COMP108
Data Structures and Algorithms

Graphs (Part I)

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2022-23

Outline

Graphs

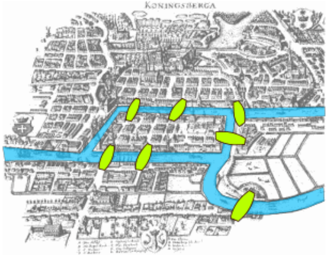
- Basic terminologies
- Undirected and directed graphs
- Euler circuits
- Graph traversals using queues & stacks

Learning outcomes:

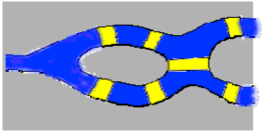
- Be able to tell what a graph is
- Be able to represent a graph using matrix and list
- Be able to describe different algorithms to traverse a graph

Graph theory

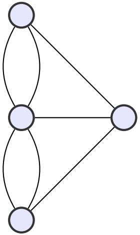
- introduced in the 18th century
- an old subject with many modern applications
- Mathematician Euler in Konigsberg
- Can we go around the city by crossing each bridge exactly once?



Map of Konigsberg



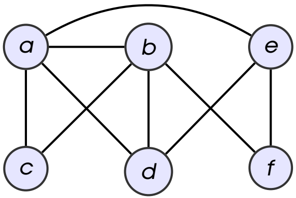
bridges and river banks



The graph

Graphs

- An undirected graph $G = (V, E)$ consists of a set of vertices V and a set of edges E. Each edge is an unordered pair of vertices.
(E.g., $\{b, c\}$ & $\{c, b\}$ refer to the same edge.)



undirected graph

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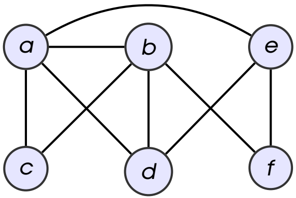
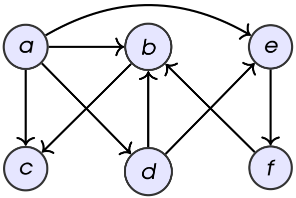


undirected graph



directed graph

- A directed graph $G = (V, E)$ consists of ... Each edge is an ordered pair of vertices.
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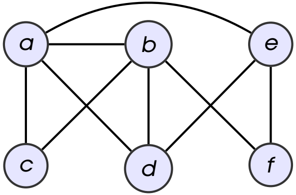
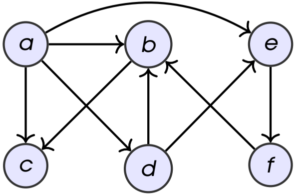


undirected graph



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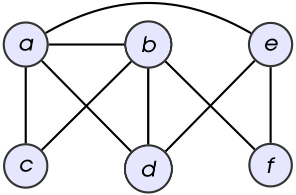
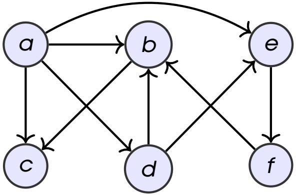
Modeling Facebook & Twitter?

Graphs

represent a set of interconnected objects



undirected graph



directed graph

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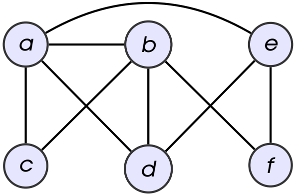
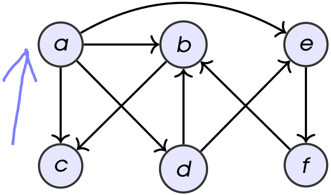




``friend'' relationship
on Facebook

``follower'' relationship
on Twitter

undirected graph

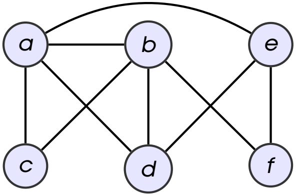
directed graph

Applications of graphs

- In computer science, graphs are often used to model
  - computer networks,
  - precedence among processes,
  - state space of playing chess (AI applications)
  - resource conflicts, $\cdots$
- In other disciplines, graphs are also used to model the structure of objects. E.g.,
  - biology - evolutionary relationship
  - chemistry - structure of molecules

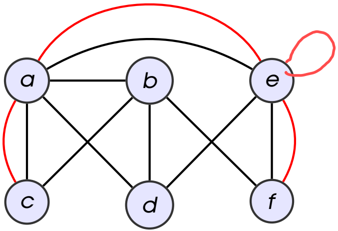
Undirected graphs

- simple graph: at most one edge between two vertices, no self loop (i.e., an edge from a vertex to itself)



Undirected graphs

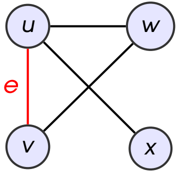
- simple graph: at most one edge between two vertices, no self loop (i.e., an edge from a vertex to itself)
- multigraph: allows more than one edge between two vertices



Undirected graphs

In an undirected graph G, suppose that $e = \{u, v\}$ is an edge of G

- u and v are said to be adjacent and called neighbors of each other.



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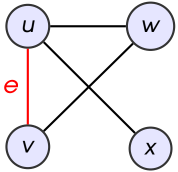
- u and v are said to be adjacent and called neighbors of each other.
    - u and v are called endpoints of e.
    - e is said to be incident with u and v.
    - e is said to connect u and v.



Undirected graphs

u 3
v 2
w 2
x 1
8

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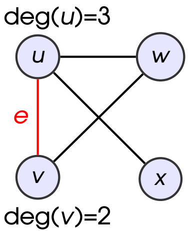
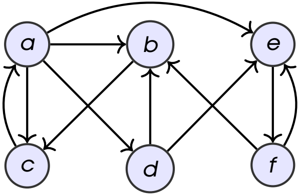


- The degree of a vertex v, denoted by deg(v), is the number of edges incident with it (a loop contributes twice to the degree)
- The degree of a graph is the maximum degree over all vertices.

sum of degrees
= 2 x number of edges

Directed graph

- A directed graph $G = (V, E)$ consists of ... Each edge is an ordered pair of vertices.
  E.g., (b, c) refer to an edge from b to c and differs from (c, b).



(a, b) is in E but not (b, a);
both (a, c) and (c, a) are in E

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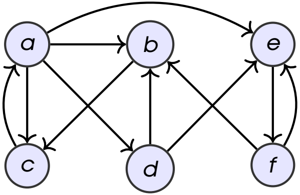
- Given a directed graph G, a vertex a is said to be connected to a vertex b if there is a path from a to b.



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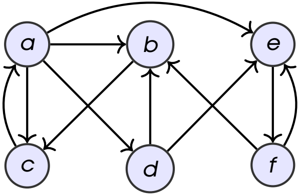
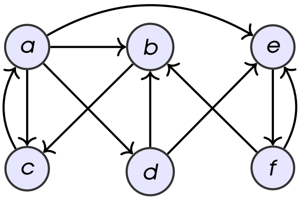
  E.g., (b, c) refer to an edge from b to c and differs from (c, b).
- Given a directed graph G, a vertex a is said to be connected to a vertex b if there is a path from a to b.
- Road network (with one way roads)



(a, b) is in E but not (b, a);
both (a, c) and (c, a) are in E

Directed graph - In/Out-degree

- in-degree of a vertex v: the number of edges leading to v
- out-degree of a vertex v: the number of edges leading away from v.



v	in-deg(v)	out-deg(v)

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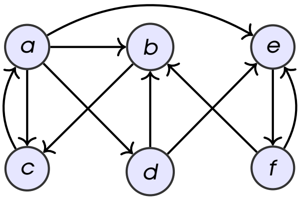

v	in-deg(v)	out-deg(v)
a	1	4

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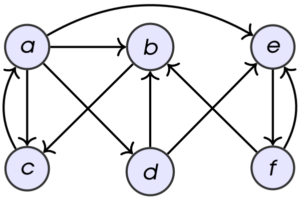

v	in-deg(v)	out-deg(v)
a	1	4
b	3	1

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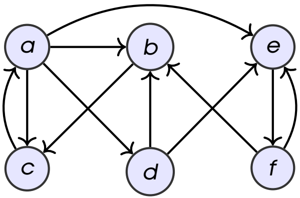

v	in-deg(v)	out-deg(v)
a	1	4
b	3	1
c	2	1

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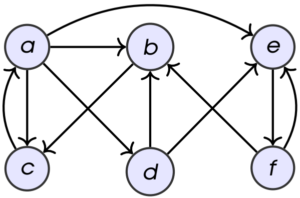

v	in-deg(v)	out-deg(v)
a	1	4
b	3	1
c	2	1
d	1	2

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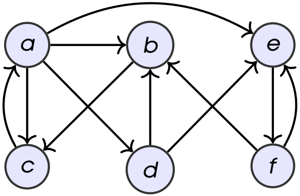

v	in-deg(v)	out-deg(v)
a	1	4
b	3	1
c	2	1
d	1	2
e	3	1

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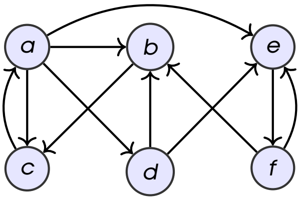

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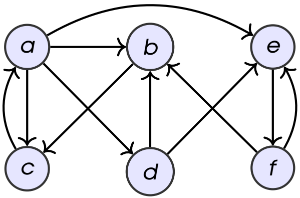

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sum:	11	11

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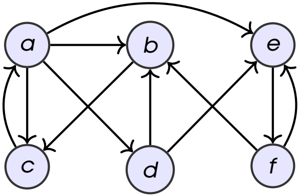

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f	1	2
sum:	11	11

```
sum of in-degree
= sum of out-degree
= number of edges
```

in/out-deg always equal?

Claim on Handshaking

- In a room full of people, some shake hands with others.
  Some people have odd number of handshakes.
  Some people have even number of handshakes.
  Claim: The number of people with odd number of handshakes must be even.
- In the context of graph
  Claim: The number of vertices with odd degree must be even.

Representation of undirected graphs

- An undirected graph can be represented by adjacency matrix, adjacency list, incidence matrix or incidence list
- Adjacency matrix/list: relationship between vertex adjacency (vertex vs vertex)
- Incidence matrix/list: relationship between edge incidence (vertex vs edge)

Matrix / 2-Dimensional Array

m-by-n matrix

- m rows
- n columns

$A_{i,j}$

- row i, column j

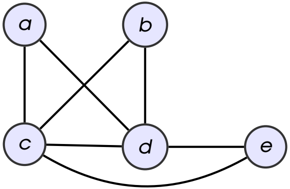
$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & A_{2,3} & \cdots & A_{2,n} \\ A_{3,1} & A_{3,2} & A_{3,3} & \cdots & A_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ A_{m,1} & A_{m,2} & A_{m,3} & \cdots & A_{m,n} \end{pmatrix}$$

Undirected graph - Adjacency matrix / list

Adjacency matrix M for a simple undirected graph with n vertices is an $n \times n$ matrix

- $M(i, j) = 1$ if vertex i and vertex j are adjacent
- $M(i, j) = 0$ otherwise

Adjacency list: each vertex has a list of vertices to which it is adjacent



$$\begin{array}{cc} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \left(\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right) \end{array}$$

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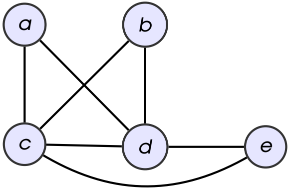

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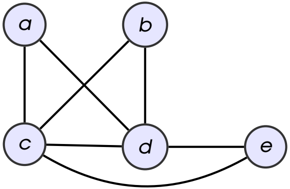

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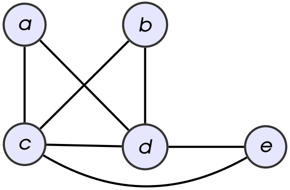

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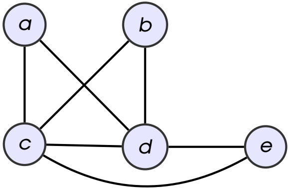

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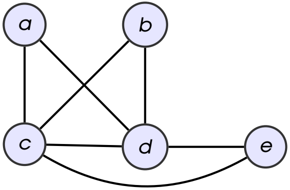

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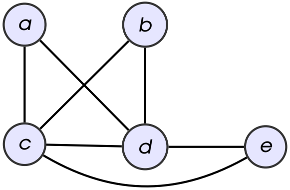

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Any property of the matrix?

Undirected graph - Adjacency matrix / list

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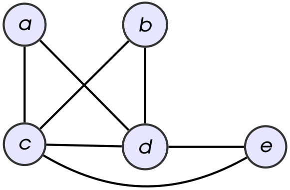
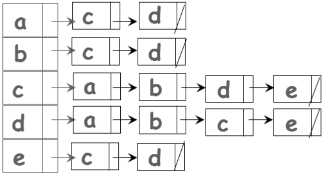


$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$



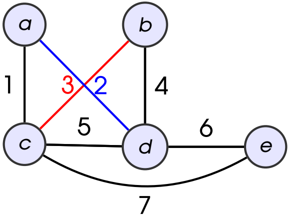
sparse graph: when there are very few edges

Undirected graph - Incidence matrix / list

Incidence matrix M for a simple undirected graph with n vertices and m edges is an $m \times n$ matrix

- $M(i,j) = 1$ if edge i and vertex j are incident
- $M(i,j) = 0$ otherwise

Incidence list: each edge has a list of vertices to which it is incident with



$$\begin{array}{cc} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \left(\begin{array}{ccccc} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right) \end{array}$$

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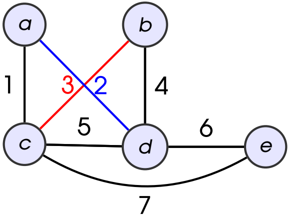

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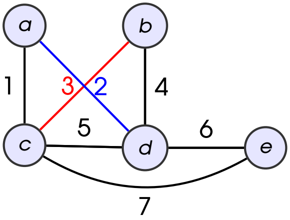

$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix} \end{array}$$

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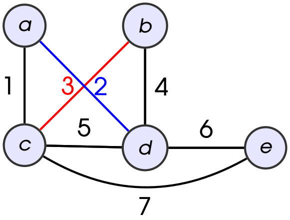

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \left(\begin{array}{ccccc} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array}\right) \end{array}$$

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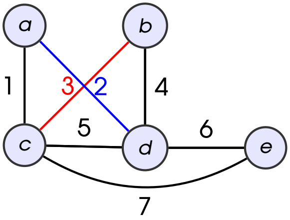

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\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
 & & & & \\
 & & & & \\
 & & & &
\end{pmatrix}
\end{array}
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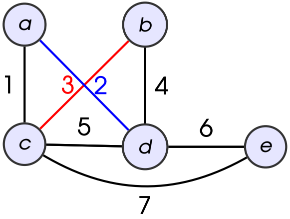

$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ & & & & \\ & & & & \end{pmatrix} \end{array}$$

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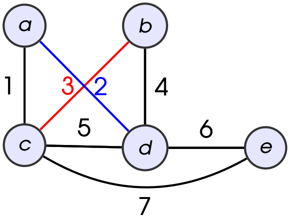

$$\begin{array}{cc} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ & & & & \end{pmatrix} \end{array}$$

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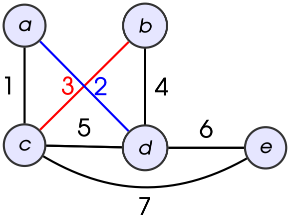

$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{pmatrix}
1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix}
\end{array}
$$

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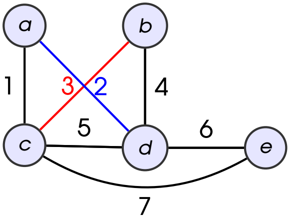

$$
\begin{array}{c}
\\
1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7
\end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & c & d & e \end{array}\\
\begin{pmatrix}
1 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 1 & 0\\
0 & 1 & 1 & 0 & 0\\
0 & 1 & 0 & 1 & 0\\
0 & 0 & 1 & 1 & 0\\
0 & 0 & 0 & 1 & 1\\
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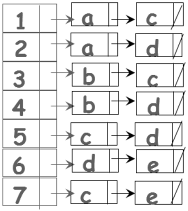
\end{pmatrix}
\end{array}
$$



Representation of directed graphs

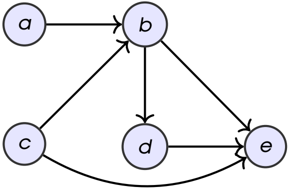
- Similar to undirected graph, a directed graph can be represented by adjacency matrix, adjacency list, incidence matrix or incidence list
- but needs to handle direction of connection of the edges

Directed graph - Adjacency matrix / list

Adjacency matrix M for a directed graph with n vertices is an $n \times n$ matrix

- $M(i, j) = 1$ if (i, j) is an edge, i.e., i points to j
- $M(i, j) = 0$ otherwise

Adjacency list: each vertex u has a list of vertices pointed to by an edge leading away from u



$$\begin{array}{cc} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix} \end{array}$$

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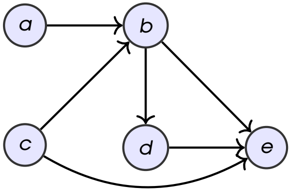

$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}
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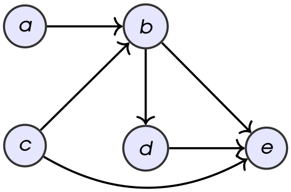

$$\begin{array}{cc} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ & & & & \\ & & & & \\ & & & & \end{pmatrix} \end{array}$$

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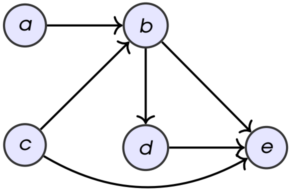

$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
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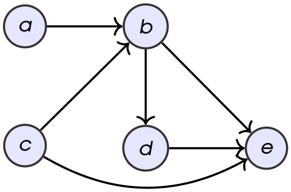

$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ & & & & \end{pmatrix} \end{array}$$

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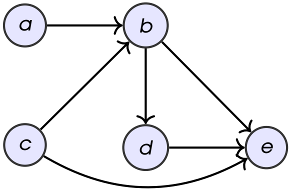

$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
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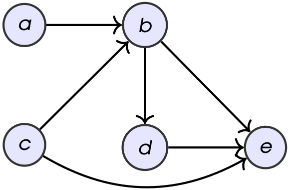

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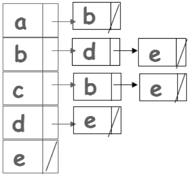

Directed graph - Incidence matrix / list

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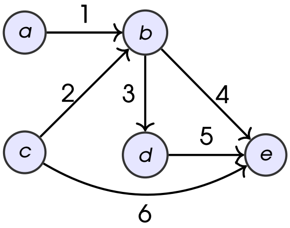

$$\begin{array}{c c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \left(\begin{array}{ccccc} \; & \; & \; & \; & \; \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{array} \right) \end{array}$$

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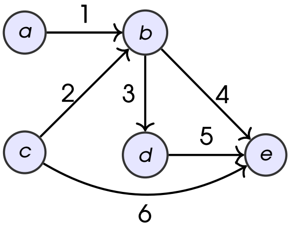

$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{pmatrix}
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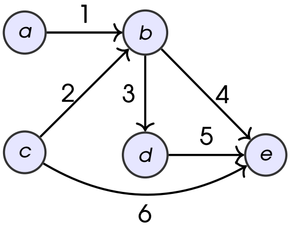

$$
\begin{array}{c c}
 & \begin{array}{ccccc} a & b & c & d & e \end{array} \\
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\begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
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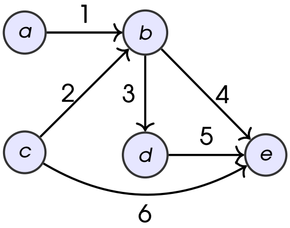

$$
\begin{array}{c}
\\
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\end{array}
\begin{array}{c}
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\begin{pmatrix}
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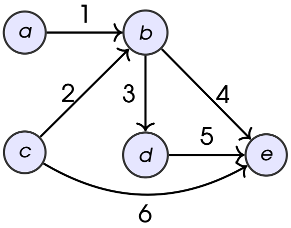

$$\begin{array}{cc} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ & & & & \\ & & & & \end{pmatrix} \end{array}$$

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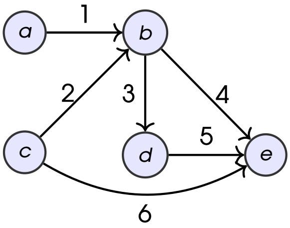

$$\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6
\end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{pmatrix}
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0 & -1 & 1 & 0 & 0 \\
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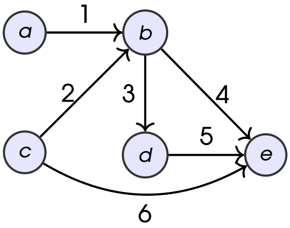
Incidence list: each vertex u has a list of vertices pointed to by an edge leading away from u



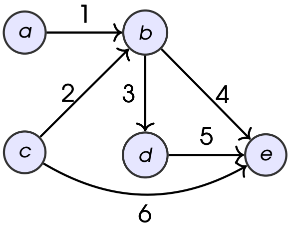
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix} \end{array}$$

Directed graph - Incidence matrix / list

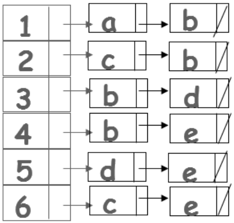
Incidence matrix M for a directed graph with n vertices and m edges is an $m \times n$ matrix

- $M(i, j) = 1$ if edge i is leading away from vertex j
- $M(i, j) = -1$ if edge i is leading to vertex j
- $M(i, j) = 0$ otherwise

Incidence list: each vertex u has a list of vertices pointed to by an edge leading away from u



$$
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6
\end{array}
\begin{array}{c}
\begin{array}{ccccc} a & b & c & d & e \end{array} \\
\begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & -1
\end{pmatrix}
\end{array}
$$



Summary

Summary: What is a graph and how to represent one?

Next: Paths, Circuits, Traversals

For note taking