Mathematical induction

Mathematical induction

- Mathematical induction is one of the more *recently* developed techniques of proof in the history of mathematics.
- It is used to check conjectures about the outcomes of processes that occur repeatedly and according to definite patterns.
- In general, mathematical induction is a method for proving that a property defined for integers n is true for all values of n that are greater than or equal to some initial integer

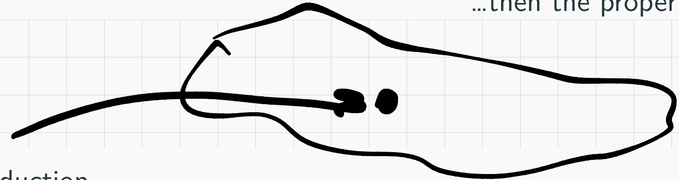
Generic particular vs induction for universal statements

- Generalisation from the generic particular:

  "Suppose that x is a particular but arbitrarily chosen ..."

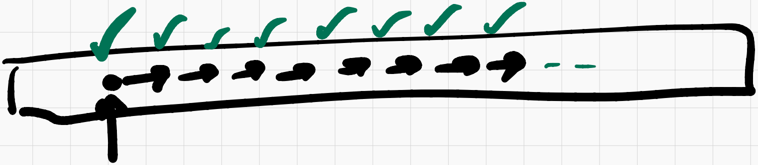
  ..."property holds for this x"...

  "...then the property holds for all x"



- Induction

  Some kind of a process that goes over the elements of a set



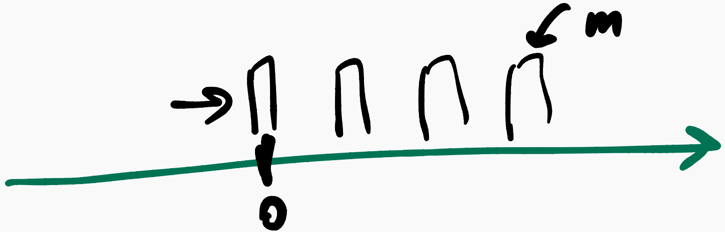
Example: Domino effect

One domino for each natural number, arranged in order.

- I will push domino 0 (the one at the front of the picture) towards the others.
- For every natural number m, if the m'th domino falls, then the $(m+1)$st domino will fall.

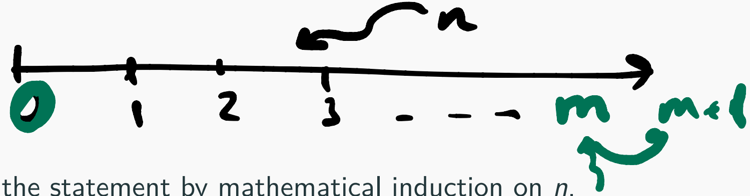
Conclude: All of the Dominoes will fall.

Proving by induction that a property holds for every natural number n



- **Prove** that the property holds for some initial value (e.g. $n = 0$).
- **Prove** that if the property holds for $n = m$ (for any natural number m) then it holds for $n = m + 1$.

A proof of a property by induction looks like this



We prove the statement by mathematical induction on n.

Base Case: Show that the property holds for some initial value (e.g. $n = 0$).

Inductive Step: Assume that the property holds for $n = m$. Show that it holds for $n = m + 1$.

Conclusion: You can now conclude that the property holds for every natural number n.

Carl Friedrich Gauss (1777-1855)

$$1 + \cdots + 100 = 5050$$

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 \ldots & 49 & 50 \\ 100 & 99 & \ldots & & 52 & 51 \\ \hline \end{array}$$

$$\underbrace{101 \quad 101 \quad \cdots \quad 101 \quad 101}_{50}$$

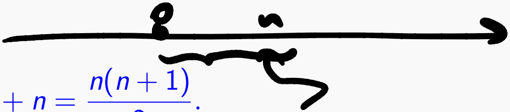
$$50 \times 101 = 5050$$

$$\boxed{1+2+\ldots+n=\frac{n(n+1)}{2}}$$

$$n=100 \Rightarrow \frac{100\times 101}{2}=50\times 101$$

Example: Proof by induction

For every natural number n,



$$0 + 1 + \cdots + n = \frac{n(n+1)}{2}.$$

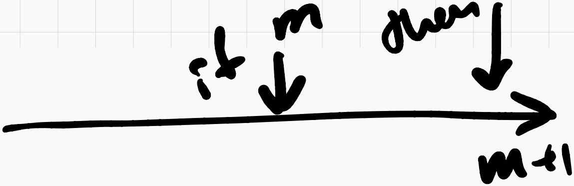
Base Case:

$n = 0$

$LHS = 0$

$RHS = \frac{0 \cdot 1}{2} = 0$

Inductive Step:



Suppose that the property holds for m Then it holds for m+1

Proof continued

Expand this:

Suppose that $$1+2+\dots+m = \frac{m(m+1)}{2}$$

Then we need to prove that

$$\underbrace{1+2+\dots+m+(m+1)}_{LHS} = \frac{(m+1)((m+1)+1)}{2}$$

$$LHS = \underbrace{1+2+\dots+m}+(m+1) = \frac{m(m+1)}{2} + (m+1)$$

$$= \frac{m(m+1)+2(m+1)}{2} = \frac{(m+1)(m+2)}{2}$$

Induction step Suppose that the property holds for $n = m$ prove that it holds for $n = m+1$

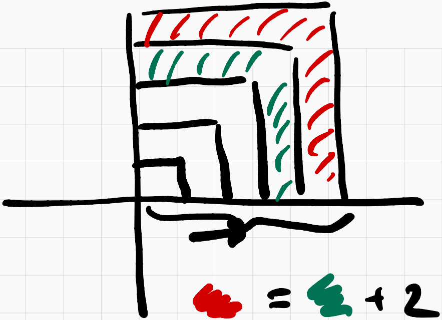
$1 + 3 + \cdots + (2n+1) = (n+1)^2$ **for every natural number** n

$n=1$

$n=2$

$n=3$

$\vdots$



Guess the answer and prove that it is correct.

Prove that $1+3+5+\ldots(2n+1) = (n+1)^2$

Base case $n=1$

$LHS = 1+(2\cdot 1+1) = 4$

$RHS = (1+1)^2 = 4$

$(a+b)^2 = a^2+2ab+b^2$

$a = m+1,\ b=1$

Induction step

Suppose that $1+3+\ldots+(2m+1) = (m+1)^2$

Consider the sum $\underbrace{1+3+\ldots+(2m+1)}+(2(m+1)+1)$

$(m+1)^2+2(m+1)+1 = ((m+1)+1)^2$ ✓

Proof continued

For all integers $n \geq 8$, $n¢$ can be obtained using $3¢$ and $5¢$ coins

Base Case: For $n = 8$

Inductive Step: Suppose that $m¢$ can be obtained using $3¢$ and $5¢$ coins for any $m \geq 8$. We must show that $(m+1)¢$ can be obtained using $3¢$ and $5¢$ coins.

Consider cases

- There is a $5¢$ coin among those used to make up the $m¢$.

- There is no $5¢$ coin among those used to make up the $m¢$.

Proving properties of programs

Using induction to show that a program is correct

What does the following program do?

```
mylist = [1, 2, 6, 3, 5, 6]
i = 0
M = mylist[0]
while i < len(mylist):
    M = max(M, mylist[i])
    i = i+1
print(M)
```

1 2 6 3 5 6

0 i 5

M=1

i	M
0	1
1	2
2	6
3	6
4	6

i	M
5	6
6	

Using induction to show that a program is correct

```
mylist = [1, 2, 6, 3, 5, 6]
i = 0
M = mylist[0]
while i < len(mylist):
    M = max(M, mylist[i])
    i = i+1
print(M)
```

Property: After the statement M = max(M , mylist[i]) gets executed, the value of M is max(mylist[0],...,mylist[i]).