Infinite sets

Sets A and B have the same cardinality iff there is a bijection from A to B.

- $\mathbb{Z}$ and even integers

$f: \mathbb{Z} \to 2\mathbb{Z}$

$f(n) = 2n$

$f(n) = 2n$ is injective and surjective

Proof

Let $l \in 2\mathbb{Z}$. By def of even $\exists k$:

$l = 2k$

So $f(k) = 2k = l$. Hence f is surjective

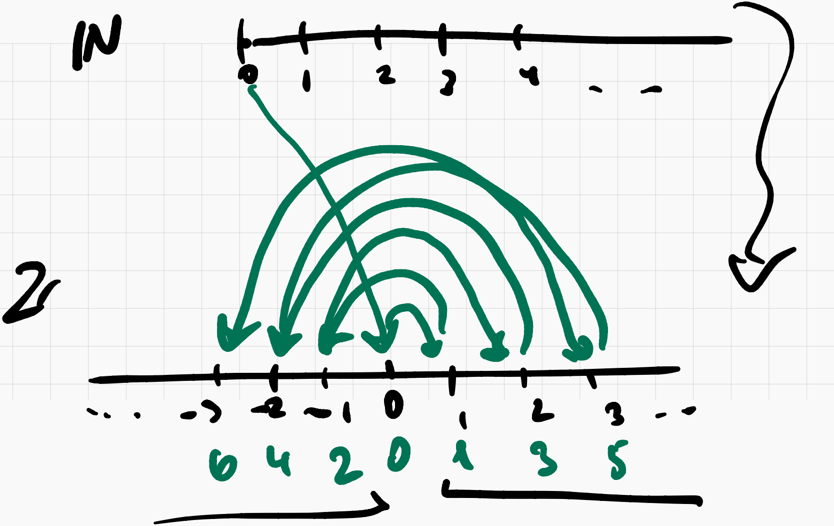
Suppose that $f(k) = f(l)$

$2k = 2l \rightarrow k = l$

Hilbert's infinite hotel

- $\mathbb{N}$ and $\mathbb{Z}$

$g(n)$ $\quad n \in \mathbb{N}$



If n is odd then

$$g(n) = \frac{n+1}{2}$$

If n is even then

$$g(n) = -\frac{n}{2}$$

$$\cancel{(-1)^{n+1}} \sin\left(\frac{n\pi}{2}\right) \frac{n}{2} \ldots$$

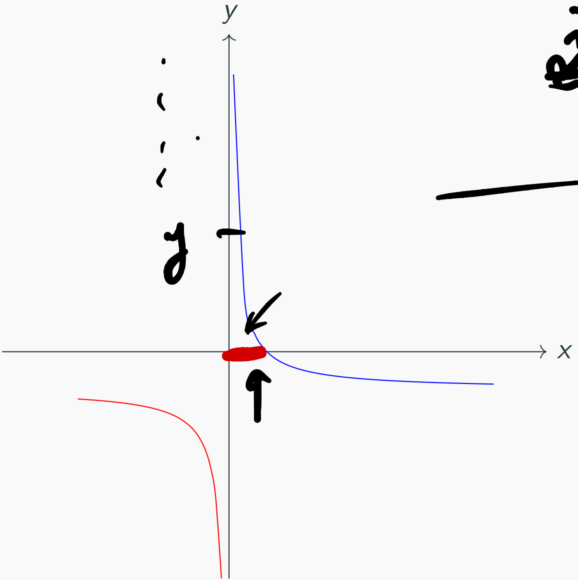
n	$g(n)$
0	0
1	1
2	-1
3	2
4	-2
5	3
6	-3
7	4
$\vdots$	

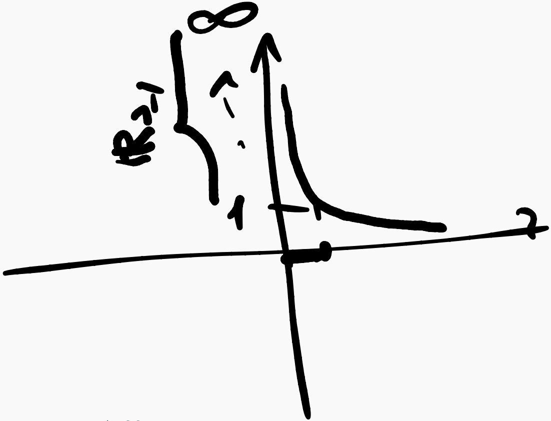
$$(-1)^n$$

$$\sin\left(\frac{n\pi}{2}\right)$$

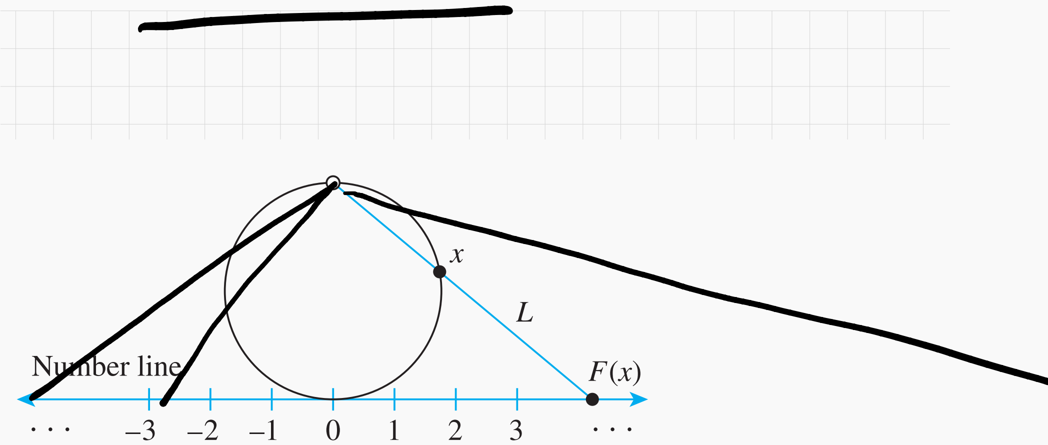
Real numbers: $\{x \in \mathbb{R} \mid 0 < x < 1\}$ and $\mathbb{R}^+$

- consider $g(x) = \frac{1}{x} - 1$





$\{x \in \mathbb{R} \mid 0 < x < 1\}$ and $\mathbb{R}$



Countable sets

A set that is either finite or has the same cardinality as $\mathbb{N}$ is called **countable**.

- $\mathbb{Z}$

$$\cdots \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots$$

Countable Sets: $\mathbb{Q}$

$\frac{1}{1}$ 0	$\frac{1}{2}$ 3	$\frac{1}{3}$ 5	$\frac{1}{4}$ 9	$\frac{1}{5}$ 11	$\frac{1}{6}$	$\cdots$
$\frac{2}{1}$ 2	$\cancel{\frac{2}{2}}$	$\frac{2}{3}$ 8	$\cancel{\frac{2}{4}}$	$\frac{2}{5}$	$\frac{2}{6}$	$\cdots$
$\frac{3}{1}$ 4	$\frac{3}{2}$ 7	$\cancel{\frac{3}{3}}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\cdots$
$\frac{4}{1}$ 6	$\cancel{\frac{4}{2}}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	$\cdots$
$\frac{5}{1}$ 10	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\cdots$
$\frac{6}{1}$	$\frac{6}{2}$	$\frac{6}{3}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\ddots$

$\frac{k}{\ell}$

$\mathbb{N}$

$0, 1, 2, 3 \ldots$

Uncountable sets

- A set that is not countable is called **uncountable**.
  - $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$ is uncountable

Cantor's diagonal argument

Suppose for a proof by contradiction that there exists a bijection $f : \mathbb{N}^+ \to S$.
Consider decimal representations of $f(n)$, for $n \in \mathbb{N}^+$:

$$
\begin{array}{l}
f(1) = 0.a_{11}\ a_{12}\ a_{13} \ldots\ a_{1n} \ldots \\
f(2) = 0.a_{21}\ a_{22}\ a_{23} \ldots\ a_{2n} \ldots \\
f(3) = 0.a_{31}\ a_{32}\ a_{33} \ldots\ a_{3n} \ldots \\
\vdots \qquad\qquad\qquad \ddots \\
f(n) = 0.a_{n1}\ a_{n2}\ a_{n3} \ldots\ a_{nn} \ldots \\
\vdots
\end{array}
$$

$0,123456\ldots$

$\frac{\pi}{10} = 0,314\ldots$

We show that there exists $d \in S$ such that for no $i \in \mathbb{N}^+$ we have $f(i) = d$.

Let $d = 0.d_1\ d_2\ d_3 \ldots d_n \ldots$ where $\quad d_i = \begin{cases} 2, & \text{if } a_{ii} = 1 \\ 1, & \text{if } a_{ii} \neq 1 \end{cases}$

Then for every $i \in \mathbb{N}^+$ d is different at position i from $f(i)$. So, for no $i \in \mathbb{N}^+$ we have $f(i) = d$, so f is not surjective. A contradiction