Distributed Systems
COMP 212

Lecture 7

Othon Michail

UNIVERSITY OF
LIVERPOOL

Leader Election

Problem Statement

- *Elect a unique leader processor from among all the processors in the distributed system*
- Leader to be interpreted as:
  - coordinator
  - master processor
- Special case of consensus/agreement
- Processors should agree eventually on who they elect

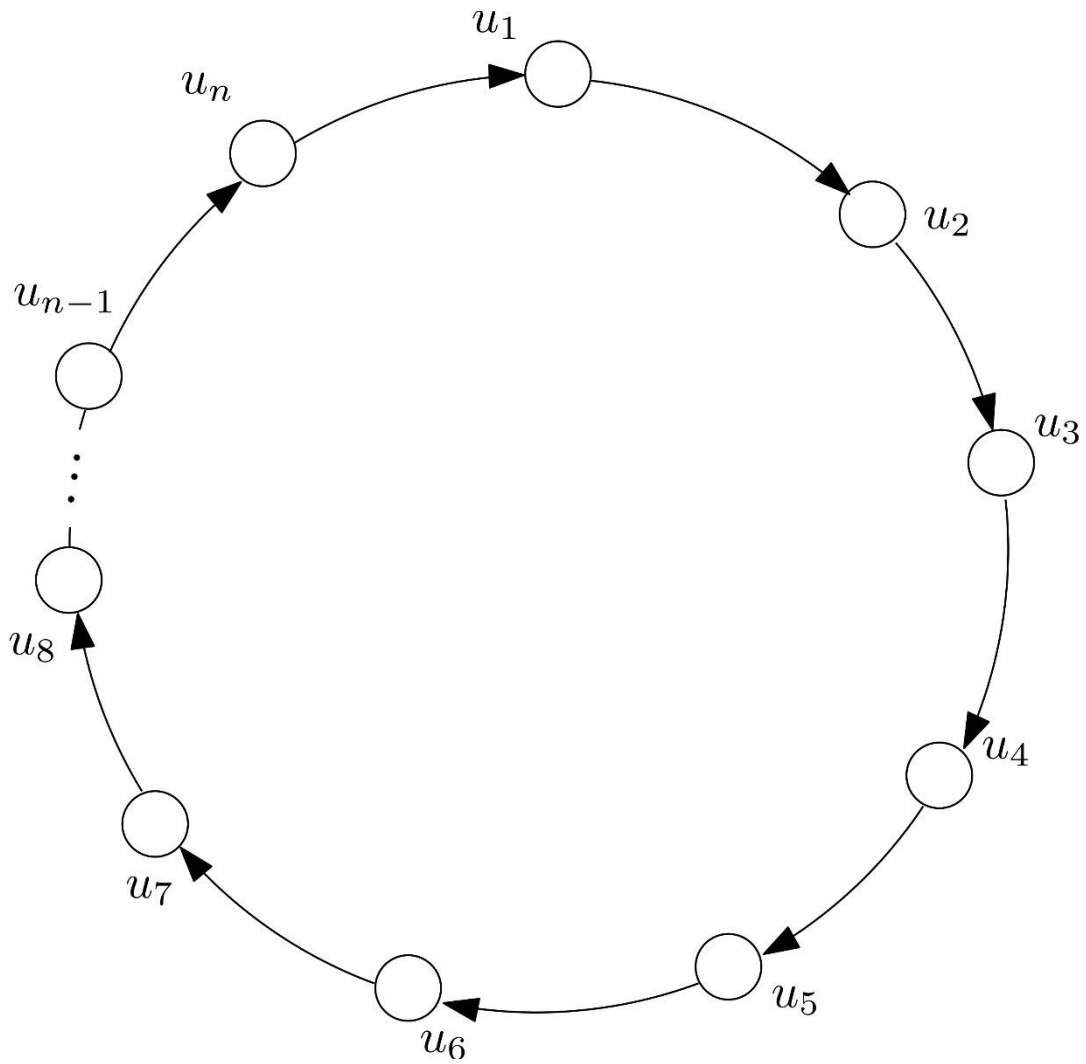
Variants of Leader Election

- General network or special type of network, e.g., a ring
- Processors can be identical or have pre-assigned unique ids
- All processors may be required to
  - know the elected processor
  - to output the fact that themselves were not elected
  - to terminate
- Processors may possess in advance some information about the network
  - e.g., the number of processors *n* in the network

Important remark: Even slightly different assumptions can completely change the algorithmic solutions required or even make a problem impossible to solve

The Ring Network

- Unidirectional (or directed)



Leader Election with ids

- Processors have unique ids and do not know *n* in advance
- LCR algorithm: solves the problem
  - Le Lann, Chang, and Roberts [1977, 1979]
- Uses only transmission and comparison of ids
- Simplest version: Only the elected processor gives "output" and terminates
  - e.g., "I am the leader"
  - The other processors never produce any output and do not terminate

LCR: Pseudocode

Algorithm LCR

State of processor u_i :

- $myID_i$: holds the processor's unique id
- $sendID_i$: holds a unique id to be sent or *null*
- $status_i \in$ {"unknown", "leader"}: indicates whether u_i has been elected ("leader") or not ("unknown")

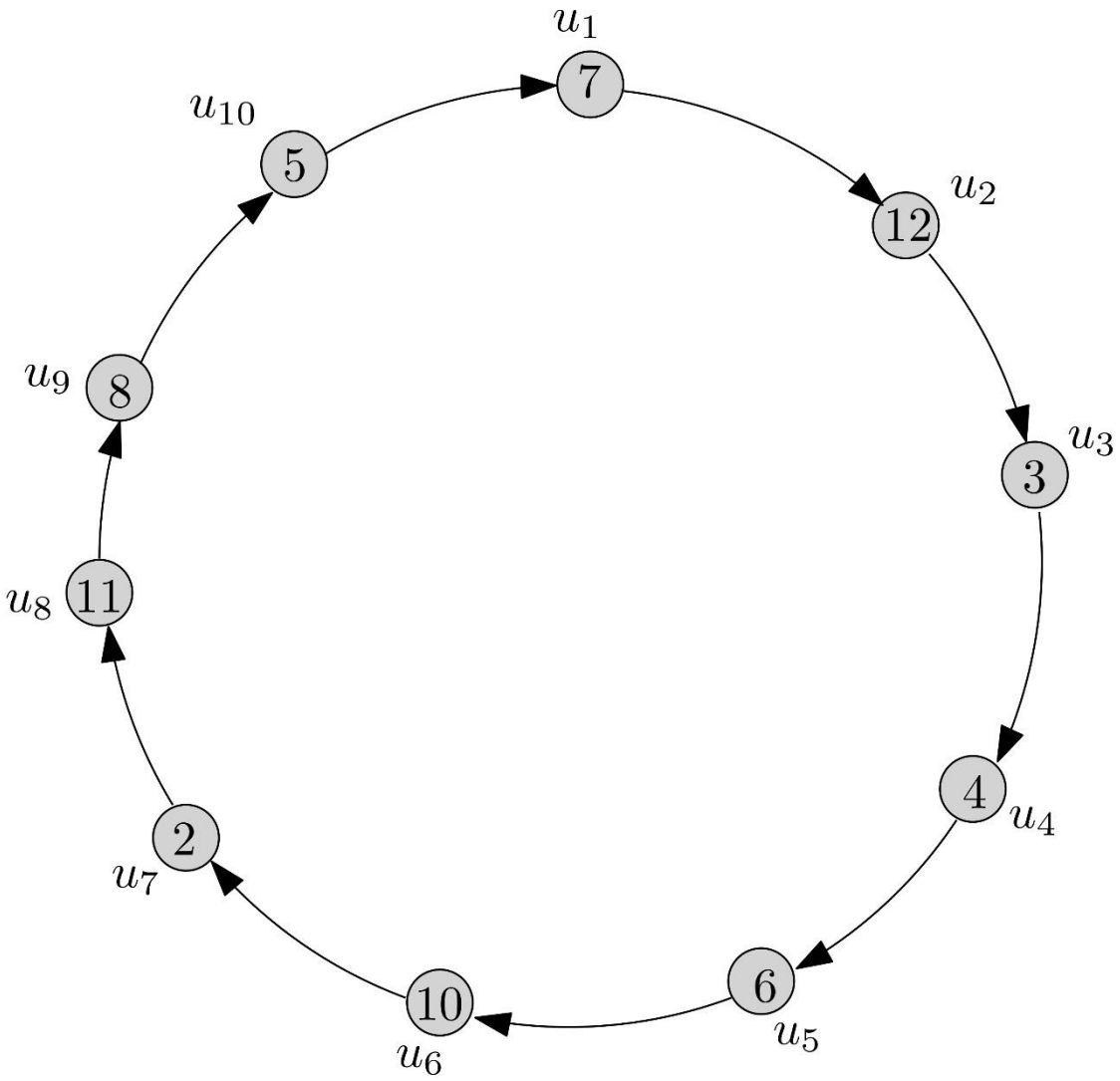
LCR: Pseudocode

Algorithm LCR
Code for processor u_i , i ∈ {1, 2,… , n}:

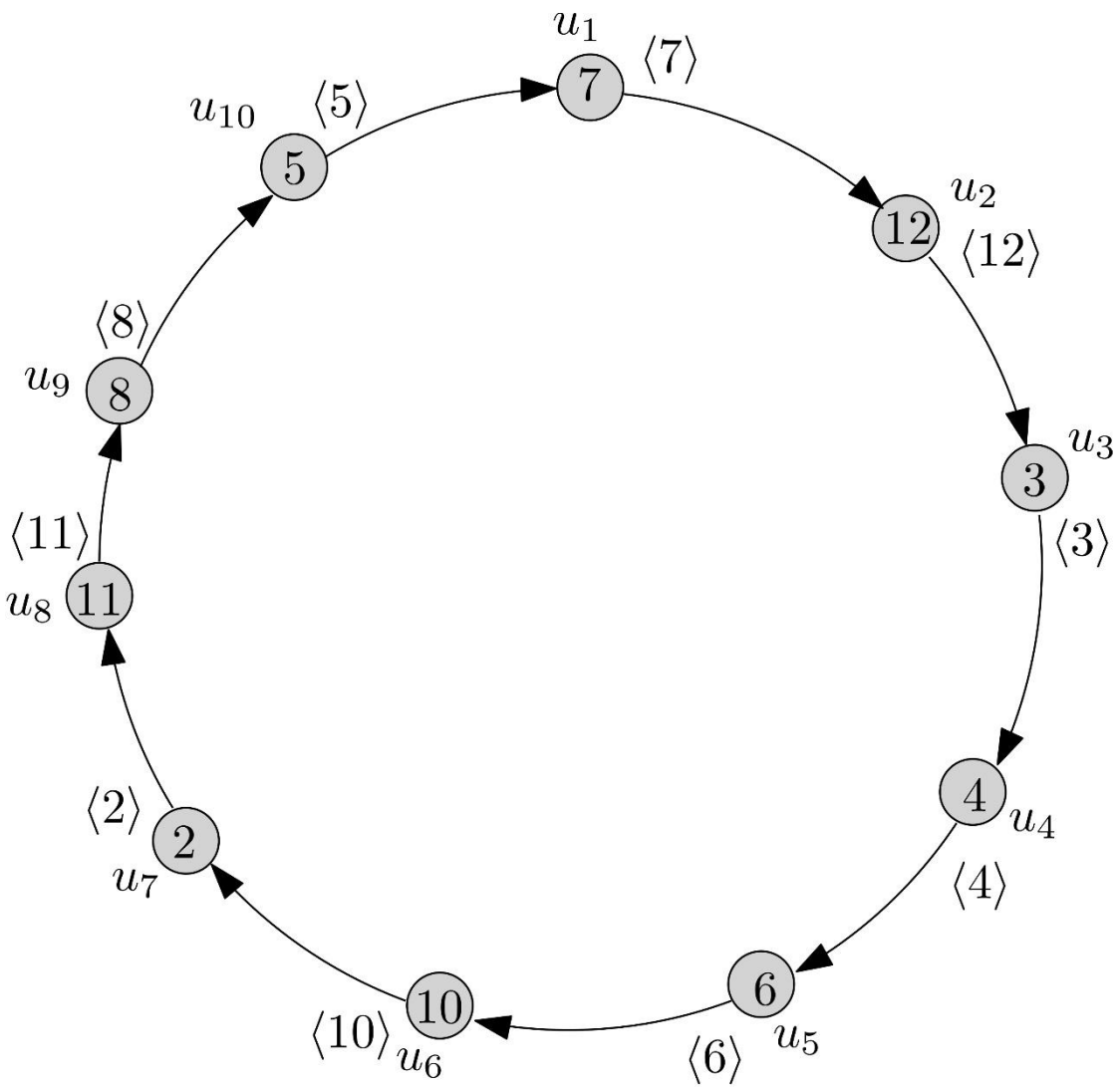
Initially:
  u_i knows its own unique id stored in $myID_i$
  $sendID_i := myID_i$
  $status_i$:= "*unknown*"

```
if round = 1 then
   send ⟨sendID_i⟩ to unique out-neighbour
else     // round > 1
   upon receiving ⟨inID⟩ from unique in-neighbour          // an id arriving from the left
   if inID > myID_i then                                   // if greater than your own
      sendID_i := inID                                     // forward it
      send ⟨sendID_i⟩ to unique out-neighbour
   else if inID = myID_i then     // if equal to your own, your id managed a complete turn
      status_i := "leader"        // therefore, elect yourself a leader
   else if  inID < myID_i then                             // if smaller than own
      do nothing                                           // ignore (discard) it
```

Example Execution

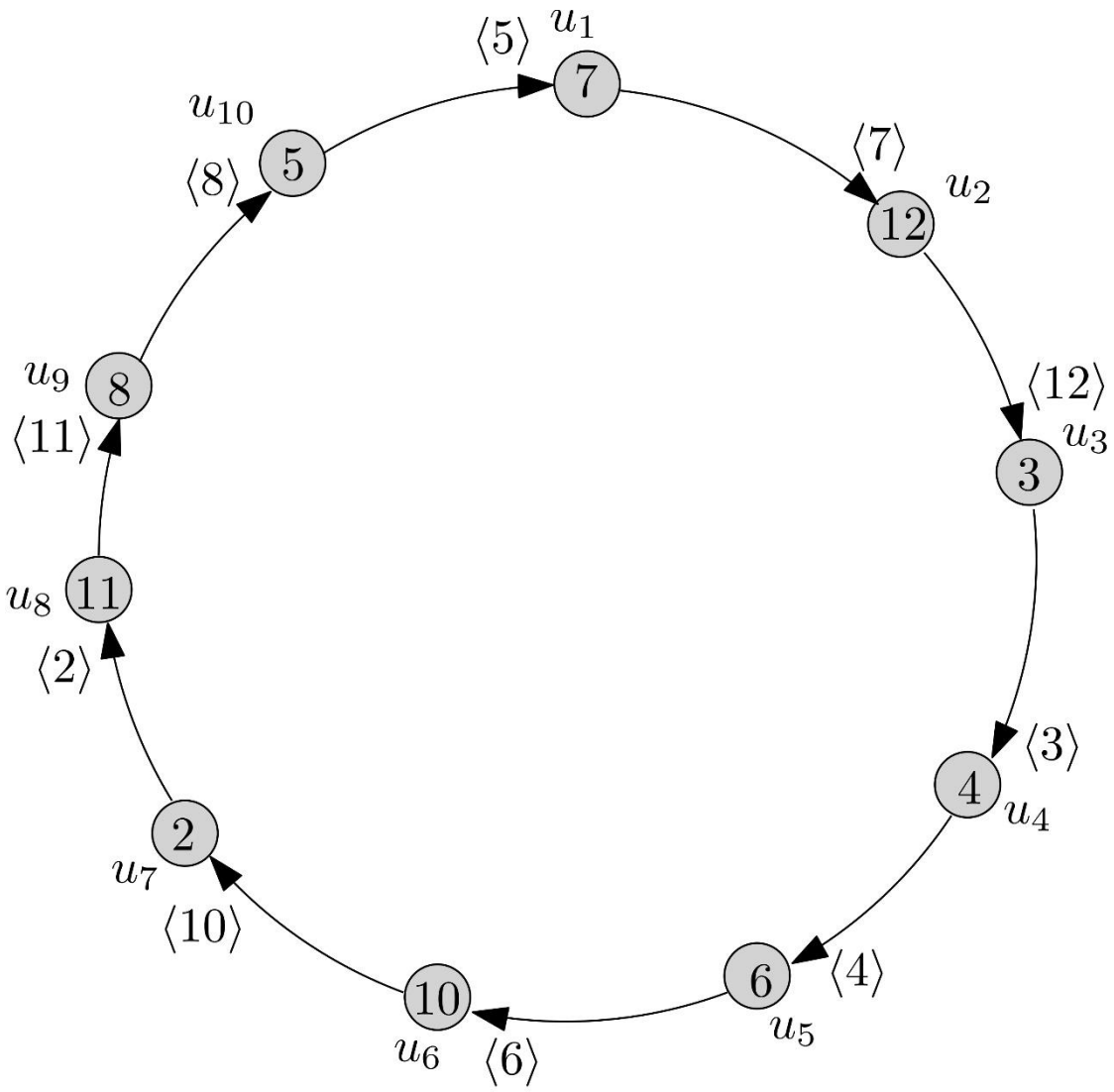


Example Execution



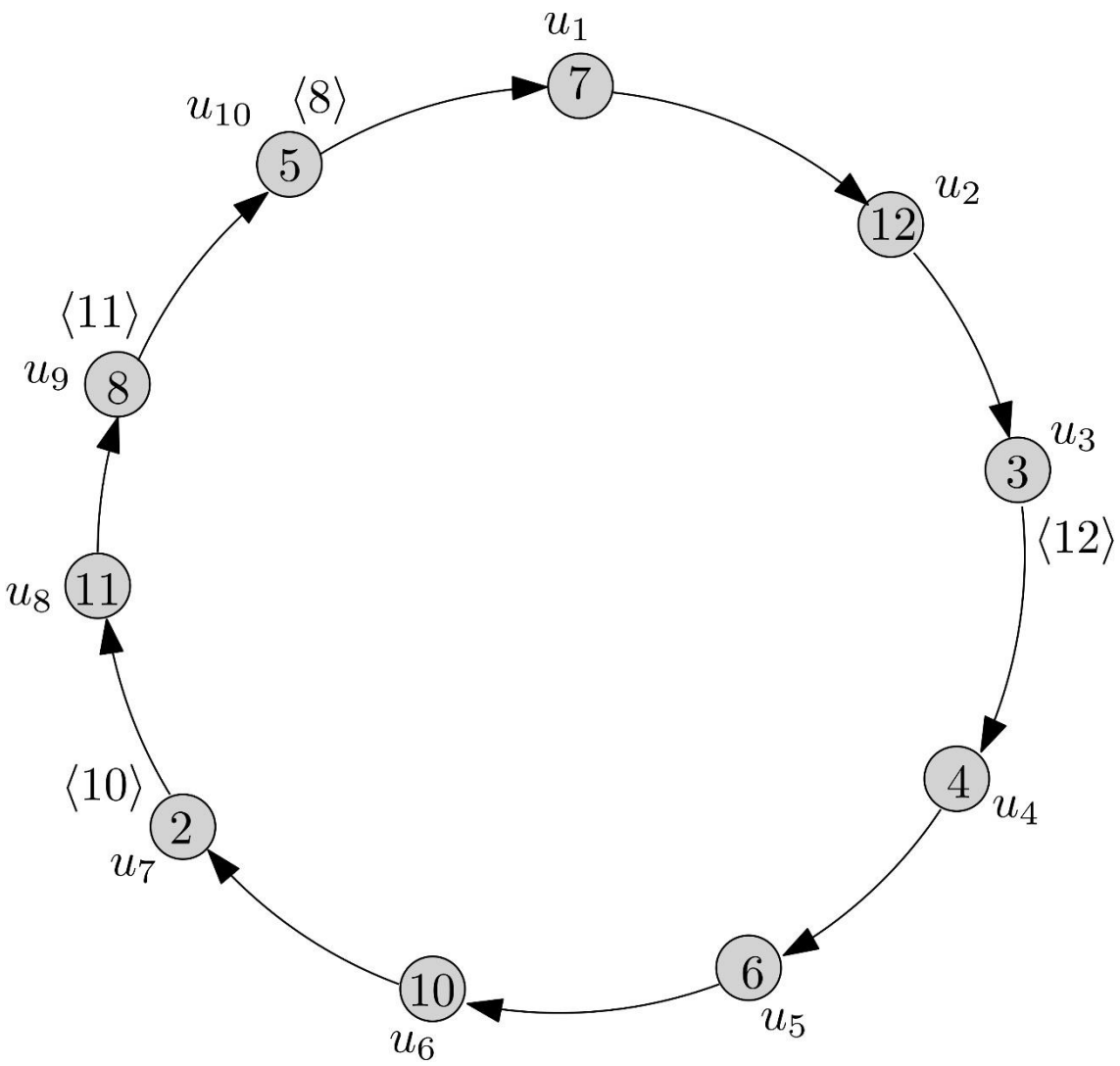
$round = 1$

Example Execution



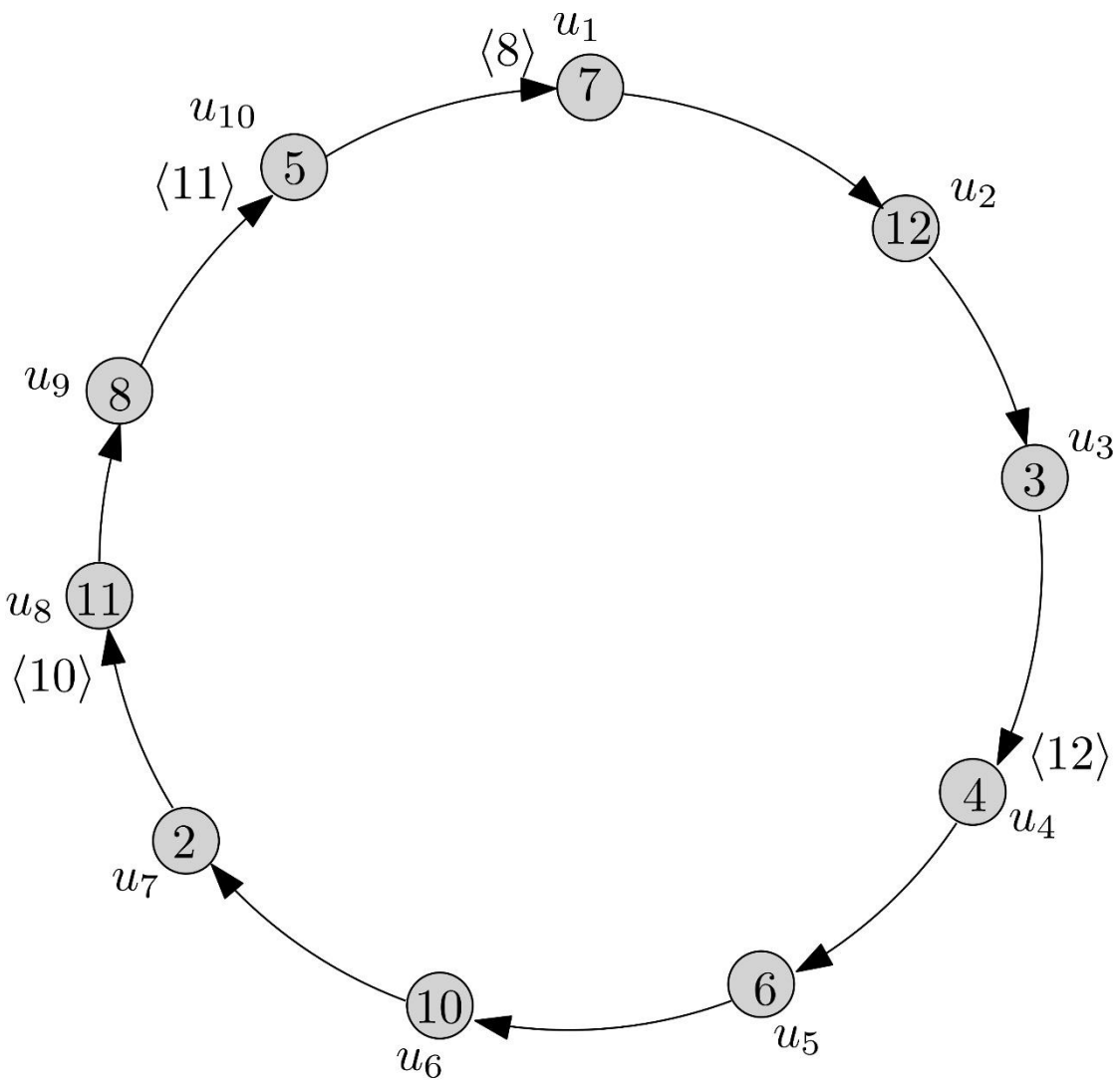
$round = 1$

Example Execution



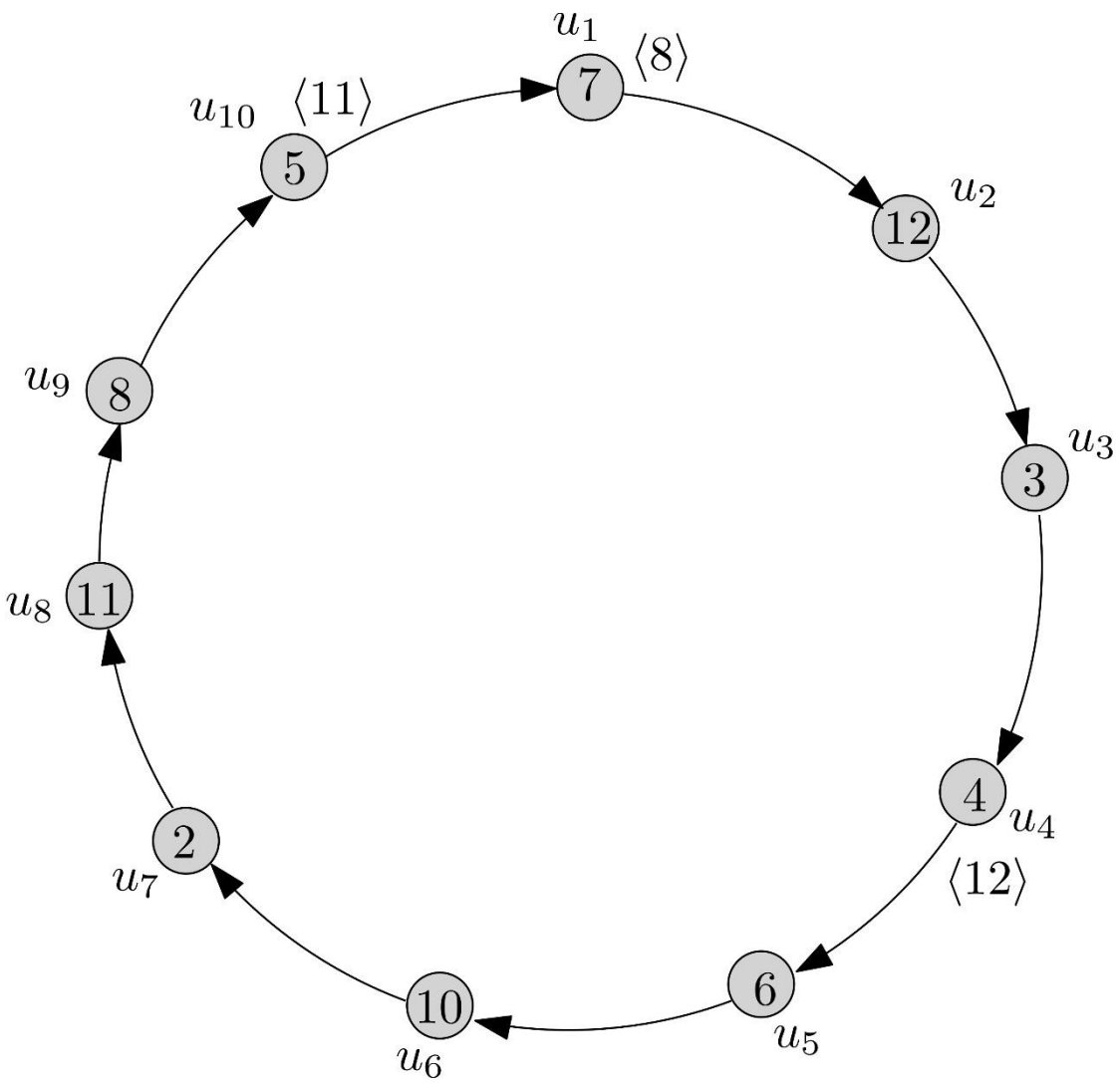
$round = 2$

Example Execution



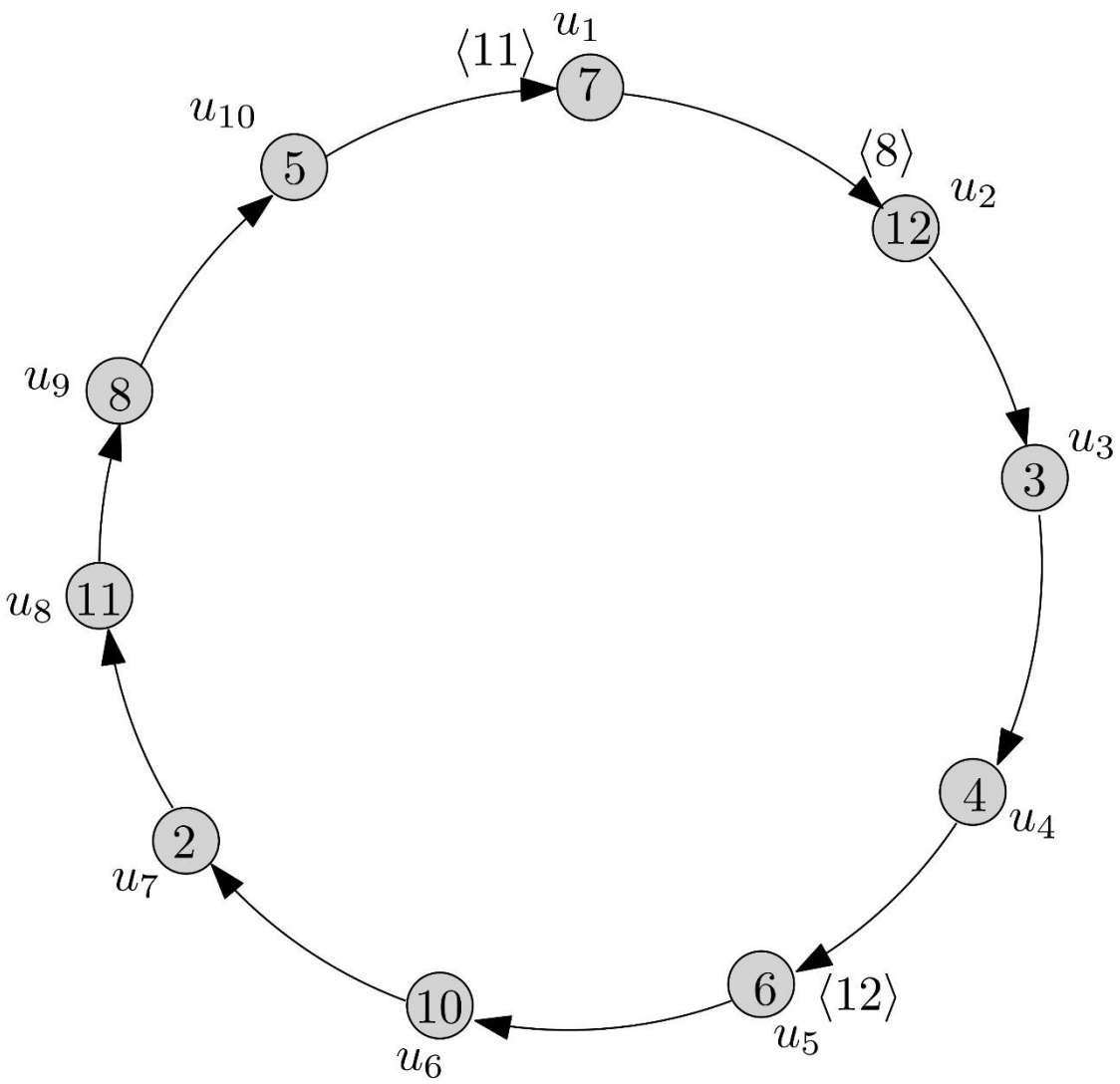
$round = 2$

Example Execution



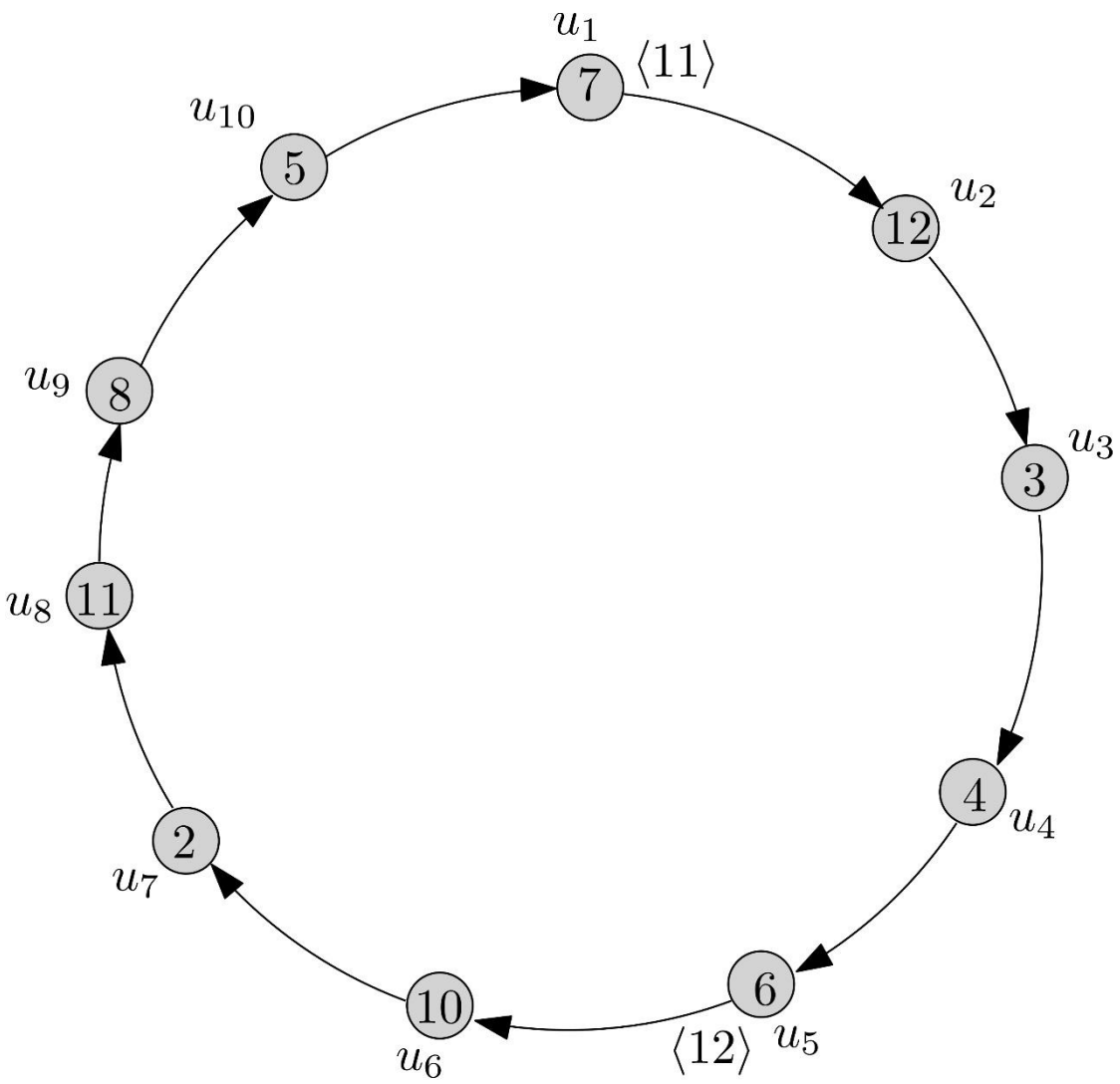
$round = 3$

Example Execution



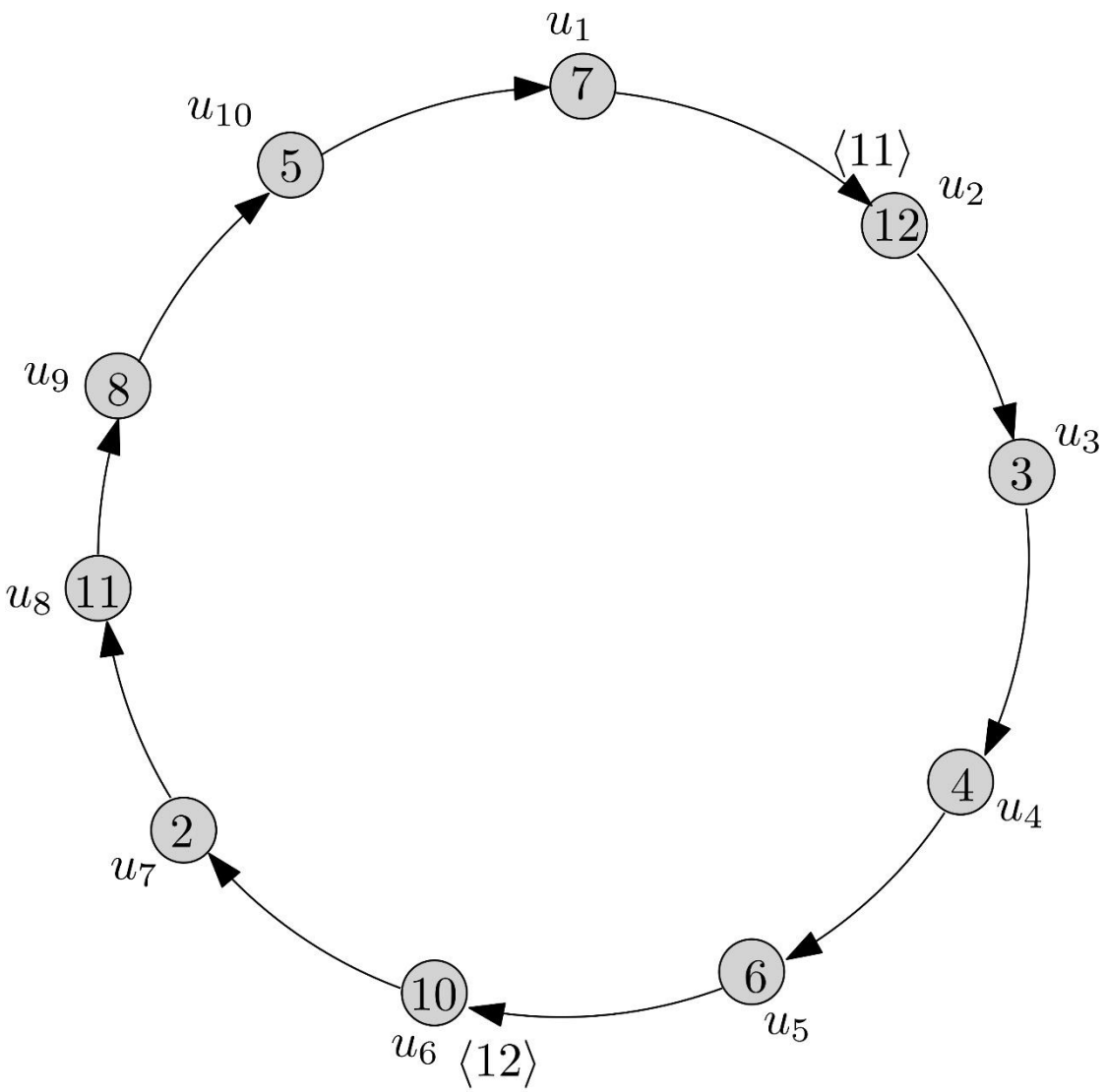
$round = 3$

Example Execution



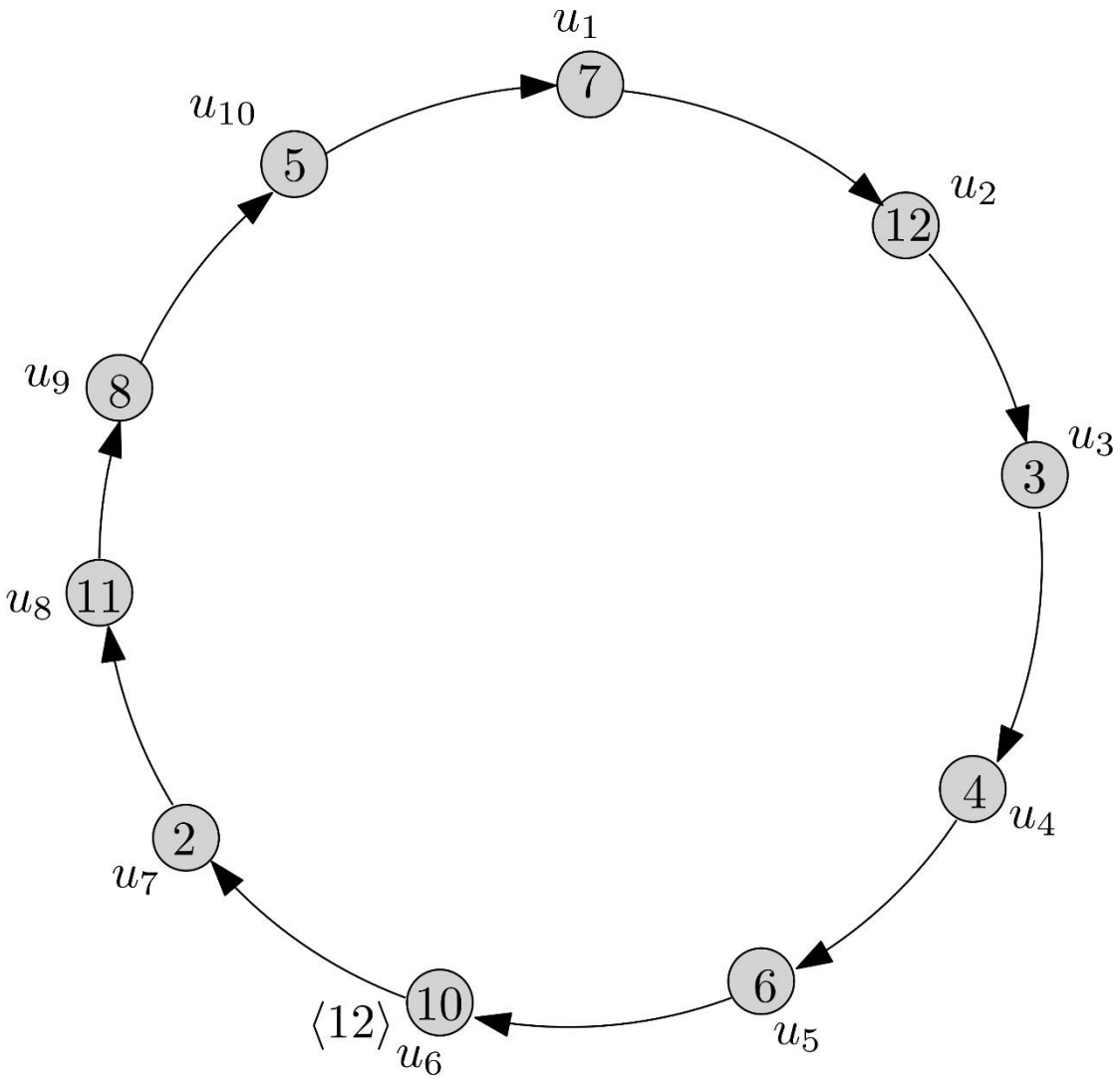
$round = 4$

Example Execution



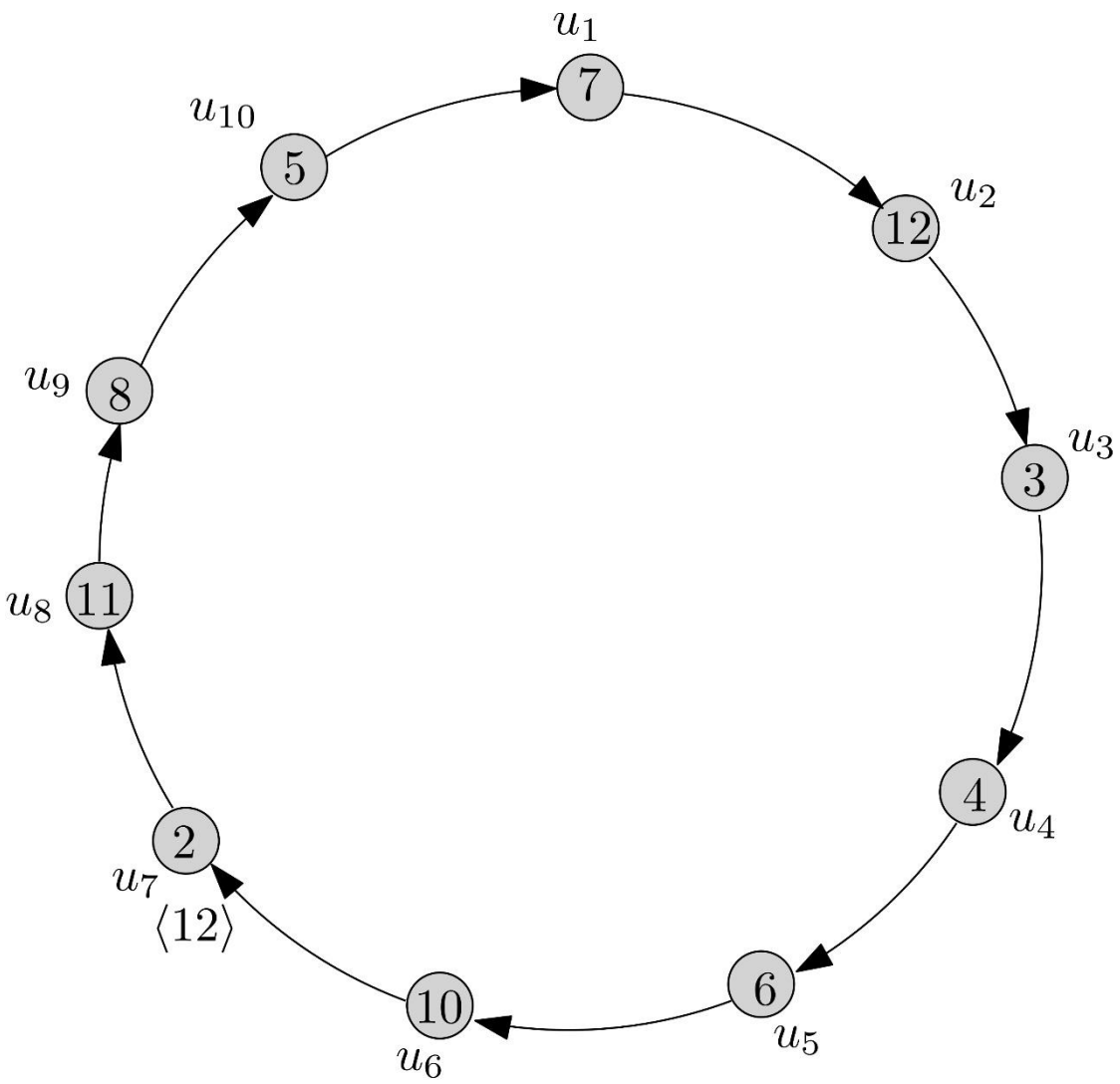
$round = 4$

Example Execution



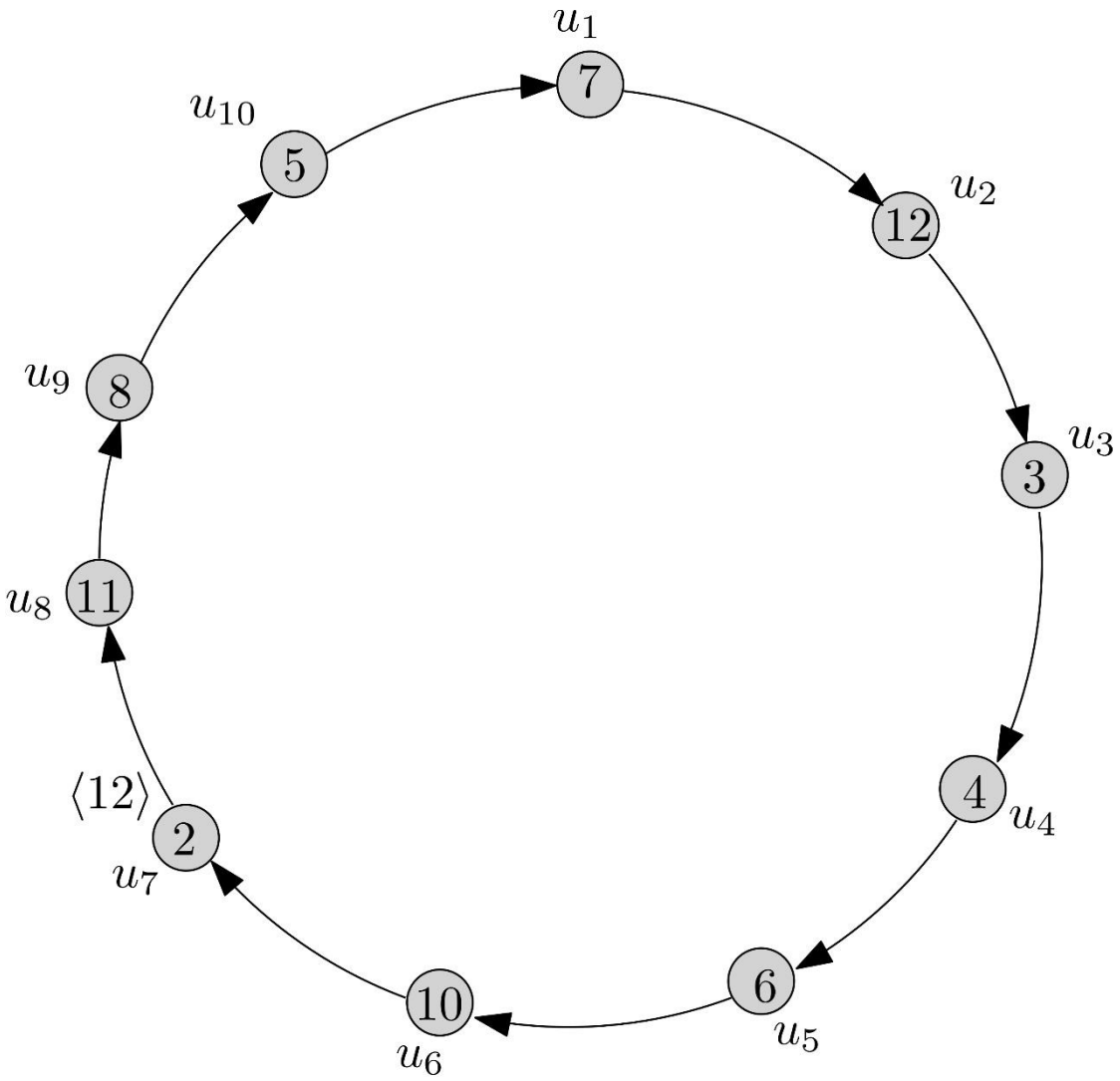
$round = 5$

Example Execution



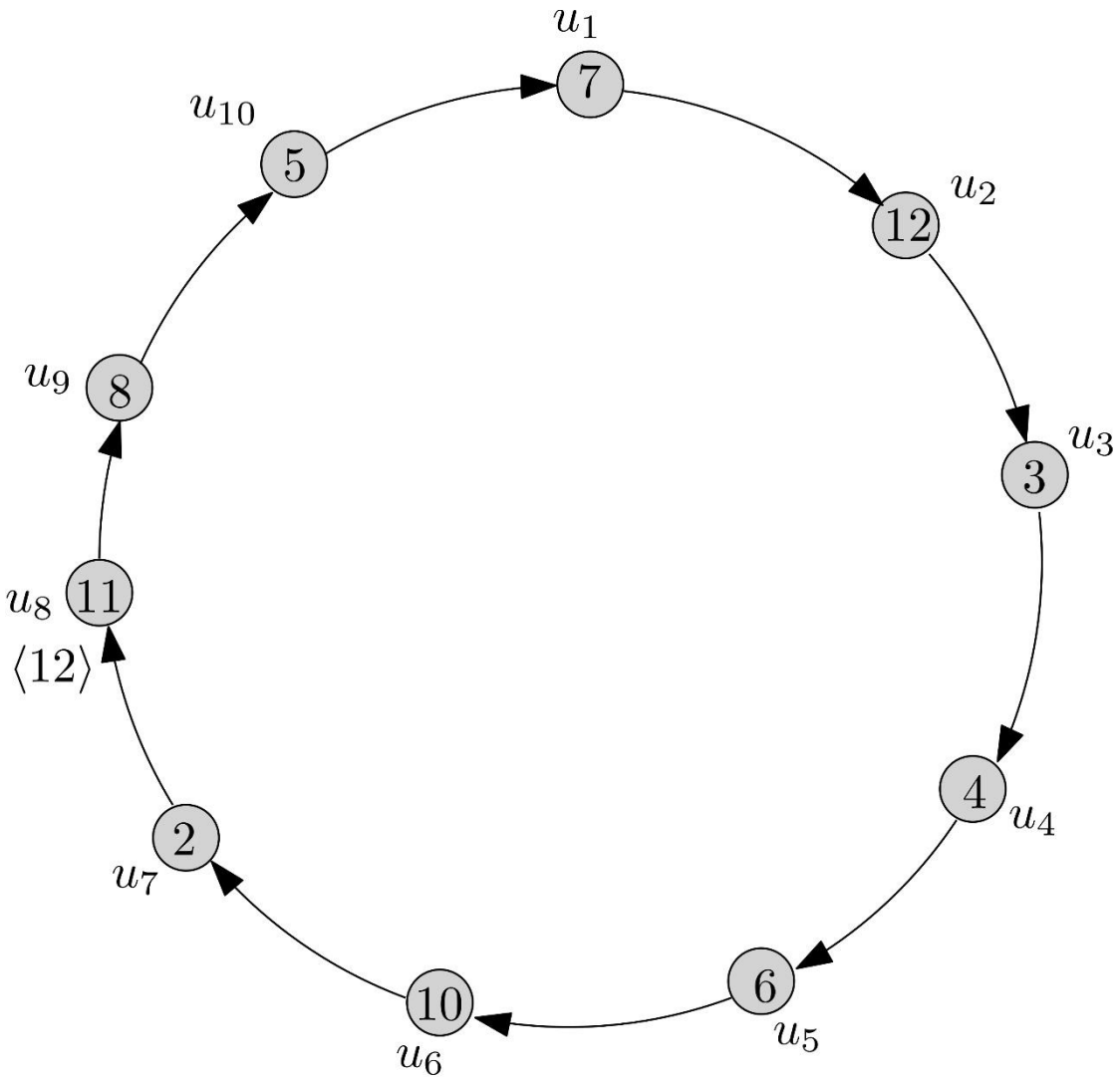
$round = 5$

Example Execution



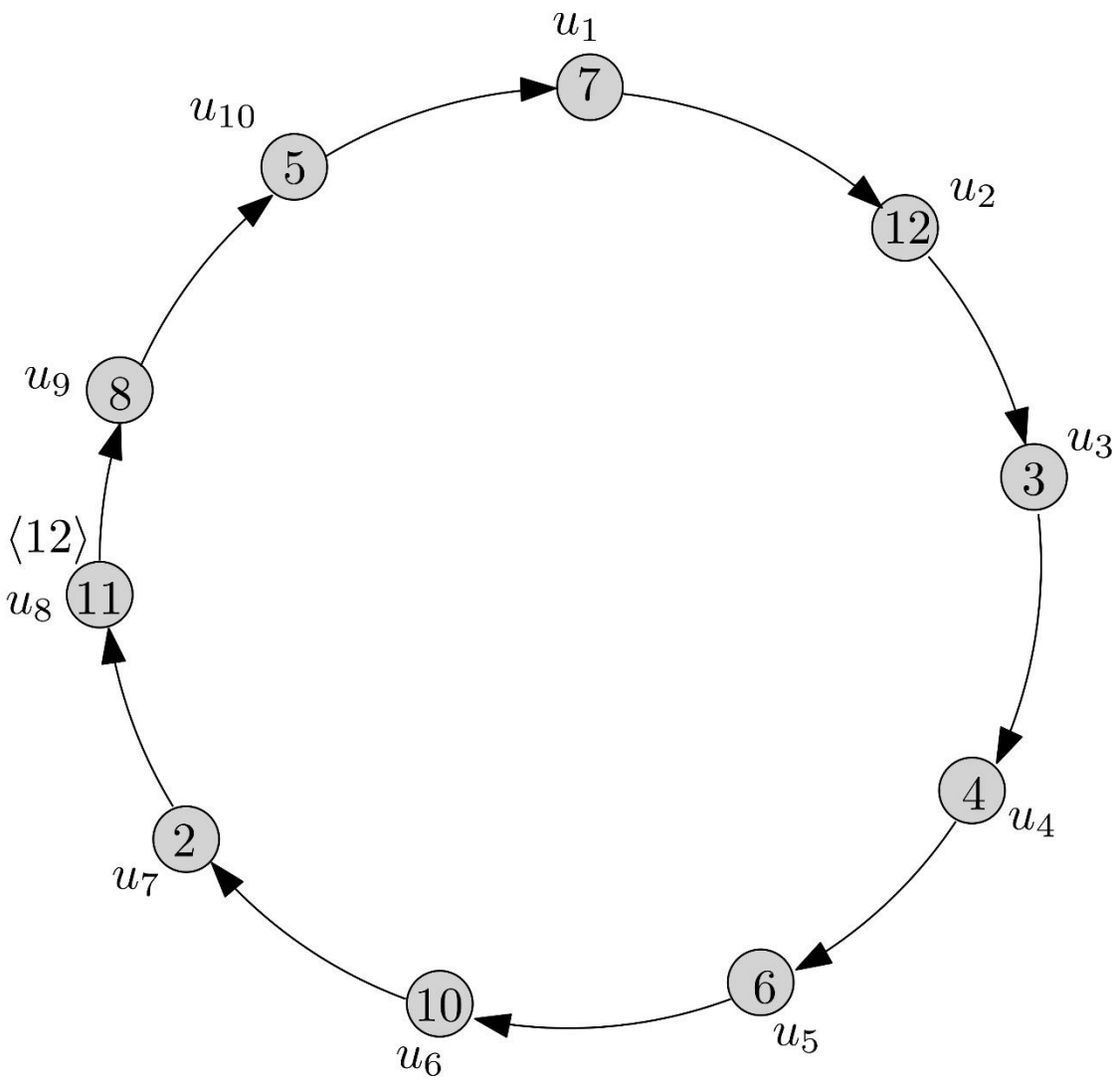
$round = 6$

Example Execution



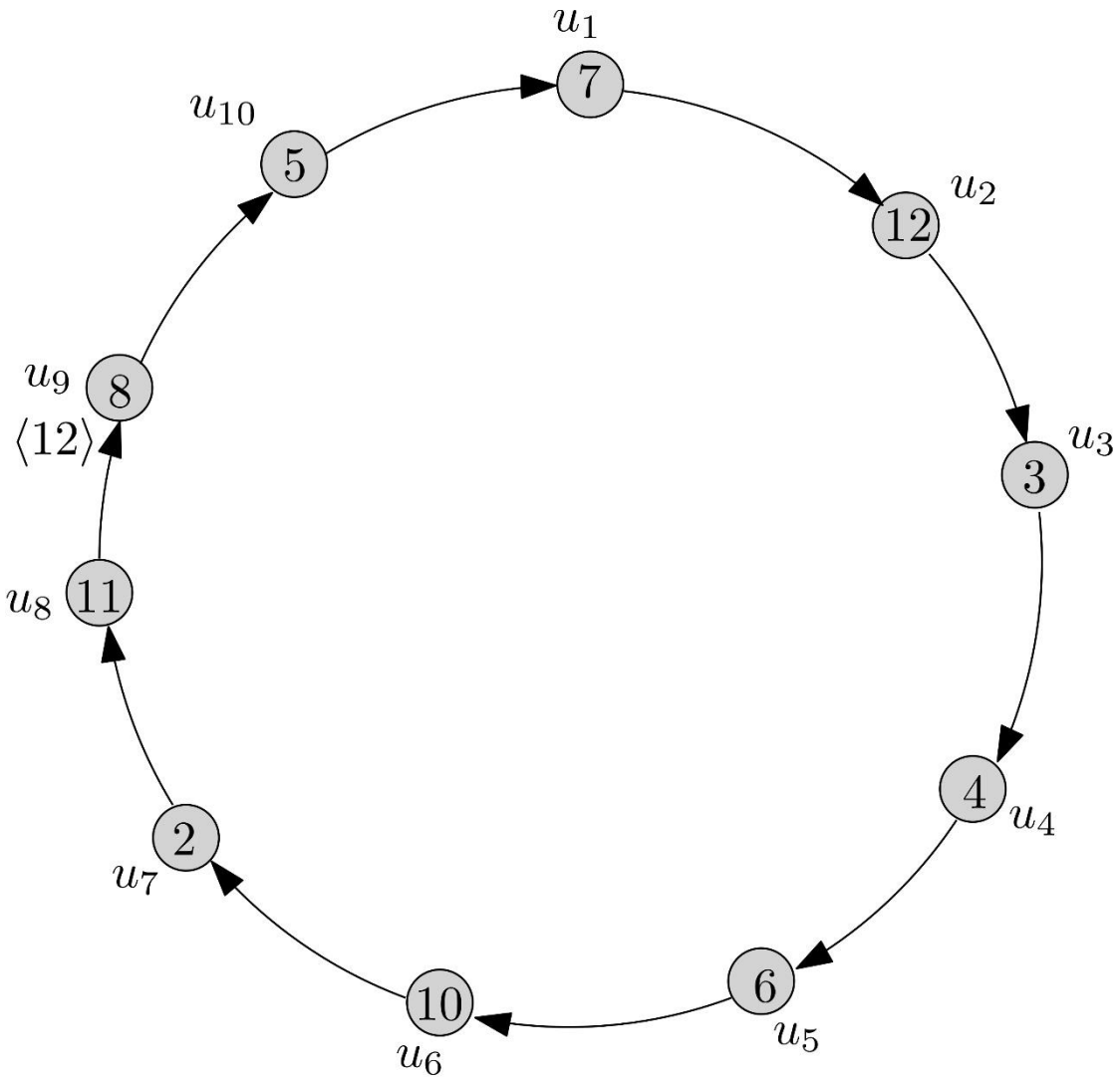
$round = 6$

Example Execution



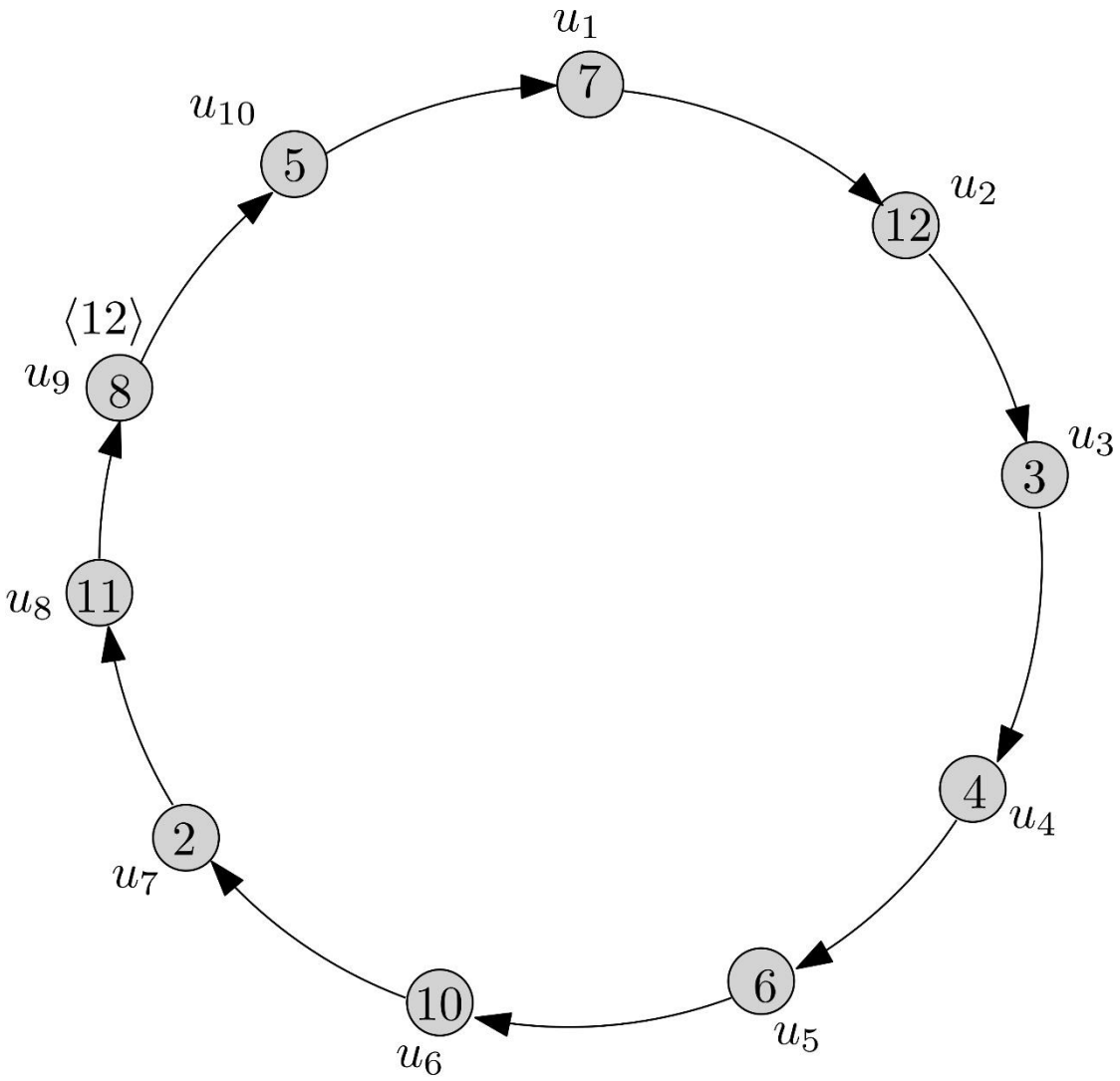
$round = 7$

Example Execution



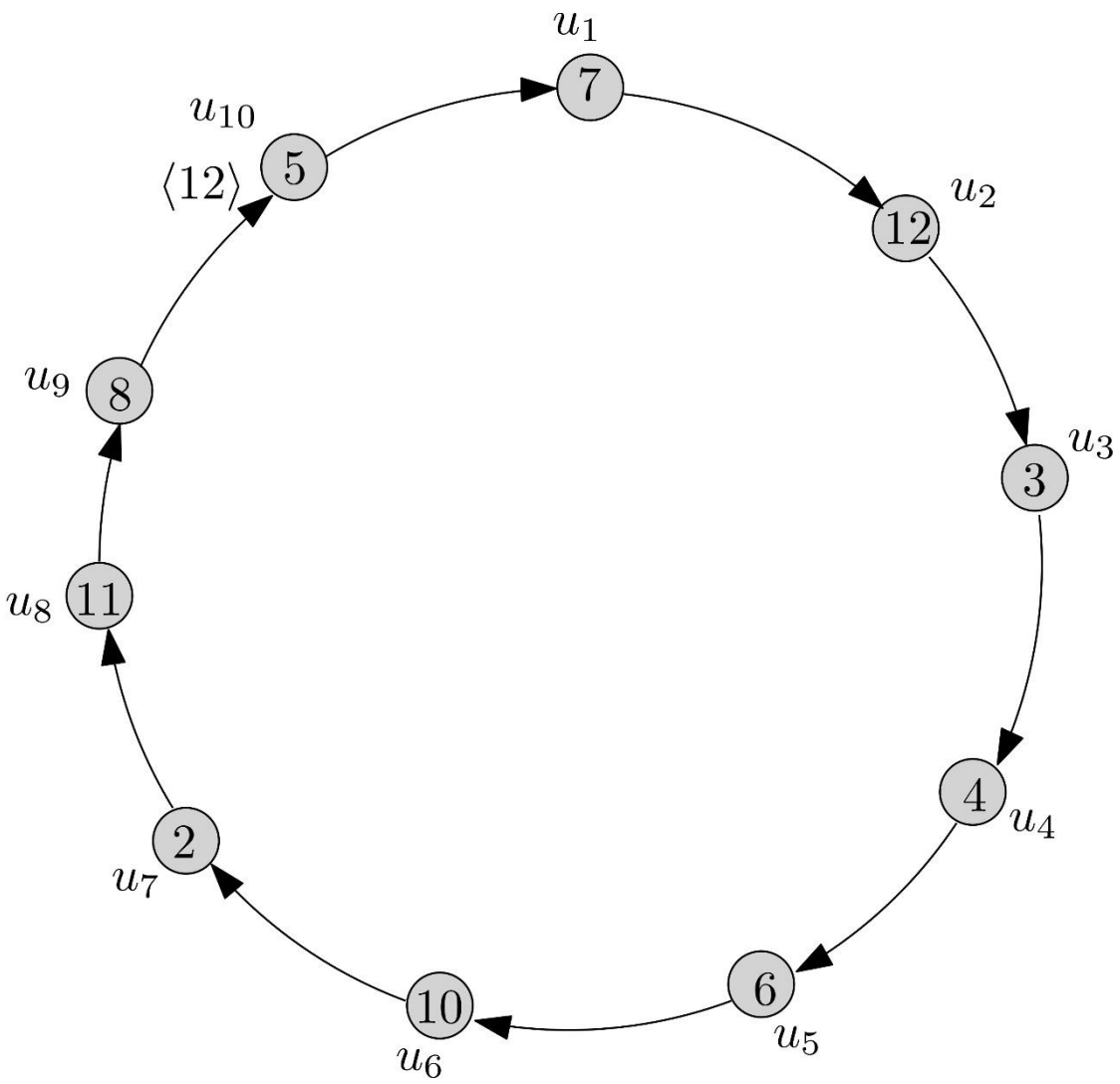
$round = 7$

Example Execution



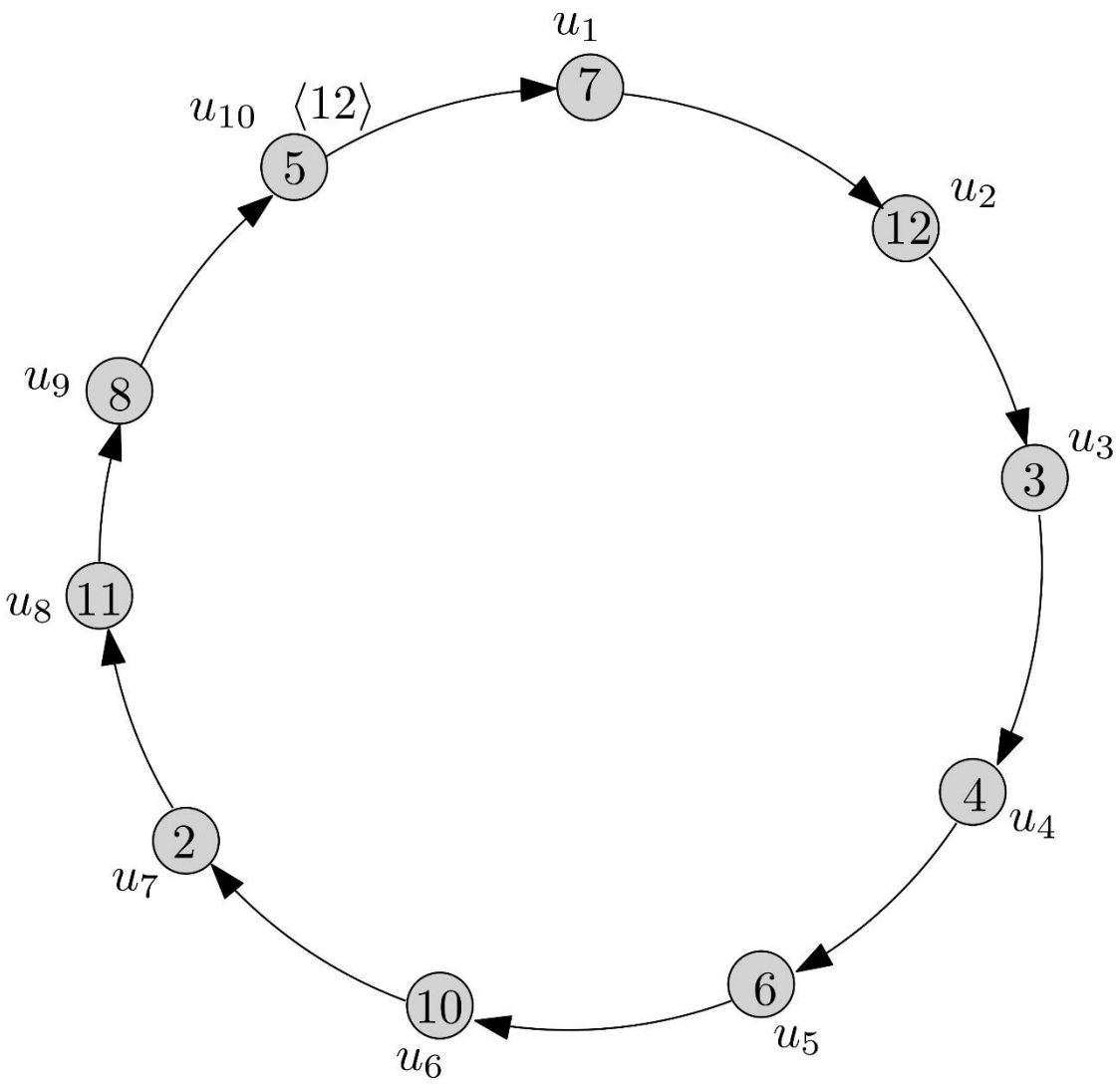
$round = 8$

Example Execution



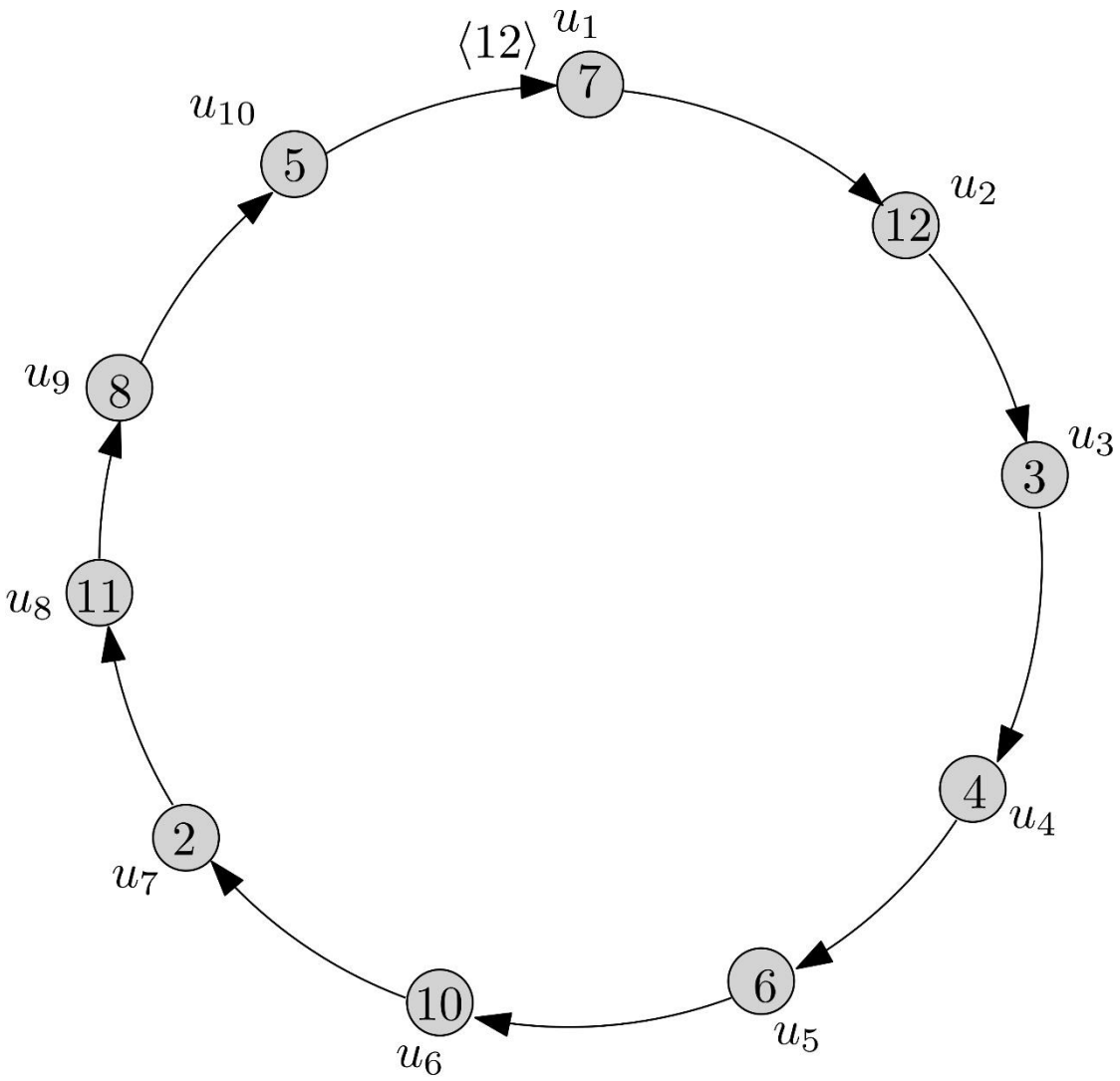
$round = 8$

Example Execution



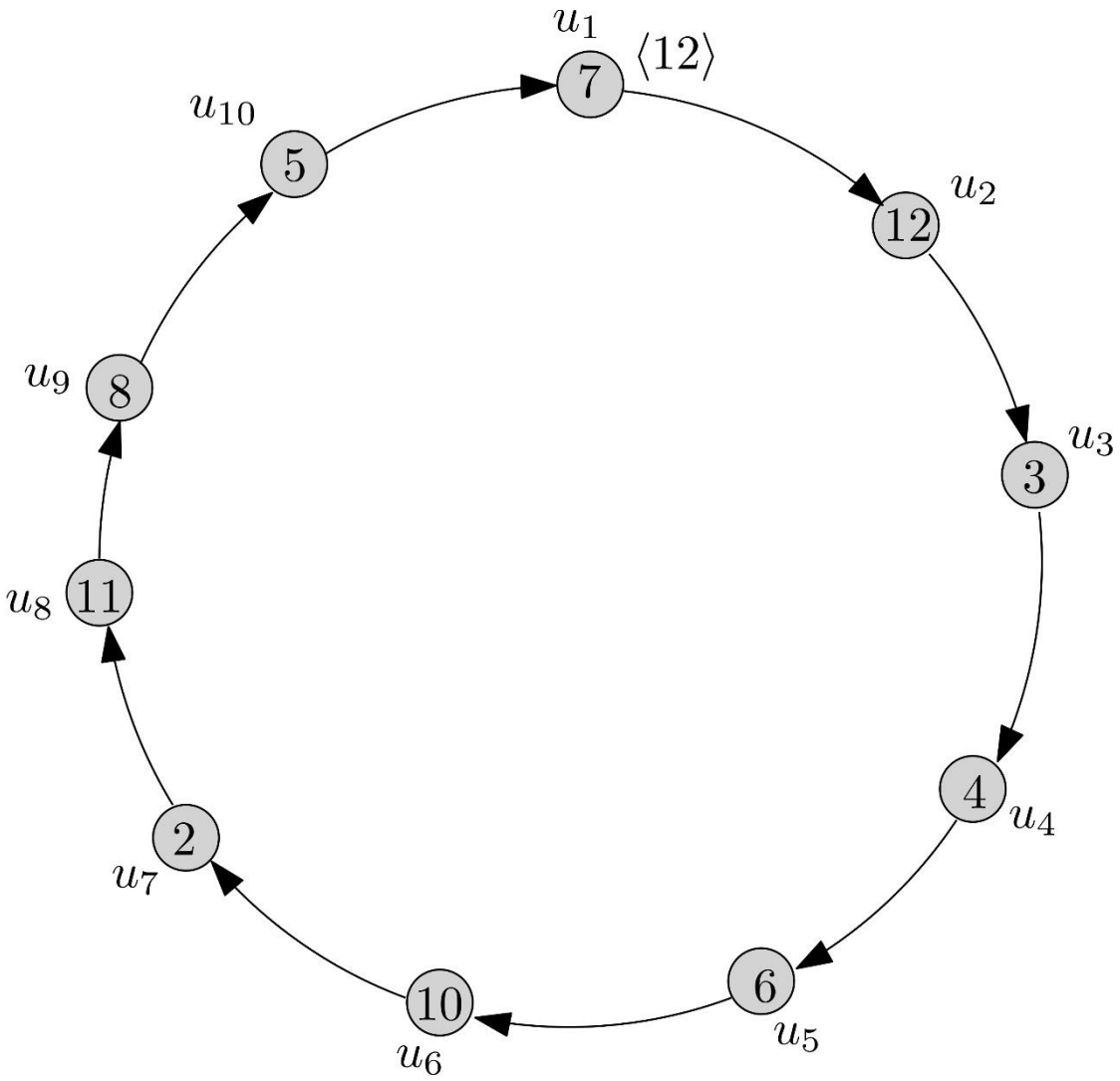
$round = 9$

Example Execution



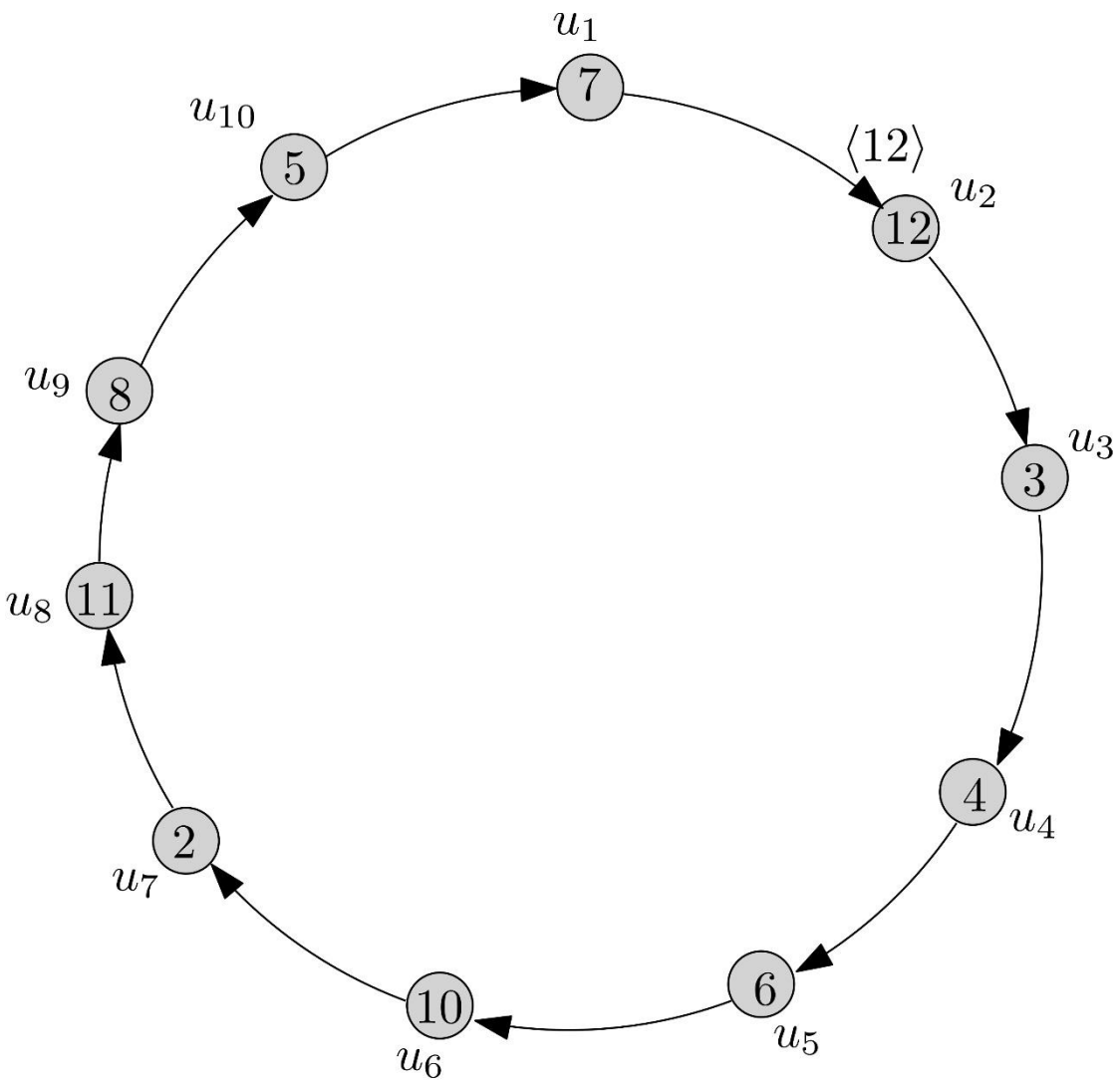
$round = 9$

Example Execution



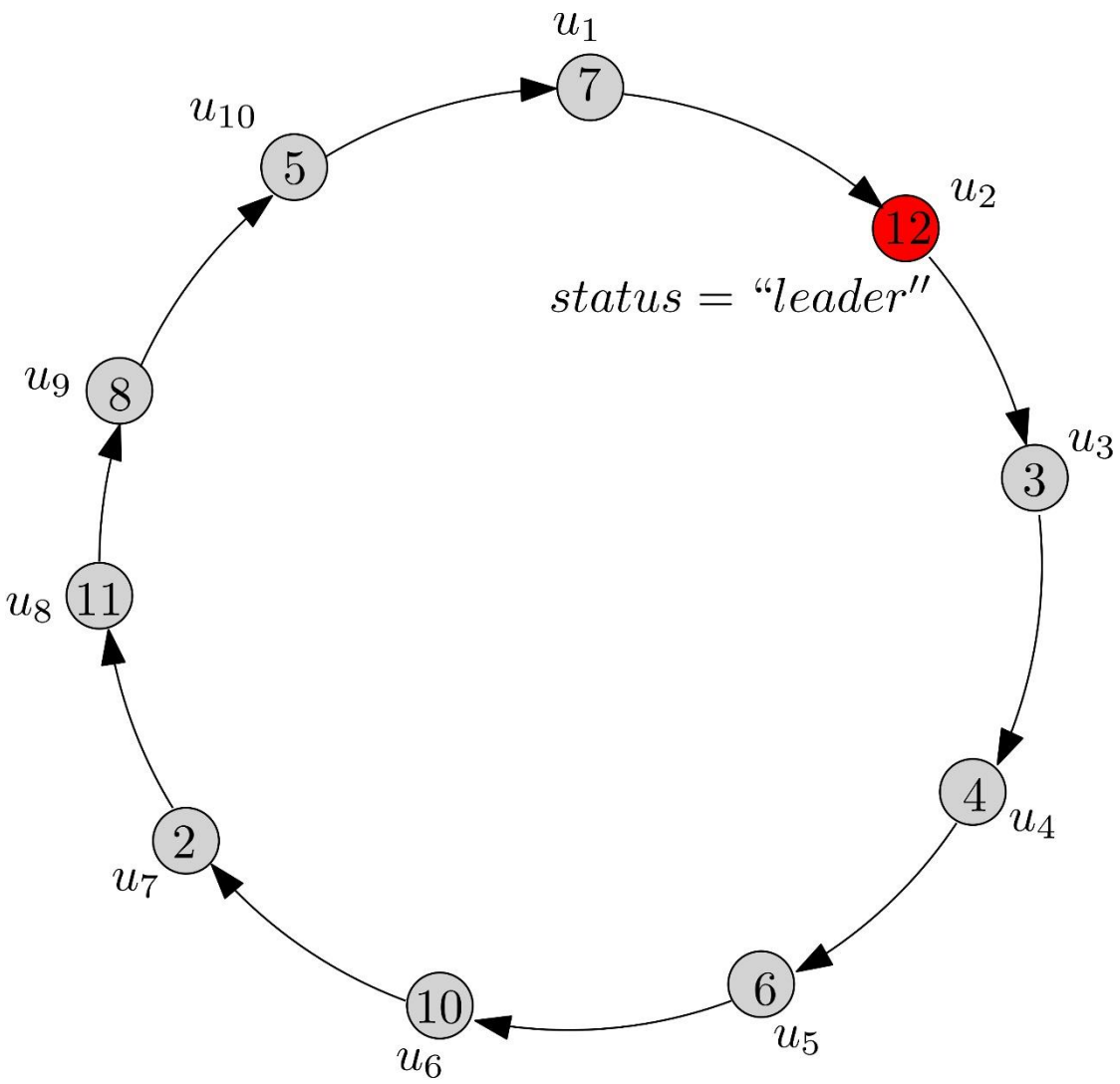
$round = 10$

Example Execution



$round = 10$

Example Execution



$round = 11$

Correctness and Complexity

- Correctness:
  - some processor is eventually elected
  - never 2 or more processors are elected
- Time complexity:
  - $n + 1$ rounds (or n depending on the round model)
- Communication complexity:
  - size of messages: encoding in bits of the maximum id
  - $O(n^2)$ messages in the worst case
    - *Can you think which is the worst case for this algorithm?*
- *How can we make all nodes terminate and know the elected leader and what will be the additional effect in performance?*

Think of these and we shall prove them in class

LCR Correctness

- We have to show that:
  - Exactly one processor u_i eventually sets $status_i$:= "*leader*"
  - We know that the algorithm aims to elect the processor with the maximum id
    - Call it u_{max}
- Suffices to show that:
  - u_{max} eventually outputs "*leader*"
  - No other u_i ever does this

LCR Correctness

Lemma. u_{max} outputs "*leader*" in round $n + 1$.

Proof. We observe that id_{max} (i.e., the id of u_{max}) makes progress in every round.

- No processor with larger id to discard it

Therefore, in round 1 it is transmitted over the 1st clockwise edge, in round 2 over the 2nd, in round n over the nth, i.e., received back by u_{max}.

□

- Can be made completely formal by induction on the number of rounds r

LCR Correctness

Lemma. No processor u_i other than u_{max} ever outputs "*leader*".

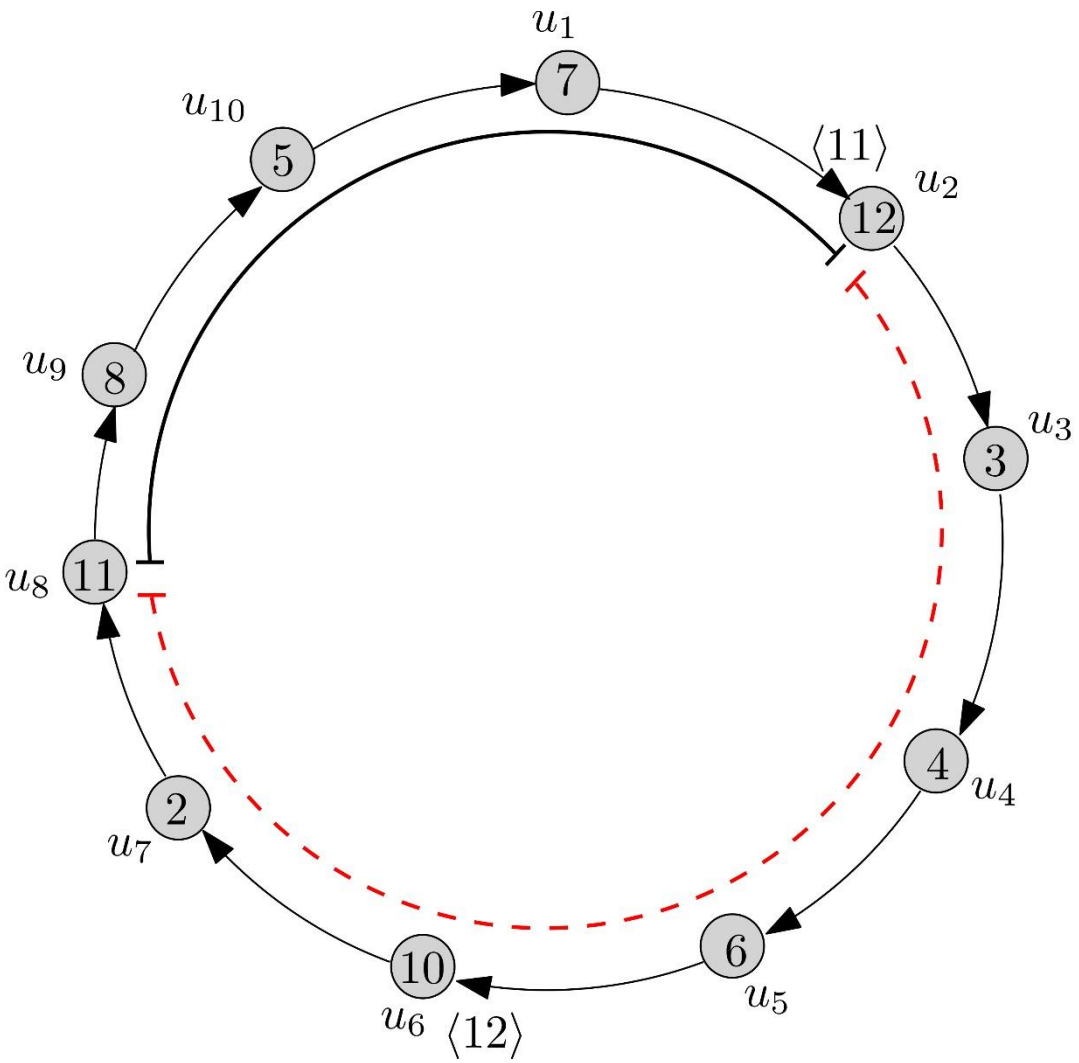
Proof.

- To output "*leader*", the id_i of u_i must perform a complete turn and get back to u_i
- But u_{max} is always in the way and
- $id_i < id_{max}$
- Therefore, id_i will meet u_{max} before performing a complete turn and will be discarded by u_{max}

□

  - Can be made completely formal by induction on the number of rounds *r*

LCR Correctness



- u_2 is u_{max} in this case
- u_8's id is $id_8 = 11$
- u_8's id can travel up to the point that it meets u_2 but not further than this
- It will always be discarded by u_2

LCR Correctness

Theorem. The LCR algorithm solves the leader election problem in any directed ring network (provided the availability of unique ids).

- Works also for undirected rings
  - provided that all processors can distinguish their clockwise from their counterclockwise neighbour
- We could easily modify the algorithm to elect the processor with the minimum id instead
  - Try this at home

LCR Time Complexity

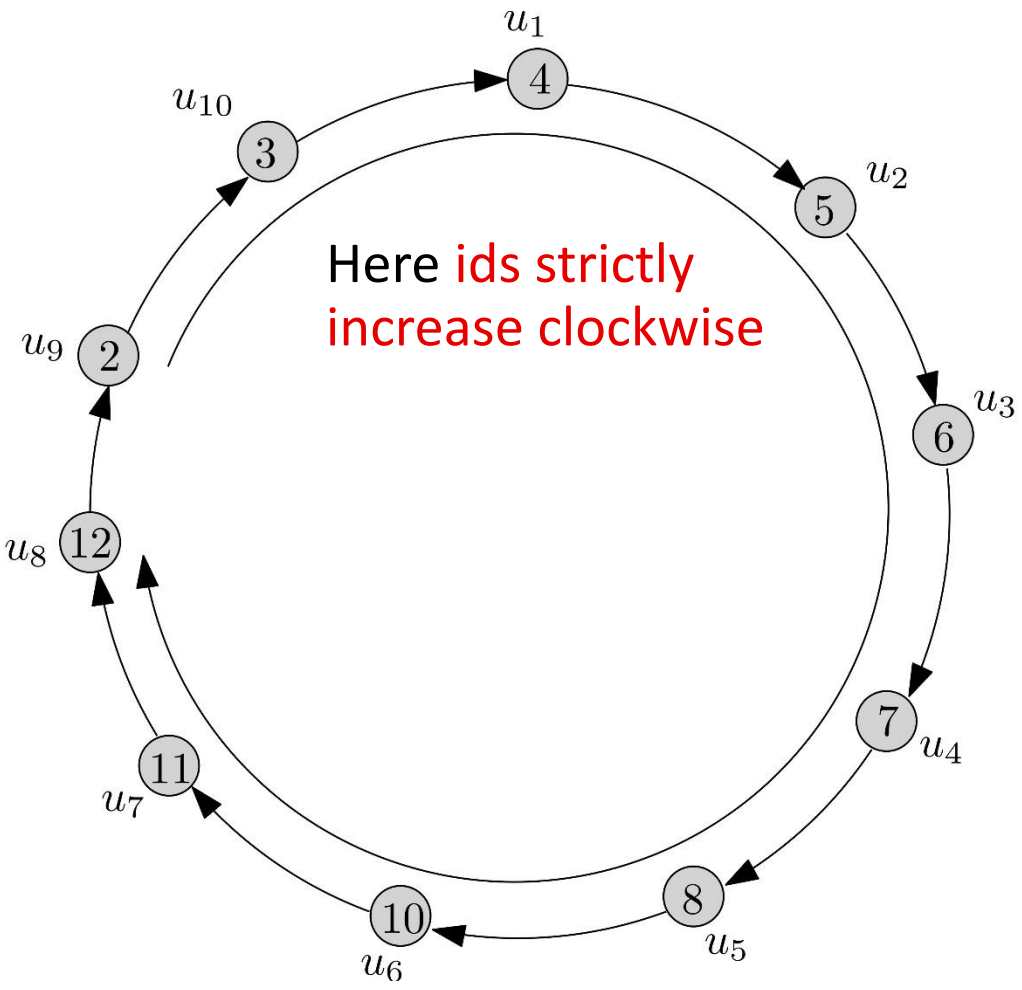
- n + 1 rounds
  - *Why?*
- How can we modify the algorithm so that all processors
  - eventually terminate and
  - know who the elected leader is
- ?
- *What will be the new time complexity then?*

LCR Communication Complexity

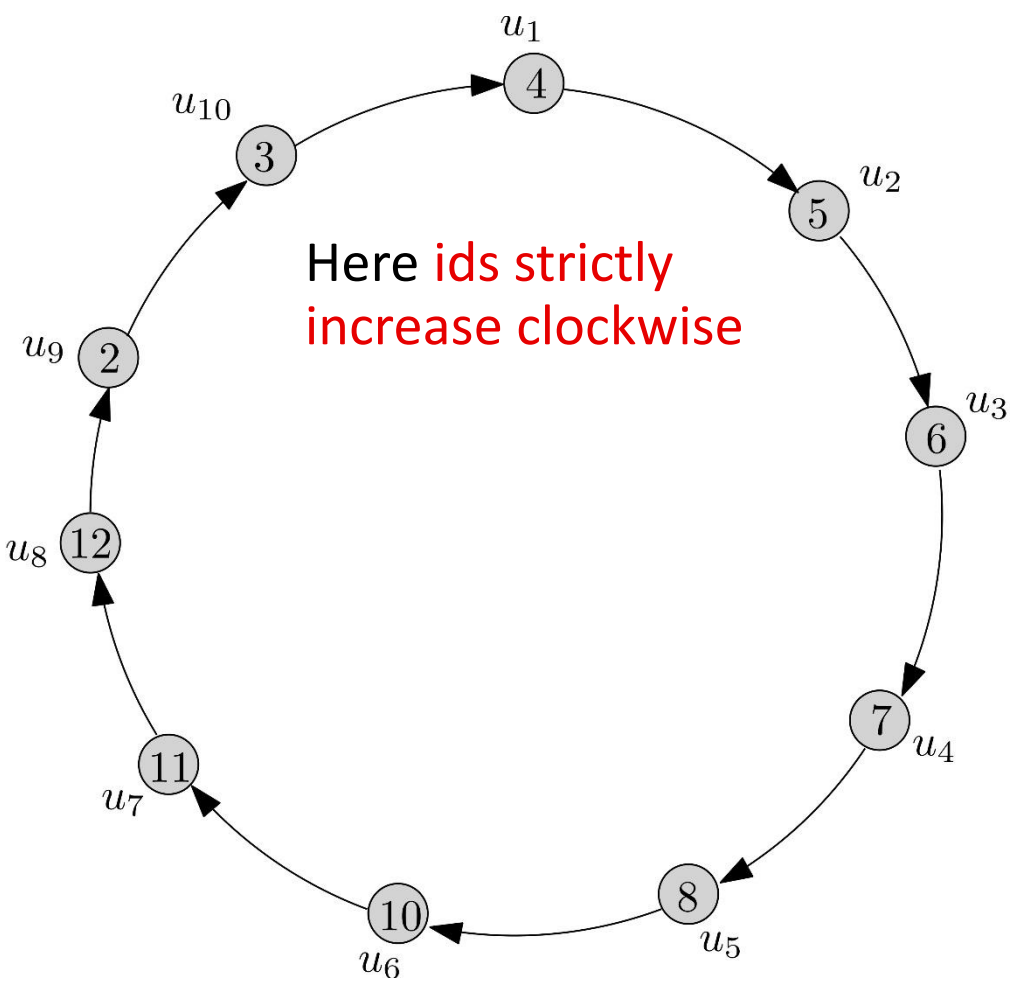
- $O(n^2)$ messages (in total) in the worst case
- The worst case in not the same for every problem and algorithm
  - *We have to think and find out the worst case of each specific algorithm*
- But what parameters it depends on?
  - i.e., what affects the #messages to be transmitted?
  1. Size n of the ring
  2. How ids are initially assigned

LCR Communication Complexity

- *Is this the worst case?*



LCR Communication Complexity



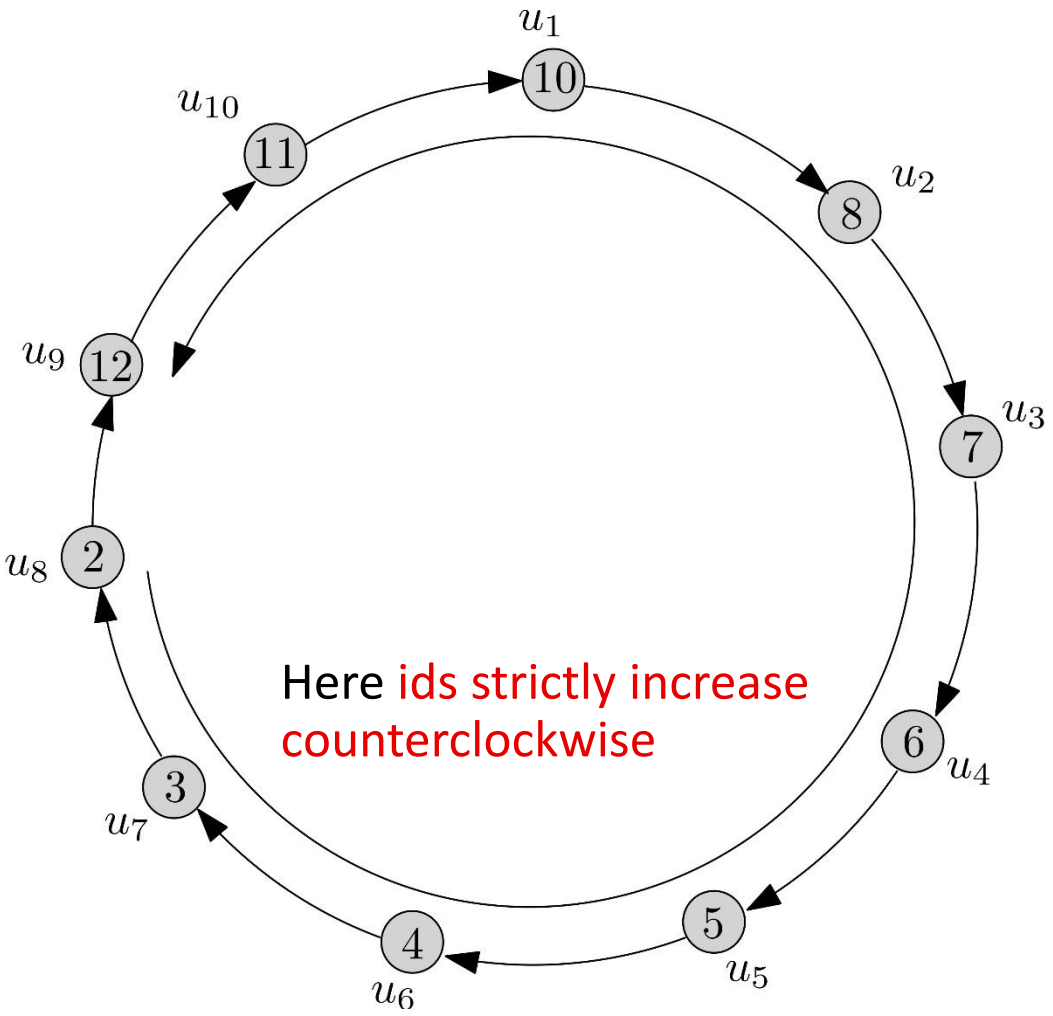
- Cannot tell yet if this is the worst case
- But we can analyse it
- Every $id_i < id_{max}$ has a greater id to its right initially
  - So, transmitted once and then discarded
  - So, *n* - 1 ids are only transmitted once
- id_{max} is always transmitted *n* times (complete turn)

In total:

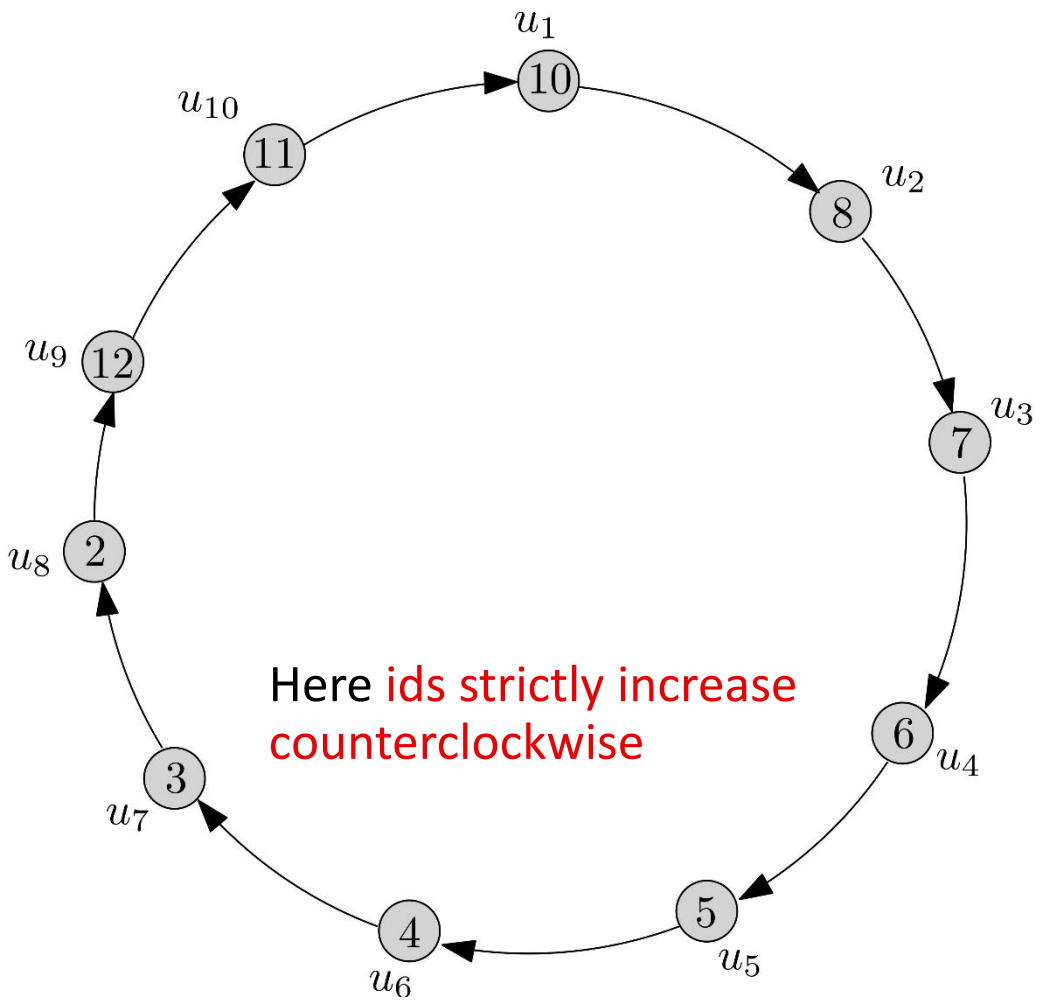
#messages = *n* - 1 + *n* = *2n - 1*

LCR Communication Complexity

- *Is this the worst case?*



LCR Communication Complexity



- Yes it is!
- Let's analyse it
- id_{max} is always transmitted n times (complete turn)
- Every $id_i < id_{max}$ will manage to reach u_{max}
  - So, one id will travel n - 1 edges, the next n - 2, … the next 2, the next 1
  - This gives:

    $1 + 2 + \ldots + (n - 2) + (n - 1)$

In total:

#messages = $1 + 2 + \ldots + n$

$= n(n + 1)/2$

$= \Theta(n^2)$

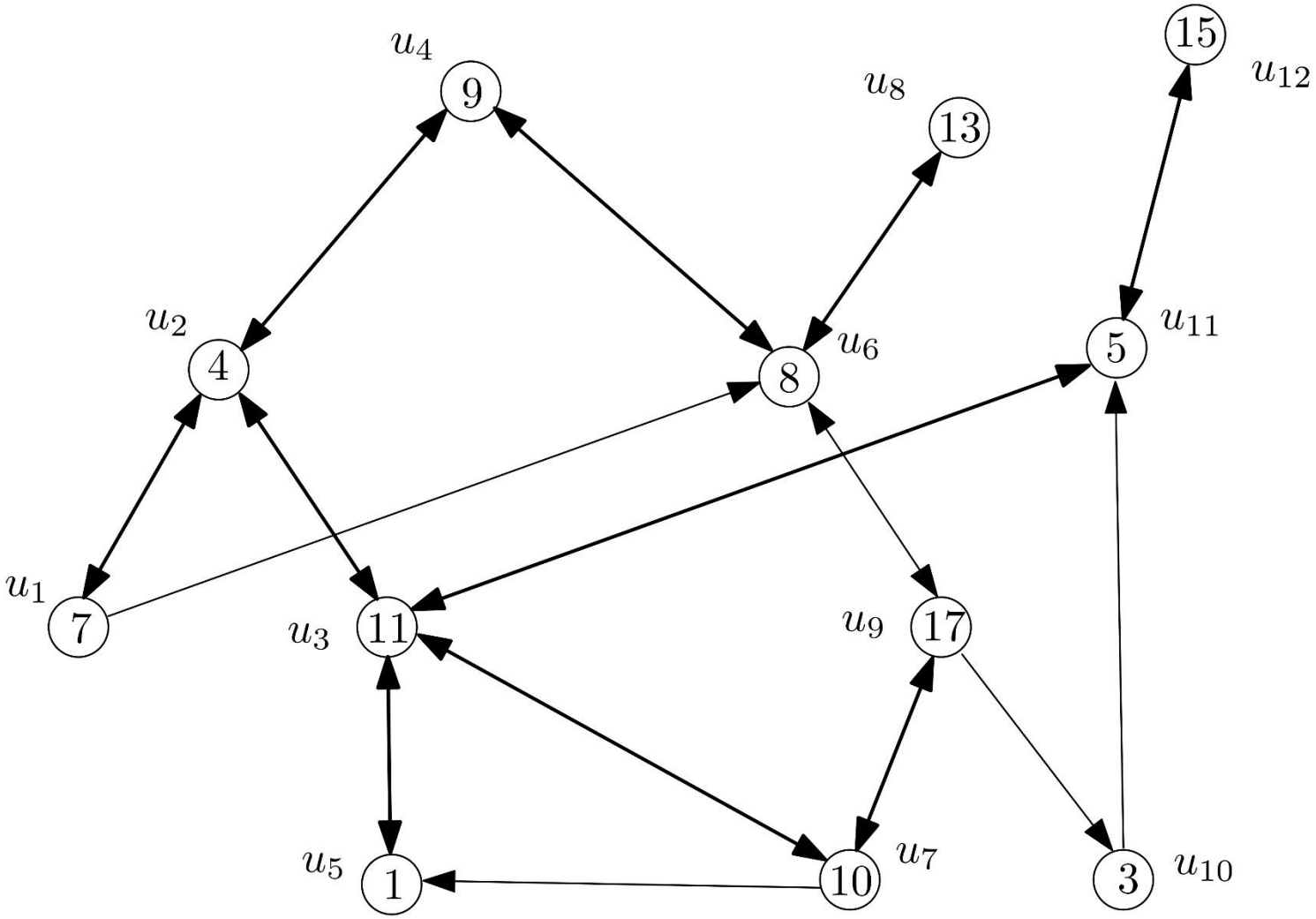
Beyond Rings
Leader Election in General Networks

Leader Election in General Networks

- *Elect a unique leader processor from among all the processors in the distributed system*
- Now the network can be any strongly connected directed network
  - Strongly connected: For every processors u, v there is a path from u to v and a path from v to u
  - e.g., the directed ring is just a special case
  - *Why can't we use LCR in this case?*
- Processors have unique ids

Leader Election in General Networks

- A strongly connected directed network



A Simple Algorithm based on Flooding

- Processors have unique ids, do not know *n* in advance, but do know the diameter *D* of the network
  - Diameter:
    - the distance between two nodes is given by the shortest path between them
    - Then the diameter of the network is determined by the pair of nodes at maximum distance (and is equal to that distance)
    - In other words, it is the maximum shortest path in the network
- FloodMax algorithm: solves the problem
- Uses transmission, comparison, and storage of ids
- Main idea: Flood the maximum id
  - LCR also does something like this but does not require knowledge of *D* and its termination condition works only for rings

FloodMax: Informal description

- All processors know the diameter *D* and their own id in advance
- All processors remember the greatest id that they have "heard" so far (initially their own)
- In every round all processors send the greatest known to all their out-neighbours
- After *D* rounds compare the largest heard to your own
  - if *greatest heard* = *own id*, declare yourself the leader
  - otherwise, declare yourself non-leader
- Intuitively:
  - The maximum id will manage to reach the whole network
  - So everyone non-maximum will know that there is a greater id and u_{max} can never receive a larger id

FloodMax: Pseudocode

Algorithm FloodMax

State of processor u_i :

- $myID_i$: holds the processor's unique id
- $maxID_i$: holds the greatest id "heard" so far
- $status_i \in$ {"unknown", "leader", "non-leader"}: indicates whether u_i has been elected ("leader"), not elected ("non-leader") or doesn' know yet ("unknown")

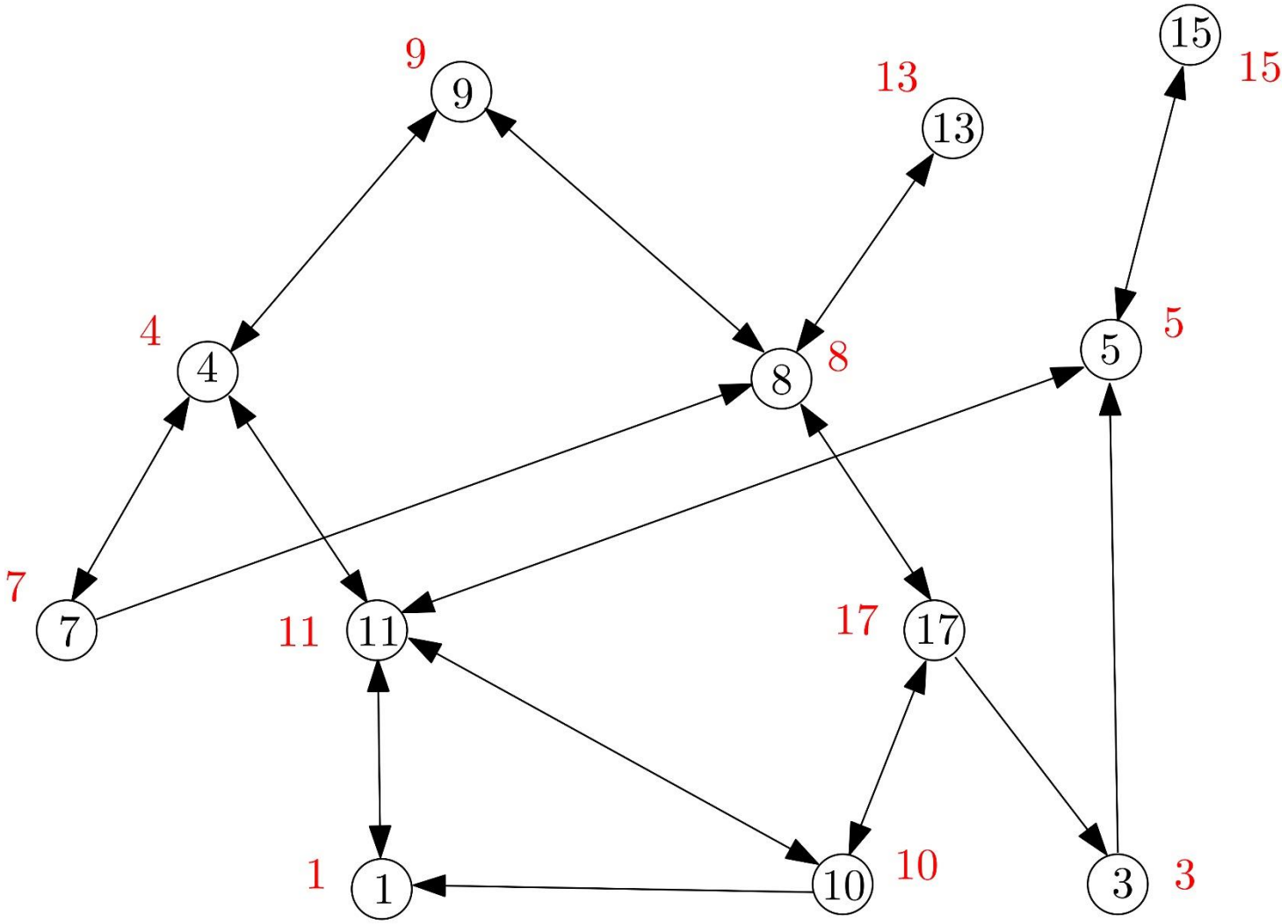
FloodMax: Pseudocode

Algorithm FloodMax
Code for processor u_i , i ∈ {1, 2,… , n}:

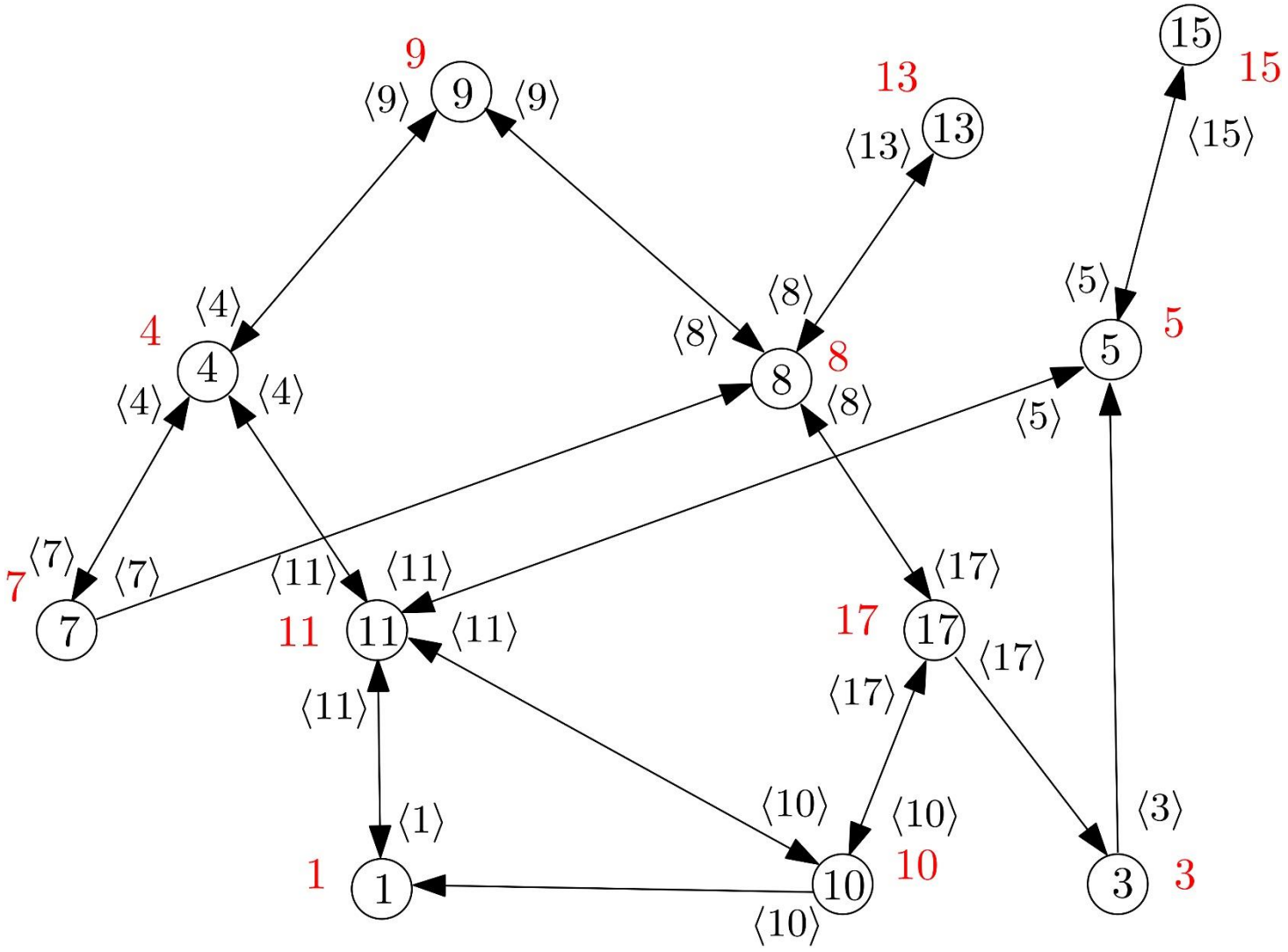
Initially:
  u_i knows its own unique id stored in $myID_i$
  $maxID_i := myID_i$
  $status_i$:= "*unknown*"
  Also has access to the current round and knows the diameter *D*

```
if round = 1 then
   send ⟨maxID_i⟩ to all out-neighbours
else
   upon receiving ⟨inIDs⟩ from in-neighbours        // one or more ids arriving from neighbours
   maxID_i := max({maxID_i} ∪ inIDs)                // remember only the maximum "heard" so far
   if round ≤ D then    // 1 < round ≤ D
      send ⟨maxID_i⟩ to all out-neighbours
   else    // round = D + 1
      if maxID_i = myID_i then        // if equal to your own, no greater id exists in the network
         status_i := "leader"         // therefore, elect yourself a leader
      else                            // greater than own
         status_i := "non-leader"     // therefore, declare yourself a non-leader
// observe that in the end all processors know the id of the elected leader, stored in their maxID_i
// variable
```

Example Execution

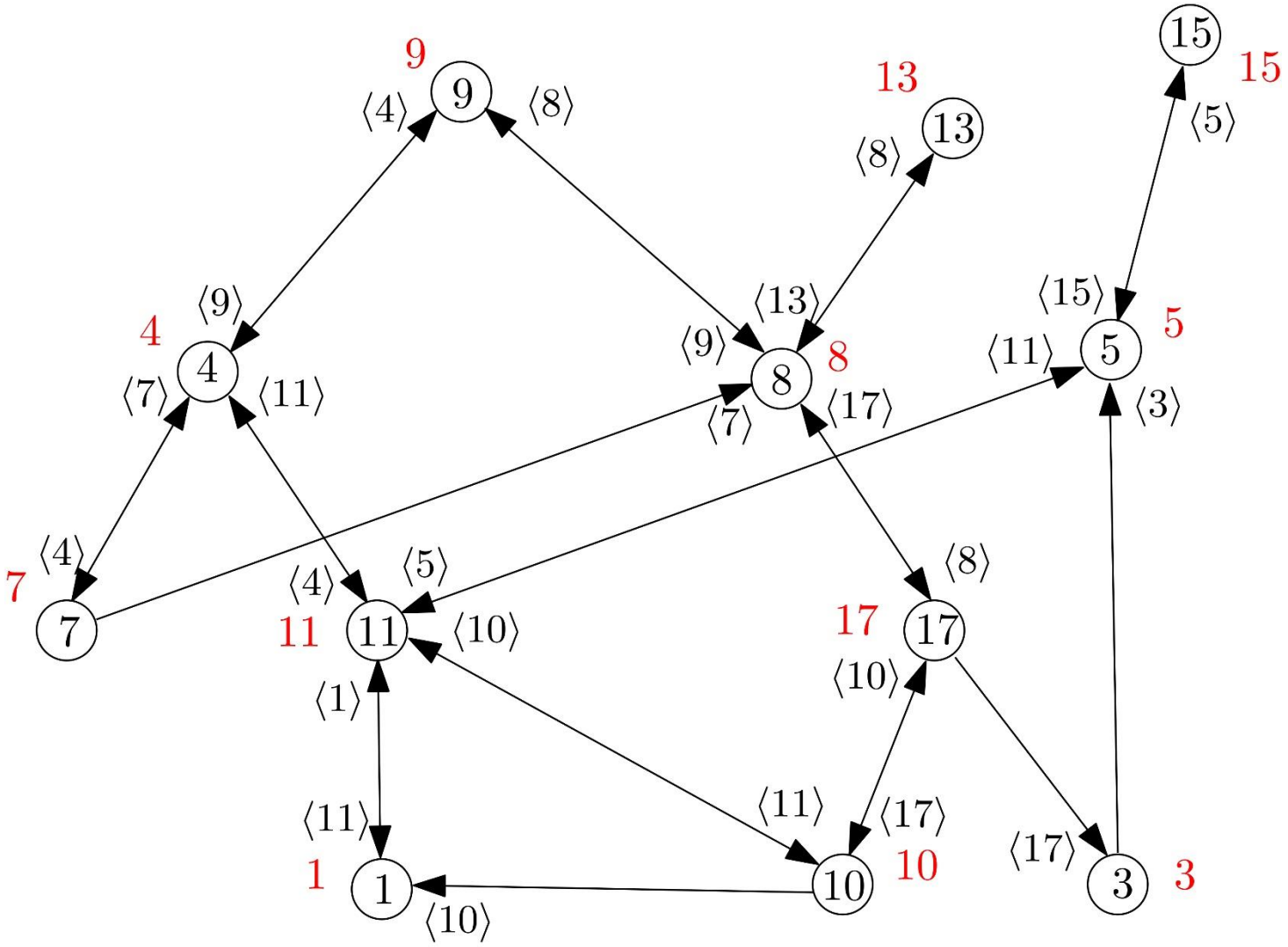


Example Execution



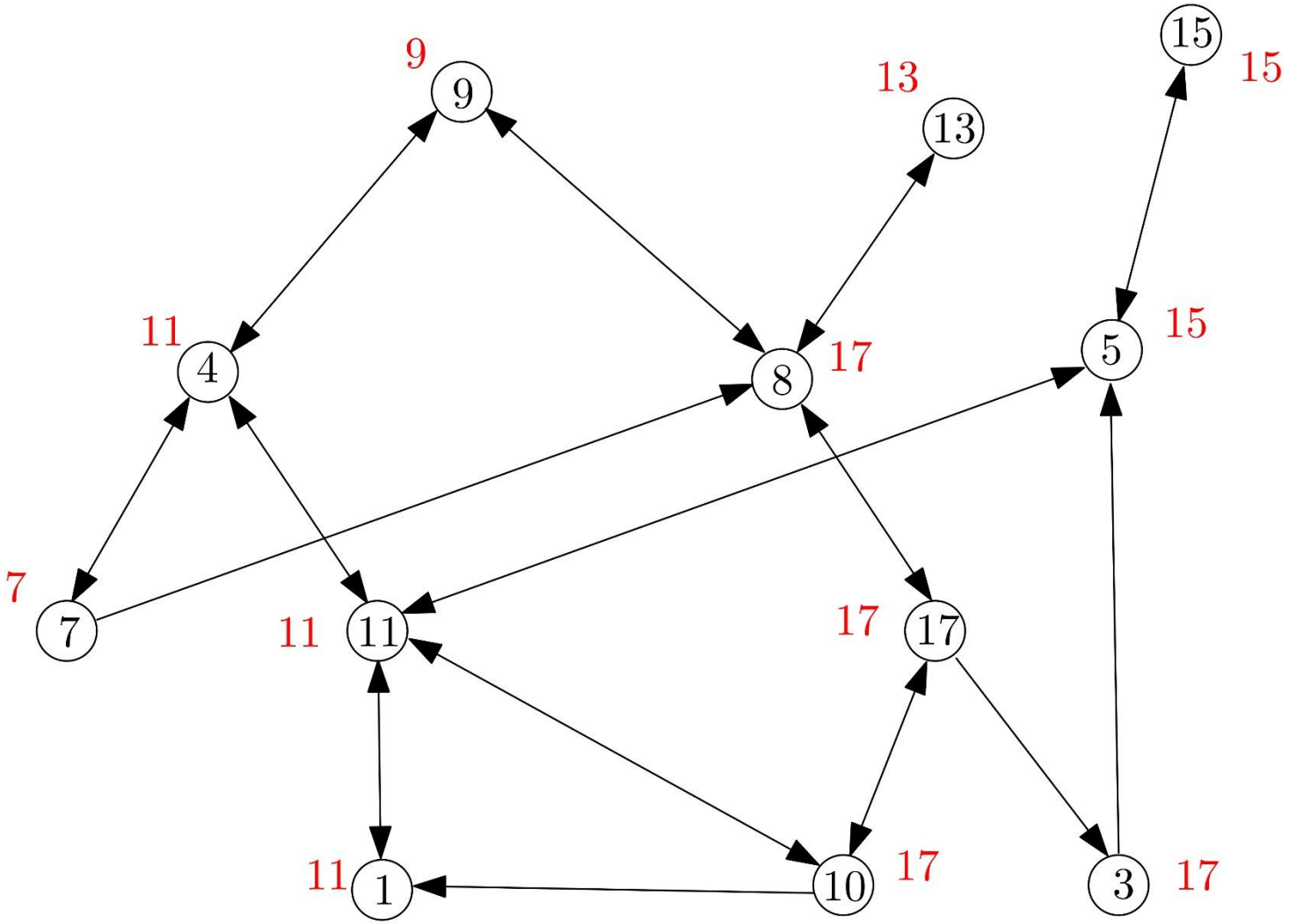
$round = 1$

Example Execution



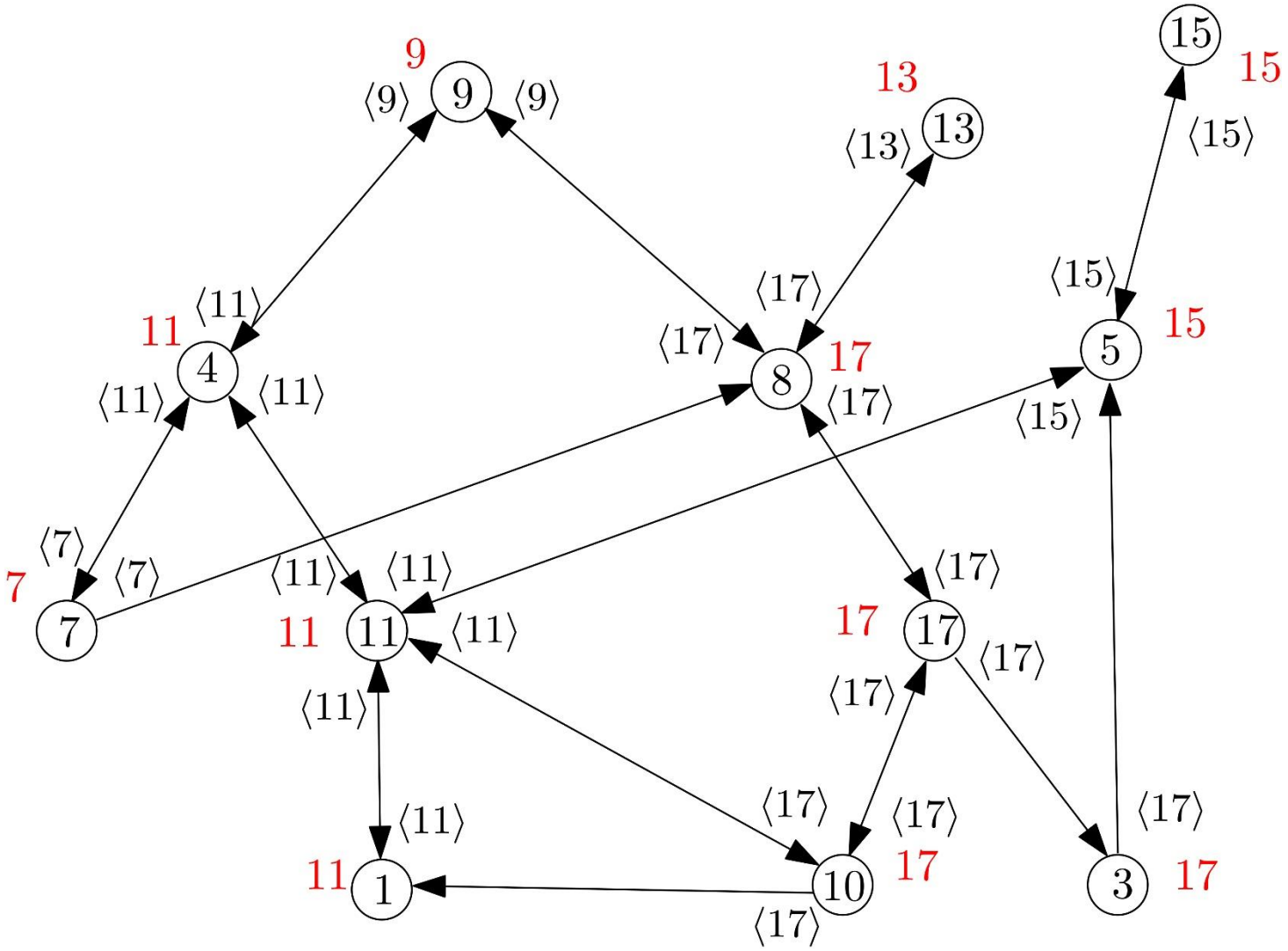
$round = 1$

Example Execution



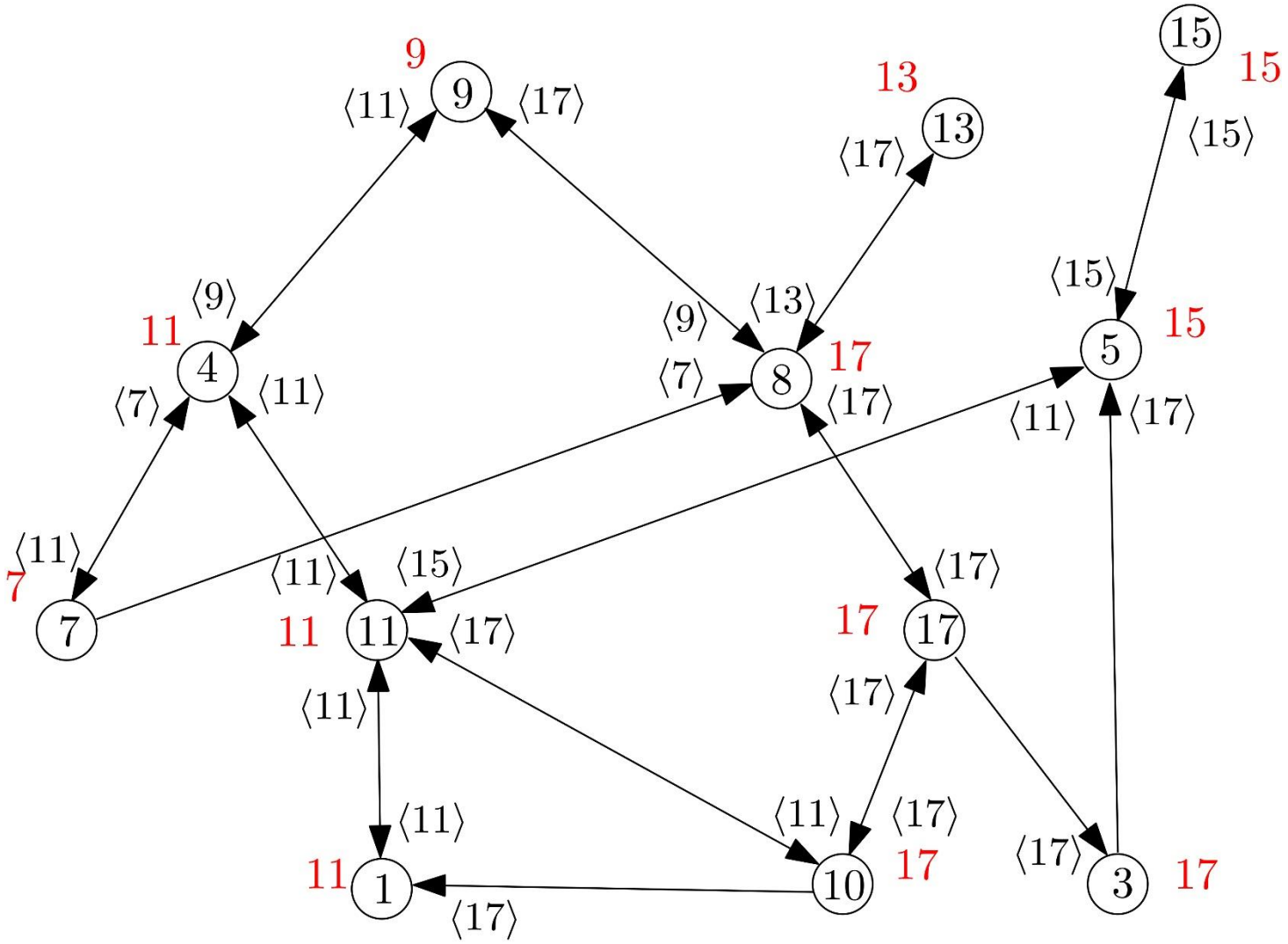
$round = 2$

Example Execution



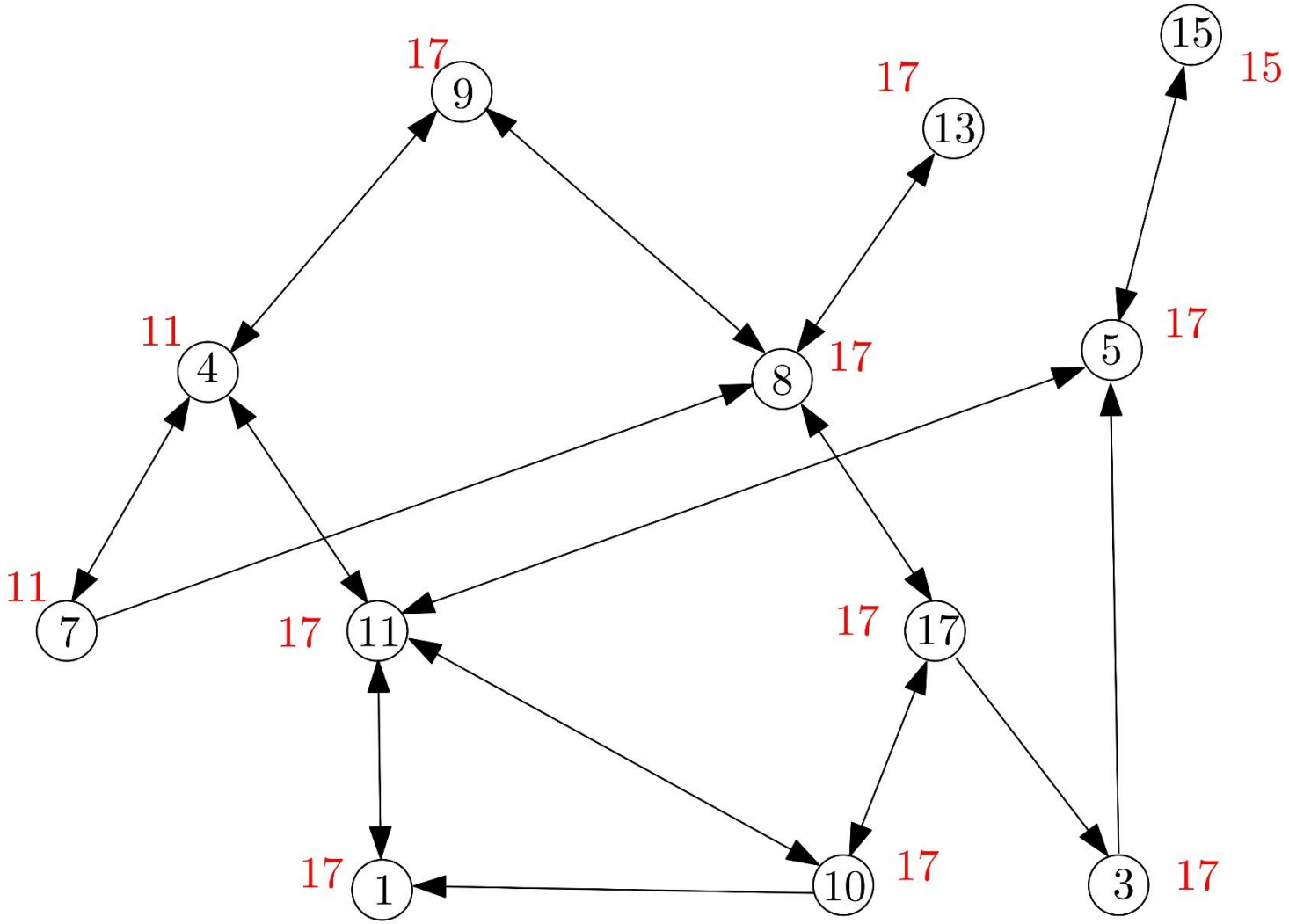
$round = 2$

Example Execution



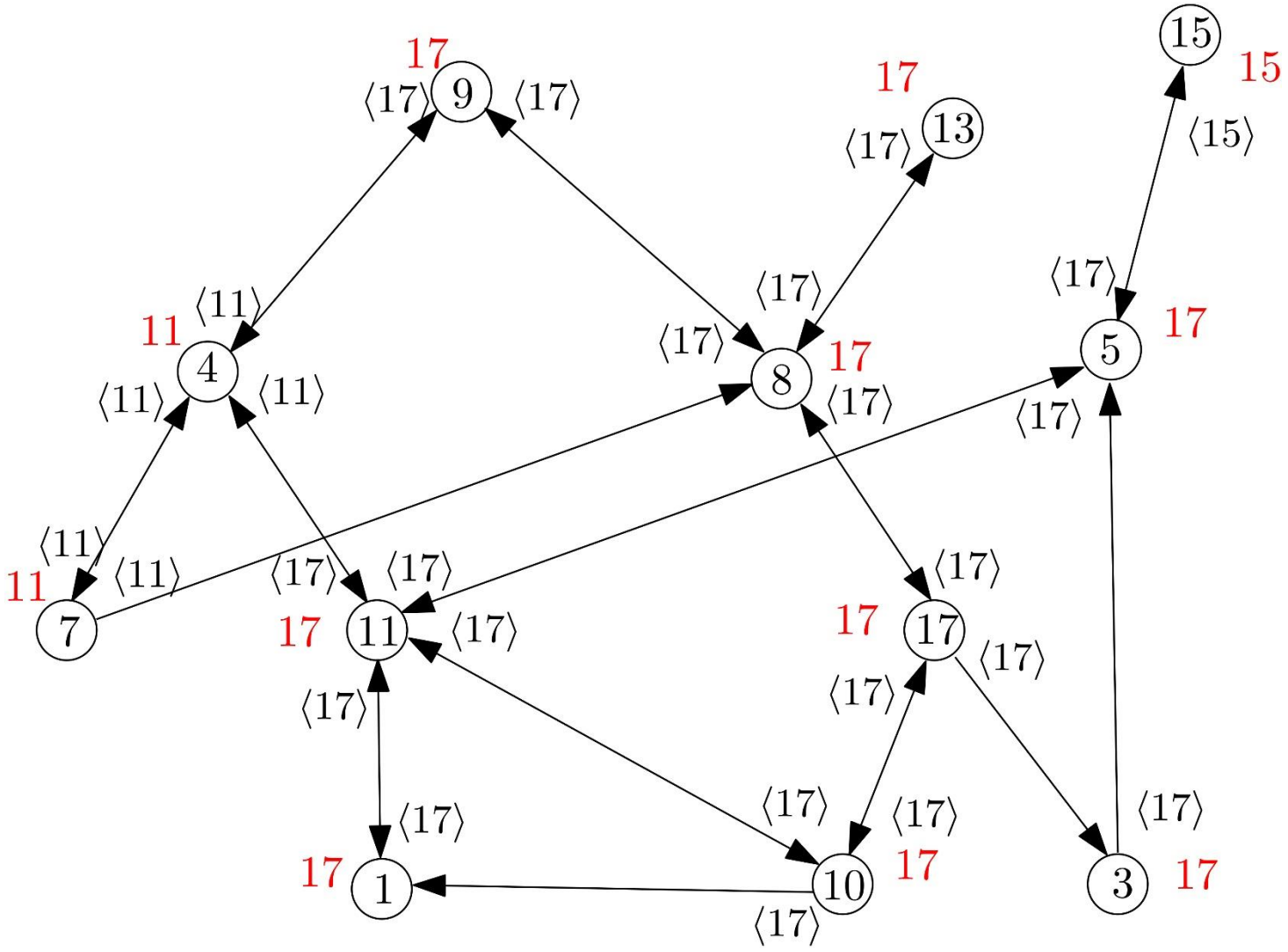
$round = 2$

Example Execution



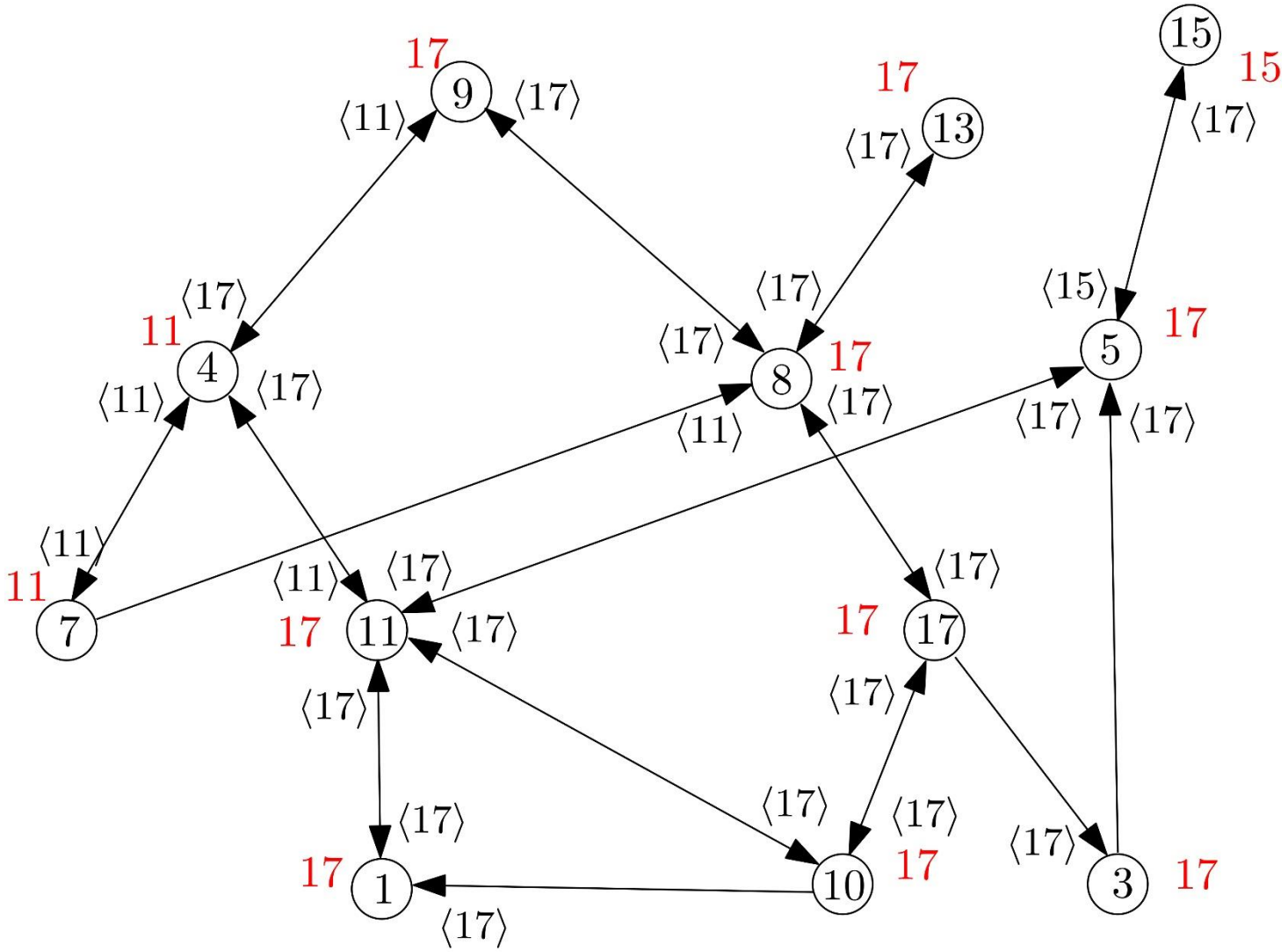
$round = 3$

Example Execution



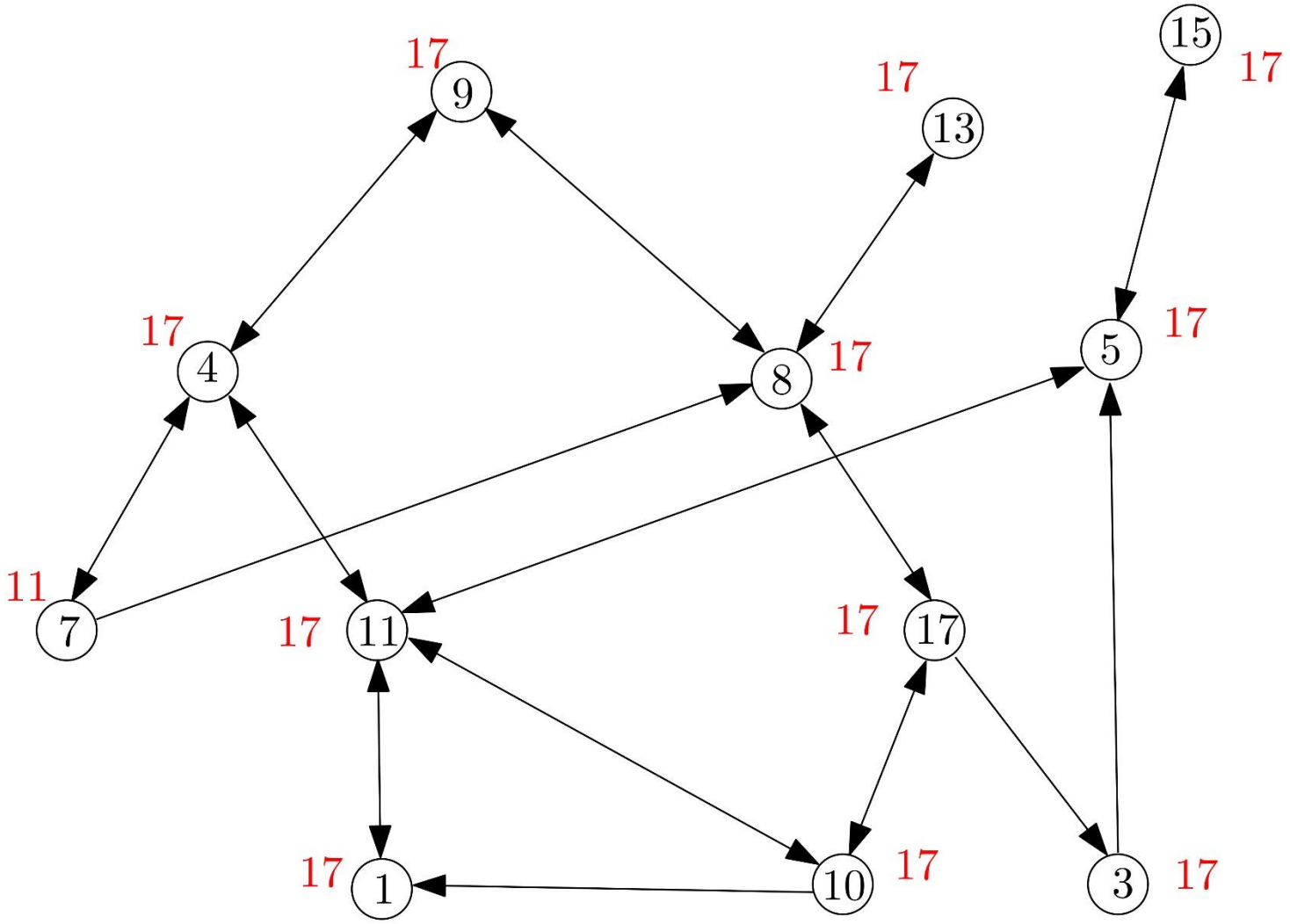
$round = 3$

Example Execution



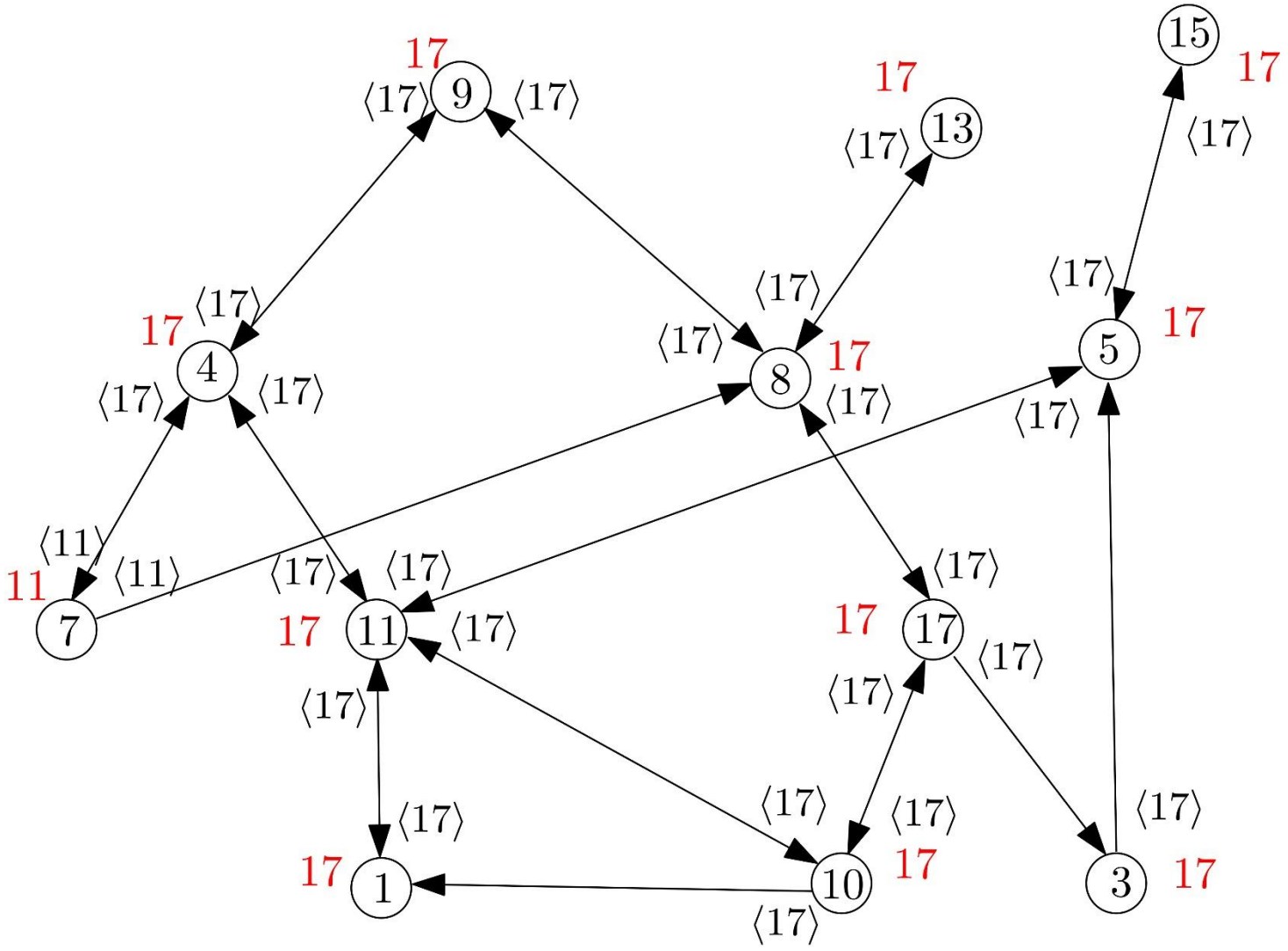
$round = 3$

Example Execution



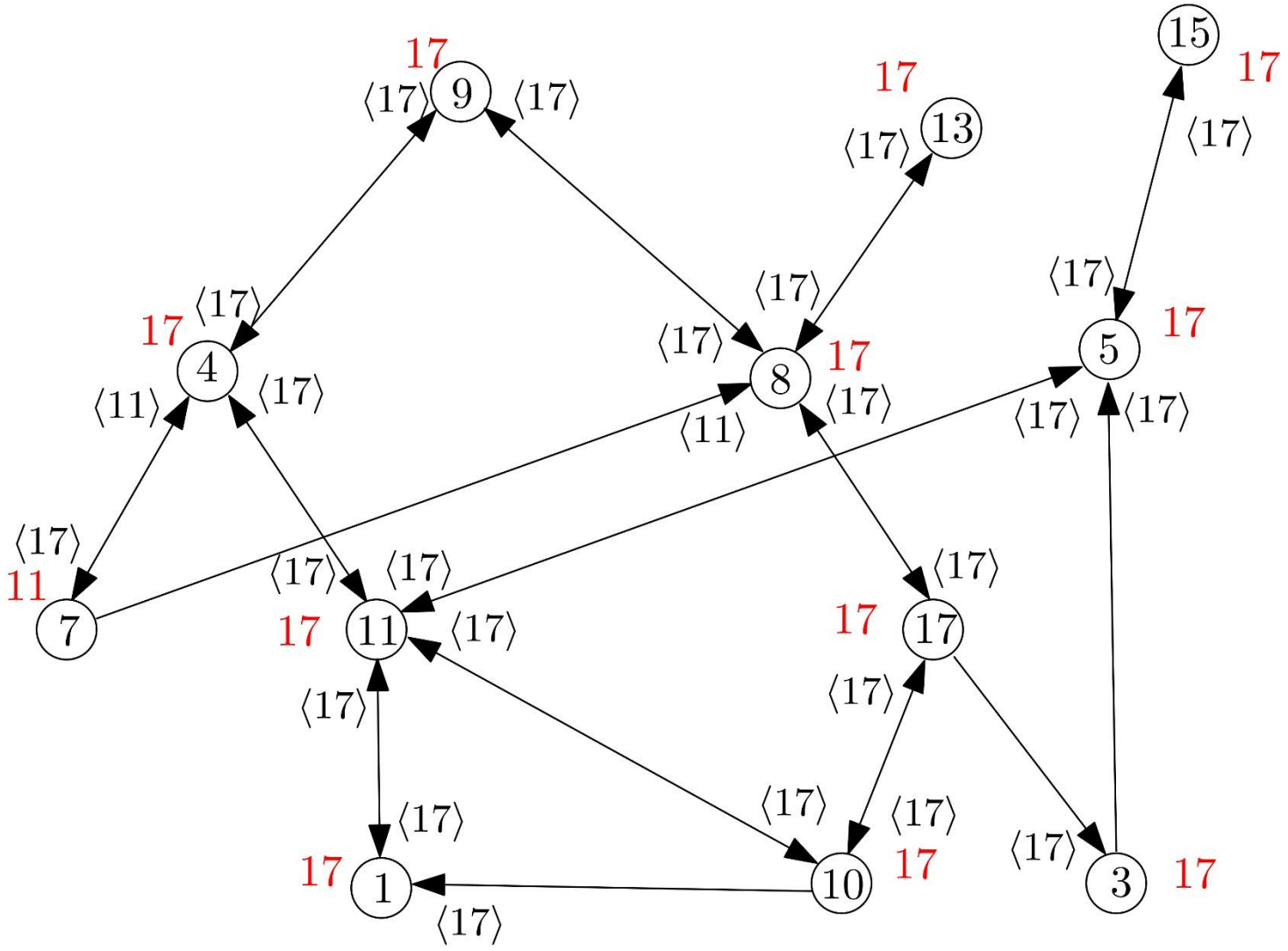
$round = 4$

Example Execution



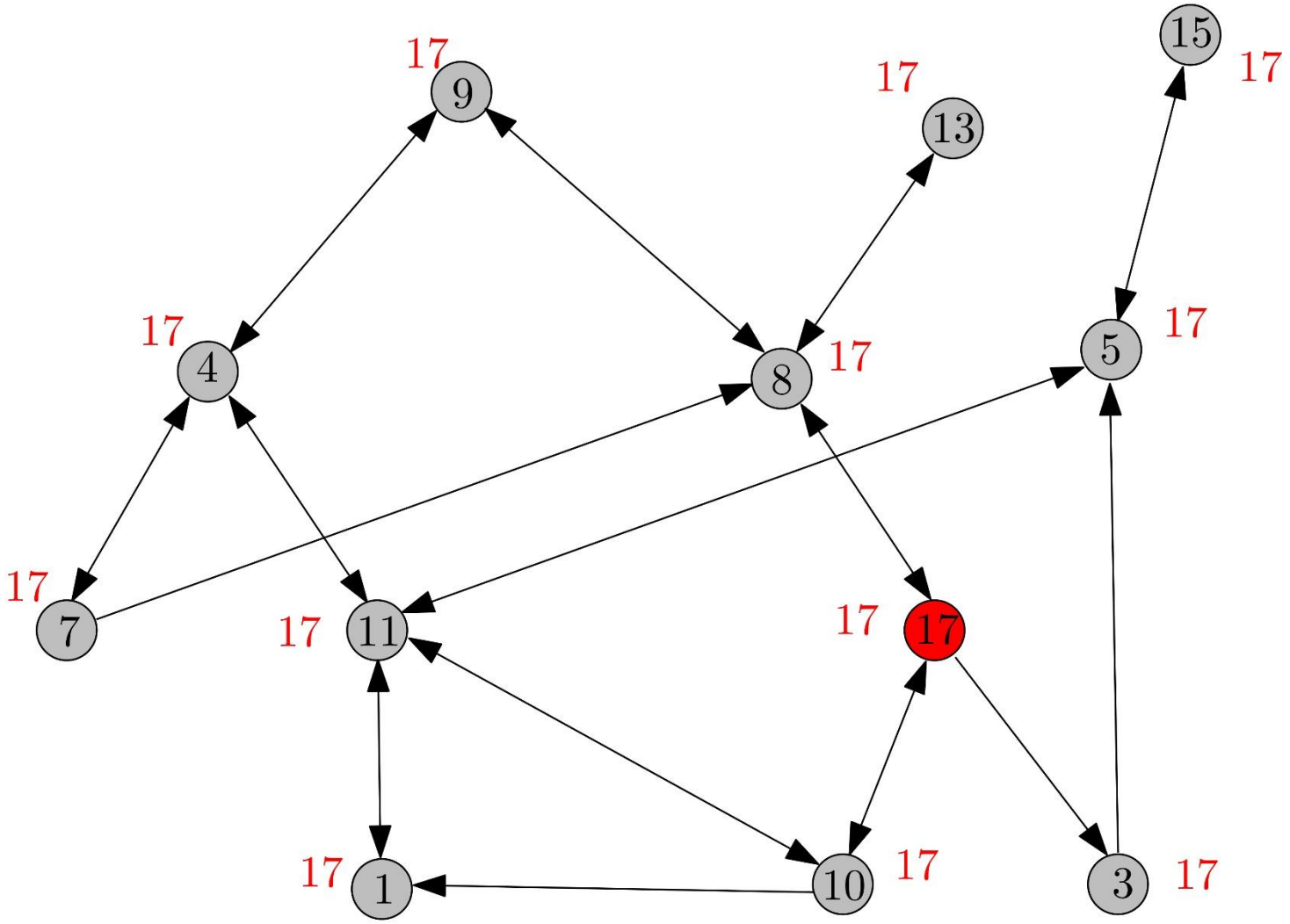
$round = 4$

Example Execution



$round = 4$

Example Execution



$round = 5$

Correctness and Complexity

- Correctness:
  - exactly one processor is elected in the last round
- Time complexity:
  - $D + 1$ rounds (or D depending on the round model)
- Communication complexity:
  - size of messages: encoding in bits of the maximum id
  - $D \cdot m$ messages always
    - m is the number of directed links in the network
- *Can you see why?*

Think of these and we shall prove them in class

Summary

- Leader election is crucial for distributed systems
  - breaks symmetry
  - allows for coordination
- If all processors are initially identical then
  - impossible to elect a leader even in very simple networks
  - e.g., a ring
- Adding unique ids breaks this inconvenient initial symmetry
- The LCR algorithm elects a leader in any ring network
  - simple conceptually, assumes unique ids
  - n rounds (or $2n$ for all to terminate), $O(n^2)$ messages
- The FloodMax algorithm elects a leader in any strongly connected network
  - like a generalisation of LCR
  - simple, assumes unique ids and knowledge of the diameter D
  - D rounds, $D \cdot m$ messages