# Foundations of Computer Science Comp109

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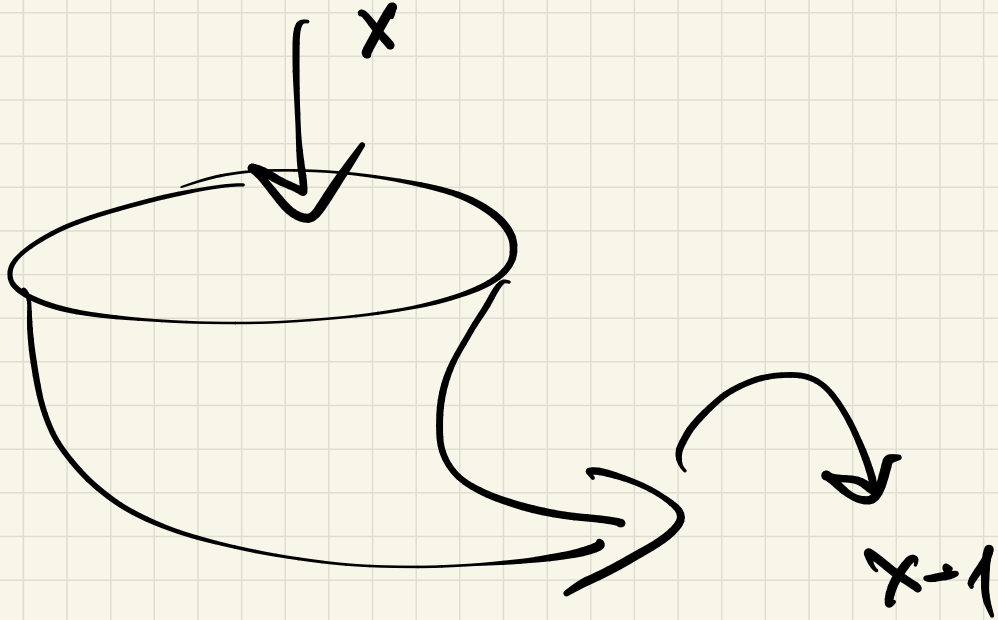
# Part 4. Function

Comp109 Foundations of Computer Science

# Reading

- Discrete Mathematics with Applications S. Epp, Chapter 7.
- **Discrete Mathematics and Its Applications** K. Rosen, Section 2.3.

# Contents

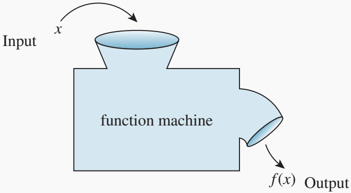



$y = x+1$

$y = x^2$

$f(x)$

# Functions




Examples:

- $y = x^2$
- $y = \sin(x)$
- first letter of your name

# Functions/methods on programming

**Java**

```java
public int f(int x) {
    return x+5;
}
```

**C/C++**

```c
int f(int x) {
    return x+5;
}
```

**Python**

```python
def f(int x):
    return x+5
```

# Definition

A **function** from a set $A$ to a set $B$ is an assignment of exactly one element of $B$ to each element of $A$.

We write $f(a) = b$ if $b$ is the unique element of $B$ assigned by the function $f$ to the element of $a$.

If $f$ is a function from $A$ to $B$ we write $f\colon A \to B$.

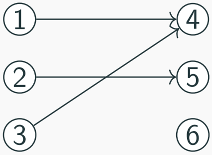



**Figure 1:** A function $f\colon \{1, 2, 3\} \to \{4, 5, 6\}$

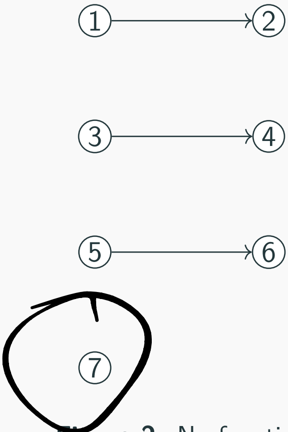



**Figure 2:** No function

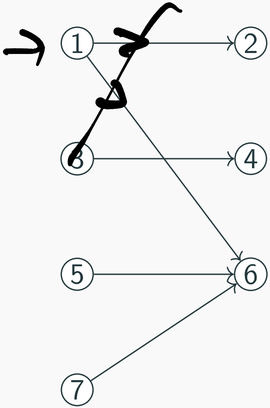



**Figure 3:** No function

# Domain, codomain, and range


Suppose $f\colon A \to B$.

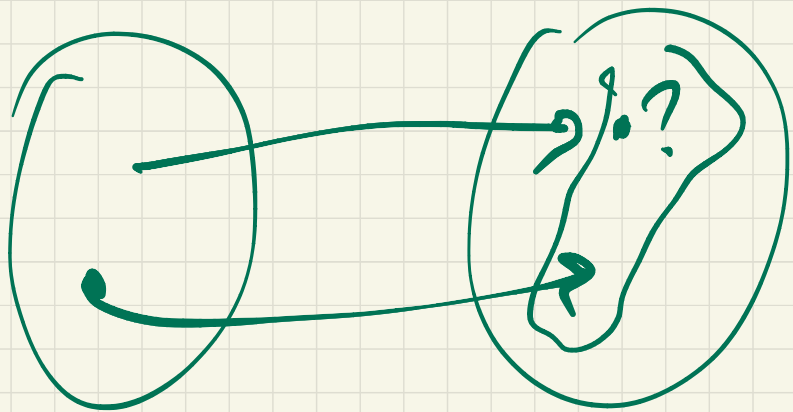
- $A$ is called the *domain* of $f$. $B$ is called the *codomain* of $f$.
- The *range* $f(A)$ of $f$ is

$$f(A) = \{f(x) \mid x \in A\}.$$

$$f(x) = x^2$$

$$\underbrace{\text{range}(f) = \mathbb{R}_{\geq 0}}$$

$$f: \underbrace{\mathbb{R}} \to \underbrace{\mathbb{R}}$$



# Codomain vs range

range(f)        f(A)

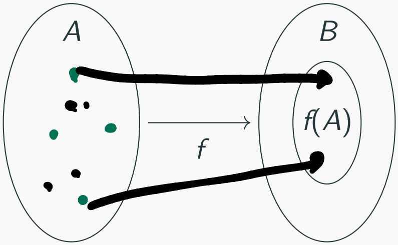
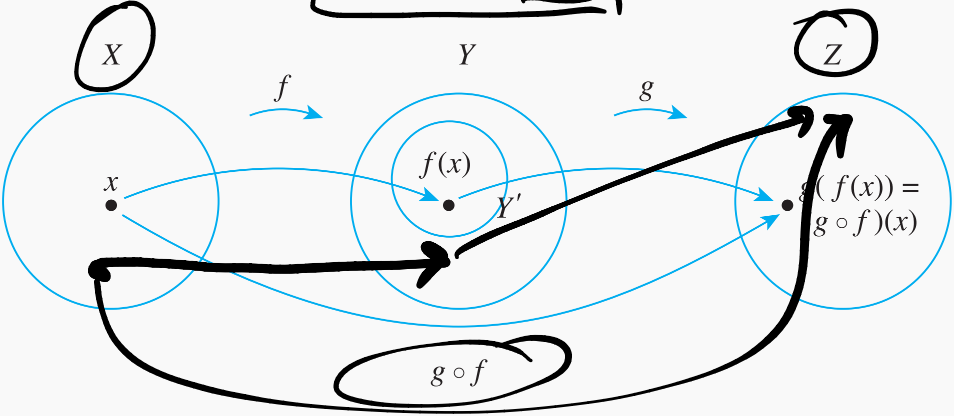



**Figure 4:** the range of $f$

# Composition of functions

If $f\colon X \to Y$ and $g\colon Y \to Z$ are functions, then their **composition** $g \circ f$ is a function from $X$ to $Z$ given by

$$\boxed{(g \circ f)(x) = g(f(x)).}$$



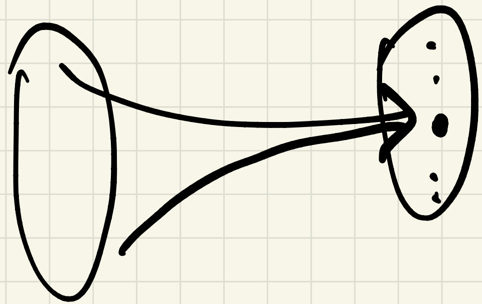
$f, g, h$

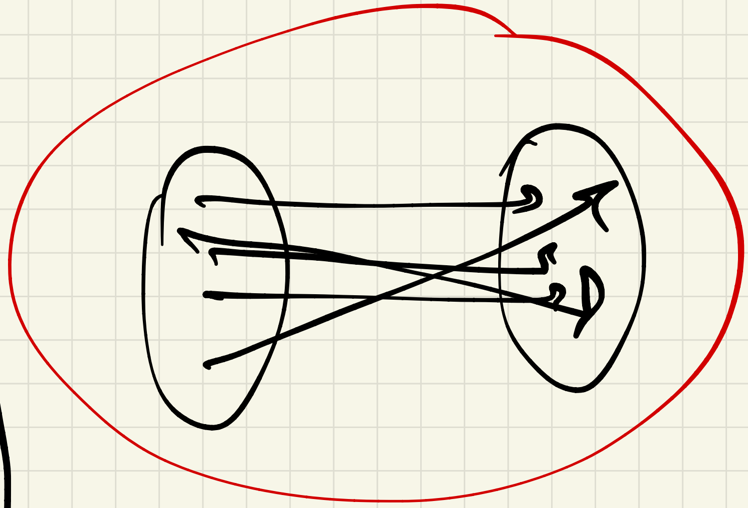
$$(f \circ g)(x) = f(g(x))$$

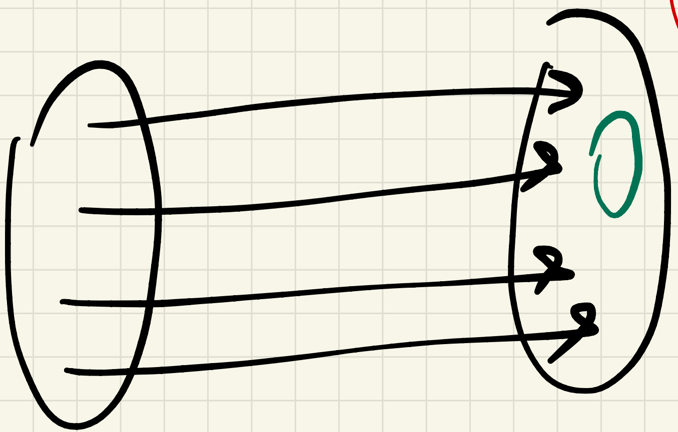
# Example

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = 4x + 3$.

- $g \circ f(x) =$ $g(f(x)) = g(x^2) = 4x^2 + 3$
- $f \circ g(x) =$ $f(g(x)) = f(4x+3) = (4x+3)^2$
- $f \circ f(x) =$ $x^4$
- $g \circ g(x) =$ $16x + 15$





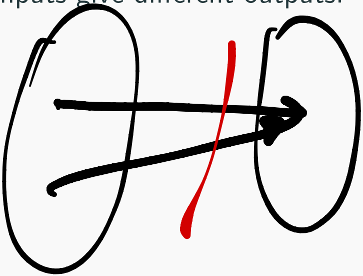


# Injective (one-to-one) functions

**Definition** Let $f\colon A \to B$ be a function. We call $f$ an *injective* (or *one-to-one*) function if

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \text{ for all } a_1, a_2 \in A.$$

This is logically equivalent to $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$ and so injective functions never repeat values. In other words, different inputs give different outputs.



*Examples*

$f\colon \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not injective.

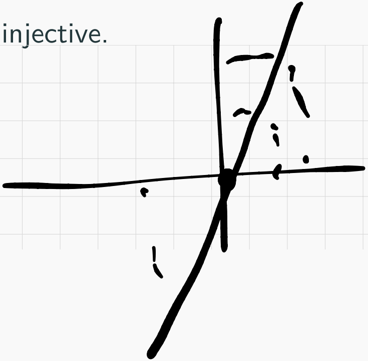
$h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is injective.

# Prove that $f(x)$ is not injective and $h(x) = 2x$ is injective

- $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not injective.

$$f(2) = f(-2)$$

- $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is injective.

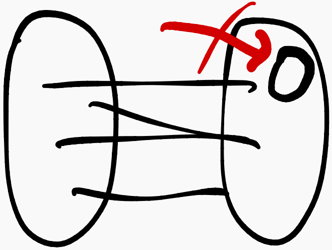


# More examples

- $first\_letter : People \to Char$
- $ID : People \to \mathbb{N}$

# Surjective (or onto) functions

**Definition** $f: A \to B$ is *surjective* (or onto) if the range of $f$ coincides with the codomain of $f$. This means that for every $b \in B$ there exists $a \in A$ with $b = f(a)$.



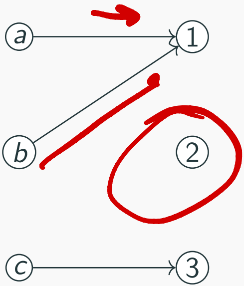
*Examples*

$f: \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is not surjective.

$h: \mathbb{Z} \to \mathbb{Z}$ given by $h(x) = 2x$ is not surjective.

$h': \mathbb{Q} \to \mathbb{Q}$ given by $h'(x) = 2x$ is surjective.

# Classify $f\colon \{a, b, c\} \to \{1, 2, 3\}$ given by

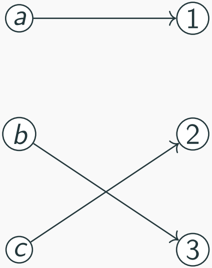



Function

Non-injective

Non-surjective

# Classify $g : \{a, b, c\} \to \{1, 2, 3\}$ given by

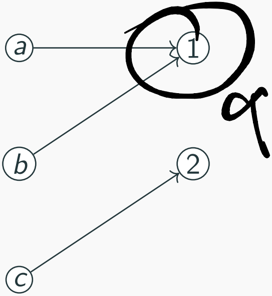



Function

Injective

Surjective

## Classify $h : \{a, b, c\} \to \{1, 2\}$ given by




Function

Not injective

Surjective

# Classify $h' : \{a, b, c\} \to \{1, 2, 3\}$ given by