Lecture 24 – D-Separation

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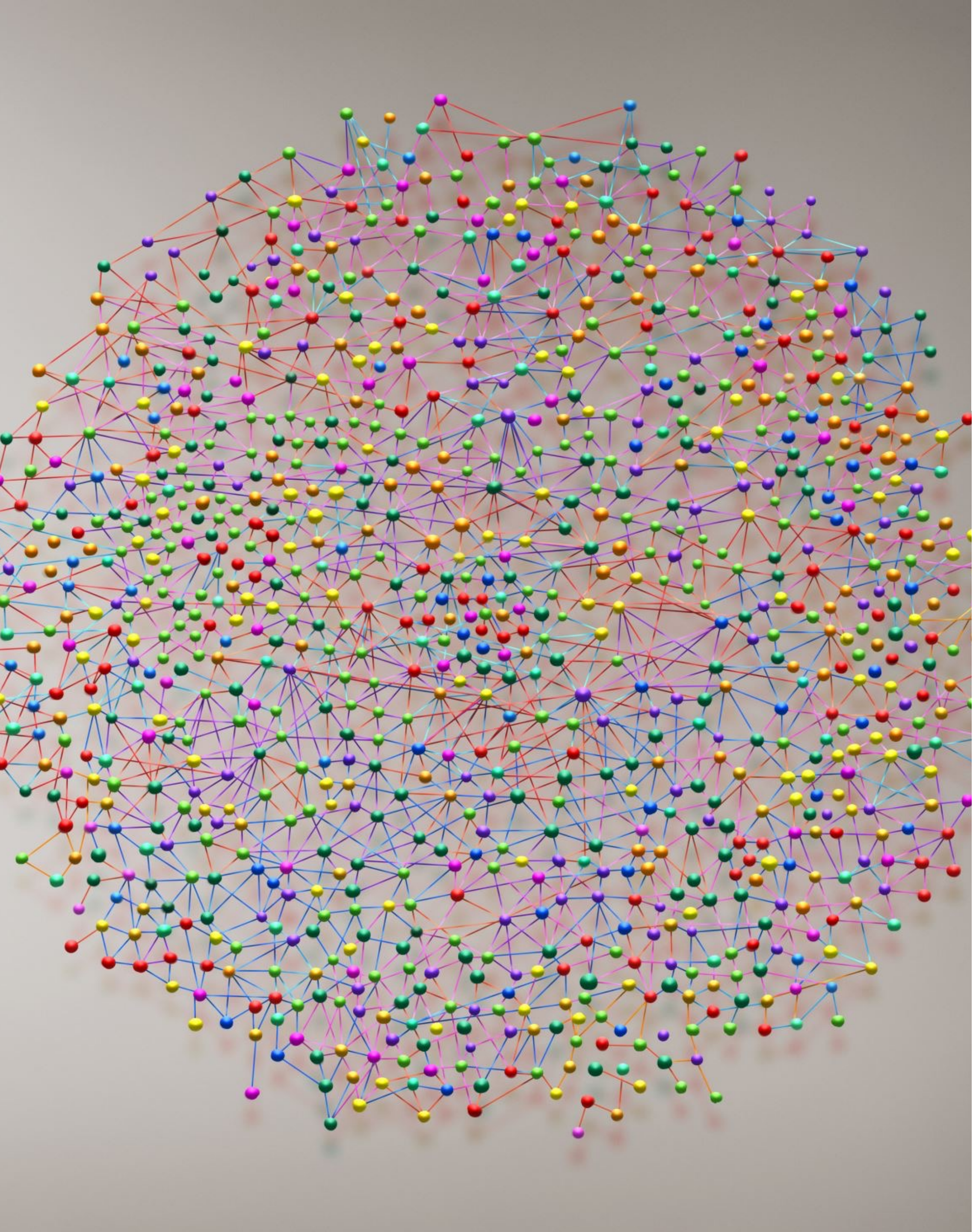
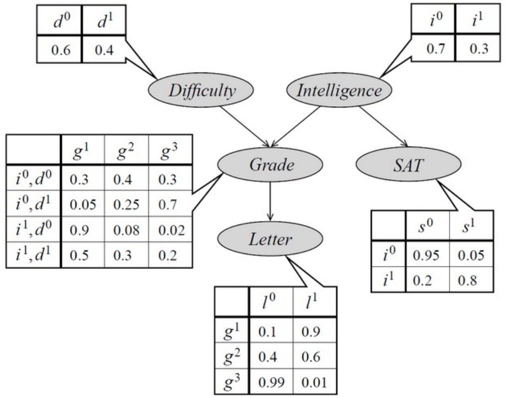
(Attendance Code: **205529**)

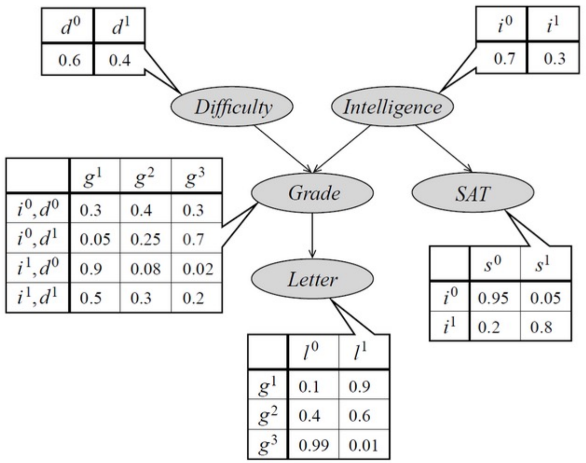


Up to now,



- Traditional Machine Learning Algorithms
- Adversarial Attack and Defence
- Deep learning
- Probabilistic Graphical Models
  - Introduction
  - Distribution <=> Graph
  - Reasoning Patterns (Causal Reasoning, Evidential Reasoning, Intercausal Reasoning)

Recap: Local Independencies



- Graph G with CPDs is equivalent to a set of independence assertions

$$P(D,I,G,S,L) = P(D)P(I)P(G \mid D,I)P(S \mid I)P(L \mid G)$$

- Local Conditional Independence Assertions (starting from leaf nodes):

$$\begin{aligned} I_{local}(G) = \{(L \perp I, D, S \mid G), &\quad \text{L is conditionally independent of all other nodes given parent G} \\ (S \perp D, G, L \mid I), &\quad S \text{ is conditionally independent of all other nodes given parent I} \\ (G \perp S \mid D, I), &\quad \text{Even given parents, G is NOT independent of descendant L} \\ (I \perp D \mid \emptyset), &\quad \text{Nodes with no parents are marginally independent} \\ (D \perp I, S \mid \emptyset)\} &\quad \text{D is independent of non-descendants I and S} \end{aligned}$$

Can we have the following conditional independence?

$$D \perp S \mid G$$
$$D \perp S \mid I$$
$$D \perp S \mid G, I$$

- Parents of a variable shield it from probabilistic influence
  - Once value of parents known, no influence of ancestors
- Information about descendants can change beliefs about a node
- $I_{local}(G) \subseteq I(G)$

Recap:

low grade drastically decreases the probability of high intelligence

low grade justifies the difficulty

$P(i^1)=0.3$

$P(i^1 \mid g^3)=0.079$

$P(d^1 \mid g^3)=0.629$

$P(i^1 \mid l^0)=0.14$

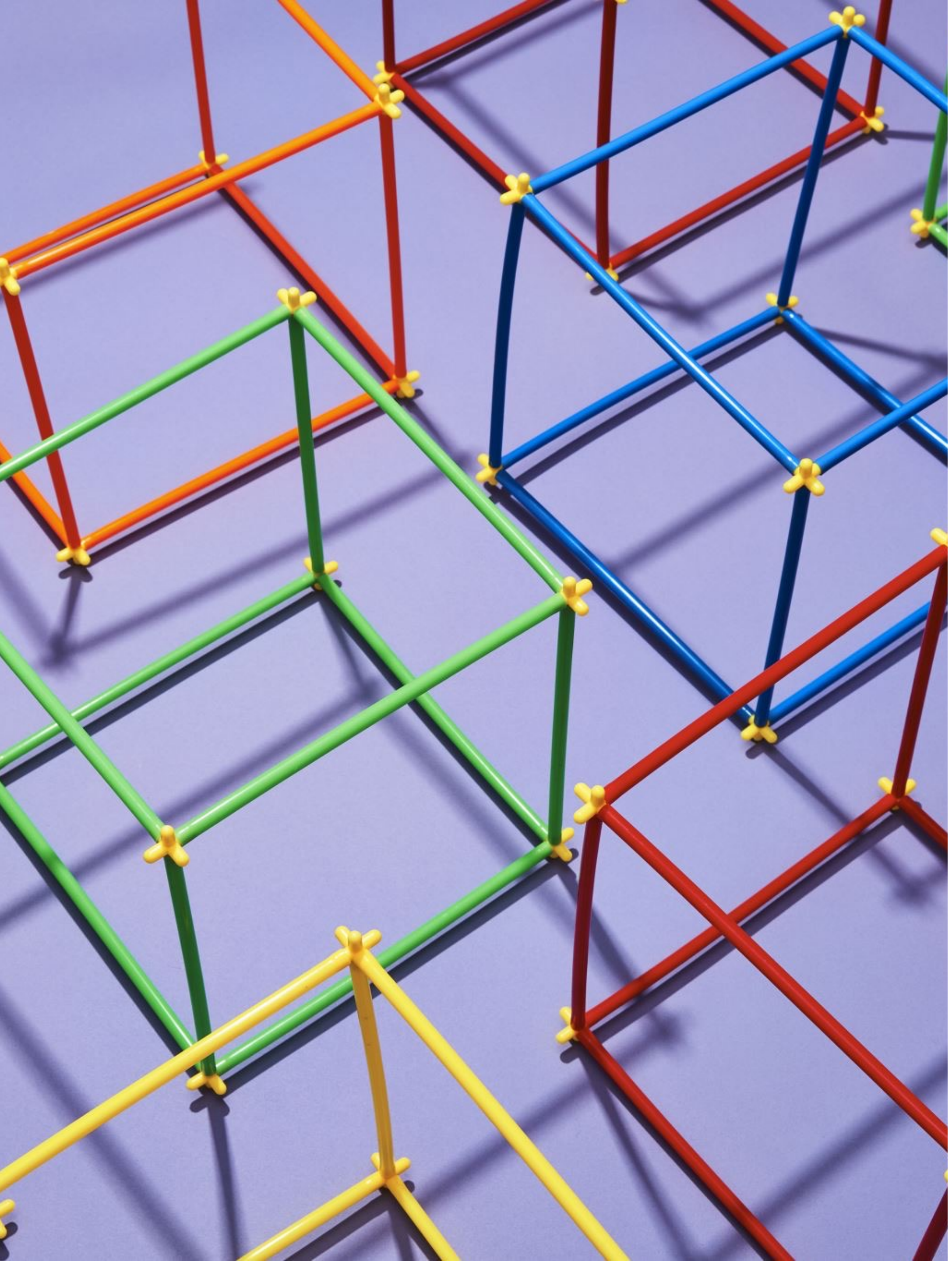
$P(i^1 \mid l^0, g^3)=0.079$

A weak letter drastically decreases the probability of high intelligence

After knowing low grade, a weak letter won't make the probability of high intelligence lower.

$$I \perp L \mid G$$

But this is not a conditional independence that we found from local Markov assumption

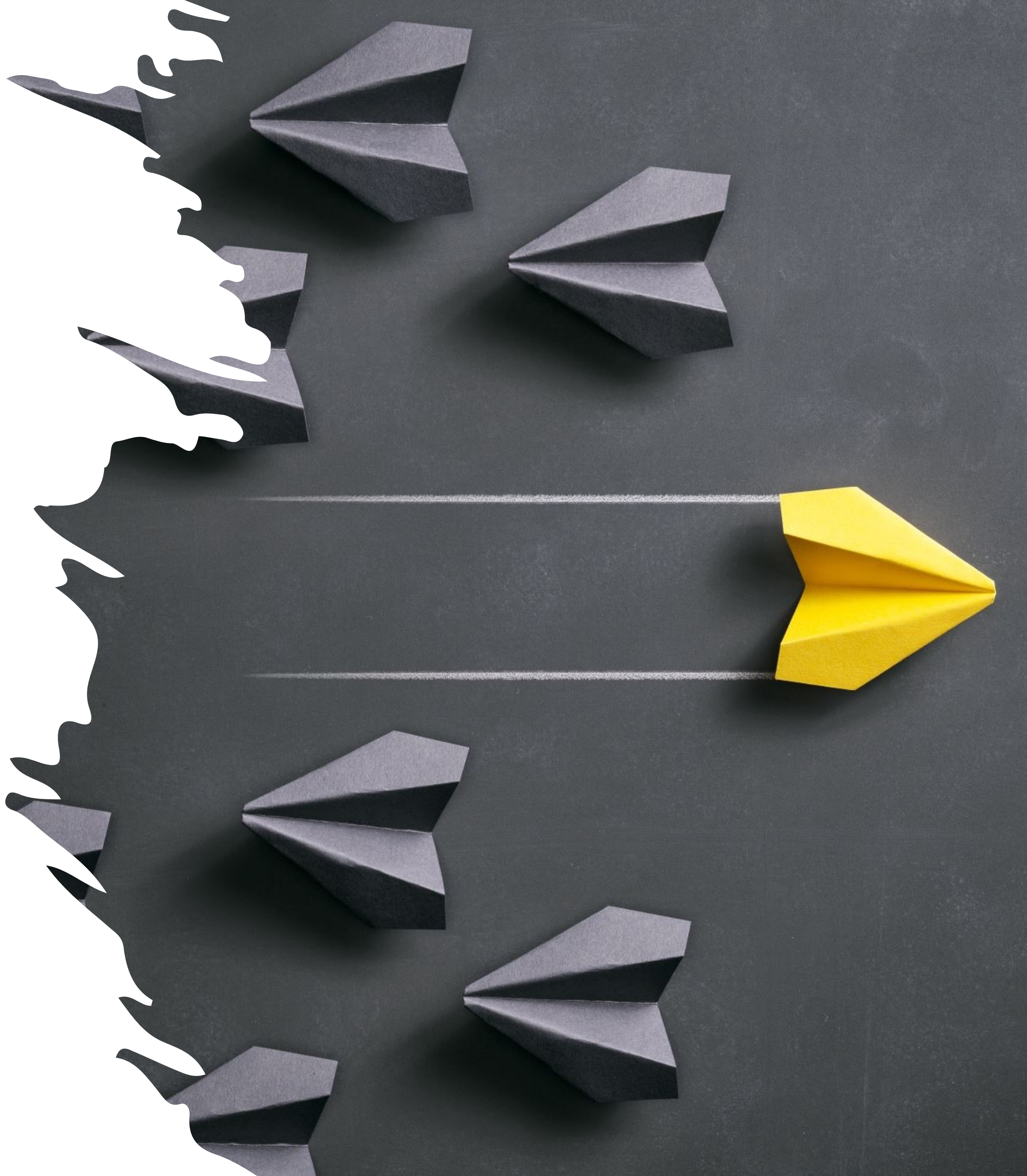


Independencies in Graphs

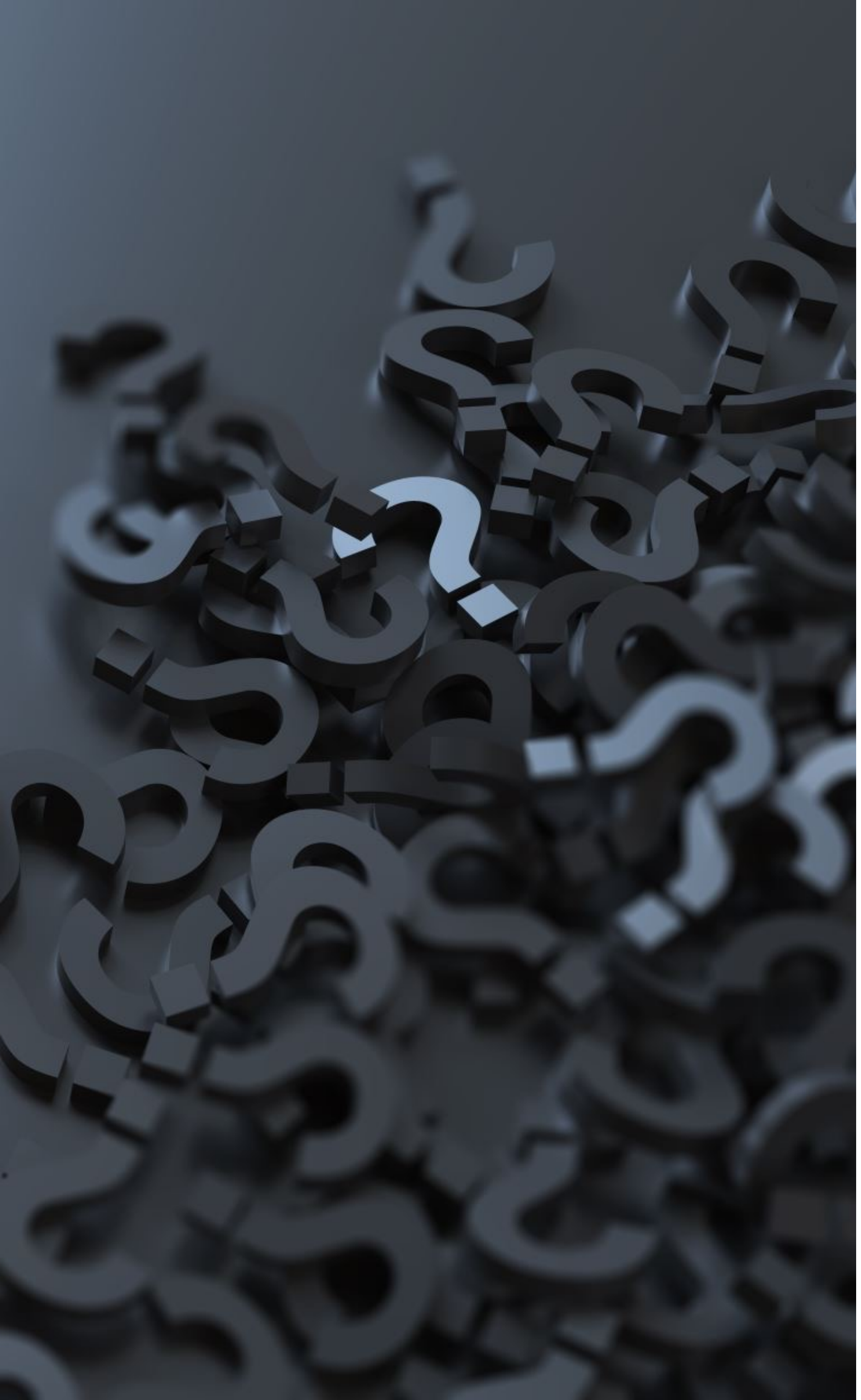
- A graph structure G encodes a set of conditional independence assumptions *I*(G)
- Are there other independencies that we can read-off?
  - i.e., are there other independencies that hold for every distribution that factorizes over G?
- D-separation holds the key

Topics

- Why D-separation?
- What is D-separation?
- Algorithm for D-separation (extended materials)
- More Example of D-separation



Why D-separation?



Dependencies and Independencies

- Crucial for understanding network behaviour
- Independence properties are important for answering queries
  - Exploited to reduce computation of inference
  - A distribution *P* that factorizes over *G* satisfies *I(G)*
- *Local Markov assumption cannot figure out all conditional independencies*

What is D-separation?

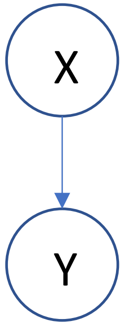
D-separation

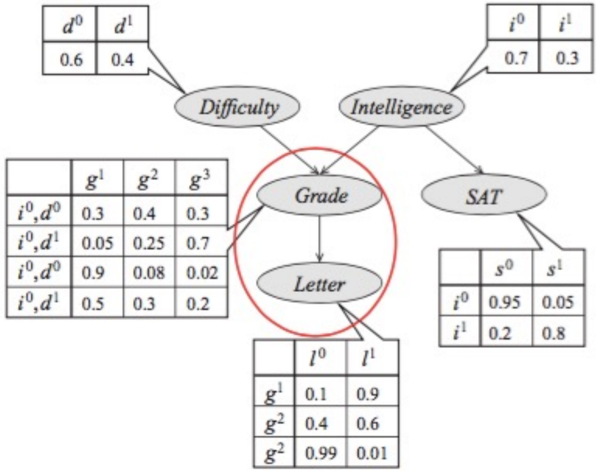
- Study independence properties for subgraphs (connected triples)
- Analyze complex cases in terms of triples along paths between variables
- ***D-separation:*** a condition / algorithm for answering such queries

- Definition: A procedure $\text{d-sep}_G(X \perp Y | Z)$ that given a DAG *G*, and sets *X*, *Y*, and *Z* returns either *yes* or *no, where* $\text{d-sep}_G(X \perp Y | Z) = \text{Yes}$ iff $(X \perp Y | Z)$ follows from *I(G)*

Direct Connection between *X* and *Y*

- *X* and *Y* are correlated regardless of any evidence about any other variables
  - E.g., Feature *Y* and character *X* are correlated
  - Grade *G* and Letter *L* are correlated
- If *X* and *Y* are directly connected we can get examples where they influence each other regardless of *Z*

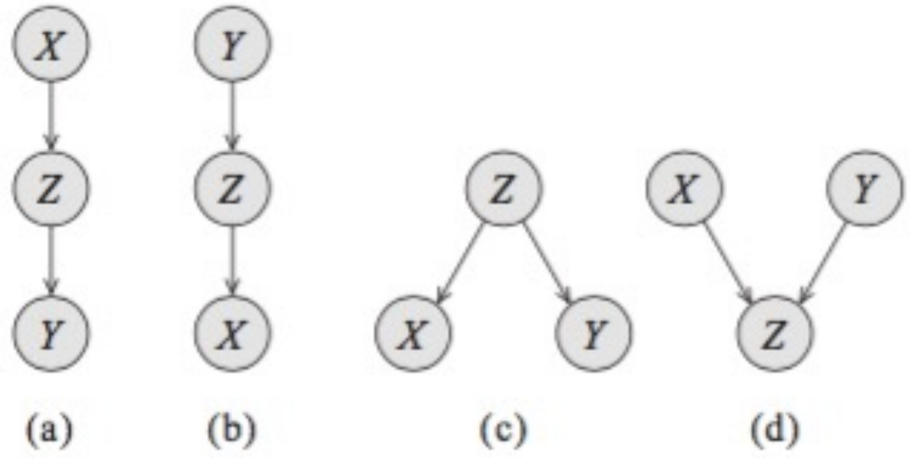




Indirect Connection between *X* and *Y*

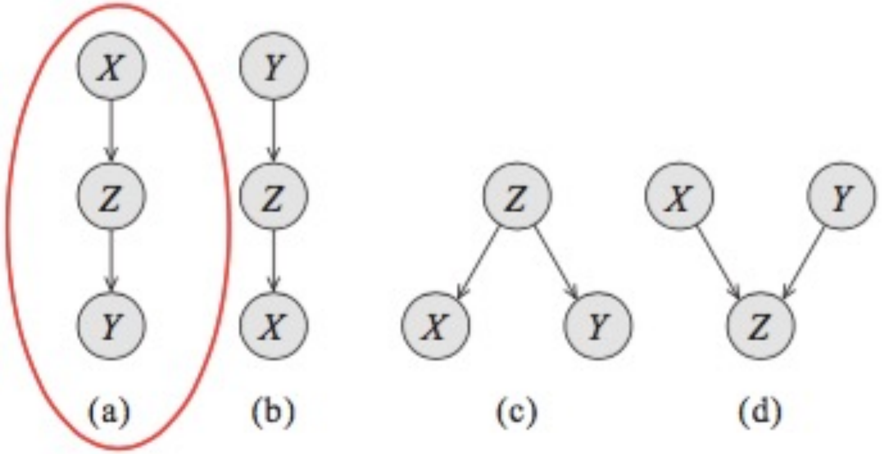
- Four cases where *X* and *Y* are connected via *Z*
- (a). Indirect causal effect
- (b). Indirect evidential effect
- (c). Common cause
- (d). Common effect

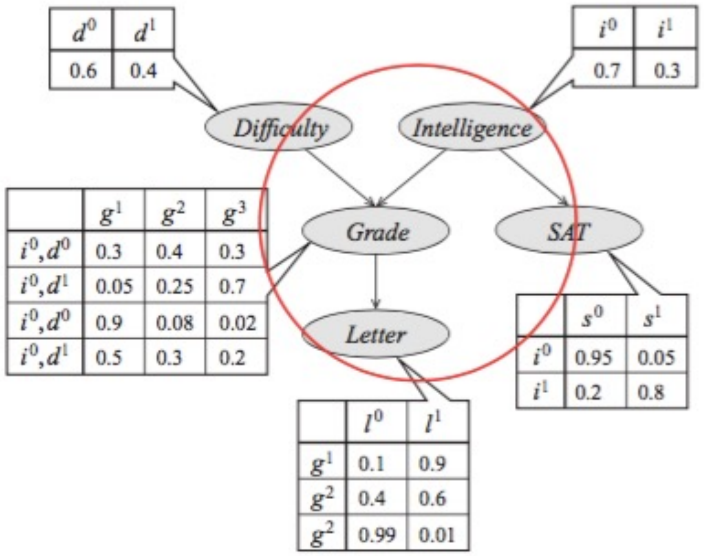
- We will see that first three cases are similar while fourth case (*V*-structure) is different



1. Indirect Causal Effect: *X->Z->Y*

- Cause *X* cannot influence effect *Y* if *Z* observed
    - Observed *Z* blocks influence
- If *Grade* observed then *I* does not influence *L*
    - *Intelligence* influences *Letter* if *Grade* is unobserved





d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0,d^0	0.3	0.4	0.3
i^0,d^1	0.05	0.25	0.7
i^0,d^0	0.9	0.08	0.02
i^0,d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^2	0.99	0.01

$Z = Grade$

$$I \perp L \mid G$$

Recap:

low grade drastically decreases the probability of high intelligence

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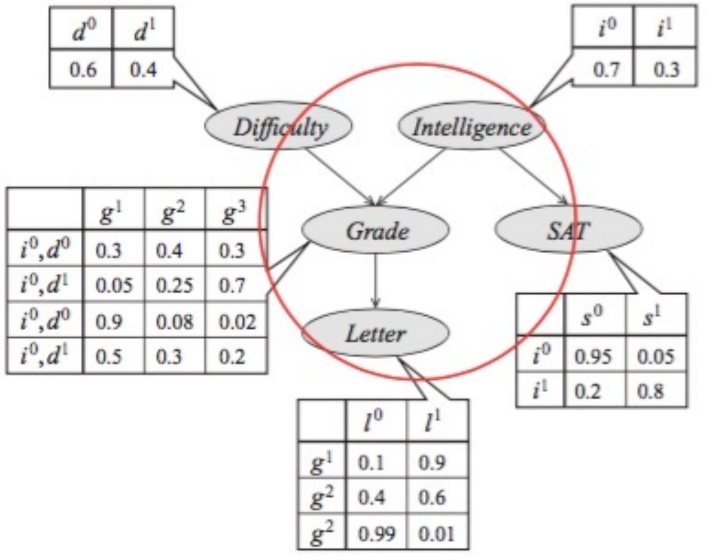

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	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^2	0.99	0.01

$$I \perp L \mid G$$

The indirect causal effect explains this conditional independence

Causal Chains

Evidence along the chain “blocks” the influence (makes “inactive”)

- This configuration is a “causal chain”

X: Low pressure Y: Rain Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

Indirect causal effect

Guaranteed X independent of Z given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

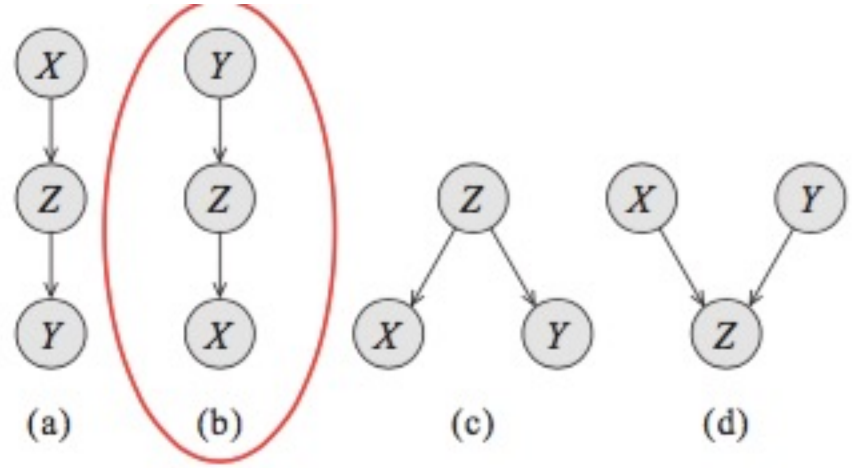
$$= \frac{\cancel{P(x)P(y|x)}P(z|y)}{\cancel{P(x)P(y|x)}}$$

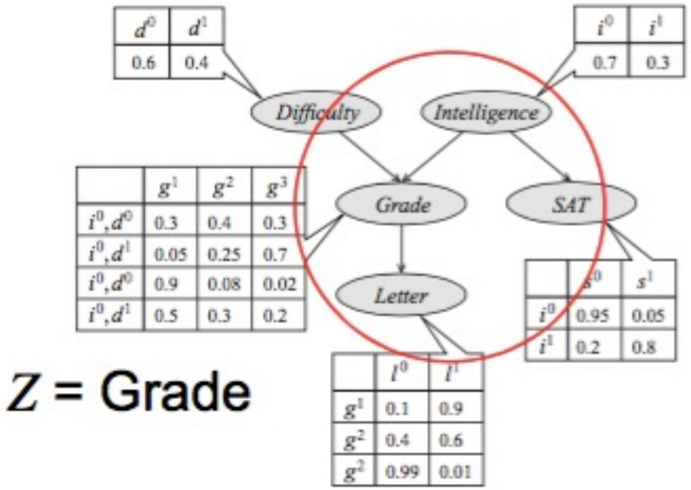
$$= P(z|y)$$

Yes! x and z are conditionally Independent upon y

2. Indirect Evidential Effect: *Y->Z->X*

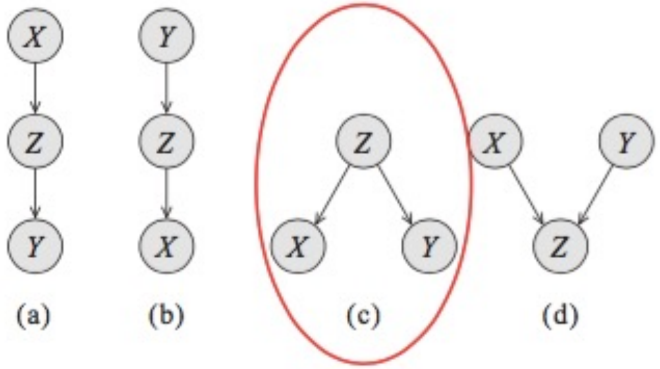
- Evidence *X* can influence *Y* via *Z* only if *Z* is unobserved
  - Observed *Z* blocks influence
- If Grade unobserved, Letter influences assessment of Intelligence
- Dependency is a symmetric notion
  - $X \perp Y$ does not hold then $Y \perp X$ does not hold either



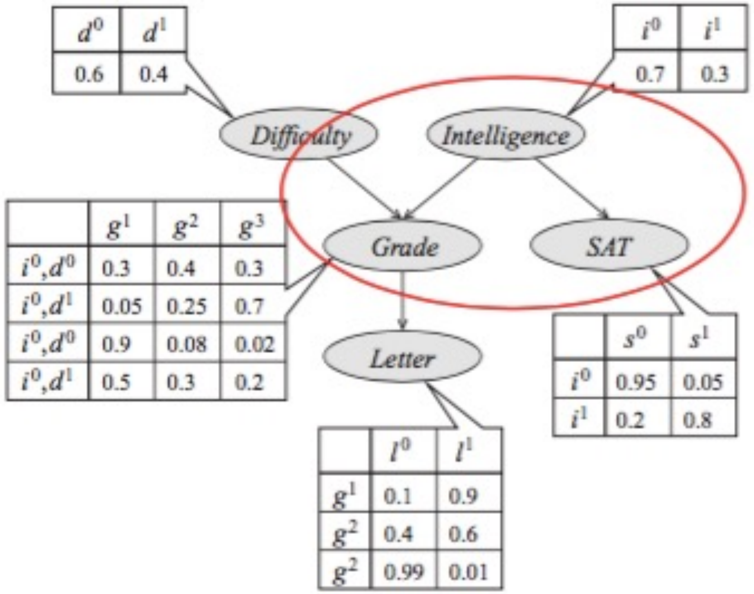


3. Common Cause: *X<-Z->Y*

- *X* can influence *Y* if and only if *Z* is not observed
  - Observed *Z* blocks
- Grade is correlated with SAT score
- But if Intelligence is observed then SAT provides no additional information



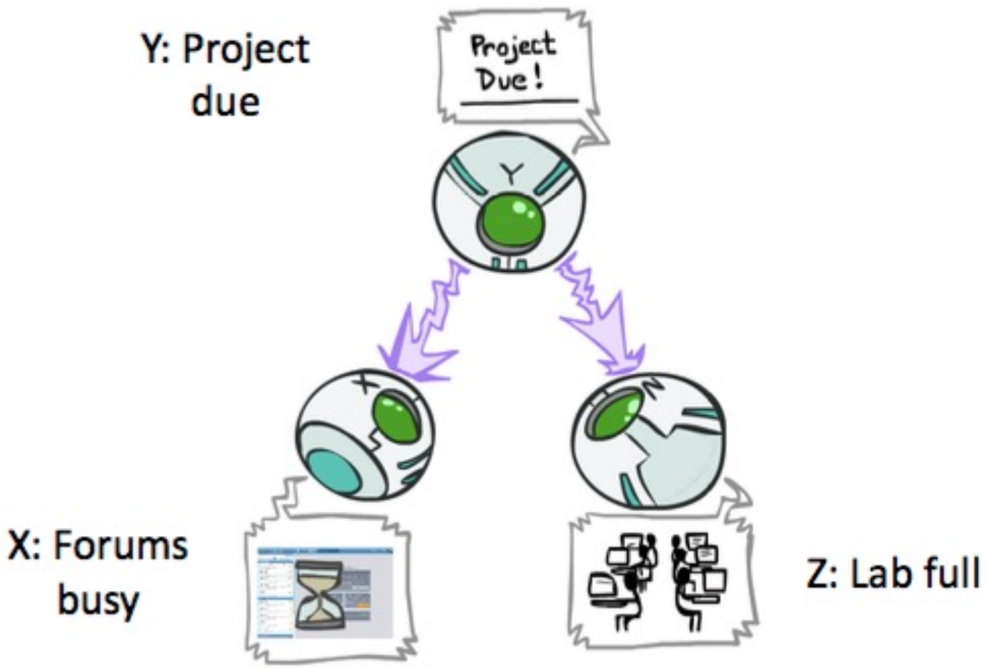
$$(S \perp G \mid I)$$



Common Cause

Observing the cause blocks influence between effects. (makes inactive)

- This configuration is a "common cause"



$$P(x, y, z) = P(y)P(x|y)P(z|y)$$

Common cause

- Guaranteed X and Z independent given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)}$$

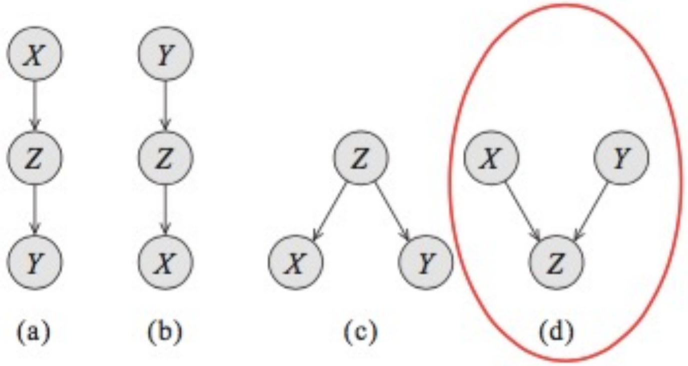
$$= \frac{\cancel{P(y)}\cancel{P(x|y)}P(z|y)}{\cancel{P(y)}\cancel{P(x|y)}}$$

$$= P(z|y)$$

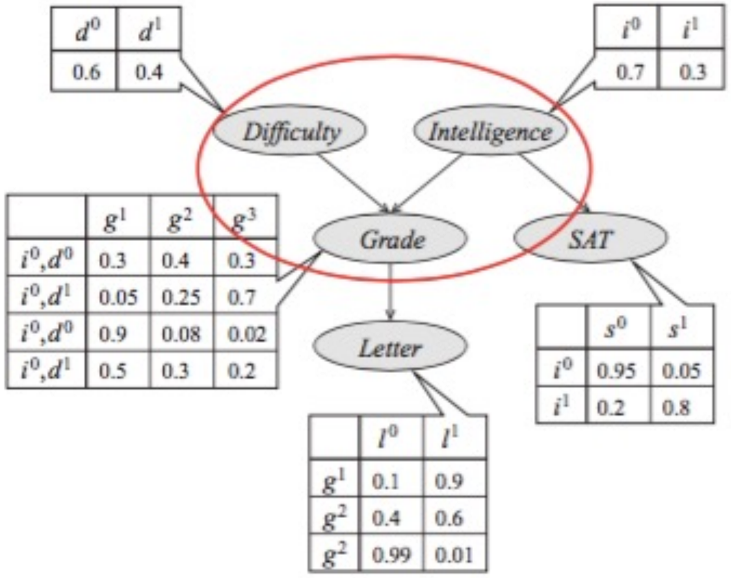
Yes! x and z are conditionally Independent upon y

4. Common Effect (V-structure) *X->Z<-Y*

- Influence cannot flow on trail *X->Z<-Y* if *Z* is not observed
  - Observed *Z* enables
  - Opposite to previous 3 cases (Observed *Z* blocks)
- When *G* not observed *I* and *D* are independent
- When *G* is observed, *I* and *D* are correlated

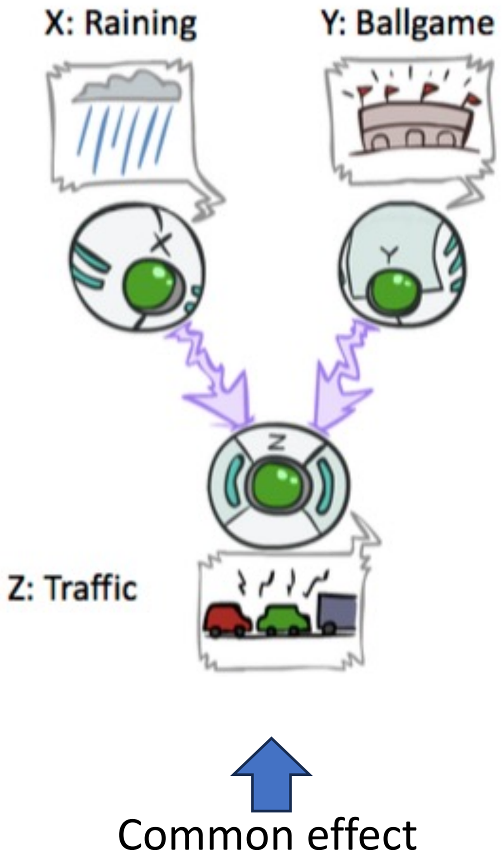


$$I \perp D | \sim G$$



Common Effect

- Last configuration: two causes of one effect (v-structures)

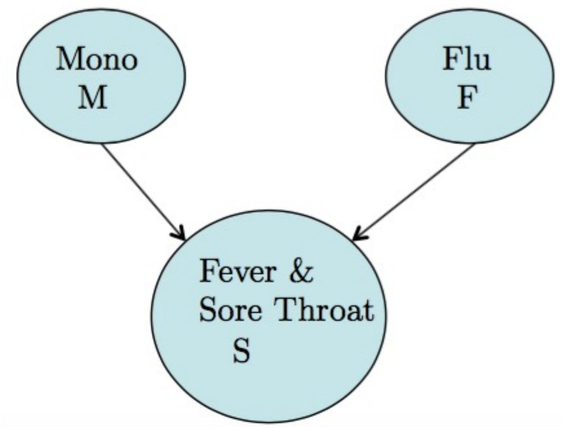


- Are X and Y independent?
  - *Yes*: the ballgame and the rain cause traffic, but they are not correlated
  - Still need to prove they must be (try it!)
- Are X and Y independent given Z?
  - *No*: seeing traffic puts the rain and the ballgame in competition as explanation.
- This is ***backwards*** from the other cases
  - Observing an effect activates influence between possible causes. (makes active!)

Recall: Common in Human Reasoning

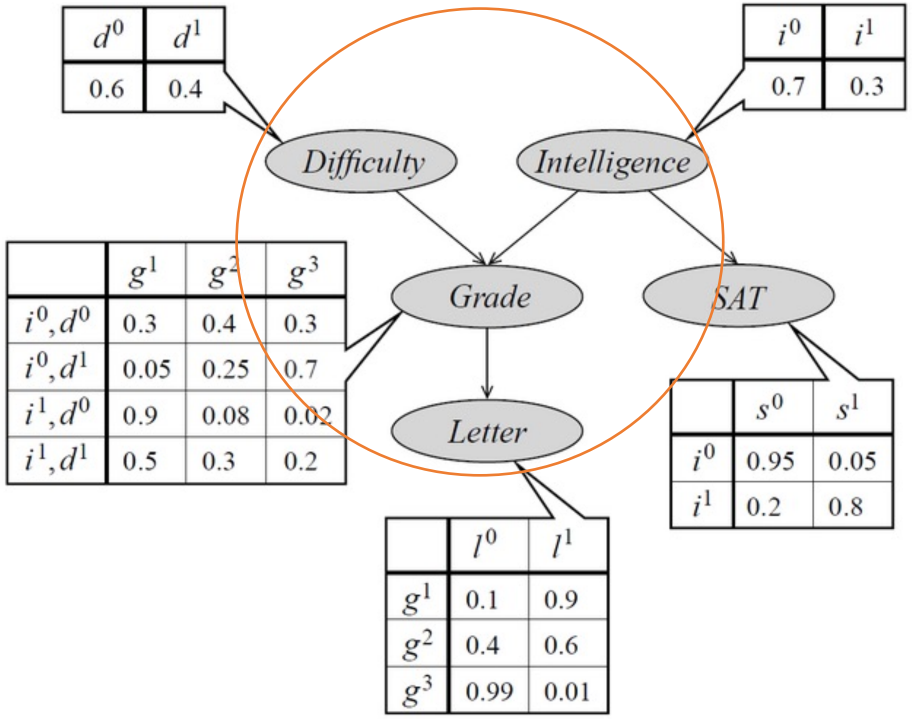
- Binary Variables
- Fever & Sore Throat can be caused by mono and flu
- When flu is diagnosed probability of mono is reduced (although mono could still be present)
- It provides an alternative explanation of symptoms

$$P(m^1 | s^1) > P(m^1 | s^1, f^1)$$



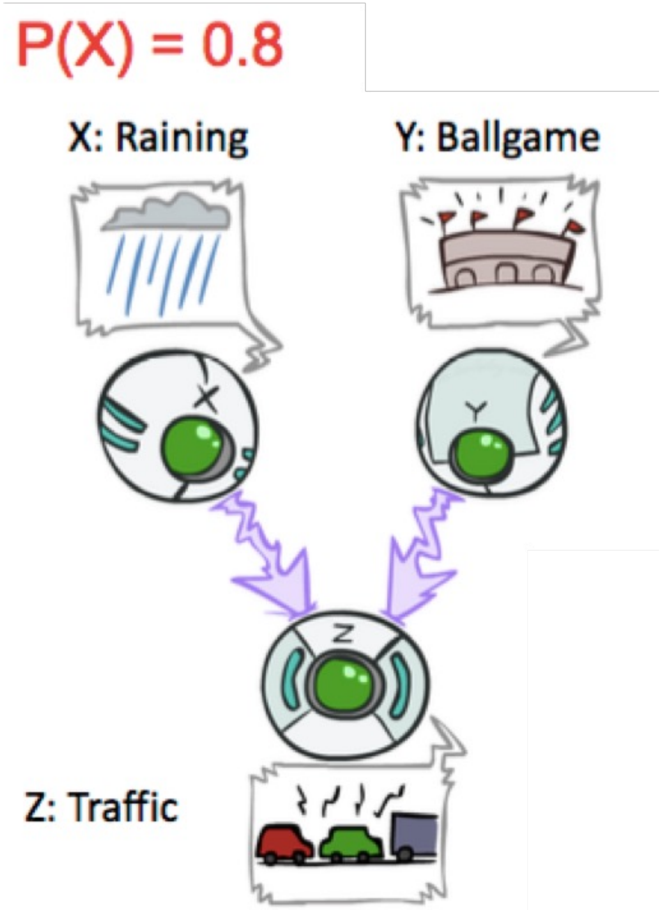
4. Common Effect (V-structure) *X->Z<-Y*

- *Grade* is not observed
- Observe weak letter *L*
  - Which indicates low *Grade*
  - Suffices to correlate *D* and *I*



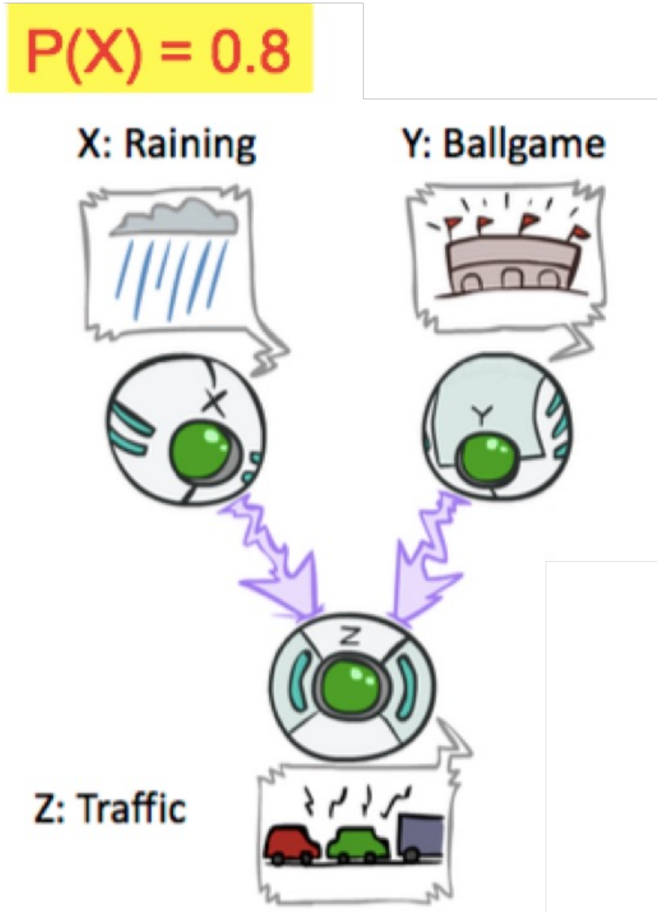
Effective for not only child,
but also descendants

Example: Common Effect – why V-structure?



X	Y	Z	P
T	T	T	0.076
T	T	F	0.004
T	F	T	0.576
T	F	F	0.144
F	T	T	0.162
F	T	F	0.002
F	F	T	0.090
F	F	F	0.009

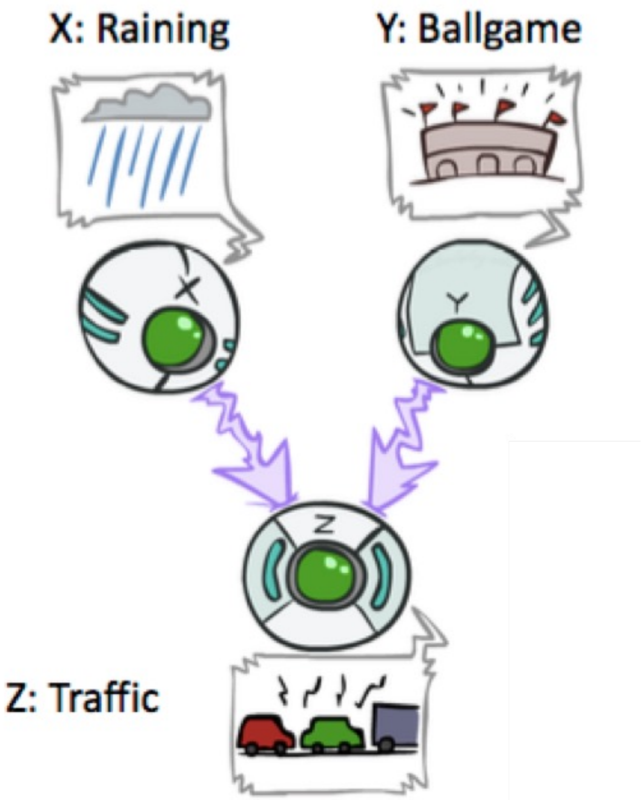
Example: Common Effect – why V-structure?



X	Y	Z	P
T	T	T	0.076
T	T	F	0.004
T	F	T	0.576
T	F	F	0.144
F	T	T	0.162
F	T	F	0.002
F	F	T	0.090
F	F	F	0.009

Example: Common Effect – why V-structure?

P(X) = 0.8



X	Y	Z	P
T	T	T	0.076
T	T	F	0.004
T	F	T	0.576
T	F	F	0.144
F	T	T	0.018
F	T	F	0.002
F	F	T	0.090
F	F	F	0.009

$$P(X|Y) = \frac{0.076+0.004}{0.076+0.004+0.018+0.002}$$

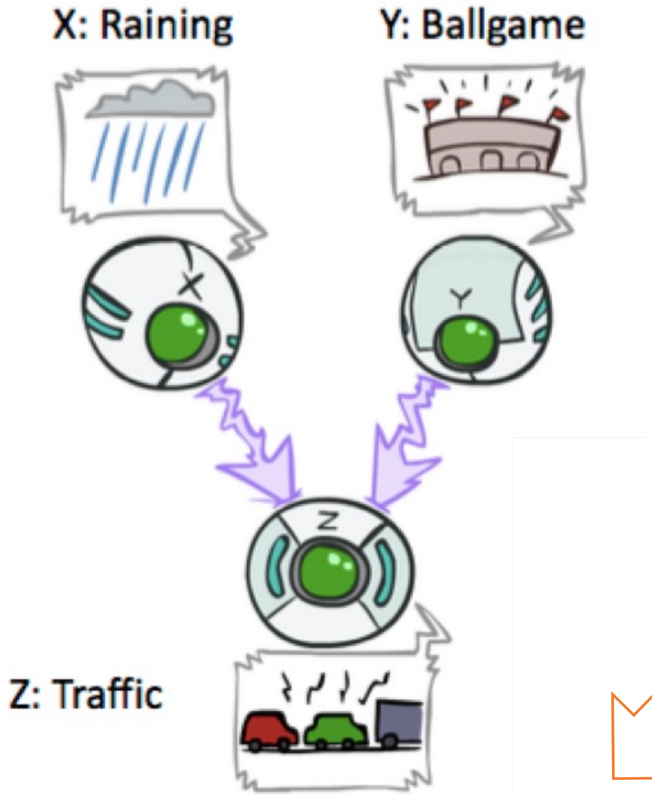
$$= 0.08 / 0.1$$

$$= 0.8$$

X and Y are independent!

But Suppose Also Know Z=T – why V-structure?

$P(X) = 0.8$



$$P(X|Z) = \frac{.076+.576}{.076+.576+.018+.090}$$
$$= 0.652/0.76$$
$$= 0.858$$

$$P(X|Y,Z) = \frac{0.076}{0.076+0.018}$$
$$= 0.8085$$

X	Y	Z	P
T	T	T	0.076
T	T	F	0.004
T	F	T	0.576
T	F	F	0.144
F	T	T	0.018
F	T	F	0.002
F	F	T	0.090
F	F	F	0.009

X and Y are not independent given Z!

So, observing Z makes X and Y dependent

Summary of Indirect Connection

- Causal trail: *X->Z->Y*: active iff *Z* not observed
- Evidential Trail: *X<-Z<-Y*: active iff *Z* is not observed
- Common Cause: *X<-Z->Y*: active iff *Z* is not observed
- Common Effect: *X->Z<-Y*: active iff either *Z* or one of its descendants is observed

What is the general case?

The General Case

- General question: in a given BN, are two variables independent (given evidence)?

- Solution: analyze the graph

- Any complex example can be broken into repetitions of the three canonical cases

D-Separation

- Query: $X_i \perp X_j | \{X_{k_1}, ..., X_{k_n}\}$?
- Check all (undirected) paths between X_i and X_j
  - If one or more paths is active, then independence not guaranteed

$$X_i \not\perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

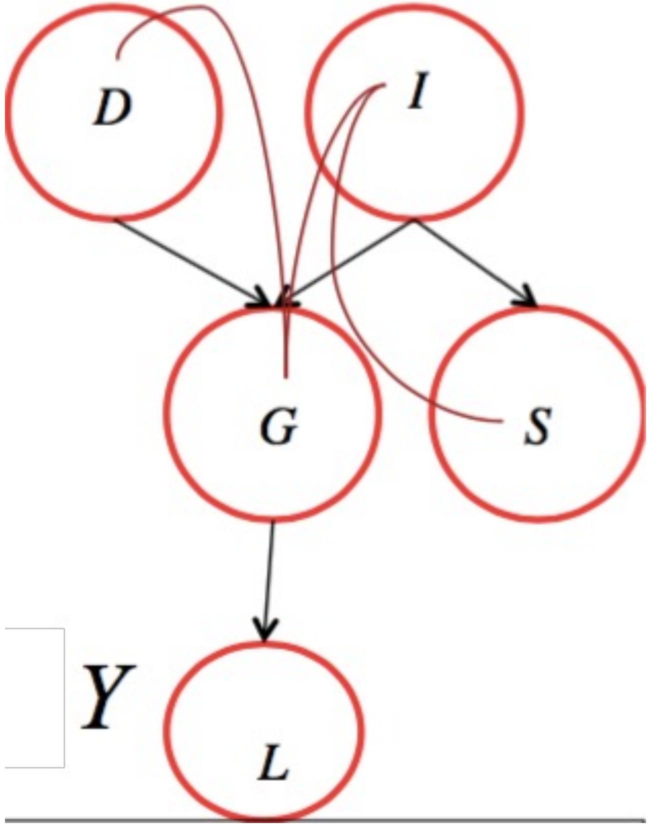
  - Otherwise (i.e. if ***all paths*** are inactive),
    then "D-separated" = independence ***is*** guaranteed

$$X_i \perp X_j | \{X_{k_1}, ..., X_{k_n}\}$$

Active Trail

- When influence can flow from *X* to *Y* via *Z* then trail *X—Z—Y* is active

- Example: Consider Trail *D->G<-I->S*
- When observed
  - $Z = \{\emptyset\}$ *:* trail is *inactive* because v-structure *D->G<-I* is inactive
  - $Z = \{L\}$ *: active* (*D->G<-I* active) since L is descendant of G
  - $Z = \{L, I\}$ *: inactive* because observing I blocks *G<-I->S*



D-separation definition

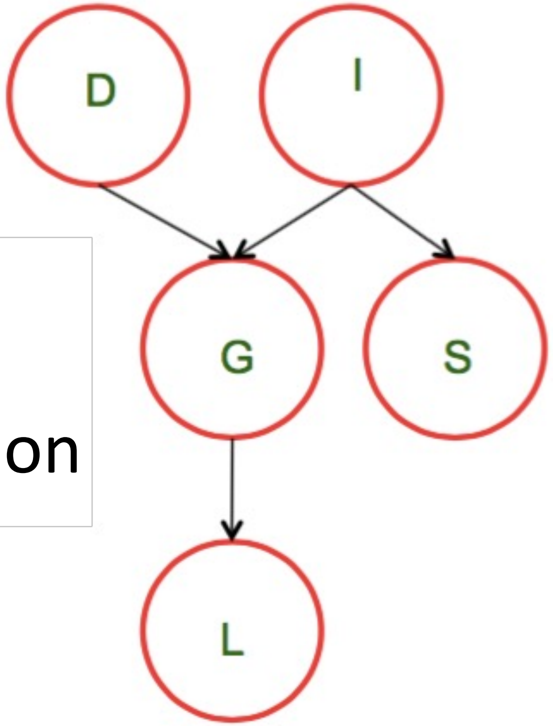
- Let ***X*,*Y*** and ***Z*** be three sets of nodes in G.
- ***X*** and ***Y*** are d-separated given ***Z*** denoted $\text{d-sep}_G(X \perp Y | Z)$ if there is no active trail between any node *X*∈***X*** and *Y*∈***Y*** given ***Z***
- That is, nodes in ***X*** cannot influence nodes in ***Y***
- Provides notion of separation between nodes in a directed graph ("directed" separation)

Independencies from D-separation

- Definition

$$I(G) = \{(X \perp Y | Z) : \text{d-sep}_G(X \perp Y | Z)\}$$

  - Also called *Global Markov independencies*
  - Note: Derived purely from graph (using trails)
- Example: *Global* independence using D-separation



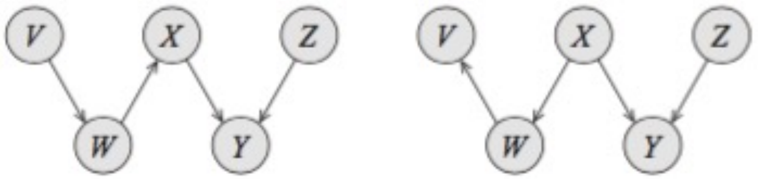
- Compare with *local* independencies
  - For each X_i: ($X_i \perp$ Non-Desc(X_i) | $Pa(X_i)$)

$$\{(L \perp I, D, S | G), (S \perp D, G, L | I), (G, S | D, I), (D \perp I, S | \emptyset)\} \subseteq I(G)$$

I-Equivalence

I-Equivalence

- Conditional independence assertion statements can be the same with different structures
- Two graphs G_1 and G_2 are *I- equivalent* if $I(G_1)=I(G_2)$
- Skeleton of a BN graph G is an undirected graph with an edge for every edge in G
- If two BN graphs have the same set of skeletons and v-structures then they are *I-equivalent*



Same skeleton
Same v-structure $X \rightarrow Y \leftarrow Z$

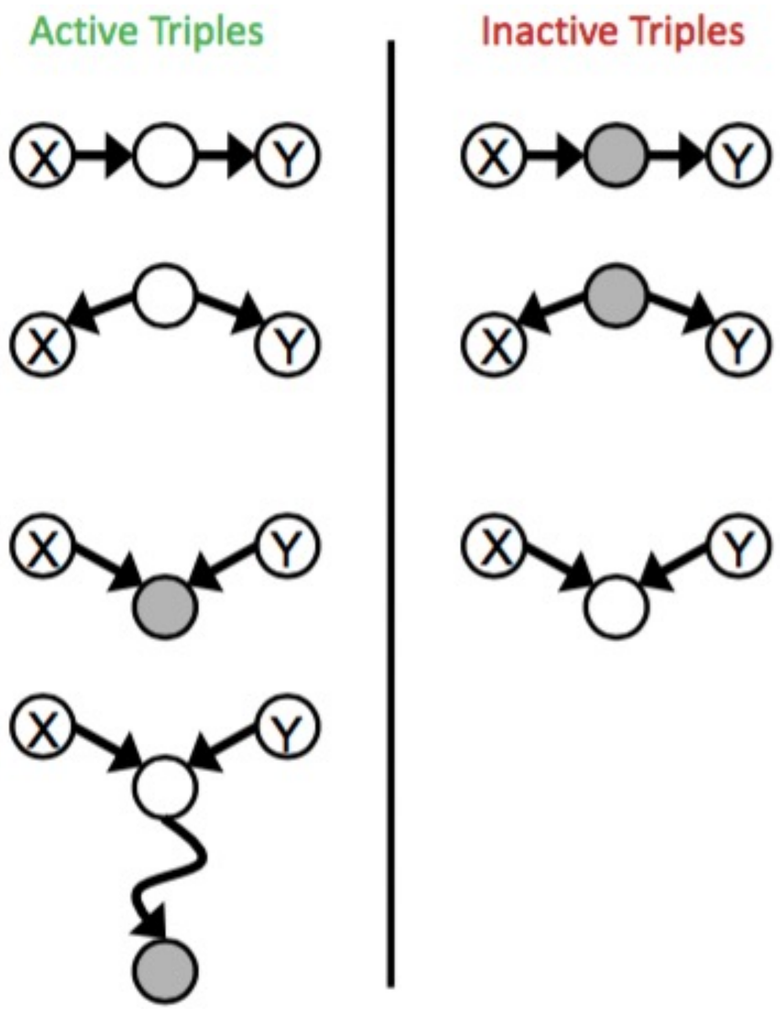
Bayes Nets Representation Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- D-separation gives precise conditional independence guarantees from graph alone
- A Bayes' net's joint distribution may have further (conditional) independence that is not detectable until you inspect its specific distribution

More Example of D-separation

Active / Inactive Paths

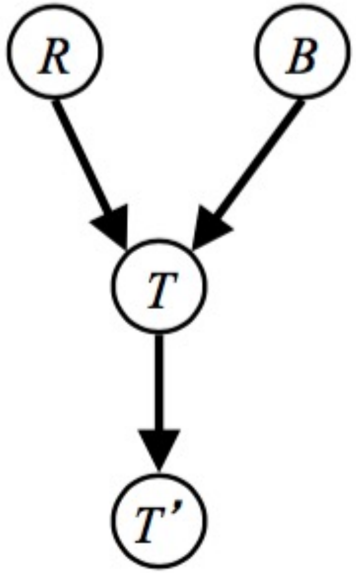
- Question: Are X and Y conditionally independent given evidence variables {Z}?
  - Yes,if X and Y “d-separated” by Z
  - Consider all (undirected) paths from X to Y
  - If no path is active -> independence!
- A ***path*** is active if every triple in path is active:
  - Causal chain A -> B -> C where B is unobserved (either direction)
  - Commoncause A <- B -> C where B is unobserved
  - Common effect (aka v-structure)
    A -> B <- C where B *or one of its descendants* is observed
- All it takes to block a path is a ***single*** inactive segment
  - (But ***all*** paths must be inactive)

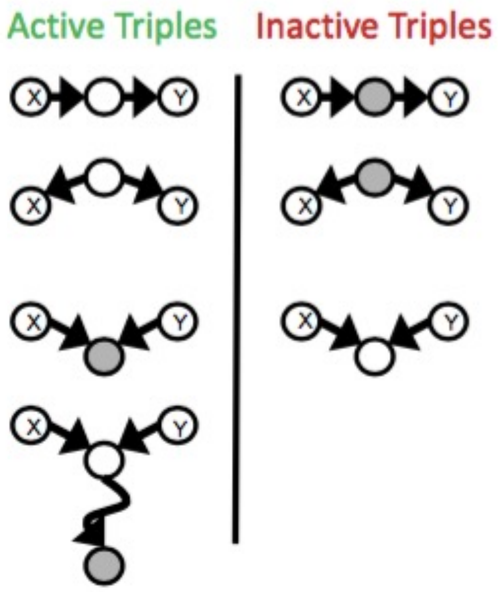


Example

$R \perp\!\!\!\perp B$

Yes, Independent!

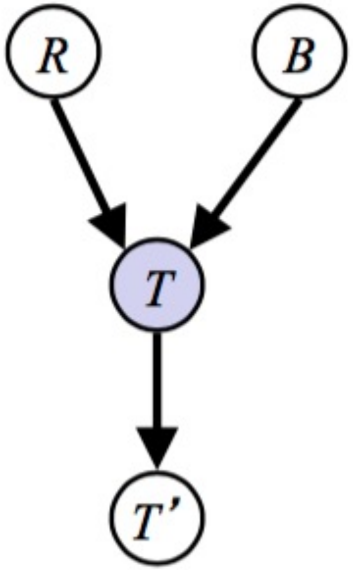


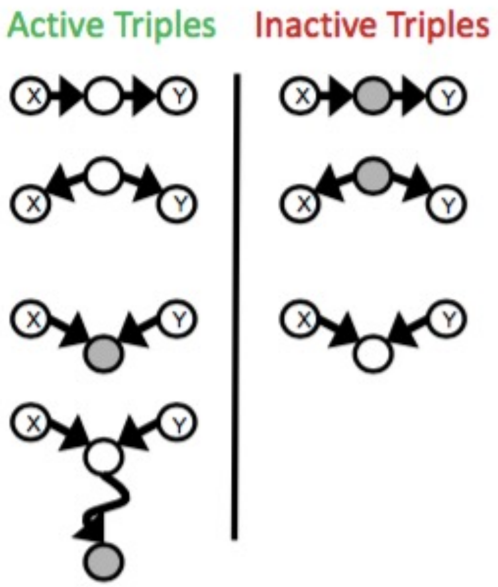


Example

$R \perp\!\!\!\perp B$ *Yes, Independent!*

$R \perp\!\!\!\perp B | T$ *No*



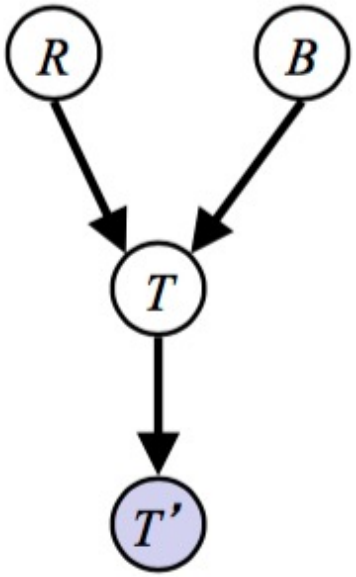


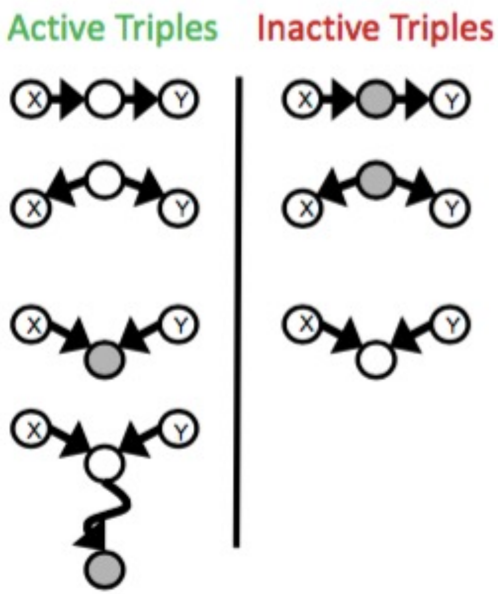
Example

$R \perp\!\!\!\perp B$ *Yes, Independent!*

$R \perp\!\!\!\perp B | T$ No

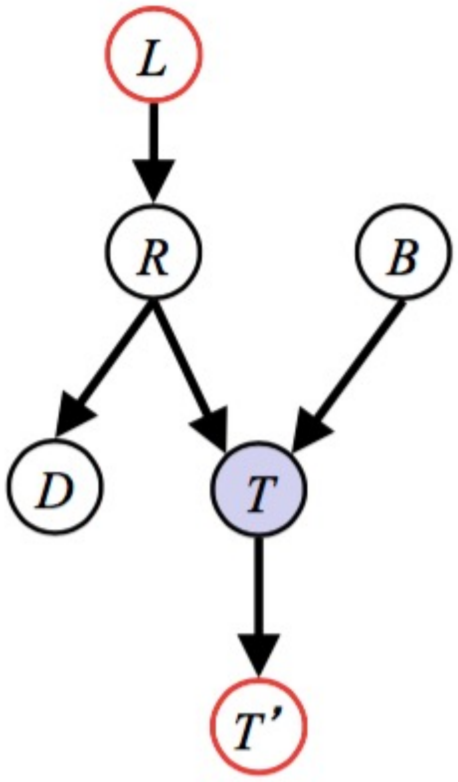
$R \perp\!\!\!\perp B | T'$ No

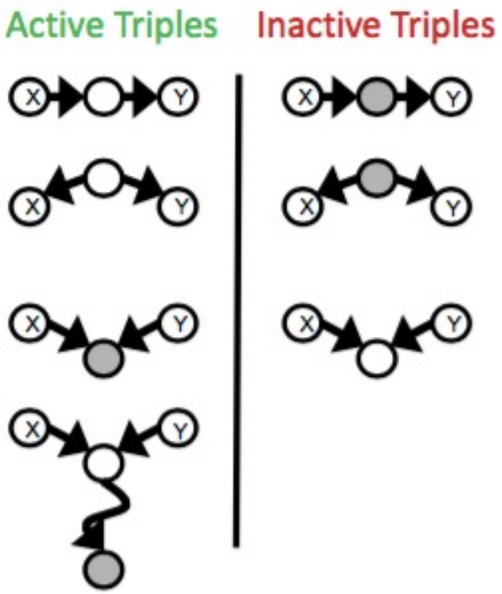




Example

$L \perp\!\!\!\perp T' | T$ *Yes, Independent*

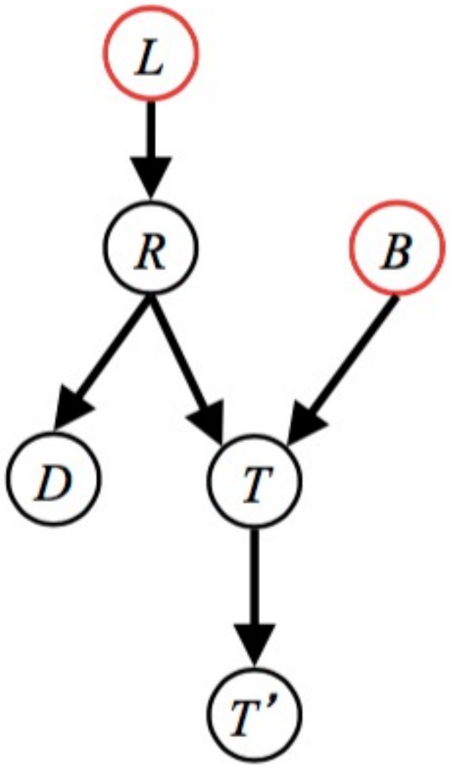


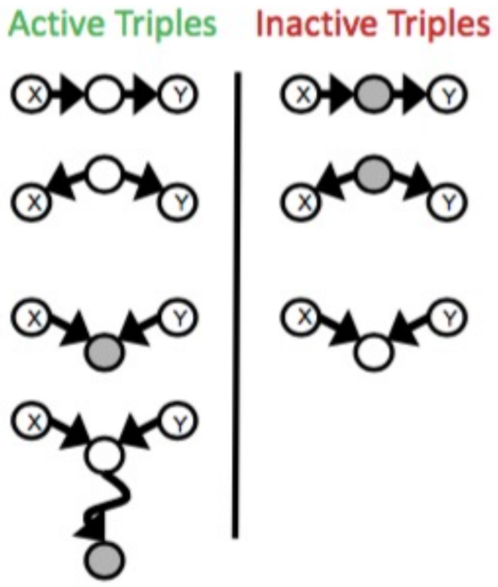


Example

$L \perp\!\!\!\perp T' | T$ *Yes, Independent*

$L \perp\!\!\!\perp B$ *Yes, Independent*



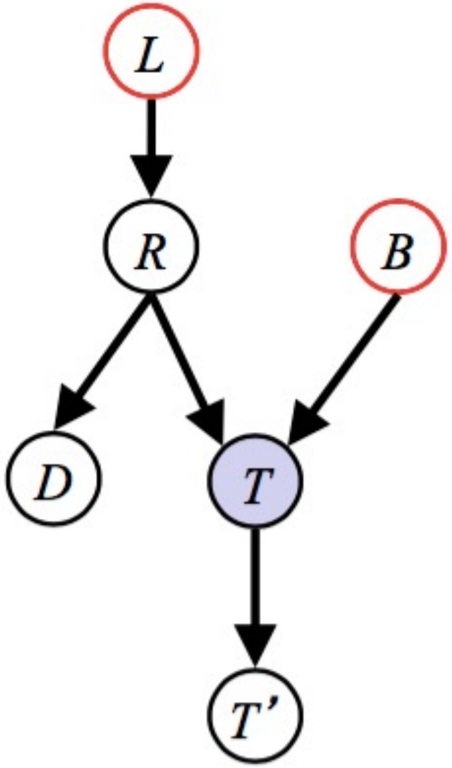


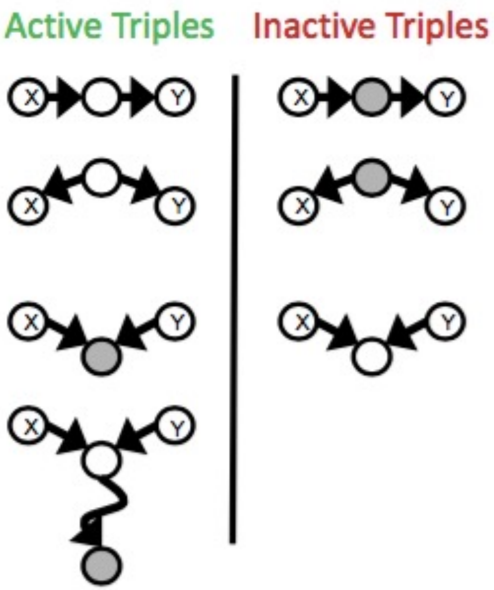
Example

$L \perp\!\!\!\perp T' | T$ *Yes, Independent*

$L \perp\!\!\!\perp B$ *Yes, Independent*

$L \perp\!\!\!\perp B | T$ *No*





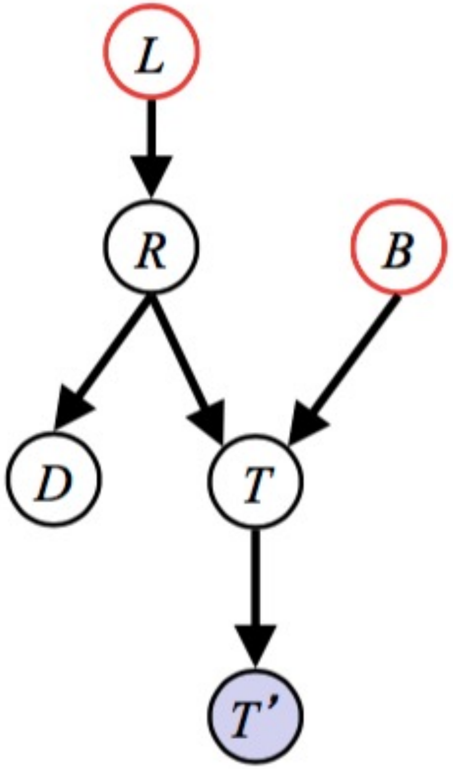
Example

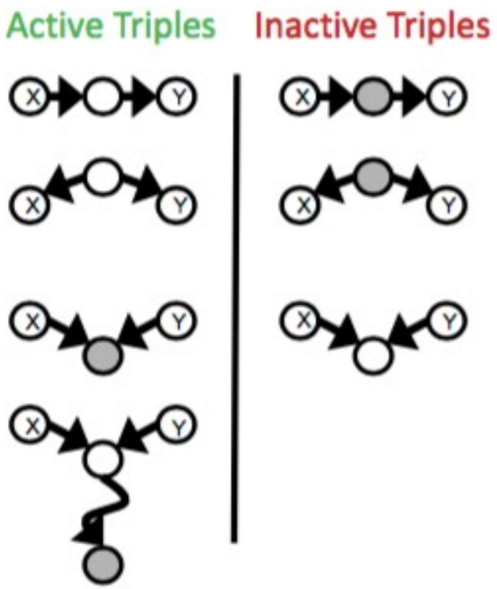
$L \perp\!\!\!\perp T' | T$ Yes, Independent

$L \perp\!\!\!\perp B$ Yes, Independent

$L \perp\!\!\!\perp B | T$ No

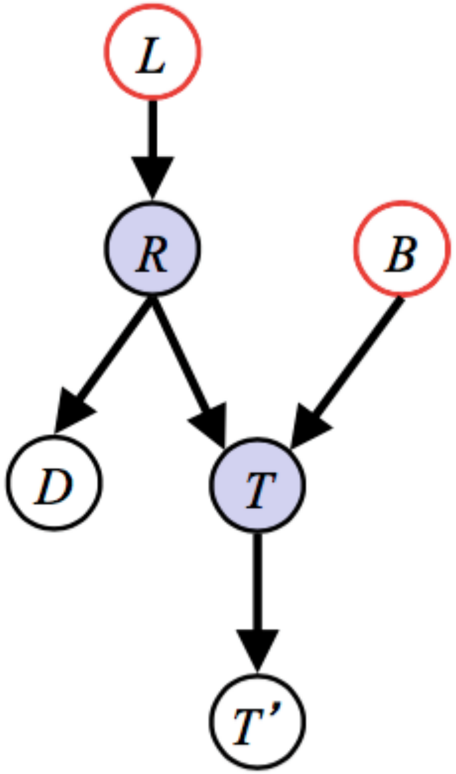
$L \perp\!\!\!\perp B | T'$ No

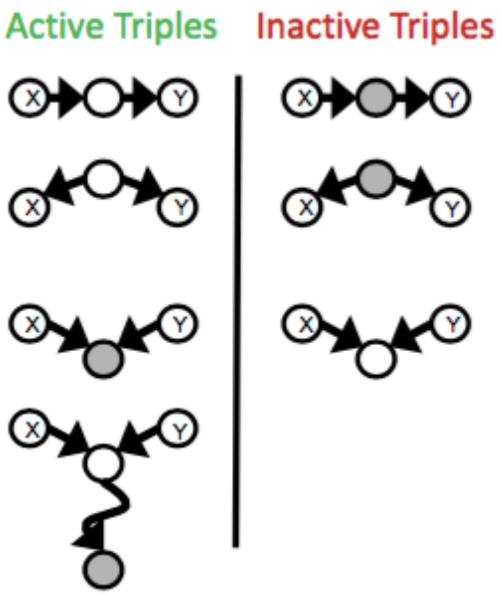




Example

$L \perp\!\!\!\perp T' \mid T$	*Yes, Independent*
$L \perp\!\!\!\perp B$	*Yes, Independent*
$L \perp\!\!\!\perp B \mid T$	*No*
$L \perp\!\!\!\perp B \mid T'$	*No*
$L \perp\!\!\!\perp B \mid T, R$	*Yes, Independent*





Example

$L \perp\!\!\!\perp T' | T$ *Yes, Independent*

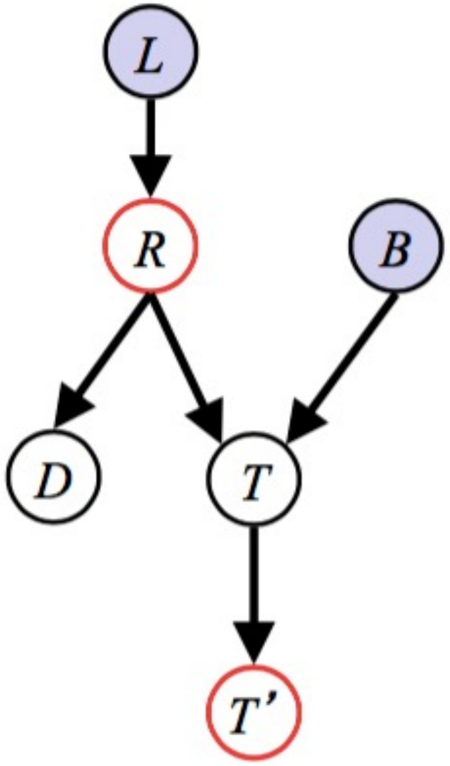
$L \perp\!\!\!\perp B$ *Yes, Independent*

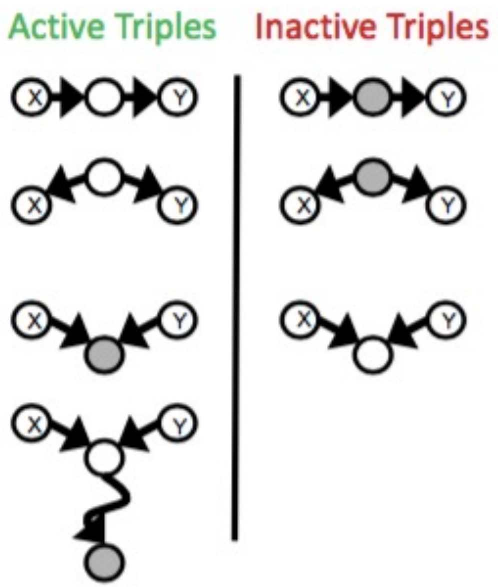
$L \perp\!\!\!\perp B | T$ *No*

$L \perp\!\!\!\perp B | T'$ *No*

$L \perp\!\!\!\perp B | T, R$ *Yes, Independent*

$R \perp\!\!\!\perp T' | L, B$ *No*

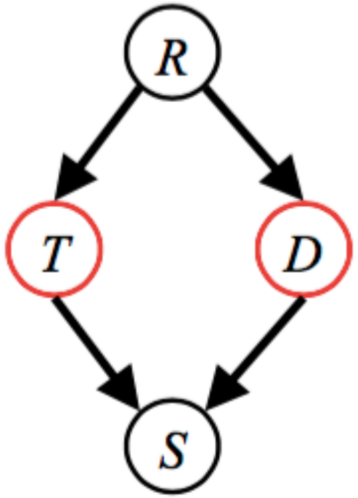


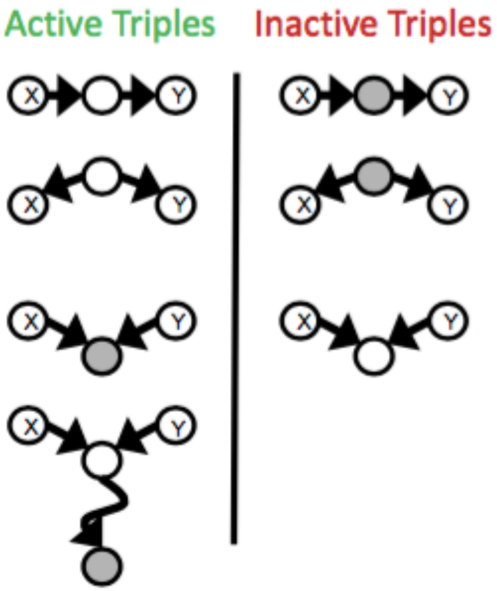


Example

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I'm sad
- Questions:

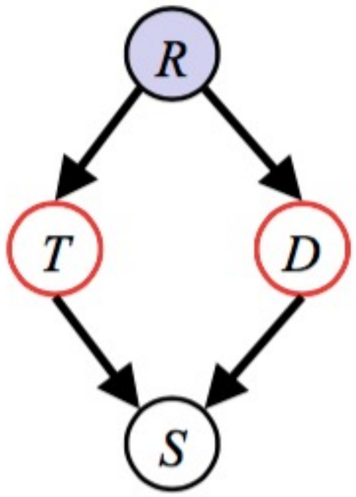
  $T \perp\!\!\!\perp D$ *No*





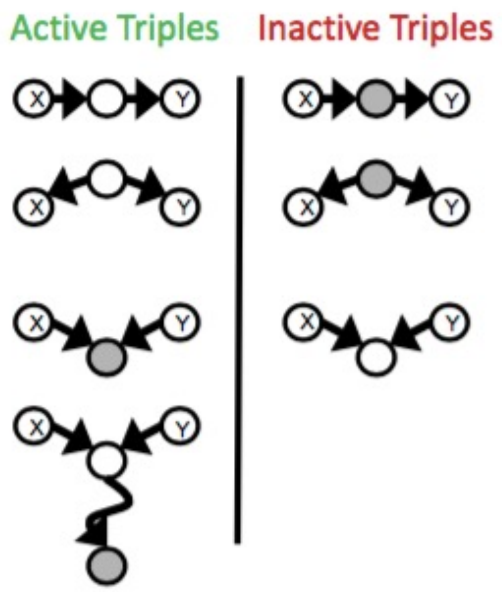
Example

- **Variables:**
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I'm sad
- **Questions:**



$T \perp\!\!\!\perp D$ *No*

$T \perp\!\!\!\perp D | R$ *Yes, Independent*



Example

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I'm sad
- Questions:

$T \perp\!\!\!\perp D$ *No*

$T \perp\!\!\!\perp D | R$ *Yes, Independent*

$T \perp\!\!\!\perp D | R, S$ *No*