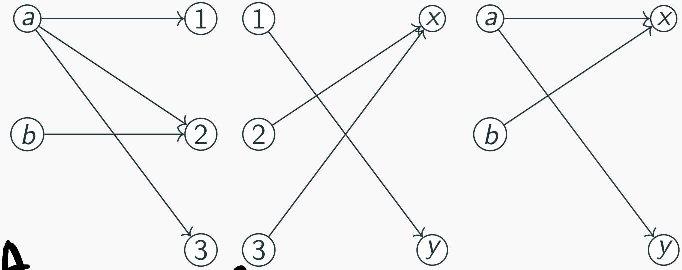
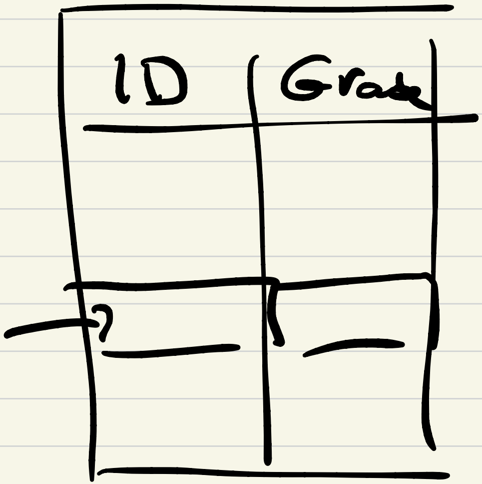
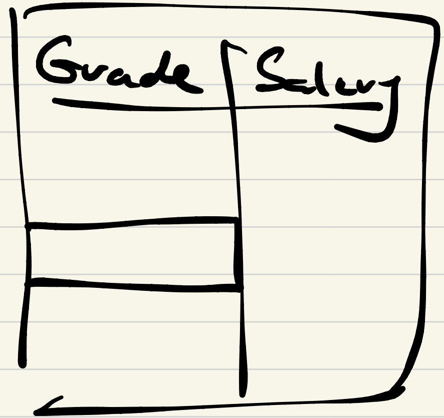
Now let's go back and see how this works for matrices representing relations



A B C

$$R : \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline a & \\ b & \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad S : \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \end{array} \qquad S \circ R : \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$





$S \circ R$

$(id, grade) \in R$

$(grade, salary) \in S$

$R \subseteq \{1, \dots 10^6\} \times \{1, \dots, 50\}$

$S \subseteq \{1, \dots 50\} \times \mathbb{N}$

Simplify the example

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 1 \\
3 & 0 & 0 & 1 & 0 & 0
\end{array}
$$

Matrix for R

$$
\begin{array}{c|ccccc}
 & £1 & £2 & £5 & £10 & £100 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 \\
4 & 0 & 0 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

Matrix for S

$$
\begin{array}{c|ccccc}
 & £1 & \cdots & £5 & \cdots & £100 \\
\hline
1 & - & - & 1 & 1 & \\
2 & & & 0 & & 1 \\
3 & & & & &
\end{array}
$$

The formal description

Given two matrices with entries "1" and "0" representing the relations we can form the matrix representing the composition. This is called the *logical (Boolean) matrix product*.

Let $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\}$ and $C = \{c_1, \ldots, c_p\}$.

The logical matrix M representing R is given by:

$$M(i,j) = \begin{cases} 1 & \text{if} \quad (a_i, b_j) \in R \\ 0 & \text{if} \quad (a_i, b_j) \notin R \end{cases}$$

The logical matrix N representing S is given by

$$N(i,j) = \begin{cases} 1 & \text{if} \quad (b_i, c_j) \in S \\ 0 & \text{if} \quad (b_i, c_j) \notin S \end{cases}$$

Matrix representation of compositions

Then the entries $P(i, j)$ of the logical matrix P representing $S \circ R$ are given by

- $P(i, j) = 1$ if there exists l with $1 \leq l \leq m$ such that $M(i, l) = 1$ and $N(l, j) = 1$.
- $P(i, j) = 0$, otherwise.

We write $P = MN$.

The example from before

Let R be the relation between $A = \{a, b\}$ and $B = \{1, 2, 3\}$ represented by the matrix

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Similarly, let S be the relation between B and $C = \{x, y\}$ represented by the matrix

$$N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$S \circ R$

$M \cdot N$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Example

Then the matrix $P = MN$ representing $S \circ R$ is

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Detour: Boolean multiplication in Python

```
def booleanMM(m1, m2):
    # creating a zero matrix
    res = [ [0 for i in range(len(m2[0]))]
            for j in range(len(m1)) ]
    # computing the result
    for i in range(len(m1)):
        for j in range(len(m2[0])):
            for k in range(len(m2)):
                res[i][j] = (res[i][j] or
                    (m1[i][k] and m2[k][j]))
    return res

print booleanMM([[0,0,1],[1,0,1]], [[1,0],[0,1],[0,0]])
```

(but numpy does it better!)

Properties of relations on a set

Infix notation for binary relations

$(1,2) \in R$, where $R \subseteq \{\mathbb{Z}\times\mathbb{Z} \mid x<y\}$

$1<y$

If R is a binary relation then we write xRy whenever $(x, y) \in R$. The predicate xRy is read as x is R-related to y.

xRy $\qquad$ $(x,y) \in R$

Motivating example: comparing strings

Consider relations R, S and L on the set of all strings:

- R—lexicographic ordering;
- uSv if, and only if, u is a substring of v;
- uLv if, and only if, $len(u) \leq len(v)$.

|a|

s s < a

ana   banana   bob

abe

ann

bob

ana L bob

bob L ana

Properties of binary relations (1)

$R, S, \subset$

$1 \leq 1$

$1 \not< 1$

A binary relation R on a set A is

- *reflexive* when xRx for all $x \in A$.

$$\forall x\, A(x) \implies xRx$$

ana R bob

? bob $\cancel{R}$ ana

- *symmetric* when xRy implies yRx for all $x, y \in A$;

$$\forall x, y\; xRy \implies yRx$$

Properties of binary relations (2)

R, S, L

ana R ana

$\boxed{\text{ana L bob} \quad \text{bob L ana}}$

$\not\Downarrow$

ana = bob

A binary relation R on a set A is

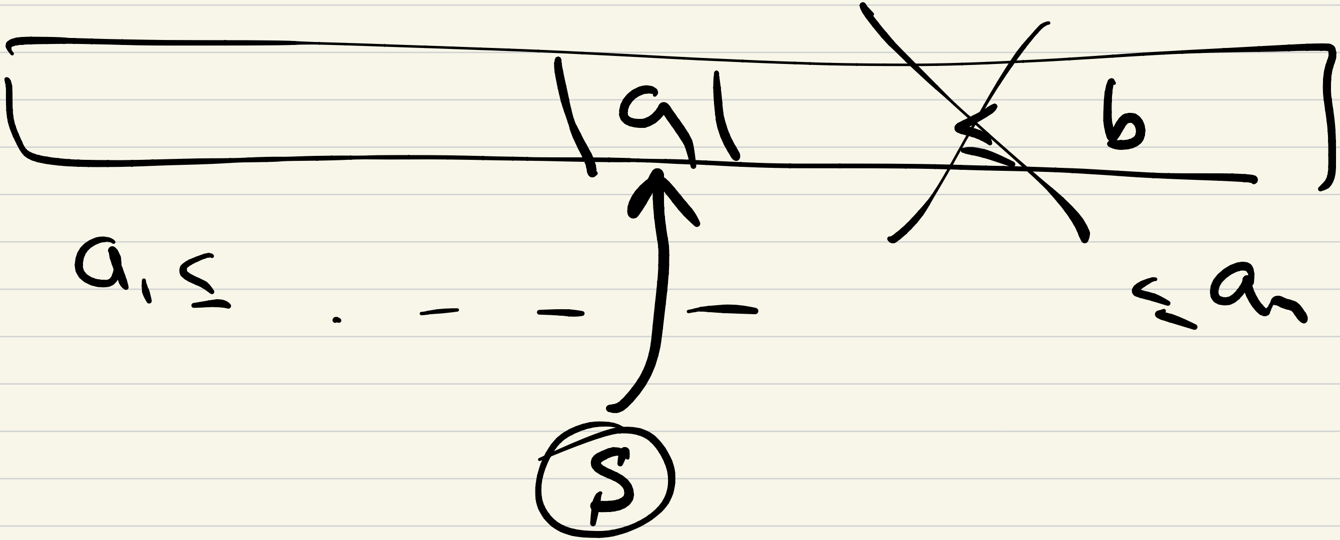
- *antisymmetric* when xRy and yRx imply $x = y$ for all $x, y \in A$;

$$\forall x, y \; xRy \text{ and } yRx \implies y = x$$

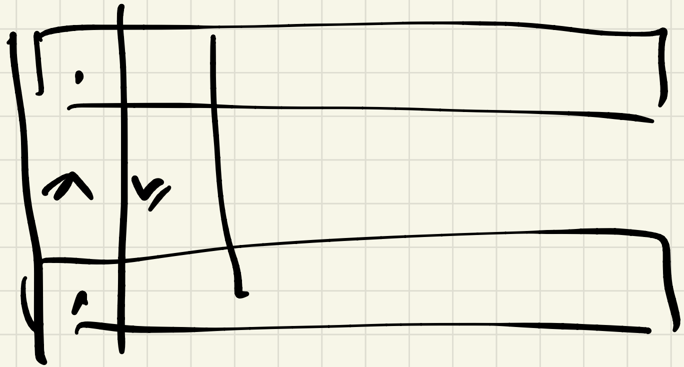
- *transitive* when xRy and yRz imply xRz for all $x, y, z \in A$.

$$\forall x, y, z \; xRy \text{ and } yRz \implies xRz$$

ana banana lambanana



$s \le a$



word

word

a b c d ... z

alphabetical order