Foundations of Computer Science Comp109

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Part 3. Relations

Comp109 Foundations of Computer Science

Reading

- Discrete Mathematics with Applications S. Epp, Chapter 8.
- **Discrete Mathematics and Its Applications** K. Rosen, Chapter 9

Contents

Motivation

- Intuitively, there is a "relation" between two things if there is some connection between them.
  E.g.
  - 'friend of'
  - $a < b$
  - m divides n
- Relations are used in crucial ways in many branches of mathematics
  - Equivalence
  - Ordering
- Computer Science

Databases and relations

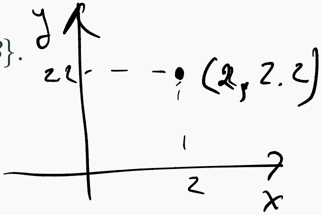
A database table $\approx$ relation

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Definition The Cartesian product $A \times B$ of sets A and B is the *set* consisting of all pairs (a, b) with $a \in A$ and $b \in B$, i.e.,

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$



Note that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

Note

- $\{1, 2\} = \{2, 1\}$ but $(1, 2) \neq (2, 1)$.

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ \underline{(x,y)} \mid x \in \mathbb{R}, \, y \in \mathbb{R} \}$$

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{ (x,y,z) \mid x \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R} \}$$

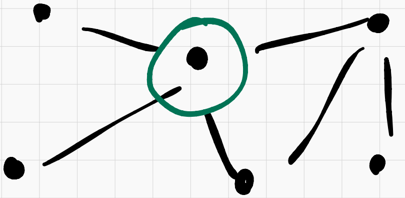
$$\mathbb{R}^1$$

Example

- Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (2, a), (1, b), (2, b), (1, c), (2, c)\}.$$

- $B \times A =$ $\{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$

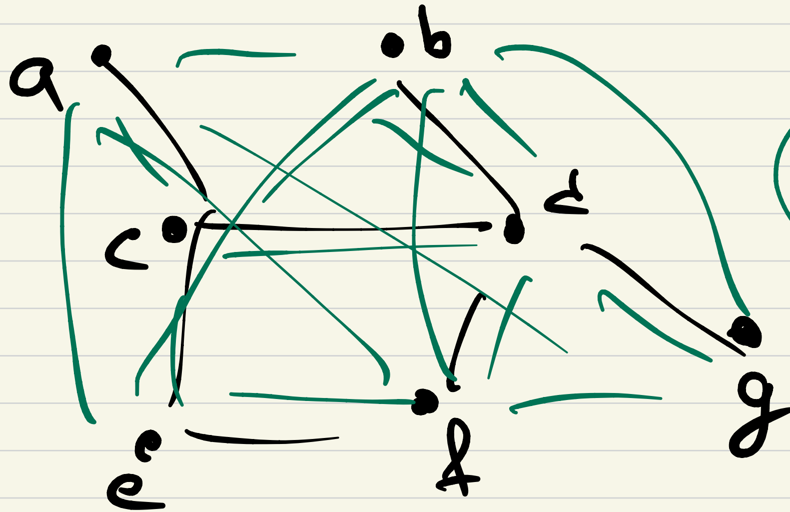


Relations

$$\emptyset \subseteq A \times B$$

Definition A binary relation between two sets A and B is a subset R of the Cartesian product $A \times B$.

If $A = B$, then R is called a binary relation on A.



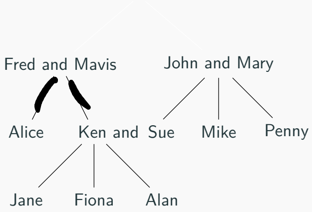
$A = \{a, b, c, d, e, f, g\}$

$A \times A = \{(x, y) \mid x \in A, y \in A\}$

$R = \{(a,c), (c,d), (b,d), (e,a) \ldots\}$

$R \subseteq A \times A$

Example: Family tree



Write down

$\text{Parent} = \{(\text{Fred}, \text{Alice}), (\text{Mavis}, \text{Alice})$

- $R = \{(x, y) \mid\ x \text{ is a grandfather of } y\}$;

$R = \{(\text{Fred}, \text{Jane}), (\text{Fred}, \text{Fiona}), (\text{Fred}, \text{Alan}), (\text{John}, \text{Jane}), (\text{John}, \text{Fiona}), (\text{John}, \text{Alan})\}$

- $S = \{(x, y) \mid\ x \text{ is a sister of } y\}$.

$\{(\text{Alice}, \text{Ken}), \ldots$

Example 2

Write down the ordered pairs belonging to the following binary relations between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$:

- $U = \{(x, y) \in \underline{A \times B} \mid x + y = 9\}$;

$U = \{(3,6), (5,4), (7,2)\}$

- $V = \{(x, y) \in A \times B \mid x < y\}$.

$\{(1,2), (1,4), (1,6), (3,4), (3,6),$
$(5,6)\}$

Example 3

Let $A = \{1, 2, 3, 4, 5, 6\}$. Write down the ordered pairs belonging to

$$R = \{(x, y) \in A \times A \mid \ x \text{ is a divisor of } y\ \}.$$

$\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,4), (2,6), (2,2)$
$(3,6), (3,3)$
$(4,4), (5,5), (6,6)\}$

Representations by digraphs

Representation of binary relations: directed graphs

- Let A and B be two finite sets and R a binary relation between these two sets (i.e., $R \subseteq A \times B$).
- We represent the elements of these two sets as vertices of a graph.
- For each $(a, b) \in R$, we draw an arrow linking the related elements.
- This is called the directed graph (or digraph) of R.

Example

Consider the relation V between $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6\}$ such that $V = \{(x, y) \in A \times B \mid x < y\}$.

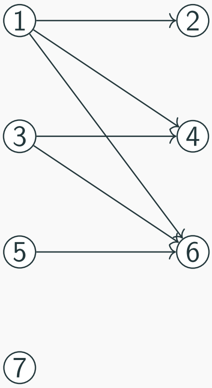
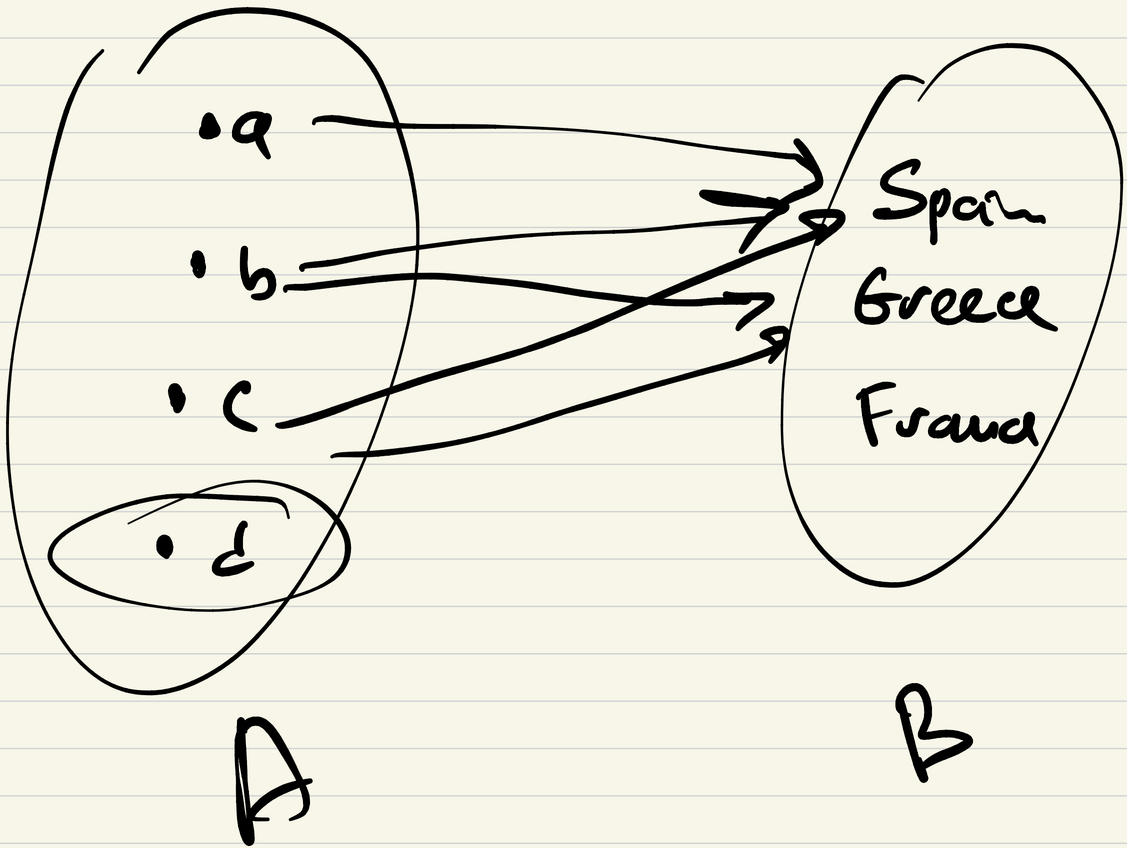
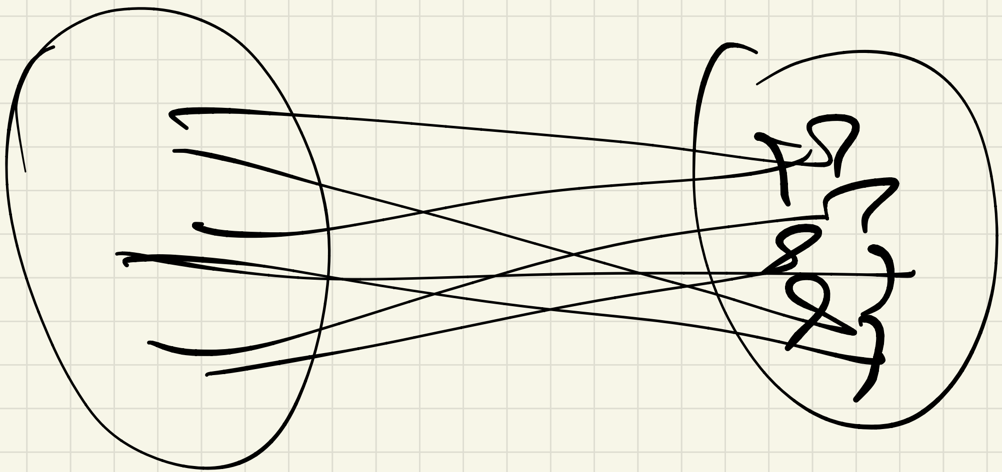


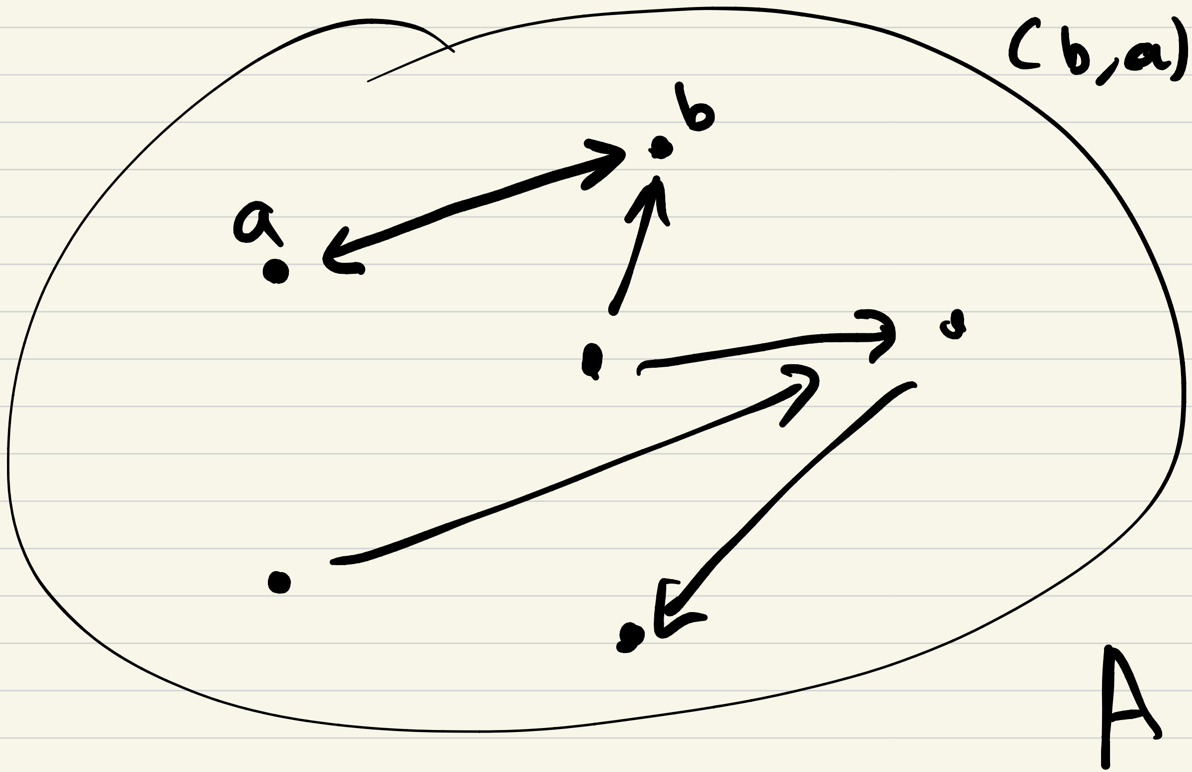
Figure 1: digraph of V


a
b
c
d
Spain
Greece
France
A
B



A $\qquad\qquad\qquad$ A

(a, b) ...
(b, a) ..



Digraphs of binary relations on a single set

A binary relation between a set A and itself is called "a binary relation on A".

To represent such a relation, we use a directed graph in which a single set of vertices represents the elements of A and arrows link the related elements.

Consider the relation $V \subseteq A \times A$ where $A = \{1, 2, 3, 4, 5\}$ and

$$V = \{(1,2), (3,3), (5,5), (1,4), (4,1), (4,5)\}.$$

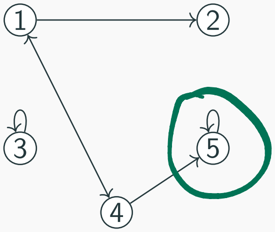
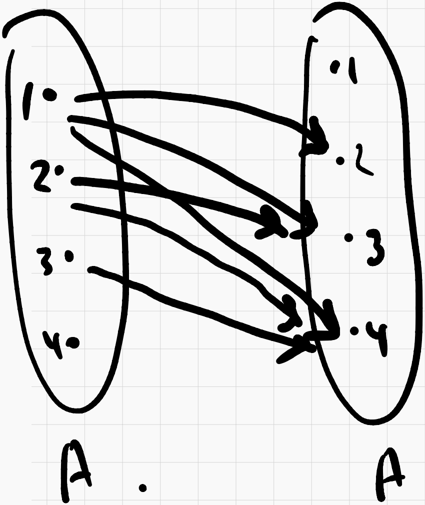
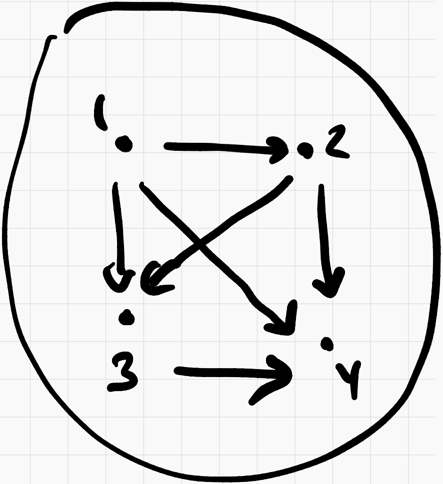


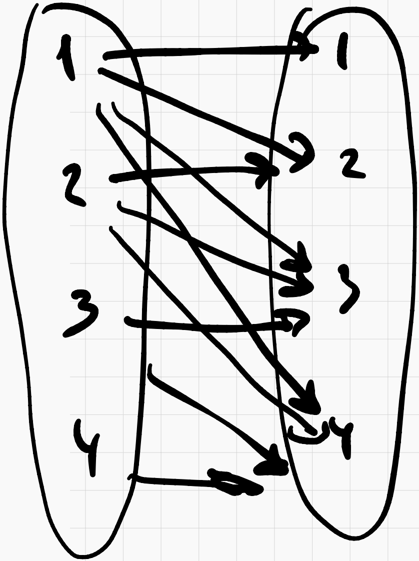
Figure 2: digraph of V

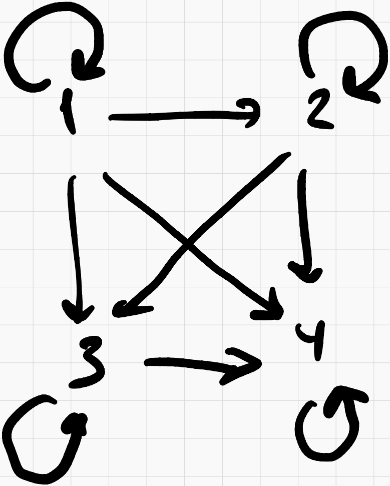
Example: $A = \{1, 2, 3, 4\}$, $R = \{(x, y) \in A \times A \mid x < y\}$





Example: $A = \{1, 2, 3, 4\}$, $R = \{(x, y) \in A \times A \mid x \leq y\}$





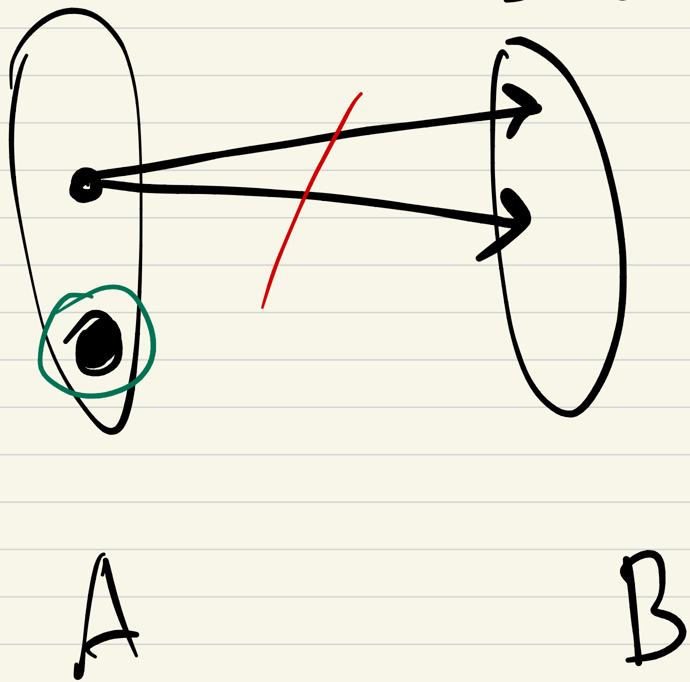
Example: functional relations

- Recall that a function f from a set A to a set B assigns exactly one element of B to each element of A.
  - Gives rise to the relation $R_f = \{(a, b) \in A \times B \mid b = f(a)\}$

- If a relation $S \subseteq A \times B$ is such that for every $a \in A$ there exists at most one $b \in B$ with $(a, b) \in S$, relation S is functional.

- (Sometimes in the literature, functions are introduced through functional relations.)

Relations - ok



Functions - NOT