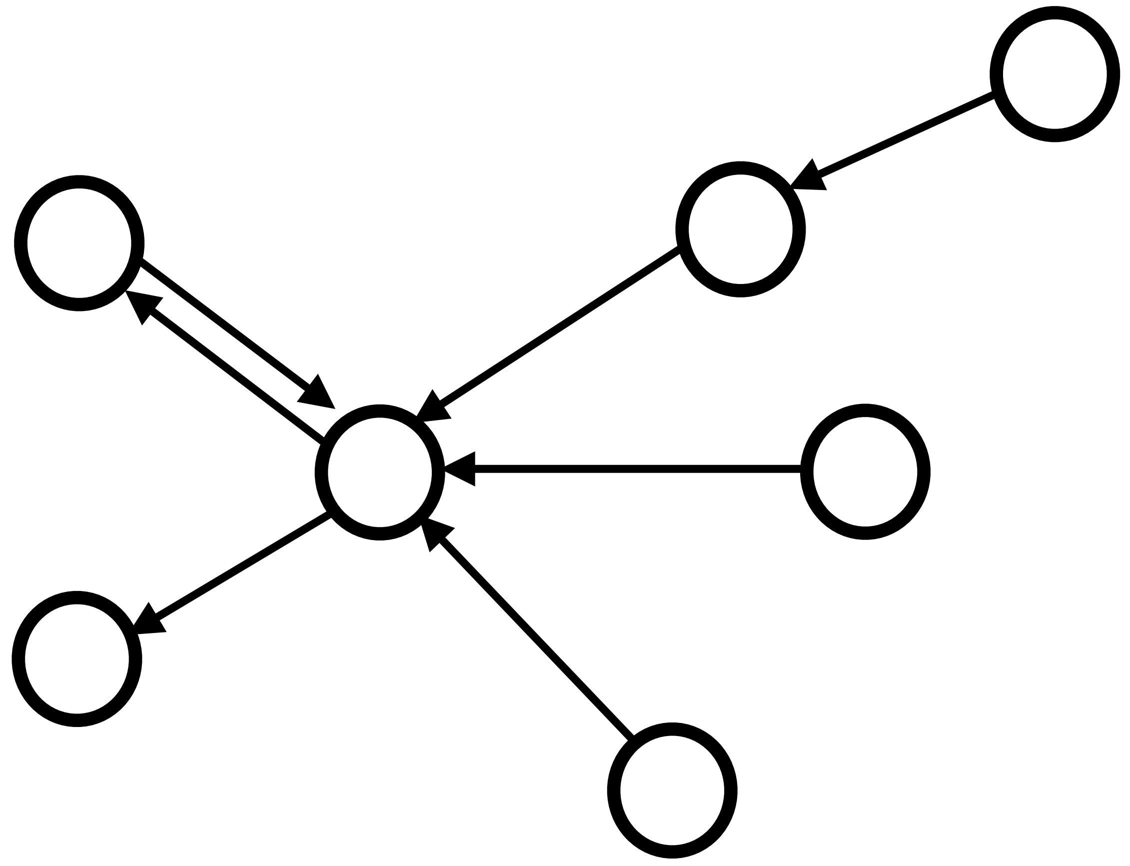
- nodes are points (circles)
- edges are lines connecting the endpoints

Undirected vs Directed graphs

Undirected graphs: edges are unordered pairs of vertices (i.e. $(u, v) = (v, u)$)



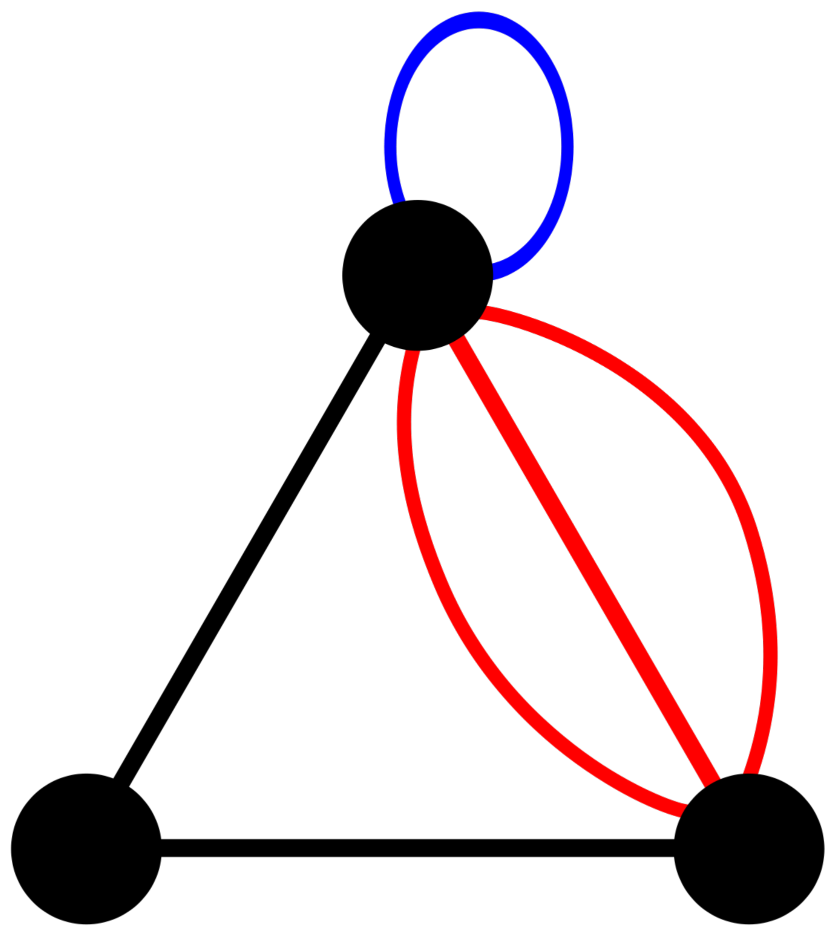
Directed graphs: edges are ordered pairs of vertices (i.e. $(u, v) \neq (v, u)$). Edges of directed graphs are sometimes called arcs.



Multi-edges and loops

Multigraph: a graph that admits multiple edges between a pair of nodes

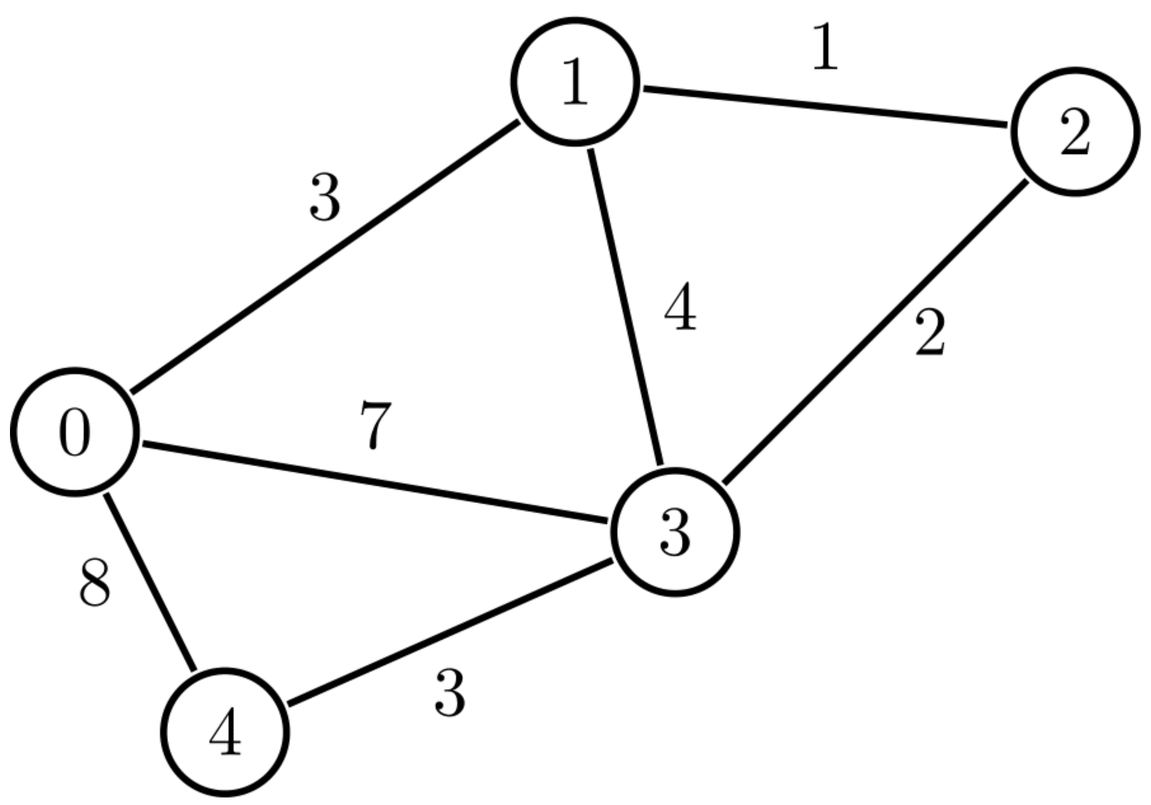
Loop: an edge that connects vertex with itself: (u, u)



Weighted graphs

Edge-weighted: every edge is assigned a number, called the weight of the edge

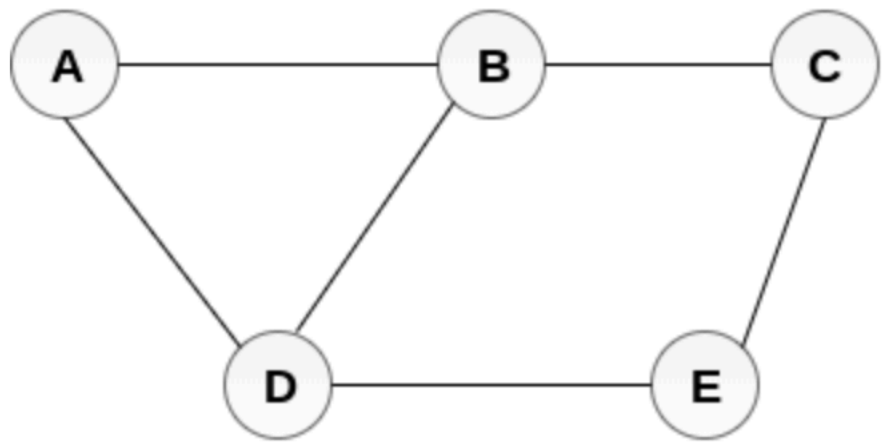
Vertex-weighted: every vertex is assigned a number, called the weight of vertex



Adjacency matrix

Adjacency matrix of an undirected n-vertex graph $G = (V, E)$ is the square $n \times n$ matrix $\overline{A}$ such that $\overline{A}_{ij} = 1$ if (i, j) is an **edge** in G and $\overline{A}_{ij} = 0$, otherwise.

Note: adjacency matrix of an undirected graph is symmetric



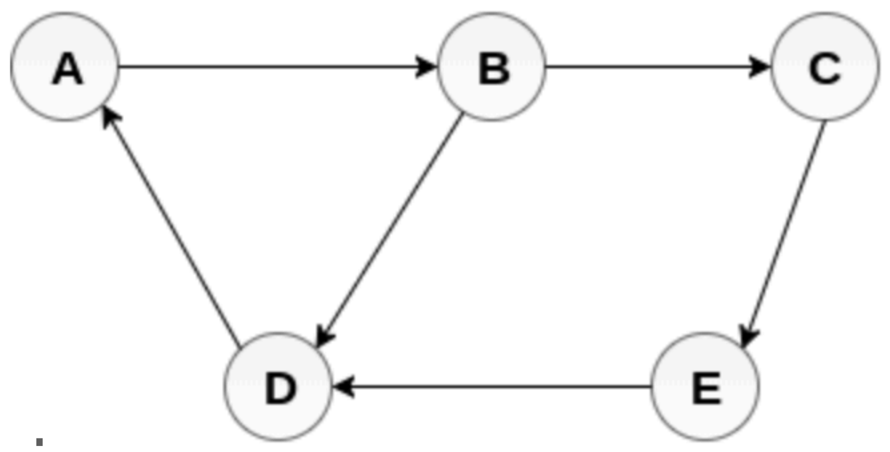
Undirected Graph

	A	B	C	D	E
A	0	1	0	1	0
B	1	0	1	1	0
C	0	1	0	0	1
D	1	1	0	0	1
E	0	0	1	1	0

Adjacency Matrix

Adjacency matrix of a directed n-vertex graph $G = (V, E)$, is the square $n \times n$ matrix $\overline{A}$ such that $\overline{A}_{ij} = 1$ if (i, j) is an **arc** in G and $\overline{A}_{ij} = 0$, otherwise.

Note: adjacency matrix of a directed graph is not necessarily symmetric

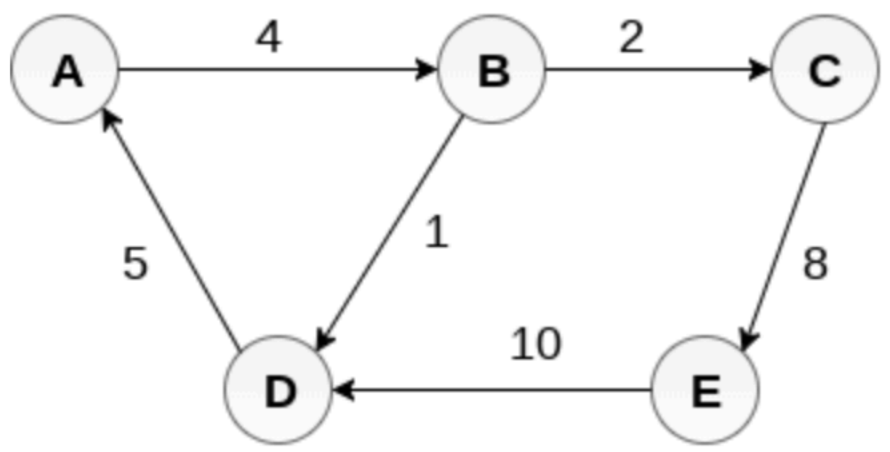


Directed Graph

	A	B	C	D	E
A	0	1	0	0	0
B	0	0	1	1	0
C	0	0	0	0	1
D	1	0	0	0	0
E	0	0	0	1	0

Adjacency Matrix

Adjacency matrix of a weighted undirected or directed n-vertex graph $G = (V, E)$, is the square $n \times n$ matrix $\overline{A}$ such that $\overline{A}_{ij} = w_{ij}$ if (i, j) is an **edge** or **arc** in G and $\overline{A}_{ij} = 0$, otherwise.
Here w_{ij} denotes the weight of an **edge/arc** (i, j).

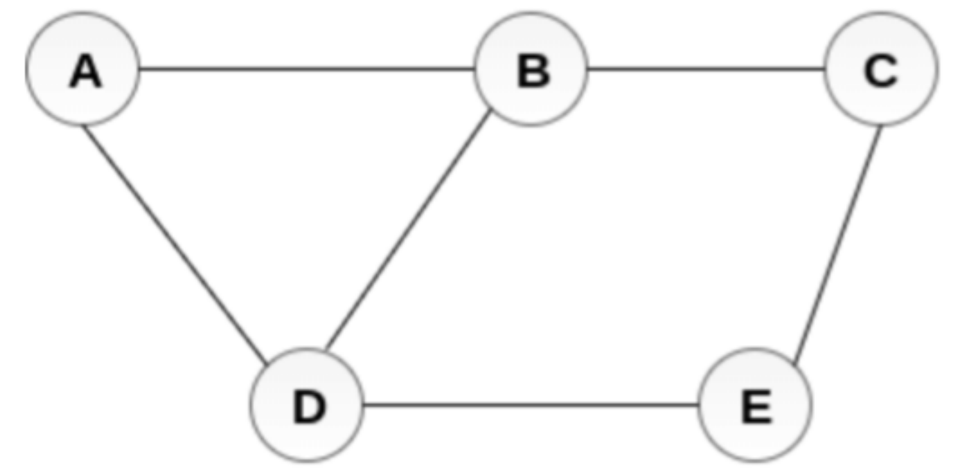


Weighted Directed Graph

	A	B	C	D	E
A	0	4	0	0	0
B	0	0	2	1	0
C	0	0	0	0	8
D	5	0	0	0	0
E	0	0	0	10	0

Adjacency Matrix

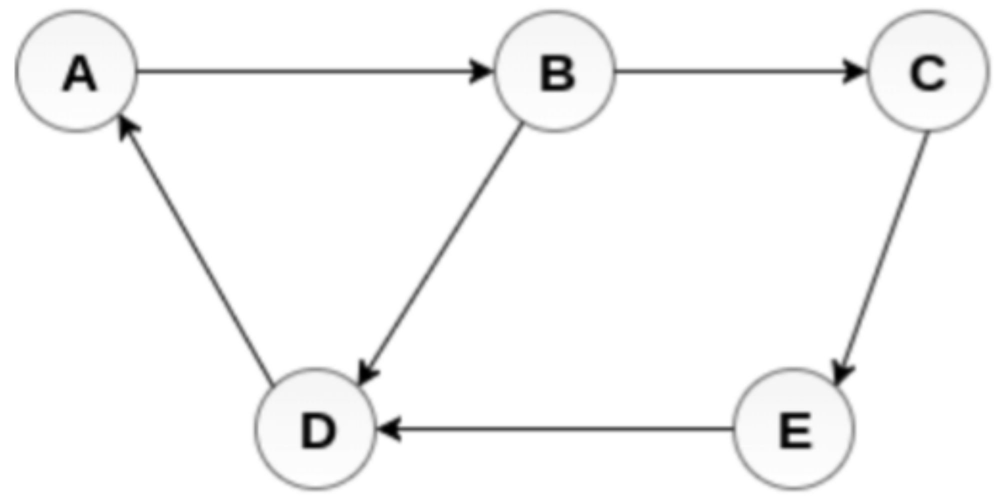
Neighbours & degree



Undirected Graph

Undirected graphs

- A vertex u is a **neighbour** of v, if (u, v) is an edge in the graph
- The set of all neighbours of v is called the **neighbourhood** of v and denoted by $N(v)$
- The **degree** of v is the number of neighbours of v, denoted by $\deg(v)$

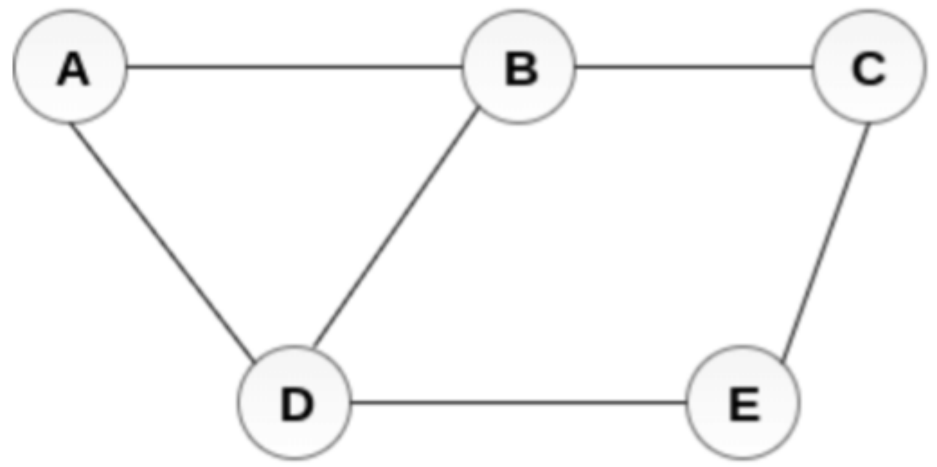


Directed Graph

Directed graphs

- A vertex u is an **in-neighbour** of v, if (u, v) is an arc in the graph (i.e. there is an arc from u to v)
- A vertex u is an **out-neighbour** of v, if (v, u) is an arc in the graph (i.e. there is an arc from v to u)
- The **in-degree** of vertex v is the number of in-neighbours of v, denoted by $\deg_+(v)$
- The **out-degree** of vertex v is the number of out-neighbours of v, denoted by $\deg\_(v)$

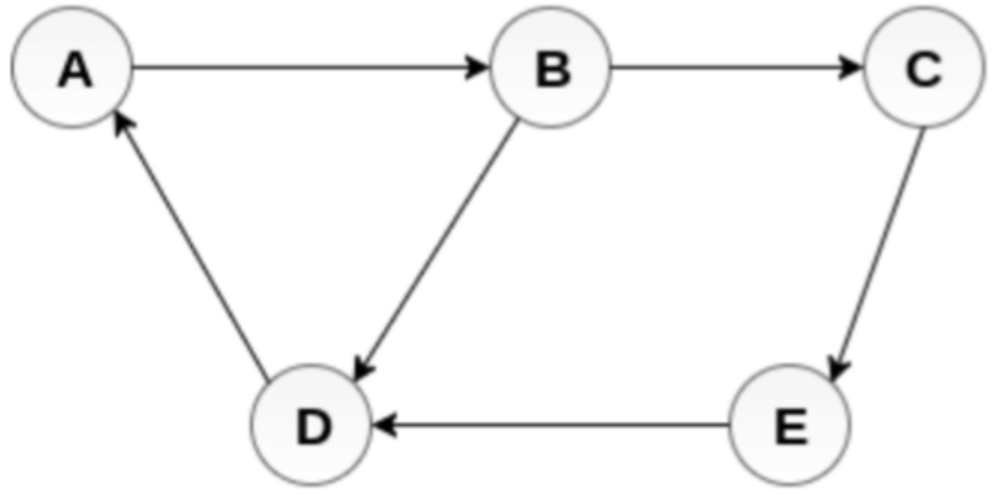
Path & distance



Undirected Graph

Undirected graphs

- A sequence of distinct vertices $v_0, v_1, \ldots, v_k$ is called a **path** between v_0 and v_k if for every $i = 1, \ldots, k$ vertices v_{i-1} and v_i are adjacent.
- The **length** of a path is the number of edges in the path.
- The **distance** between two vertices u and v is the length of a shortest path between u and v. If there is no path between u and v, then the distance between u and v is defined to be ∞.

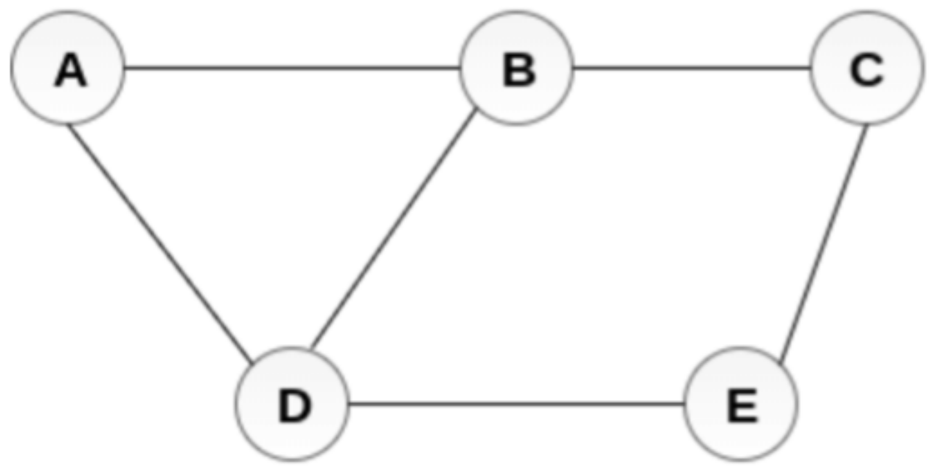


Directed Graph

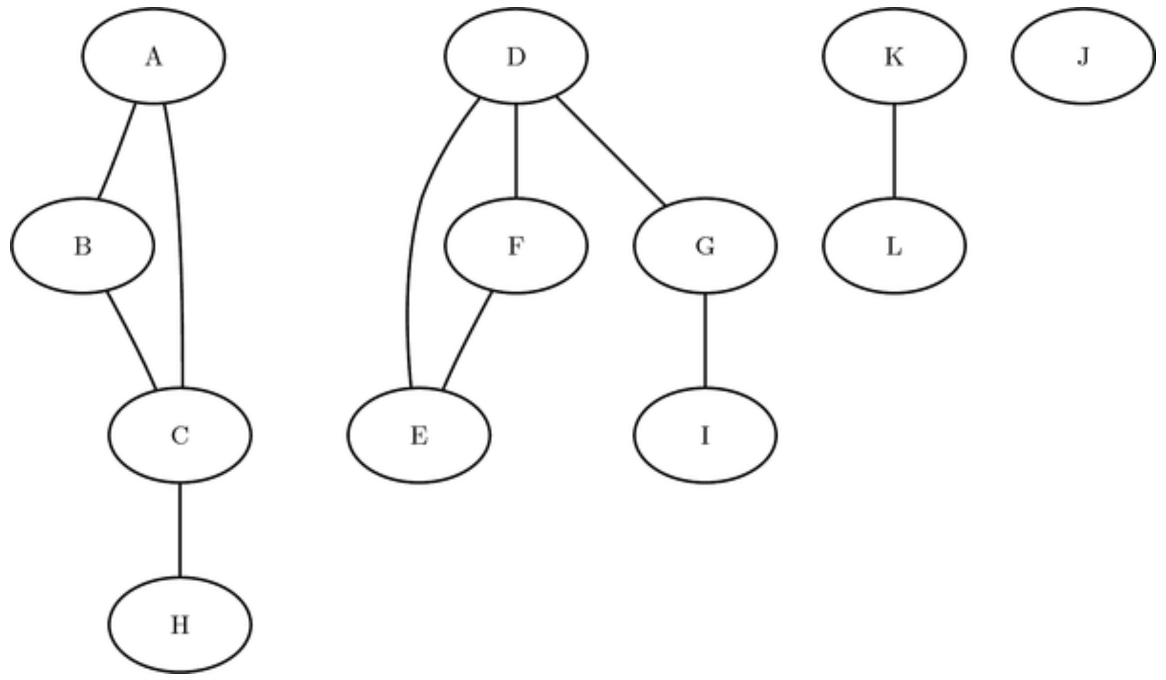
Directed graphs

- A sequence of distinct vertices $v_0, v_1, \ldots, v_k$ is called a (**directed**) **path** between v_0 and v_k if for every $i = 1, \ldots, k$ there is an arc from v_{i-1} to v_i.
- The **length** of a path is the number of arcs in the path.
- The **distance** between two vertices u and v is the length of a shortest path between u and v. If there is no path between u and v, then the distance between u and v is defined to be ∞.

Connected graphs



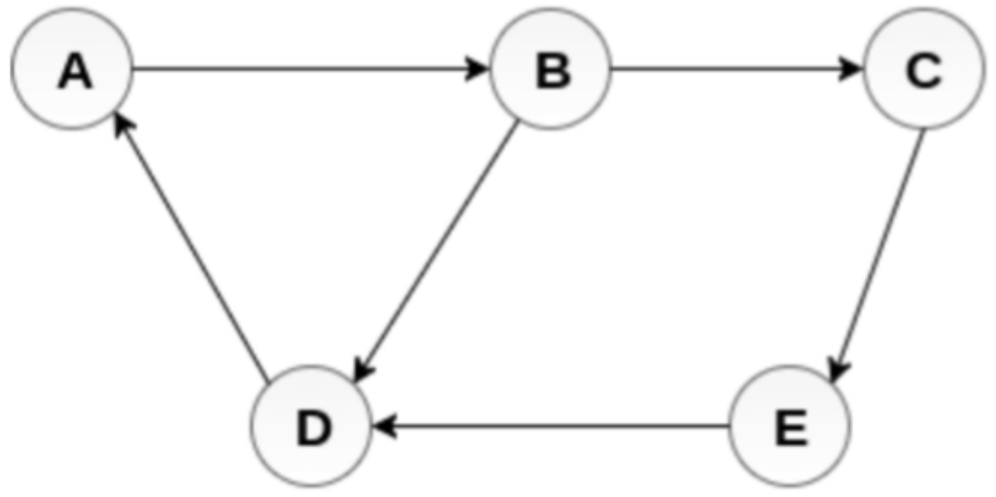
Undirected Graph



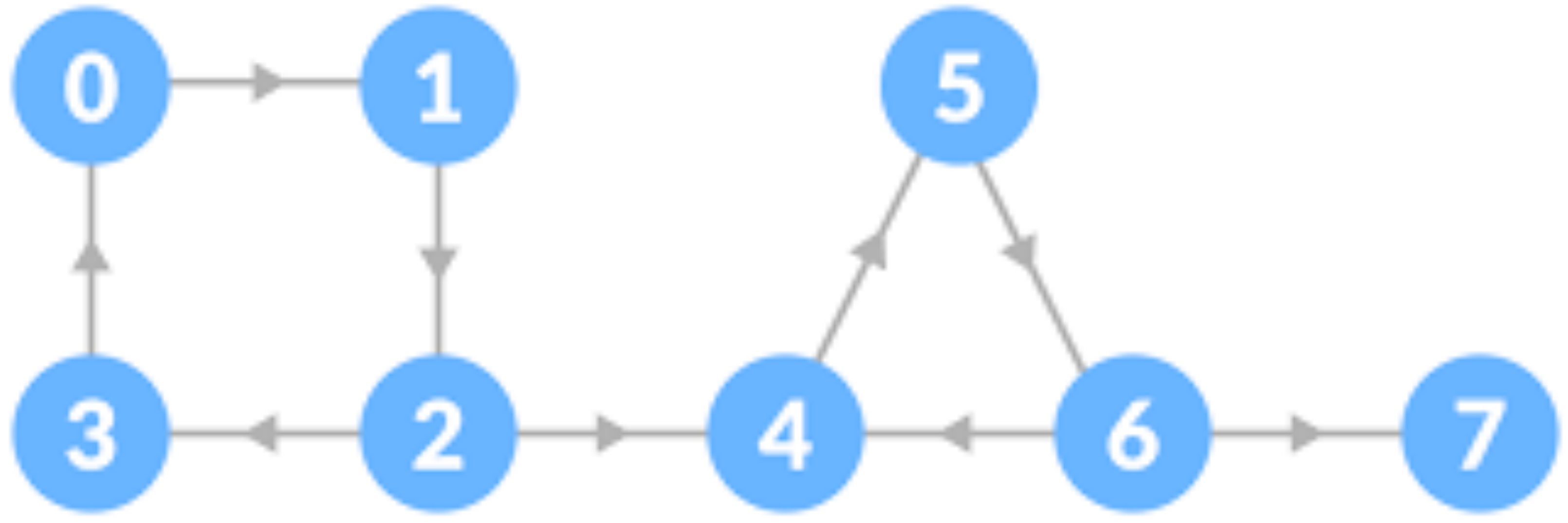
Undirected graphs

- A graph is **connected** if there is a path between any pair of vertices. Otherwise the graph is called **disconnected**.

- A maximal connected subgraph of a graph G is called a **connected component** of G



Directed Graph



Directed graphs

- A graph is **strongly connected** if for every ordered pair of vertices u, v there is a directed path from u to v.

- A maximal strongly connected subgraph of a (directed) graph G is called a **strongly connected component** of G.