COMP229: Introduction to Data Science
Lecture 24: Areas of planar polygons

Olga Anosova, O.Anosova@liverpool.ac.uk
Autumn 2023, Computer Science department
University of Liverpool, United Kingdom

Lecture plan & learning outcomes

On this lecture we should learn

- how to obtain signed area formulas from segment lengths,
- how to compute the area of a polygon from coordinates of vertices,
- what is a convex hull,
- what is the geometric meaning of a determinant of a matrix.

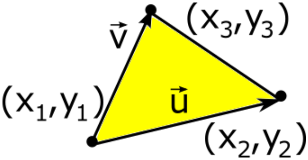
Reminder: Determinant and its uses

- The *determinant* of any $m \times m$ matrix A is $\det A = \sum_{j=1}^{m} (-1)^{i+j} a_{ij} \det A_{ij}$ for any fixed i.
- A linear map $f : \vec{v} \mapsto A\vec{v}$ in $\mathbb{R}^m$ is bijective if and only if $\det A \neq 0$ (then there is the inverse linear map that has A^{-1} such that $AA^{-1} = I = A^{-1}A$).
- Vectors are linearly independent iff their $\det A \neq 0$.
- m inearly independent vectors span the $\mathbb{R}^m$ space.

More invariants of data clouds

In addition to distances ("almost complete" invariants), other useful isometric invariants are areas and volumes.

Definition 24.1. If vertices A, B, C of the triangle $\triangle ABC$ are ordered anticlockwisely, the **signed area** of $\triangle ABC$ is the usual positive area, otherwise the area is taken with the negative sign.

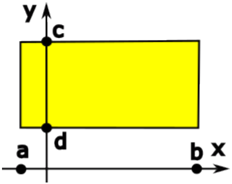


Given vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ in $\mathbb{R}^2$, we'll express the signed area of $\triangle ABC$ via coordinates x_i, y_i.

The basic definition of an area

In $\mathbb{R}$ the analogue of the area is the length.

Definition 24.2. The **length of a segment** $[a, b] \subset \mathbb{R}$ is $|a - b|$. The difference $b - a$ is the **signed length** and can be negative if $a > b$.

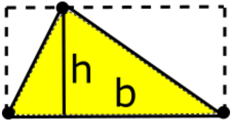


The **area of a rectangle** $[a, b] \times [c, d] \subset \mathbb{R}^2$ is the product $|a - b| \cdot |c - d|$.
For any product of segments, the **volume** is the product of all sides.

The areas of all other shapes in $\mathbb{R}^2$ can be deduced from Definition 24.2, e.g. by approximations.

The product formula for a triangle

Claim 24.3. The area S of a triangle is $\frac{bh}{2}$, where b is one side (a *base*), h is the height to this base.



Proof. If a base is parallel to a coordinate axis, S is a half of the area $b \cdot h$ of the axis-aligned rectangle.

If a triangle has no side parallel to the x-axis, one can rotate it to make one side parallel to the x-axis and apply the product formula with the base and height that are preserved under the rotation. $\square$

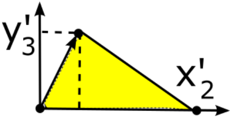
The determinant as a signed area

Claim 24.4. The signed area S of a triangle $\triangle$ with anticlockwisely ordered vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is
$$\frac{1}{2}\det\begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}.$$

Proof. Let $\vec{u} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix}$ be the vectors along two sides of the triangle. Claim 24.4 says that the area is $S = \dfrac{1}{2}\det\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$, where $\vec{u} = (u_x, u_y)$ is rotated to $\vec{v} = (v_x, v_y)$ through $\triangle$.

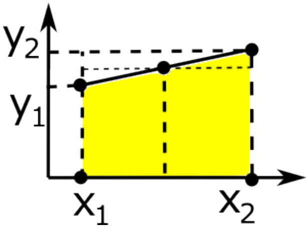
First proof for the triangle area

The determinant in Claim 24.4 is invariant under translations, hence one vertex can be fixed at the origin: $x_1 = y_1 = 0$. It remains to prove that the signed area is $S = \frac{1}{2}\det\begin{pmatrix} x_2 & x_3 \\ y_2 & y_3 \end{pmatrix}$. Apply the rotation matrix R to make the first vector (column) horizontal, i.e. $y_2 = 0$. By previous lectures this rotation multiplies the area S by $\det R = 1$.



Then $S = \frac{1}{2}\det\begin{pmatrix} x_2' & x_3' \\ 0 & y_3' \end{pmatrix} = \frac{x_2' y_3'}{2}$. $\square$

The area under a line segment



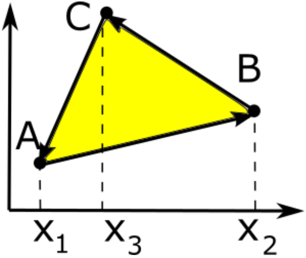
Claim 24.5. The area between the side vector connecting points $(x_1, y_1), (x_2, y_2)$ and the x-axis equals $\dfrac{y_1 + y_2}{2}(x_2 - x_1)$.

Proof. The area of the trapezium on the vertices $(x_1, 0), (x_2, 0), (x_2, y_2), (x_1, y_1)$ is the height $x_2 - x_1$ (assuming that $x_1 < x_2$) times the average of the parallel sides $\dfrac{y_1 + y_2}{2}$ (assuming that $y_1, y_2 > 0$).

The formula holds for negative areas if $x_1 > x_2$. $\square$

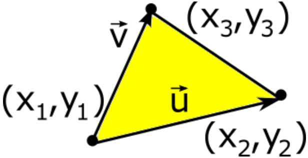
A triangle is a sum of three trapezia

Claim 24.6. The signed area of the triangle with anticlockwisely ordered vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ is the negative sum of the signed areas between the x-axis and $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$.



Proof. The signed area of $\triangle ABC$ is the sum of the signed areas under $\overrightarrow{AC}$, $\overrightarrow{CB}$ minus the area under $\overrightarrow{AB}$ (2 trapezia minus 1 trapezium in the picture). Other cases are similar. $\square$

Second proof for the triangle area



Apply Claims 24.5 and 24.6: the triangle area S equals the sum of the negative areas of trapezia.

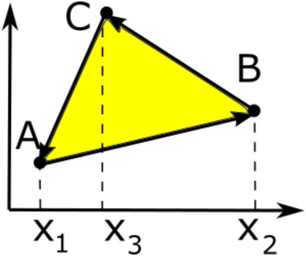
Then expand brackets, collect terms

$$2S = (y_1 + y_2)(x_1 - x_2) + (y_2 + y_3)(x_2 - x_3) +$$
$$+(y_3 + y_1)(x_3 - x_1) =$$
$$= y_2x_1 - y_1x_2 + y_3x_2 - y_2x_3 + y_1x_3 - y_3x_1 =$$
$$= (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1) =$$
$$\det \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} = \det \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}. \qquad \square$$

Gauss (shoelace) area formula

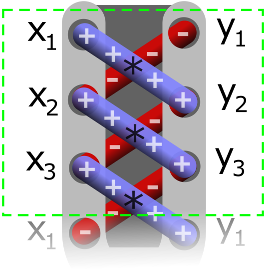
Theorem 24.7. In $\mathbb{R}^2$ the signed area of any polygon with anticlockwisely ordered vertices $(x_1, y_1), \ldots, (x_n, y_n)$ is

$$\sum_{i=1}^{n} \frac{y_i + y_{i+1}}{2}(x_i - x_{i+1}) = \frac{1}{2}\sum_{i=1}^{n} \begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1} \end{vmatrix},$$ where we set $(x_{n+1}, y_{n+1}) = (x_1, y_1)$.



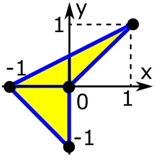
Proof for a triangle. As in Claim 24.6, the signed area of a triangle is obtained by adding the negative signed areas of the trapezia between the x-axis and vectors $\overrightarrow{(x_i, y_i), (x_{i+1}, y_{i+1})}$.

Shoelaces



$$2S = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix}$$

Problem 24.8. Find the area of the polygon with the vertices $(0,0)$, $(1,1)$, $(-1,0)$, $(0,-1)$ in $\mathbb{R}^2$.

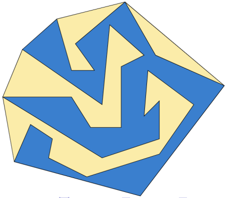


Solution 24.8. Theorem 24.6 holds for any non-convex polygon: $2S = (0+1)(0-1) + (1+0)(1-(-1)) + (0-1)(-1-0) + (-1+0)(0-0) = 0+1+1+0 = 2$. $\square$

The convex hull of a set

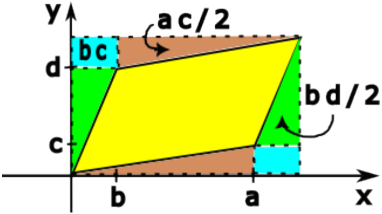
Definition 24.9. A set $C \subset \mathbb{R}^m$ is called **convex** if for any points $p, q \in C$, the set C contains the line segment $[p, q]$.
The **convex hull**(C) is the smallest convex set containing C.
The convex hull can also be found as
- the intersection of all convex sets containing C,
- the set of all convex combinations of points in C,
- the union of all *simplices* (generalisation of a triangle or tetrahedron to larger dimensions) with vertices in C.

Hull(a circle) $=$ a disk.
Complex hull (yellow) of a simple polygon (blue):



A meaning of the determinant in $\mathbb{R}^m$

Claim 24.10. For any $m \times m$ matrix A, $\det A$ equals the signed volume of the parallelepiped (called a *unit cell*) spanned by the columns of A.

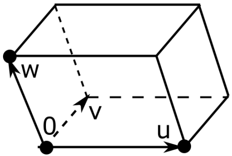


In $\mathbb{R}^2$ the parallelepiped spanned by vectors $\overrightarrow{(a,c)}$ and $\overrightarrow{(b,d)}$ has area $S = (a+b)(c+d) - ac - bd - 2bc = ad - bc = det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Linearly dependent vectors collapse into one line, making the spanned area 0, hence determinant of linearly dependent vectors is 0.

Why *signed* area? See here from 5min.

Unit cell in $\mathbb{R}^3$



In $\mathbb{R}^3$ the parallelepiped **spanned** by vectors $\vec{u}, \vec{v}, \vec{w}$ is the set $x\vec{u} + y\vec{v} + z\vec{w}$ for coordinates $x, y, z \in [0, 1]$.

In solid state physics, a periodic crystal is defined by a set of atoms, ions or molecules in a unit cell.

When vectors are linearly dependant and span a 2d plane, the parallelepiped degenerates to a flat polygon and has volume 0.

Scaling of areas and volumes

In $\mathbb{R}^2$ the signed area of the triangle T spanned by the columns of a matrix B equals to $\frac{1}{2} \det B$.
If A is a rotation matrix, the columns of AB span the image of T. Since $\det A = 1$,
$\det(AB) = \det(B)$ and the area is preserved under rotations.

Claim 24.11. In $\mathbb{R}^m$, the signed volume of a body under a linear map $\vec{v} \mapsto A\vec{v}$ is multiplied by $\det A$.

The claim follows from simpler claims for triangles and tetrahedra that can approximate any good shape (proof is not needed for the exam).

Time to revise and ask questions

- The determinant of $\begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$ is the signed area of the parallelogram spanned by the vectors $\vec{u} = (u_x, u_y)$ and $\vec{v} = (v_x, v_y)$.
- The signed area of any polygon with counter-clockwisely ordered vertices can be found form a shoelace formula.
- The signed volume of a body under a linear map $\vec{v} \mapsto A\vec{v}$ is multiplied by $\det A$.

Problem 24.12. (from the exam in January 2019) Find the area of the triangle with the vertices $(1,1), (3,2), (2,3)$ in $\mathbb{R}^2$.

Additional links

- The complete 3Blue1Brown video on determinants.
- Isoperimetric inequality between perimeter and area (or volume).
- Shapes with zero (Sierpinski triangle) or finite (Koch snowflake) areas, but infinite perimeters.