COMP229: Introduction to Data Science
Lecture 26: Eigenvalues and eigenvectors

Olga Anosova, O.Anosova@liverpool.ac.uk
Autumn 2023, Computer Science department
University of Liverpool, United Kingdom

Lecture plan & learning outcomes

On this lecture we should learn

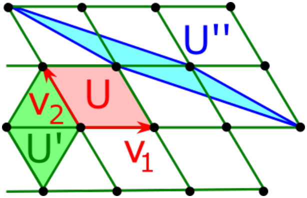
- what is the conjugation and the trace of matrices,
- how to find eigenvalues and eigenvectors,
- why eigenthings are important for the PageRank algorithm,
- when an eigenbasis exists.

Reminder: change of basis

- When an old basis is replaced by a new basis, the columns of the transition matrix B are the new basis vectors (written in the old basis).
- A linear map with a matrix A in the old basis has the matrix $B^{-1}AB$ in the new basis.
- Any linear map defines the same geometric action (such as a rotation), but can be algebraically represented by many different matrices.

The whole matrix is not enough

The last lecture has shown that the same linear map can be represented by different matrices. Hence one matrix of a linear map is always ambiguous and should be used only together with a linear basis.



The different unit cells (or bases) U, U', U'' generate the same lattice. A comparison up to isometries must use only invariants preserved under a change of a basis.

What is preserved in a matrix under such a change?

The conjugation of matrices

Definition 26.1. Matrices A, B are **conjugated** if there is a matrix D such that $B = D^{-1}AD$, e.g. A, B are matrices of one map in different bases.

Claim 26.2. The conjugation (sometimes called *similarity*) is an equivalence relation on matrices.

A proof is one of the tutorial problems.

Claim 26.2 implies that all matrices split into well-defined equivalence classes. Each of these classes represents *one* geometric map that can be expressed by different matrices in different bases.

The trace of a matrix

Definition 26.3. The **trace** of a matrix A is the sum of the diagonal elements $\operatorname{tr}(A) = \sum_{i=1}^{m} a_{ii}$, which will be proved to be an invariant of a linear map.

Claim 26.4. $\operatorname{tr}(A+B) = \operatorname{tr} A + \operatorname{tr} B$, $\operatorname{tr} A^T = \operatorname{tr} A$, $\operatorname{tr}(cA) = c \cdot \operatorname{tr} A$ for any $c \in \mathbb{R}$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

A proof is another tutorial problem.

Notice that matrices A, B don't commute in general, however $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $\det(AB) = \det(A)\det(B) = \det(BA)$ always hold by Claim 26.4 and Theorem 23.14, respectively.

Invariants of linear maps

Claim 26.5. The trace and the determinant are invariants of a map $\vec{v} \mapsto A\vec{v}$ up to conjugation.

Proof. If we change the basis by a transition matrix B, the new matrix $B^{-1}AB$ from Theorem 25.2 has $\det(B^{-1}AB) = \det(B^{-1}) \det A \det B = \det(A)$ by Theorem 23.14, because $\det(B^{-1}) = 1/\det B$.

By Claim 26.4 we can swap two matrices under the trace: $\mathrm{tr}\,(B^{-1}AB) = \mathrm{tr}\,(BB^{-1}A) = \mathrm{tr}\,A$. $\square$

Are $\det A, \mathrm{tr}\,A$ complete conjugation invariants?

Eigenvalues and eigenvectors

Definition 26.6. If $\vec{v}$ satisfies $A\vec{v} = \lambda\vec{v}$, $\lambda \in \mathbb{R}$, then λ is an *eigenvalue*, $\vec{v}$ is an *eigenvector* of A. Any $c\vec{v}$ for $c \in \mathbb{R}$ is also an eigenvector.

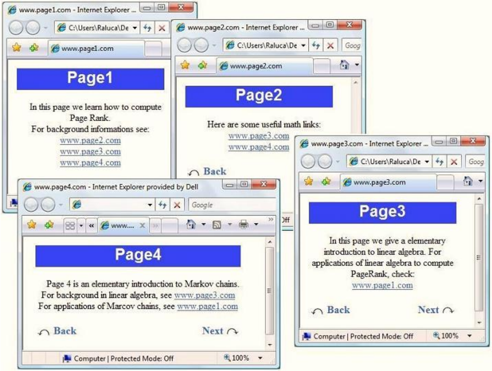
$A\vec{v} = \lambda\vec{v}$ means that geometrically the map scales all vectors in the direction of $\vec{v}$ by the same factor λ.

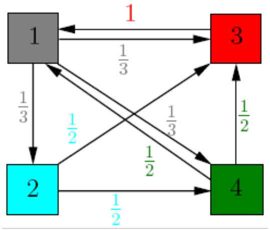
If we are lucky to have basis vectors $\vec{e}_i$ equal to eigenvectors, the map has the diagonal matrix with eigenvalues on the diagonal:

if $A\vec{e}_i = \lambda_i\vec{e}_i$, then $A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{pmatrix}$. This property is extremely important for multiple applications of the same map A, i.e. for computing A^n.

Google PageRank uses eigenvectors

Google PageRank finds a rank of each webpage by using ranks of other pages that link to this page.





$$A = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Ranks of pages form a vector satisfying $A\vec{v} = \lambda\vec{v}$.

Image from The mathematics of web seasrch

How to find eigenvalues

3Blue1Brown video

Claim 26.7. All eigenvalues λ of A are solutions of the **characteristic** equation $\det(A - \lambda I) = 0$.

Proof. $A\vec{v} = \lambda\vec{v}$ is equivalent to $(A - \lambda I)\vec{v} = \vec{0}$, where I is the identity $m \times m$ matrix, $\vec{0} \in \mathbb{R}^m$ is the zero vector.
$(A - \lambda I)\vec{v} = \vec{0}$ means that a linear combination of the columns of $A - \lambda I$ (with the coefficients equal to the coordinates of $\vec{v}$) is $\vec{0}$.

$\det(A - \lambda I) = 0$ since the determinant is preserved when we add a multiples of columns by Claim 23.12. $\square$

How to find eigenvectors

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has the characteristic equation

$0 = \det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 =$
$\lambda^2 - 4\lambda + 3$, eigenvalues $\lambda = 1$ and $\lambda = 3$.

Now we find eigenvectors as solutions $\vec{v}$ of $(A - \lambda I)\vec{v} = \vec{0}$, e.g.
$(A - I)\vec{v} = \vec{0}$ is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which has solution $\vec{v}_1 = (1, -1)$ and infinitely many other solutions parallel to this vector.

$(A - 3I)\vec{v} = \vec{0}$ has solutions parallel to $\vec{v}_2 = (1, 1)$.

Diagonalise a linear map

In the new **eigenbasis** $\vec{v}_1 = (1,-1)$, $\vec{v}_2 = (1,1)$ with the transition matrix $B = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, the map with $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has the new diagonal matrix

$$B^{-1}AB = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} =$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$ with the eigenvalues $\lambda = 1$ and $\lambda = 3$ on the diagonal.

Existence of eigen-things

Consider rotation by $\pi/2$: $0 = \det(A - \lambda I) =$ $\det\left(\begin{pmatrix} -\lambda & -1 \\ 1 & \lambda \end{pmatrix}\right) = \lambda^2 + 1$ with solutions from complex numbers, i.e. no eigenvalues.

Consider sheer $\vec{e}_1 \to \vec{e}_1, \vec{e}_2 \to (1,1)$, then $0 = \det(A - \lambda I) =$ $\det\left(\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix}\right) = (1-\lambda)^2$ with solution $\lambda = 1$, so all eigenvectors are parallel to $\vec{e}_1$.

Consider scale with $\vec{e}_1 \to 2\vec{e}_1, \vec{e}_2 \to 2\vec{e}_2$, then $0 = \det(A - \lambda I) = \det\left(\begin{pmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix}\right) = (2-\lambda)^2$ with solution $\lambda = 2$, so eigenvectors are all vectors.

Diagonalisable matrices

Definition 26.8. A matrix A is **diagonalisable** if A is conjugated to a diagonal matrix D, i.e. there exists C so that $D = C^{-1}AC$ has zeros outside the diagonal.

A diagonal matrix is a simple representation of a linear map, because any basis vector maps to a scaled version of itself.

The claim below easily follows from Definitions 26.6 and 26.8.

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Claim 26.9. Any $m \times m$ matrix A is diagonalisable if and only if $\mathbb{R}^m$ has a basis of eigenvectors of A.

Eigenvalues are invariants

Claim 26.10. Eigenvalues of matrices are invariants of linear maps (or of matrices up to conjugation).

Proof. Any conjugated matrix $B = C^{-1}AC$ has the characteristic equation $0 = \det(C^{-1}AC - \lambda I) = \det(C^{-1}(A - \lambda I)C) = \det(C^{-1})\det(A - \lambda I)\det C$. Since $\det(C^{-1}) = 1/\det C$, the characteristic equation is equivalent to $\det(A - \lambda I) = 0$,

So A and $B = C^{-1}AC$ have equal eigenvalues (up to a permutation) as roots of the same equation. $\square$

Symmetric positive-definite matrices

Definition 26.11. A symmetric matrix A is called **positive-definite** if $\vec{v}^T A \vec{v} > 0$ for any $\vec{v} \neq \vec{0}$.

For $m = 2$, a symmetric matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive-definite if and only if the polynomial

$$f = (x, y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2 > 0$$

for all $(x, y) \neq (0, 0)$, i.e. when $a > 0$ (or $c > 0$) and $0 < \det A = ac - b^2 = -\frac{1}{4} \times$ (discriminant of f), where discriminant of a bivariate quadratics is similar to the usual quadratic discriminant, and is related to the conic sections .

A basis of orthogonal eigenvectors

Theorem 26.12. Any symmetric positive-definite matrix A has an orthogonal basis of eigenvectors, hence can be diagonalised, i.e. there is a transition matrix C such that the matrix $C^{-1}AC$ is diagonal.

Consequently, the eigenvalues are complete invariants for any symmetric positive-definite matrix, which is conjugated to the diagonal matrix with the eigenvalues $\lambda_1 \geqslant \cdots \geqslant \lambda_m$ on the diagonal.

Time to revise and ask questions

- *Eigenvalues* λ and *eigenvectors* $\vec{v}$ of A are solutions of $A\vec{v} = \lambda\vec{v}$, hence $\det(A - \lambda I) = 0$.
- Any symmetric positive-definite matrix A has an orthogonal basis of eigenvectors.

Problem 26.13. Find a basis such that the map with the matrix $A = \begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}$ becomes diagonal.

Additional links

- 3Blue1Brown on eigenvalues and eigenvectors
- Page Rank as an intersection of various subjects
- Page Rank with explanation of computational part (and Python code). It comes from "Linear Algebra, Geometry and Computation" book that is a picturesque insight of many things from this module.