Foundations of Computer Science Comp109

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Part 2. (Naive) Set Theory

Comp109 Foundations of Computer Science

Reading

- S. Epp. *Discrete Mathematics with Applications* Chapter 6
- K. H. Rosen. *Discrete Mathematics and Its Applications* Chapter 2

Contents

- Notation for *sets*.
- Important sets.
- What is a *subset* of a set?
- When are two sets *equal*?
- *Operations* on sets.
- *Algebra* of sets.
- Bit strings.
- *Cardinality* of sets.
- Russell's paradox.

Notation

A *set* is a collection of objects, called the *elements* of the set. For example:

- $\{7, 5, 3\}$; $\{5,3,7\}$, $\{3,5,7\}$
- $\{\text{Liverpool}, \text{Manchester}, \text{Leeds}\}$.

We have written down the elements of each set and contained them between the *braces* $\{\ \}$. $\emptyset$

We write $a \in A$ to denote that the object a is an element of the set A:

$$7 \in \{7, 5, 3\}, \quad 4 \notin \{7, 5, 3\}.$$

$x = 4$

$x \neq 5$

Notes

- The order of elements does not matter
- Repeatations do not count

$$\{3,5,7\} = \{5,7,3\} = \{3,3,7,5,5\}$$

Notation

For a large set, especially an infinite set, we cannot write down all the elements. We use a predicate P instead.

$$A = \{x \in S \mid P(x)\}$$

denotes the set of objects x from S for which the predicate $P(x)$ is true.

Examples: Let $A = \{1, 3, 5, 7, \ldots\}$. Then

$$A = \{x \in \mathbb{Z} \mid \text{ } x \text{ is odd}\}$$

Very informal notation:

$$A = \{2n - 1 \mid \text{ } n \text{ is a positive integer } \} = \{m \in \mathbb{Z} \mid m = 2n - 1 \text{ for some integer } n\}.$$

More examples

Find simpler descriptions of the following sets by listing their elements:

- $A = \{x \in \mathbb{Z} \mid \; x^2 + 4x = 12\}$;
- $B = \{n^2 \mid \; n \text{ is an integer } \}$.
- $C = \{x \mid \; x \text{ a day of the week not containing "}u\text{" }\}$;

Important sets (notation)

The empty set has no elements. It is written as $\emptyset$ or as $\{\}$.

We have seen some other examples of sets in Part 1.

$\mathbb{N}$ $\mathbb{Q}$

$\mathbb{Z}$ $\mathbb{R}$

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ (the natural numbers)
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (the integers)
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ (the positive integers)
- $\mathbb{Q} = \{x/y \mid x \in \mathbb{Z}, y \in \mathbb{Z}, y \neq 0\}$ (the rationals)
- $\mathbb{R}$: (real numbers)
  - $[a, b] = \{x \in \mathbb{R} \mid \underline{a \leq x \leq b}\}$ the set of real numbers between a and b (inclusive)

$[3, 5]$

Detour: Sets in python

Sets are the ‘most elementary’ data structures (though they don’t always map well into the underlying hardware).

Some modern programming languages feature sets.

- For example, in Python one writes

```
empty = set()
m = {'a', 'b', 'c'}
n = {1, 2}
print 'a' in m
```

(Computer) representation of sets

Only finite sets can be represented

- Number of elements not fixed: List (?) Java&Python do differently

- All elements of A are drawn from some ordered sequence $S = \langle s_1, \ldots, s_n \rangle$: the characteristic vector of A is the sequence $[b_1, \ldots, b_n]$ where

$$b_i = \begin{cases} 1 & \text{if} \quad s_i \in A \\ 0 & \text{if} \quad s_i \notin A \end{cases}$$

Sequences of zeros and ones of length n are called *bit strings* of length n. AKA *bit vectors* AKA *bit arrays*

Example

Let $S = \langle 1, 2, 3, 4, 5 \rangle$, $A = \{1, 3, 5\}$ and $B = \{3, 4\}$.

- The characteristic vector of A is $[1, 0, 1, 0, 1]$.
- The characteristic vector of B is $[0, 0, 1, 1, 0]$.

- The set characterised by $[1, 1, 1, 0, 1]$ is $\{1, 2, 3, 5\}$.
- The set characterised by $[1, 1, 1, 1, 1]$ is $\{1, 2, 3, 4, 5\}$.
- The set characterised by $[0, 0, 0, 0, 0]$ is $\ldots$

$S = \langle a, b, c, e, f, z \rangle$

$A = \{a, c, z\}$

$X_A = [1, 0, 1, 0, 0, 1]$

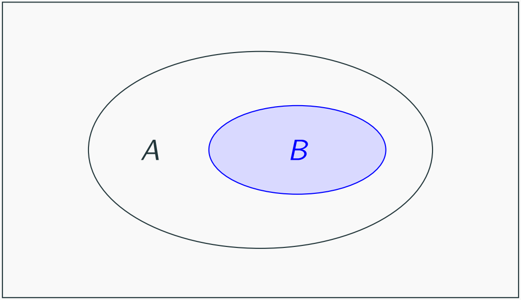
$X_B = [0, 1, 0, 1, 1, 0]$ $\quad B = \{b, 1, f\}$

Subsets

Definition A set B is called a *subset* of a set A if every element of B is an element of A. This is denoted by $B \subseteq A$.

Examples:

$$\{3, 4, 5\} \subseteq \{1, 5, 4, 2, 1, 3\}, \ \{3, 3, 5\} \subseteq \{3, 5\}, \ \{5, 3\} \subseteq \{3, 5\}.$$



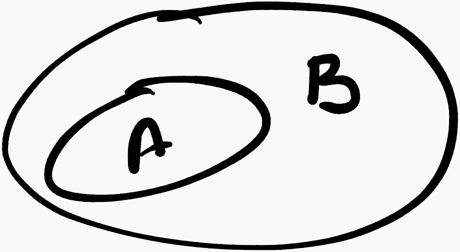
$\{5,3\} \subseteq$

$\{5,3\}$

Figure 1: Venn diagram of $B \subseteq A$.

Detour: Subsets in Python

```
def isSubset(A, B):
    for x in A:
        if x not in B:
            return False
    return True
```



Testing the method:

```
print isSubset(n,m)
```

But then there is a built-in operation:

```
print n<m
```

Subsets and bit vectors

Let $S = \langle 1, 2, 3, 4, 5 \rangle$, $A = \{1, 3, 5\}$ and $B = \{3, 4\}$.

- Is $A \subseteq B$?

$\chi_A = [1, 0, 1, 0, 1]$

$\chi_B = [0, 0, 1, 1, 0]$

- Is the set C, represented by $[1, 0, 0, 0, 1]$, a subset of the set D, represented by $[1, 1, 0, 0, 1]$?

$\chi_C = [1, 0, 0, 0, 1]$ $\qquad C = \{1, 5\}$

$\chi_D = [1, 1, 0, 0, 1]$ $\qquad D = \{1, 2, 5\}$

Equality

Definition A set A is called *equal* to a set B if $A \subseteq B$ and $B \subseteq A$. This is denoted by $A = B$.

Examples:

$$\{1\} = \{1, 1, 1\},$$

$$\{1, 2\} = \{2, 1\},$$

$$\{5, 4, 4, 3, 5\} = \{3, 4, 5\}.$$

Equality and bit vectors

Let $S = \langle 1, 2, 3, 4, 5 \rangle$, $A = \{1, 3, 5\}$ and $B = \{3, 4\}$.

- Is $A = B$?

$\chi_A = [1, 0, 1, 0, 1]$

$\chi_B = [0, 0, 1, 1, 0]$

- Is the set C, represented by $[1, 0, 0, 0, 1]$, equual to the set D, represented by $[1, 1, 0, 0, 1]$?

$[0, 1, 1, 0, 0]$

$[0, 1, 1, 0, 0]$