COMP108
Data Structures and Algorithms

Greedy Algorithm (Part III Single-Source Shortest Paths)

Professor Prudence Wong

pwong@liverpool.ac.uk

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Single-source shortest-paths

Consider a (un)directed connected graph G

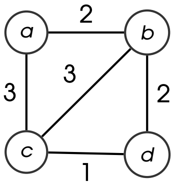
- The edges are labelled by weight

Given a particular vertex called the **source**

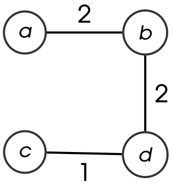
- Find **shortest** paths from the source to all other vertices (shortest path means the total weight of the path is the smallest)

Single-source shortest-paths - Examples

Graph G (edge label is weight)



MST (global property)



Single-source shortest-paths - Examples

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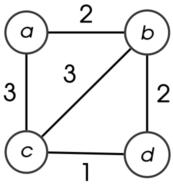

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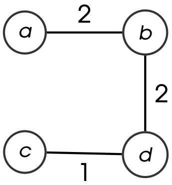
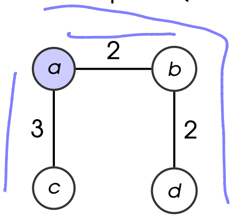


Shortest paths (local property):



from a

Single-source shortest-paths - Examples

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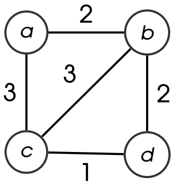

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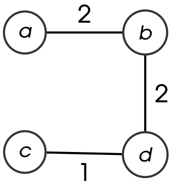

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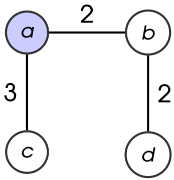

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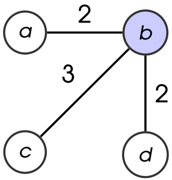

from b

Single-source shortest-paths - Examples

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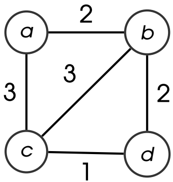

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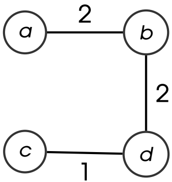
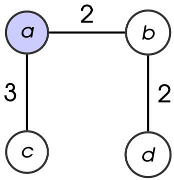
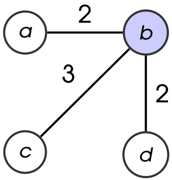


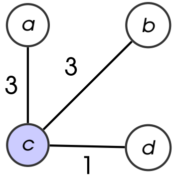
Shortest paths (local property):



from a



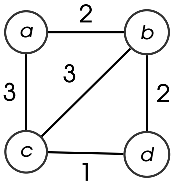
from b



from c

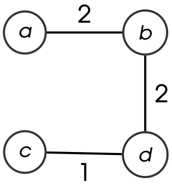
Single-source shortest-paths - Examples

Graph G (edge label is weight)



All pairs shortest paths

MST (global property)



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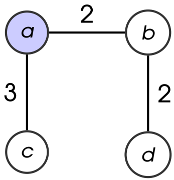
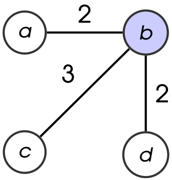
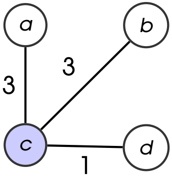


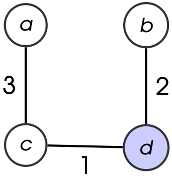
from a



from b



from c



from d

Single-source shortest-paths - Examples

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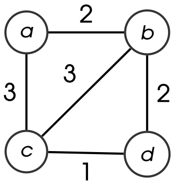

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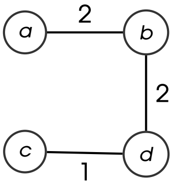

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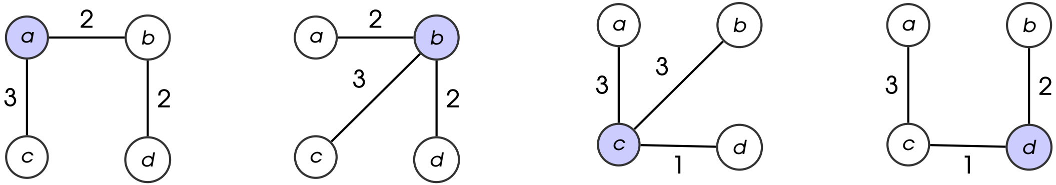


from a from b from c from d

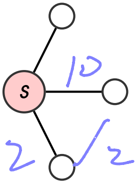
So it is source dependent.

Algorithms for shortest paths

- There are many algorithms to solve this problem
- One of them is **Dijkstra's algorithm**, which assumes the weights of edges are non-negative

Idea of Dijkstra's algorithm

Choose the edge adjacent to any chosen vertices such that cost of path to source is minimum



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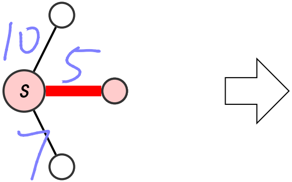

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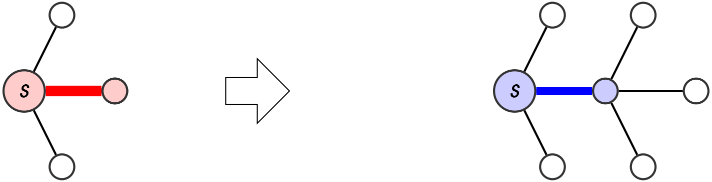

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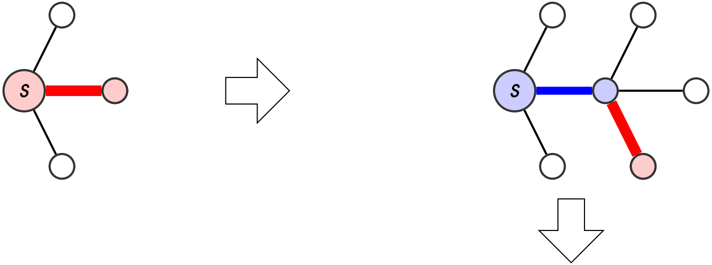

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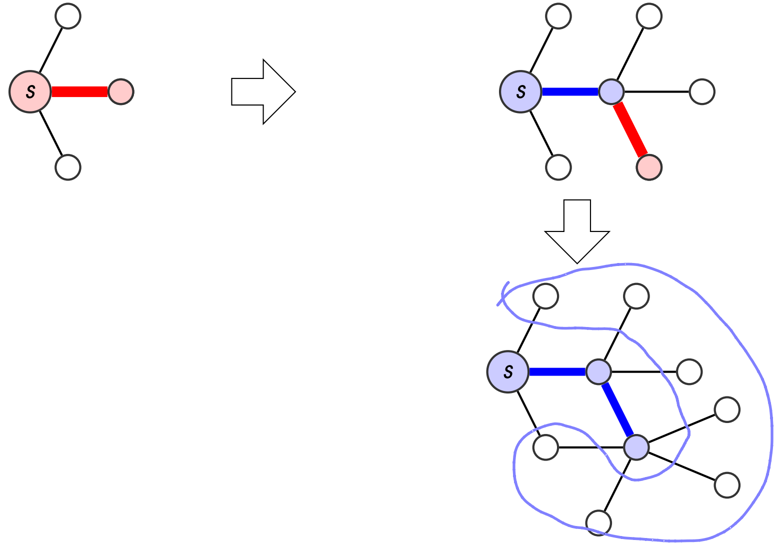

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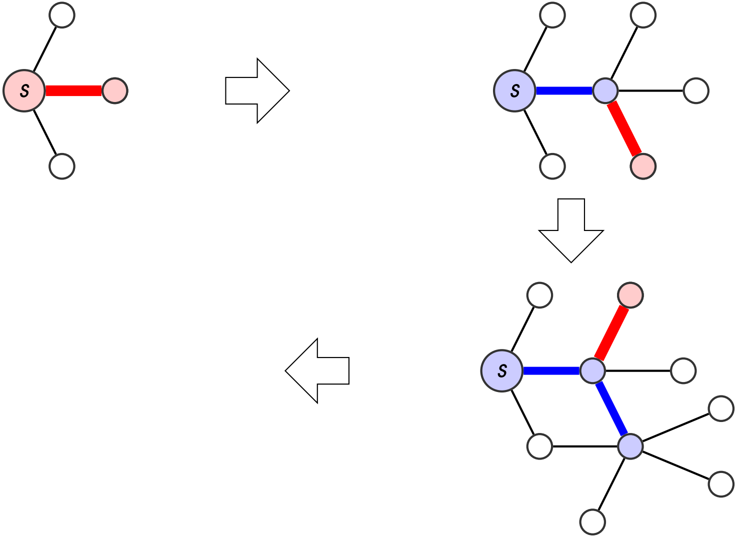

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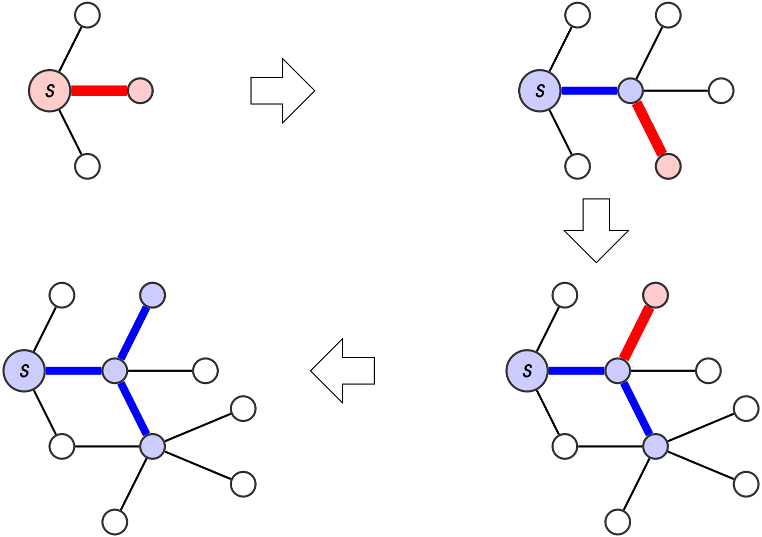

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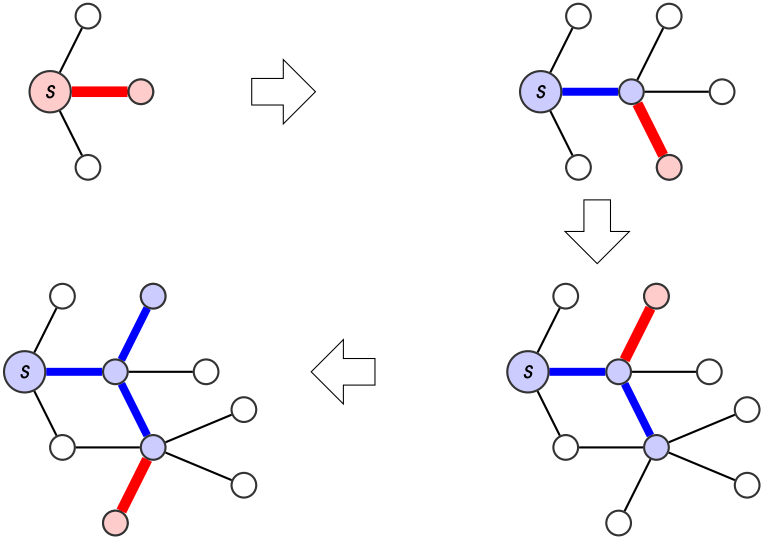
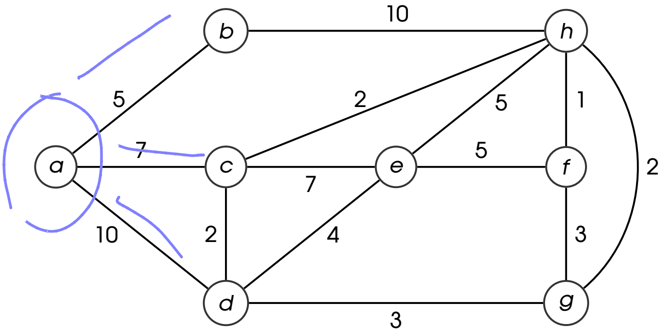
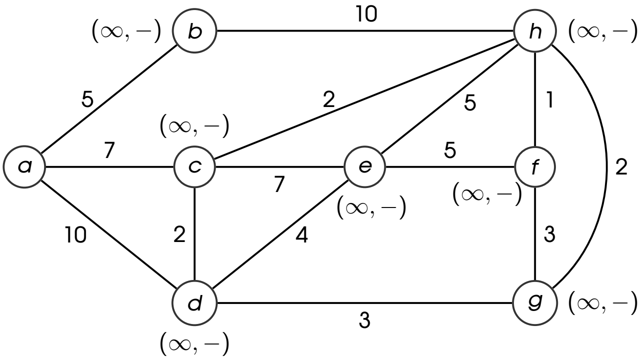


Dijkstra's algorithm - suppose *a* is the source



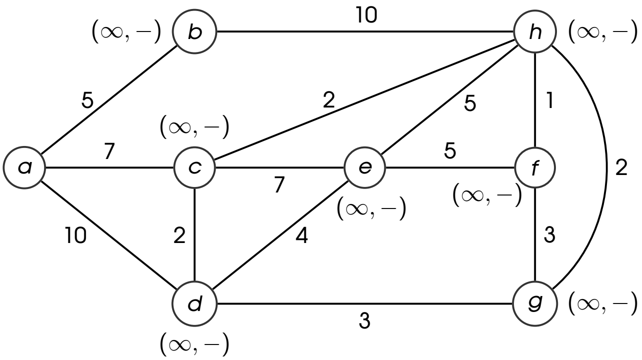
Dijkstra's algorithm - suppose a is the source

Every vertex v keeps 2 labels, initially as $(\infty, -)$: (1) weight of current shortest path from a, (2) the vertex preceding v on that path



Dijkstra's algorithm - suppose a is the source

Each round: (1) a vertex is picked, neighbours' labels updated, (2) pick from **ALL unchosen vertices** the one with smallest weight for next round



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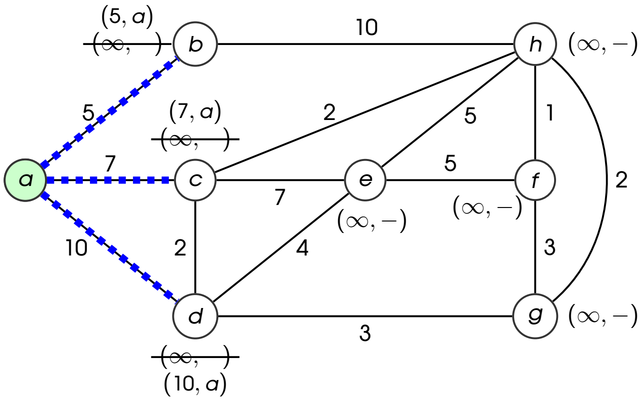

Update *b*, *c*, *d*

Choose from *b*, *c*, *d*:

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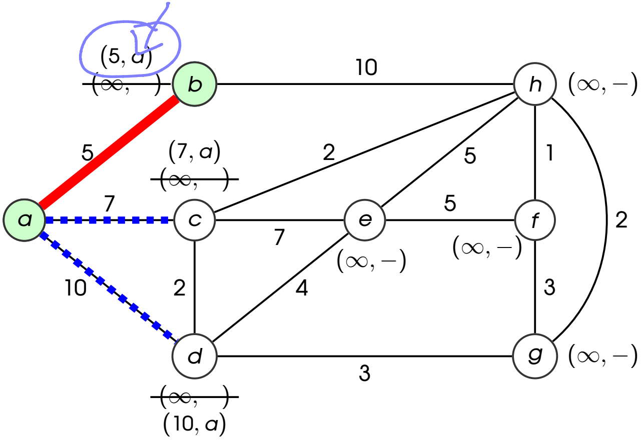

Update *b*, *c*, *d*

Choose from *b*, *c*, *d*: **(*a*, *b*) is chosen**

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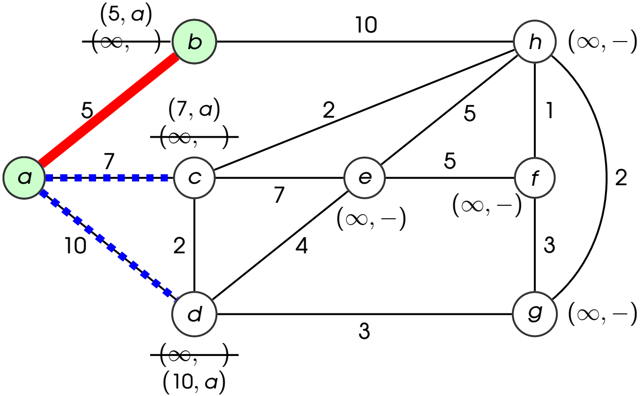

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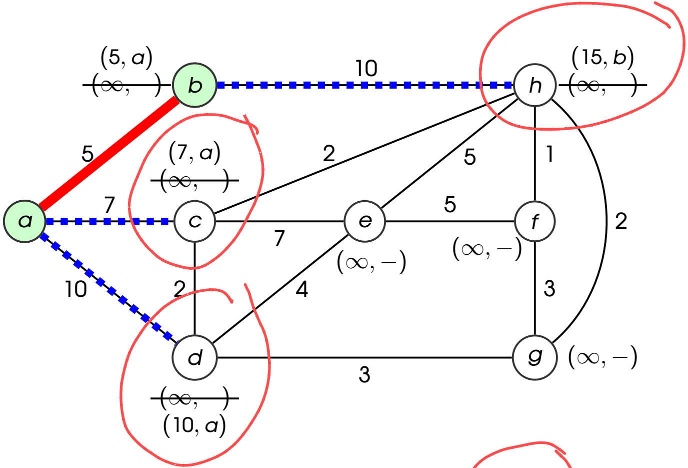

Update *h*

Choose from *c*, *d*, *h*:

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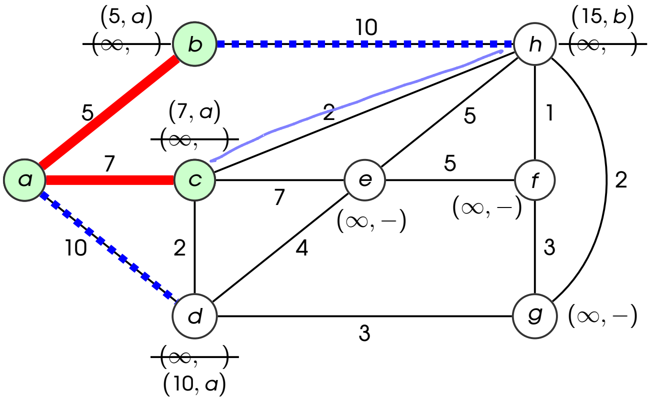

Update h Choose from c, d, h: **(a, c) is chosen**

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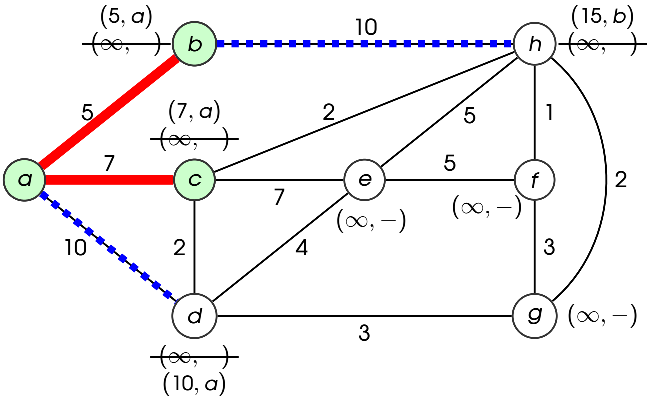

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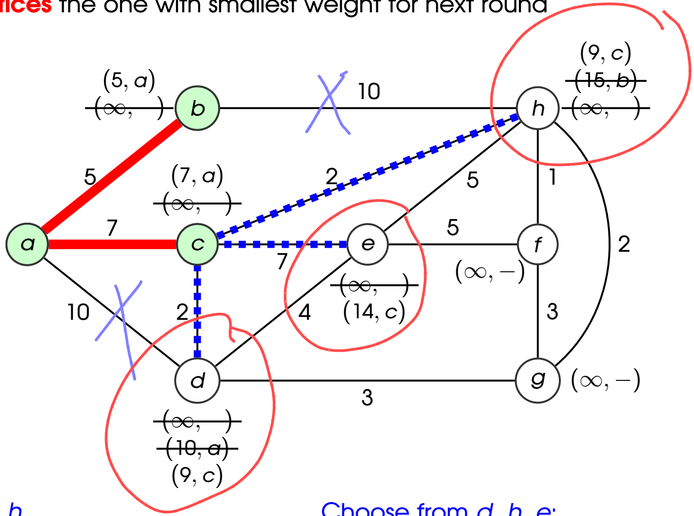

Update d, e, h

Choose from d, h, e:

Dijkstra's algorithm - suppose a is the source

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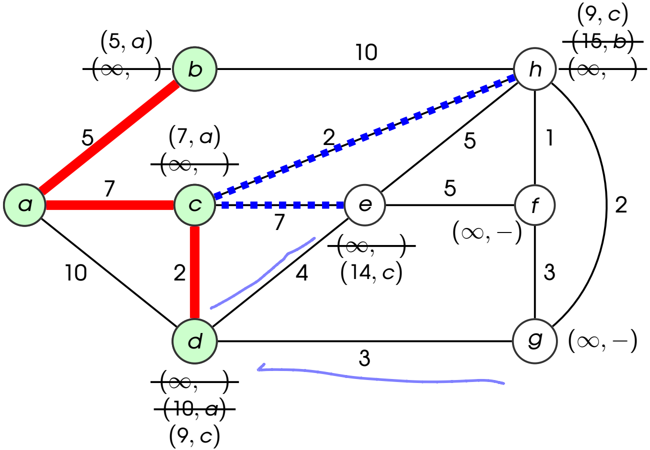

Update d, e, h

Choose from d, h, e: **(c, d) is chosen**

Dijkstra's algorithm - suppose *a* is the source

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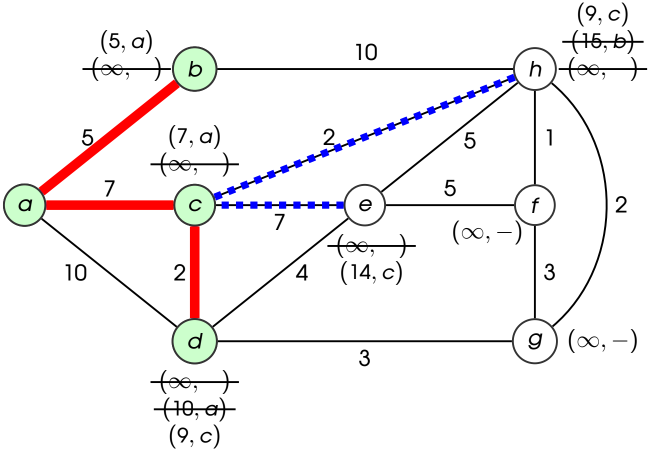

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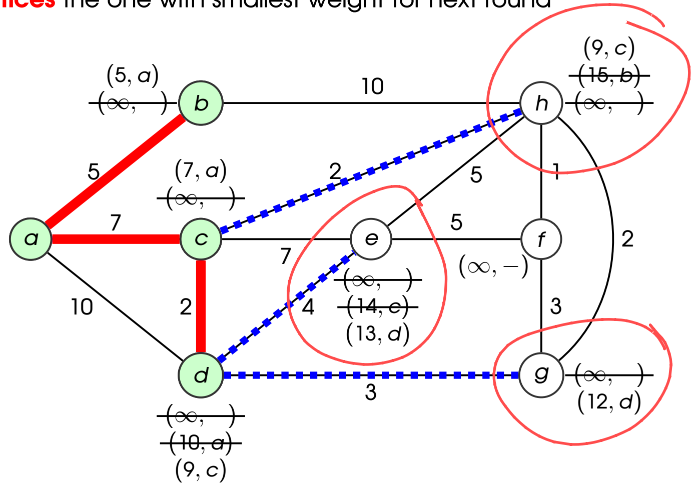

Update *e*, *g*

Choose from *h*, *e*, *g*:

Dijkstra's algorithm - suppose *a* is the source

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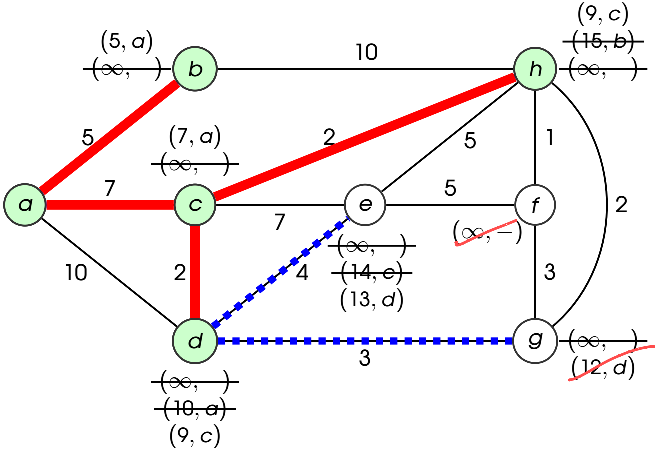

Update *e*, *g*

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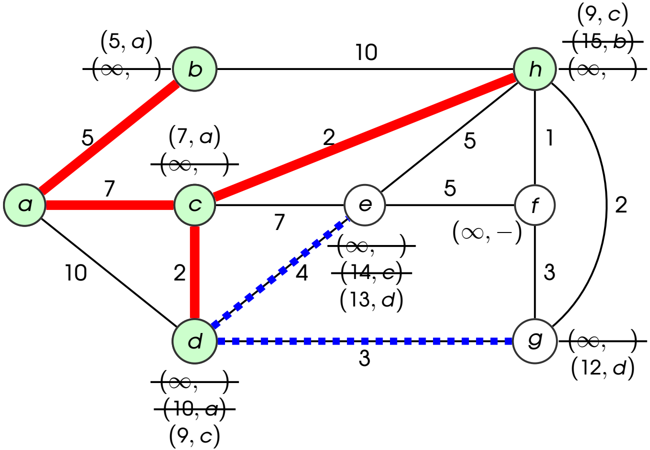

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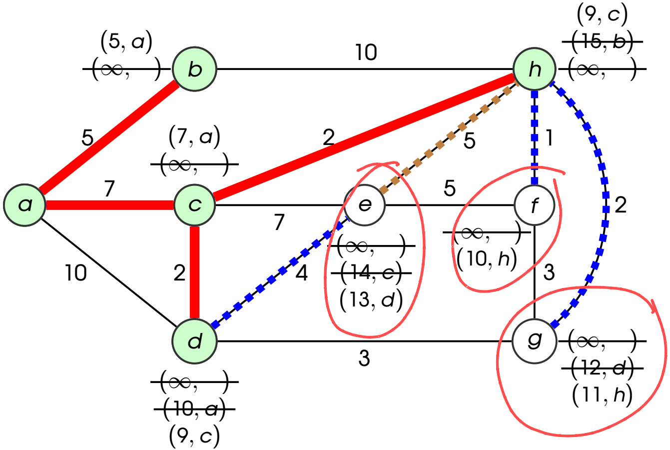
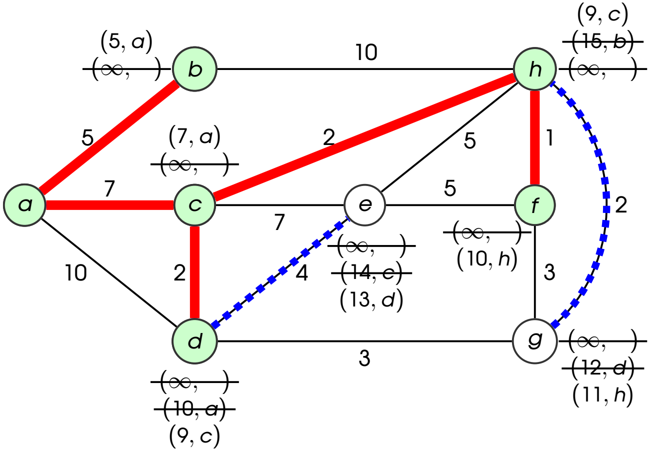


Update *g*, *f*; Unchanged: *e*

Choose from *e*, *g*, *f*:

Dijkstra's algorithm - suppose a is the source

Each round: (1) a vertex is picked, neighbours' labels updated, (2) pick from **ALL unchosen vertices** the one with smallest weight for next round



Update g, f; Unchanged: e

Choose from e, g, f: **(h, f) is chosen**

Dijkstra's algorithm - suppose *a* is the source

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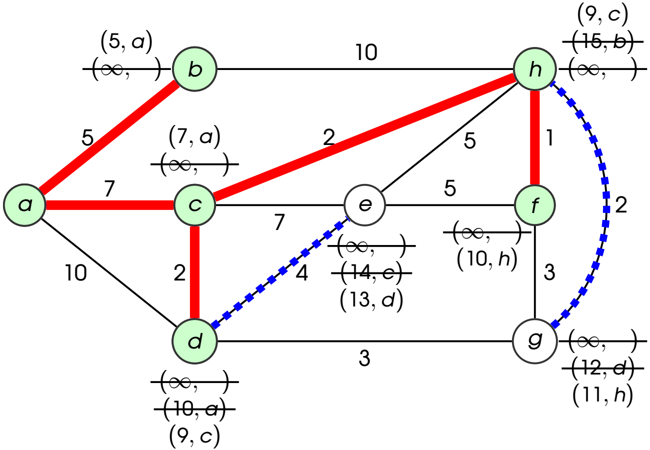

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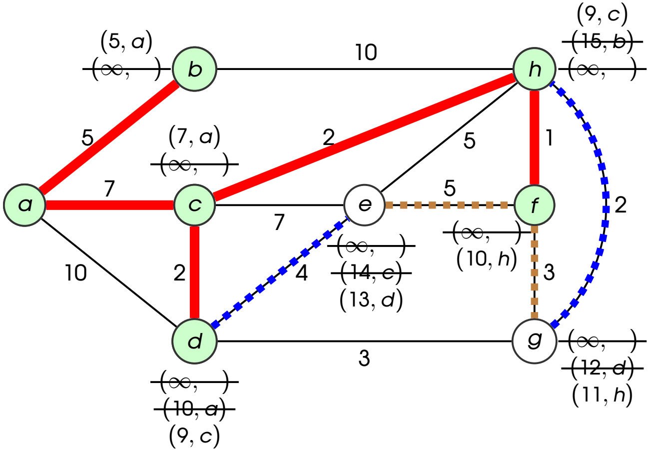

Unchanged: *e*, *g*

Choose from *e*, *g*:

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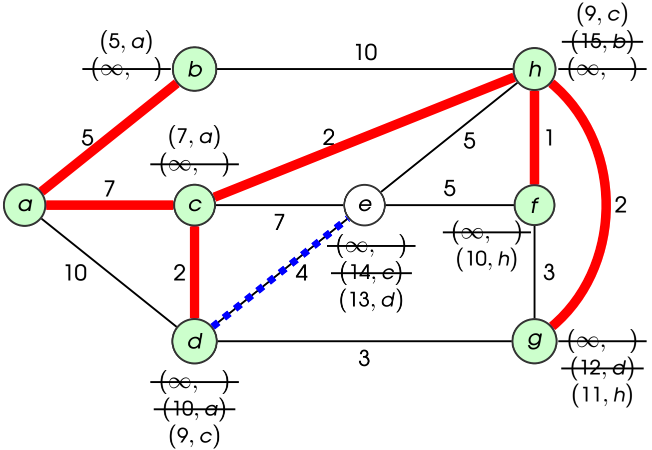

Unchanged: *e*, *g*

Choose from *e*, *g*: **(*h*, *g*) is chosen**

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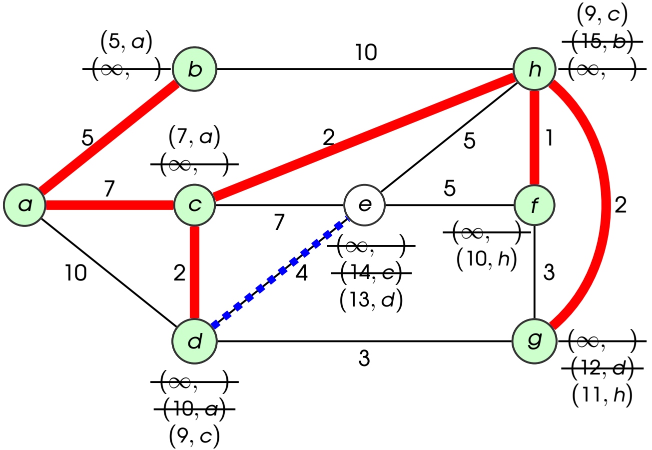

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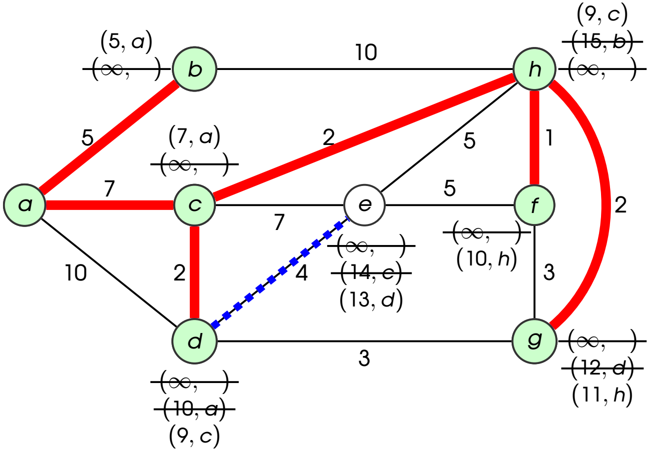

g has no unchosen neighbour $\Longrightarrow$ No change

Choose from *e*:

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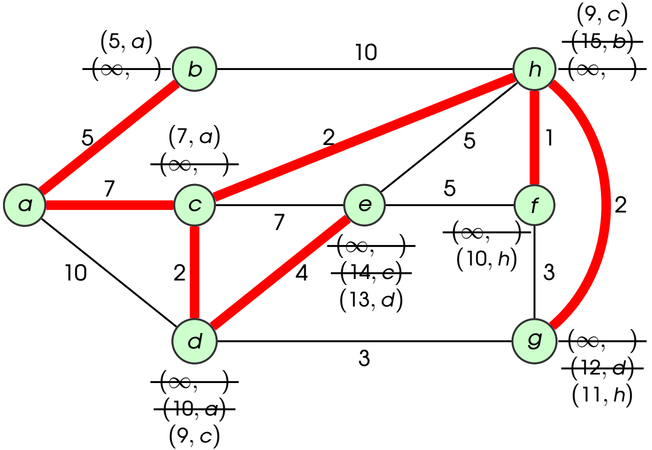

g has no unchosen neighbour $\Longrightarrow$ No change

Choose from *e*: **(*d*, *e*) is chosen**

Alternative presentation

b	c	d	e	f	g	h	chosen
$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$
$(5, a)$	$(7, a)$	$(10, a)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	(a, b)
	$(7, a)$	$(10, a)$	$(\infty, -)$	$(\infty, -)$	$(\infty, -)$	$(15, b)$	(a, c)
		$(9, c)$	$(14, c)$	$(\infty, -)$	$(\infty, -)$	$(9, c)$	(c, d)
			$(13, d)$	$(\infty, -)$	$(12, d)$	$(9, c)$	(c, h)
			$(13, d)$	$(10, h)$	$(11, h)$		(h, f)
			$(13, d)$		$(11, h)$		(h, g)
			$(13, d)$				(d, e)

Dijkstra's algorithm

To describe the algorithm using pseudo code, we give some notations

Each vertex v is labelled with two labels:

- a **numeric label** $d(v)$ indicates the length of the shortest path from the source to v found so far
- another label $p(v)$ indicates next-to-last vertex on such path, i.e., the vertex immediately preceding v on that shortest path

Dijkstra's algorithm - Pseudo code

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while $V \setminus V_T \neq \emptyset$ do   // there is still some vertex left

begin

end

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  choose the vertex u in $V \setminus V_T$ with minimum $d(u)$

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    choose the vertex u in $V \setminus V_T$ with minimum $d(u)$

    set $V_T \leftarrow V_T \cup \{u\}$ and $E_T \leftarrow E_T \cup \{(p(u), u)\}$

end

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    for every vertex v in $V \setminus V_T$ that is a neighbour of u do

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    for every vertex v in $V \setminus V_T$ that is a neighbour of u do

        if $d(u) + w(u, v) < d(v)$ then   // a shorter path is found

end

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end

Dijkstra's algorithm - Pseudo code

$|V| = n \quad |E| = m$

// Given a graph $G = (V, E)$ and a source vertex s

for every vertex v in the graph do

    set $d(v) \leftarrow \infty$ and $p(v) \leftarrow null$

set $d(s) \leftarrow 0$ and $V_T \leftarrow \emptyset$ and $E_T \leftarrow \emptyset$

while $V \setminus V_T \neq \emptyset$ do   // there is still some vertex left $O(n)$ iterations

begin

    choose the vertex u in $V \setminus V_T$ with minimum $d(u)$ — $O(n)$

    set $V_T \leftarrow V_T \cup \{u\}$ and $E_T \leftarrow E_T \cup \{(p(u), u)\}$ — $O(1)$

    for every vertex v in $V \setminus V_T$ that is a neighbour of u do $O(n)$

        if $d(u) + w(u, v) < d(v)$ then   // a shorter path is found

            set $d(v) \leftarrow d(u) + w(u, v)$ and $p(v) \leftarrow u$ $\}\ O(1)$

end

Time complexity?

$O(n^2)$

Summary

Summary: Dijkstra's algorithm for Single-Source Shortest-Paths

Next week: Divide and Conquer algorithms

For note taking