COMP229: Introduction to Data Science
Lecture 23: Determinant of a matrix

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Lecture plan & learning outcomes

On this lecture we should learn

- how to distinguish invariants,
- what is linear dependance,
- what is a determinant of a matrix,
- some properties of a determinant and its uses.

Reminder: Invariants

- *Invariant* $\mathcal{I}$: A is equivalent to $B \Rightarrow \mathcal{I}(A) = \mathcal{I}(B)$.
  In practice, invariants can be found by checking that they do not change under this equivalence relation.
- *Complete* $\mathcal{I}$: $\mathcal{I}(A) = \mathcal{I}(B) \Rightarrow A$ is equivalent to B.
- The distribution of all pairwise distances is an isometry invariant of clouds, ‘almost’ complete.

Invariant or non-invariant?

Put the following items in the relevant column:

colour of clothes, eye colour, position of vertices, full name, angles, centroid (barycenter), fingerprint, DNA, area (or volume), pairwise lengths

Non-invariant	Invariant	Complete invariant

Invariant or non-invariant of a person

For a person

	Non-invariant	**Invariant**	**Complete invariant**
Colour of clothes	in a large group/with time	in teams/schools	in a small group photo
Eye colour	with lenses	from 1-2m distance	Iris recognition
Name	with time	cross-sectional	in a class
Fingerprint	occasionally	practically	Historically
DNA	in a tiny sample size	in a population	DNA profiling

Invariant or non-invariant in geometry

For a shape

	Non-invariant	Invariant	Complete invariant
Position of vertices	under isometries	under identical transformation	under identical transformation & if shape is known
Centroid (barycenter)	under isometries	under identical transformation & if shape is known	under identical transformation & known shape & symmetry
Angles	under non-uniform scaling	under isometry with uniform scaling	under isometry
Area (or volume), signed	under isometry with uniform scaling	under isometry	under isometry & known shape & symmetry
Pairwise distances	under isometry with uniform scaling	under isometry	under isometry for triangles & "almost all" point clouds

Everything is relative!!

Relative positions of points

To simplify or compare data, we will also view the same data from different points of view, e.g. by using different bases of vectors in the same space.

These are motivations to understand when different sets of vectors define the same space, e.g. **what three vectors** instead of $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$ **can generate the whole space** $\mathbb{R}^3$.

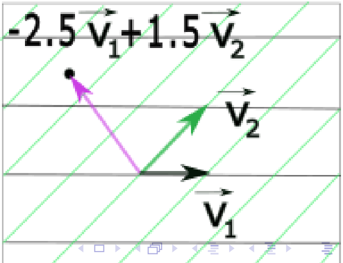
Linear independence of vectors

Definition 23.1. Vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^m$, $k \leqslant m$, are called **linearly independent** if a linear combination $\sum_{i=1}^{k} c_i \vec{v}_i$ is $\vec{0}$, $c_i \in \mathbb{R}$, only for $c_1 = \cdots = c_k = 0$. **The span** of vectors is the set of *all* their finite linear combinations $\sum_{i=1}^{k} c_i \vec{v}_i$, $c_i \in \mathbb{R}$.

For $k = 1$, linear independence of a single vector $\vec{v}_1$ means that $\vec{v}_1 \neq \vec{0}$. The span of $\vec{v}_1$ is a straight line.

For $k = 2$, linear independence of $\vec{v}_1, \vec{v}_2$ means that $\vec{v}_1, \vec{v}_2$ are not proportional to each other (not parallel).

The span of $\vec{v}_1, \vec{v}_2$ forms a plane.



The determinant of a 2×2 matrix

We can detect if m vectors are linearly independent in $\mathbb{R}^m$ by using **the determinant**.

Definition 23.2. $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$

Claim 23.3. Vectors $\vec{u} = \begin{pmatrix} a \\ c \end{pmatrix}$, $\vec{v} = \begin{pmatrix} b \\ d \end{pmatrix}$ are linearly dependent if and only if $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0.$

Proof. If $\vec{v} = t\vec{u}$, $\det \begin{pmatrix} a & ta \\ c & tc \end{pmatrix} = atc - cta = 0.$

If $ad - bc = 0$, then $\vec{v} = t\vec{u}$ for $t = \frac{b}{a} = \frac{d}{c}$. $\square$

Rotation

Problem 23.4. Find the determinants of rotation and reflection matrices from past lectures.

Solution 23.4. Rotation: The matrix of the anticlockwise rotation around 0 through an angle β is $A =$
$\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$, $\det A = \cos^2\beta + \sin^2\beta = 1$.

Reflection

Reflection: The matrix of a reflection over a line L passing through 0 is $B = I - 2\vec{v}\vec{v}^T$, where $\vec{v}$ is perpendicular to the line L of reflection and $|\vec{v}| = 1$.

Use $|\vec{v}| = 1$ to get $\vec{v}\vec{v}^T = \begin{pmatrix} v_x^2 & v_x v_y \\ v_x v_y & v_y^2 \end{pmatrix}$ and

$\det B = \det(I - 2\vec{v}\vec{v}^T) = \det \begin{pmatrix} 1 - 2v_x^2 & -2v_x v_y \\ -2v_x v_y & 1 - 2v_y^2 \end{pmatrix} =$

$(1 - 2v_x^2)(1 - 2v_y^2) - (-2v_x v_y)^2 = 1 - 2(v_x^2 + v_y^2) = -1.$ $\square$

The determinant of a 3×3 matrix

Definition 23.5. $\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33}$
$+a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$

This formula generalises to any dimension m.

The expansion formula

This formula generalises to any dimension m.

For any $m \times m$ matrix A and $i, j \in \{1, \dots, m\}$, let A_{ij} be the $(m-1) \times (m-1)$ submatrix obtained from A by removing row i, column j. For $m = 3$, $i = 1$,

$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$

Definition 23.6. (via the expansion formula)

$\det A = \sum_{j=1}^{m} (-1)^{i+j} a_{ij} \det A_{ij}$ for any fixed i,

$\det A = \sum_{i=1}^{m} (-1)^{i+j} a_{ij} \det A_{ij}$ for any fixed j.

Expansion formula for $m = 3$

Check that definitions match:

$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$

$$A_{11} = \begin{pmatrix} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{pmatrix}, \; A_{12} = \begin{pmatrix} & & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{pmatrix},$$

$$A_{13} = \begin{pmatrix} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{pmatrix},$$ substitute to get Definition 23.5:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) -$$

$$-a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Bijective linear maps

Theorem 23.7. A linear map $f : \vec{v} \mapsto A\vec{v}$ in $\mathbb{R}^m$ is bijective if and only if $\det A \neq 0$. The inverse map $f^{-1} : \mathbb{R}^m \to \mathbb{R}^m$ is linear and has the matrix that is denoted by A^{-1} and satisfies $AA^{-1} = I = A^{-1}A$.

Proof only for $m = 2$. If $\det A = ad - bc = 0$, then Claim 23.3 says that the images of the basis vectors (columns of A) are linearly dependant, i.e. parallel, hence any image $A\vec{v}$ is parallel to them, i.e. $f : \vec{v} \mapsto A\vec{v}$ is not bijective.

Now assume that $\det A \neq 0$.

The inverse of a 2×2 matrix

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, set $A^{-1} = \dfrac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Check that $AA^{-1} = I = A^{-1}A$, i.e. A^{-1} is the inverse of A.

To show that $f : \vec{v} \mapsto A\vec{v}$ is surjective, for any $\vec{u}$, set $\vec{v} = A^{-1}\vec{u}$. Then $A\vec{v} = A(A^{-1}\vec{u}) = I\vec{u} = \vec{u}$, where I is the identity matrix, so f is surjective.

To show that $f : \vec{v} \mapsto A\vec{v}$ is injective, assume that $A\vec{u} = A\vec{v}$. Applying A^{-1} to the equal vectors, we get $\vec{u} = A^{-1}(A\vec{u}) = A^{-1}(A\vec{v}) = \vec{v}$, so f is injective. $\square$

The determinant of a transpose A^T

Claim 23.8. $\det A^T = \det A$ for a square matrix A.

Proof: $m = 2$, $\det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

For $m = 3$,

$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \det \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$ (for both matrices above).

Multiplications by a scalar

Claim 23.9. For any $m \times m$ matrix A, if one column or row is multiplied by a scalar $s \in \mathbb{R}$, then $\det A$ is multiplied by s.
Hence if A has a column or row of zeros, then $\det A = 0$.
If we multiply (all elements of) A by s, then
$\det(sA) = s^m \det A$.

Proof for $m = 2$:

$$\det \begin{pmatrix} sa & b \\ sc & d \end{pmatrix} = s(ad - bc) = s \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$$\det \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix} = s^2(ad - bc) = s^2 \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Swapping rows or columns

Claim 23.10. If we swap two columns (or rows), the determinant changes its sign. Any matrix A with two identical columns (or rows) has $\det A = 0$.

Proof for $m = 2$:

$$\det \begin{pmatrix} b & a \\ d & c \end{pmatrix} = bc - ad = -\det \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

If A has two identical rows, their swap preserves A, but changes the sign of $\det A$, hence $\det A = 0$.

Adding columns or rows in 2D

Claim 23.11. If one column (or row) $\vec{v}$ of A is written as $\vec{v}_1 + \vec{v}_2$, then $\det A = \det A_1 + \det A_2$,
where A_i is obtained from A by replacing $\vec{v}$ by $\vec{v}_i$, $i = 1, 2$.

Proof for $m = 2$:

$$\det \begin{pmatrix} a & b_1 + b_2 \\ c & d_1 + d_2 \end{pmatrix} = a(d_1 + d_2) - (b_1 + b_2)c$$

$$= (ad_1 - b_1 c) + (ad_2 - b_2 c) = \det \begin{pmatrix} a & b_1 \\ c & d_1 \end{pmatrix} + \det \begin{pmatrix} a & b_2 \\ c & d_2 \end{pmatrix}.$$

Adding a multiple of another column

Claim 23.12. Adding any multiple of one column (or row) to another one preserves the determinant.

Proof for $m = 2$: $\det \begin{pmatrix} a & b+ta \\ c & d+tc \end{pmatrix} = a(d+tc) - (b+ta)c$

$= ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Claims 23.8-12 allow us to get an upper triangular matrix with zeros below the diagonal while keeping the determinant (so-called *Gaussian elimination*).

An upper triangular matrix

Claim 23.13. If a matrix A is upper triangular, i.e. $a_{ij} = 0$ for all $i > j$, then $\det A = a_{11}a_{22}\cdots a_{mm}$.

Proof for $m = 2$ *and* $m = 3$*:*

$$\det\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = ad, \ \det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33}.$$

Reduction for $m = 2$, $a \neq 0$*:*

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ 0 & d - b\frac{c}{a} \end{pmatrix} = a(d - b\tfrac{c}{a}) = ad - bc.$$

Determinant of a product

Theorem 23.14. $\det(AB) = (\det A)(\det B)$.

Proof for $m = 2$. The product of matrices is

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}.$$

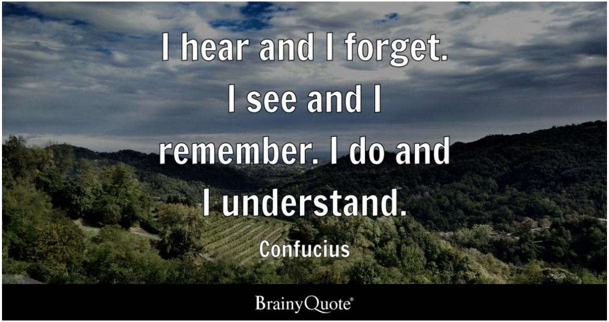
The determinants are $(a_1 d_1 - b_1 c_1)(a_2 d_2 - b_2 c_2) =$
$a_1 a_2 d_1 d_2 - a_1 b_2 c_2 d_1 - a_2 b_1 c_1 d_2 + b_1 b_2 c_1 c_2 =$
$(a_1 a_2 + b_1 c_2)(c_1 b_2 + d_1 d_2) - (a_1 b_2 + b_1 d_2)(c_1 a_2 + d_1 c_2)$.
Check that all unnecessary terms cancel. $\square$

Time to revise and ask questions

- The *determinant* of any $m \times m$ matrix A is $\det A = \sum_{j=1}^{m} (-1)^{i+j} a_{ij} \det A_{ij}$ for any fixed i.
- A linear map $f : \vec{v} \mapsto A\vec{v}$ in $\mathbb{R}^m$ is bijective if and only if $\det A \neq 0$ (then there is the inverse linear map that has A^{-1} such that $AA^{-1} = I = A^{-1}A$).
- Vectors are linearly independent iff their $\det A \neq 0$.
- m inearly independent vectors span the $\mathbb{R}^m$ space.

Problem 23.15. Is the linear map $\vec{v} \mapsto B\vec{v}$ in $\mathbb{R}^3$ with the matrix $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ bijective?

Additional references



Linear combinations and span in 3Blue1Brown videos.