Comp305

Biocomputation

Lecturer: Yi Dong

Comp305 Part I.

Artificial Neural Networks

Topic 3.

Hebb's Rules

ANN Learning Rules

- The **McCulloch-Pitts** neuron made a base for a machine (network of units) capable of
  - ***storing information and***
  - ***producing logical and arithmetical operations on it***

- The next step
  - must be to realise another important function of the brain, which is

***to acquire new knowledge through experience*, i.e. *<u>learning</u>*.**

ANN Learning Rules

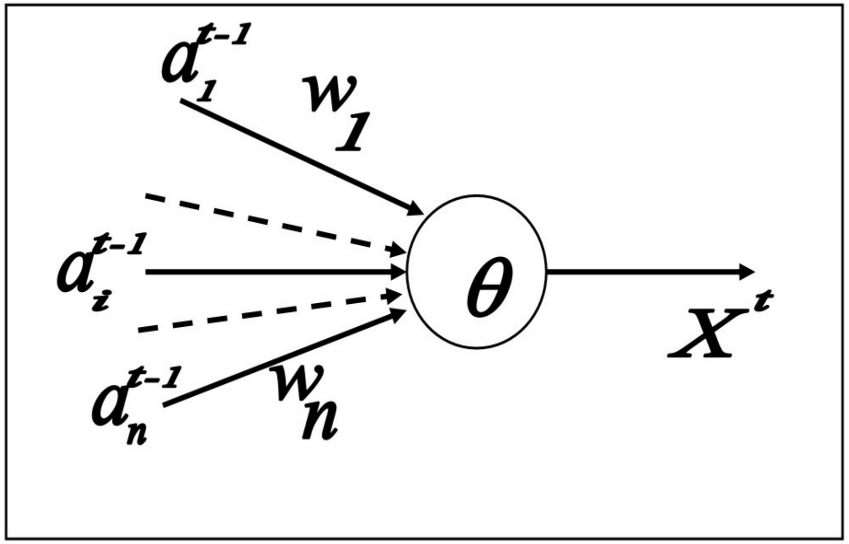
- Learning means

  <u>to change in response to experience</u>

- In an MP neural network, wights of connections are fixed.

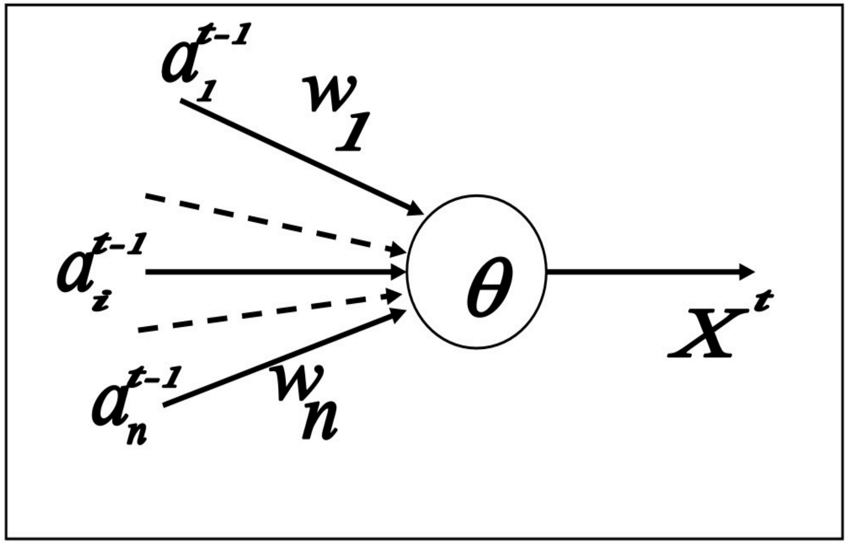
- We need a new model, of which the parameters are easily changeable (to be learnt).

Beyond Standard MP Neuron



- From now, we do **NOT** have the restriction on the weight. That is, the weight can be any real value.
- We do **NOT** check the inhibitory input.

ANN Learning Rules



Definition:

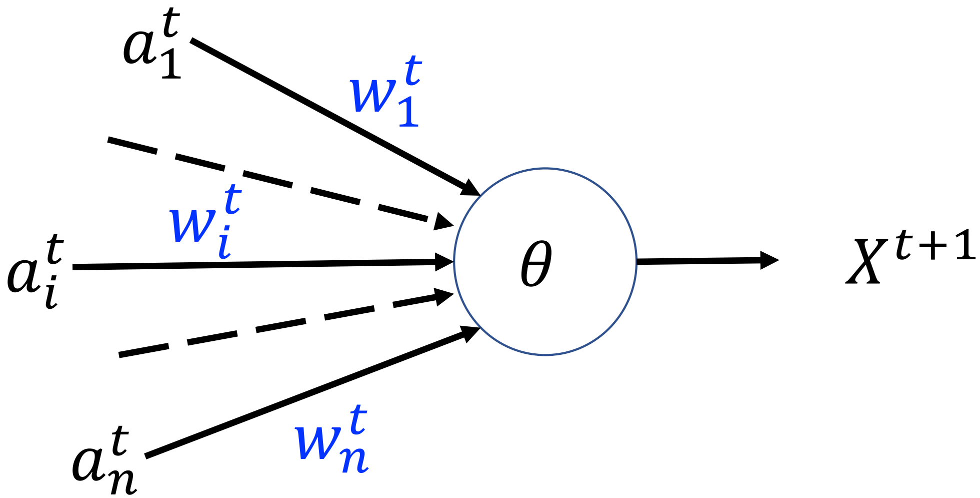
ANN learning rule is
<u>the rule how to adjust the weights of connections to get desirable output.</u>

Hebb’s Rule (1949)

Hebb proposed that

“… Cells that fire together, wire together…”

Hebb's Rule (1949)



Consider the above neuron, the weight w_i^t is what we want to learn.

Again, here we allow w_i^t could be other than -1 or 1, and we do not check inhibitory inputs. It is NOT a standard MP neuron.

Hebb’s Rule (1949)

- The simplest formulation of Hebb’s rule is to increase weight of connection at every next instant in the way:

$$w_i^{t+1} = w_i^t + \Delta w_i^t,$$

Where

$$\Delta w_i^t = C a_i^t X^{t+1}$$

Hebb's Rule (1949)

$$w_i^{t+1} = w_i^t + \Delta w_i^t,$$

Where

$$\Delta w_i^t = C a_i^t X^{t+1}$$

w_i^t is the weight of connection at instant t,

w_i^{t+1} is the weight of connection at the next instant $t + 1$,

Δw_i^t is the increment by which the weight of connection is enlarged,

C is positive coefficient which determines learning rate.

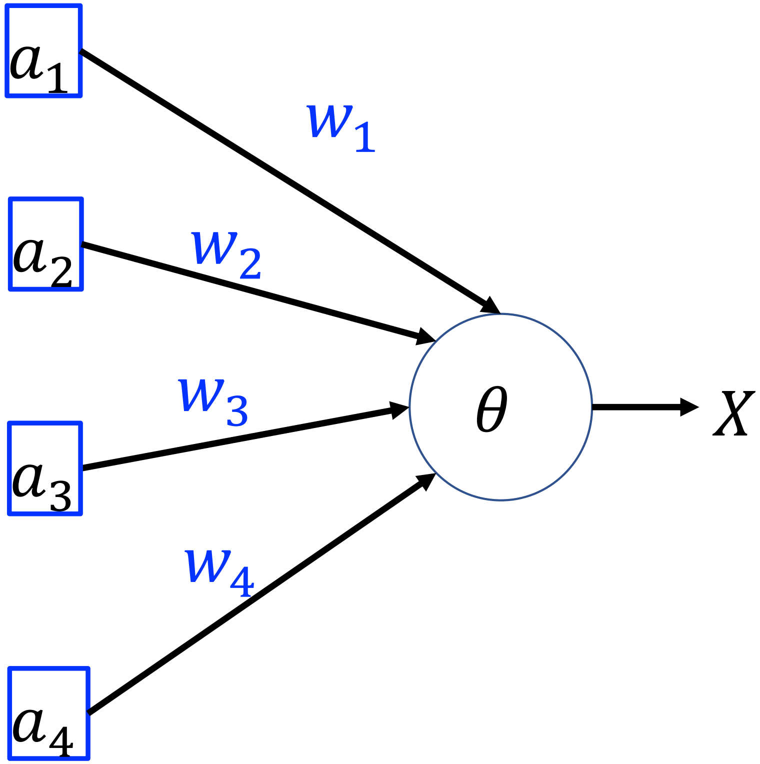
a_i^t is input value from the presynaptic neuron at instant t,

X^{t+1} is output of the postsynaptic neuron at the instant t.

Algorithm of Hebb's Rule for a Single Neuron

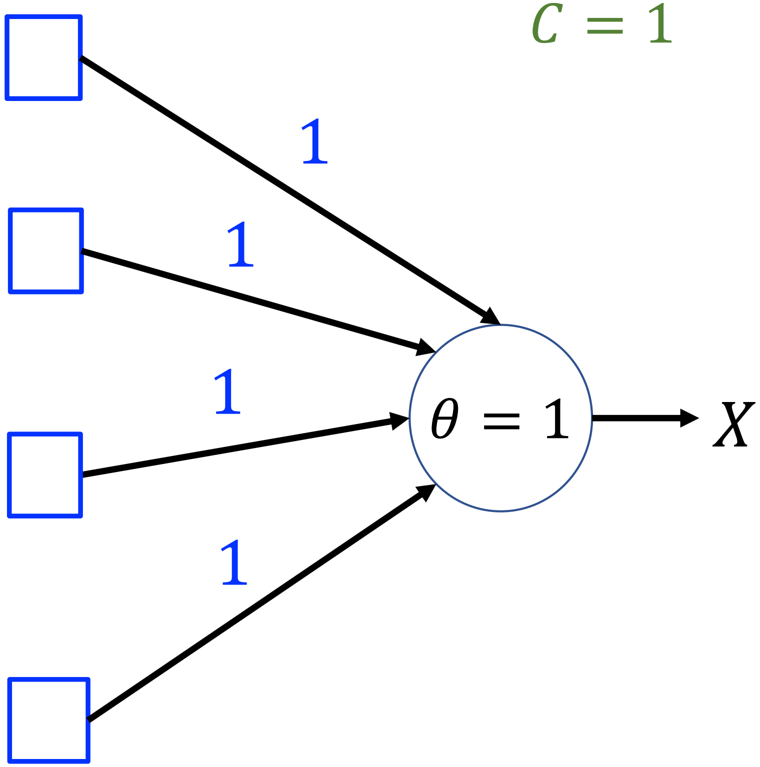
1. Set the neuron threshold value θ and the learning rate C.
2. Set *random initial values* for the weights of connections w_i^t.
3. Give instant input values a_i^t by the input units.
4. Compute the instant state of the neuron $S^t = \sum_i w_i^t a_i^t$
5. Compute the instant output of the neuron X^{t+1}
$$X^{t+1} = g(S^t) = H(S^t - \theta) = \begin{cases} 1, & S^t \geq \theta; \\ 0, & S^t < \theta. \end{cases}$$
6. Compute the instant corrections to the weights of connections $\Delta w_i^t = C a_i^t X^{t+1}$
7. Update the weights of connections $w_i^{t+1} = w_i^t + \Delta w_i^t$
8. **Go to the step 3**.

A Running Example



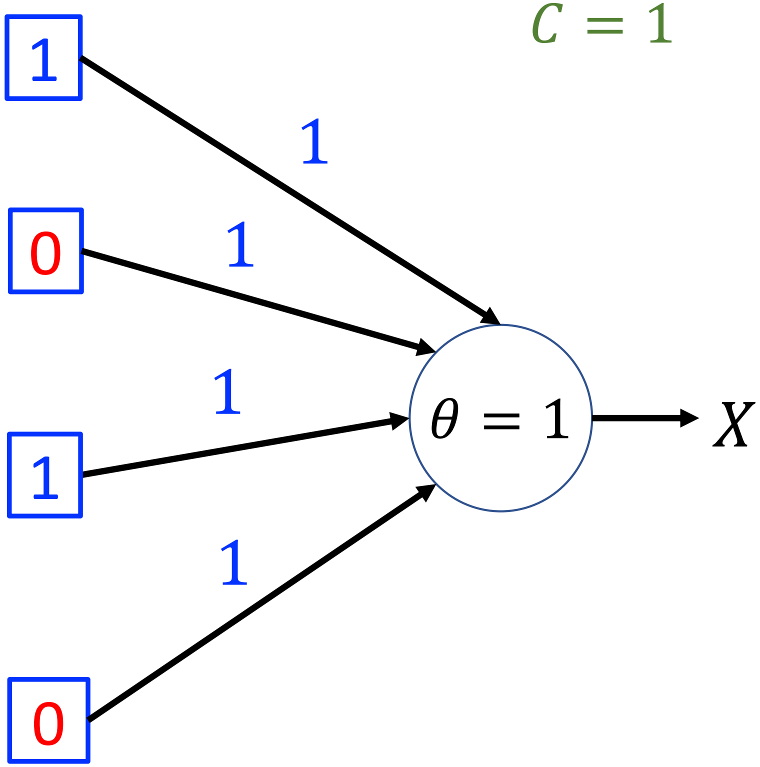
A Running Example

$C = 1$ $\quad t = 0$



w_1^0	w_2^0	w_3^0	w_4^0
1	1	1	1

A Running Example

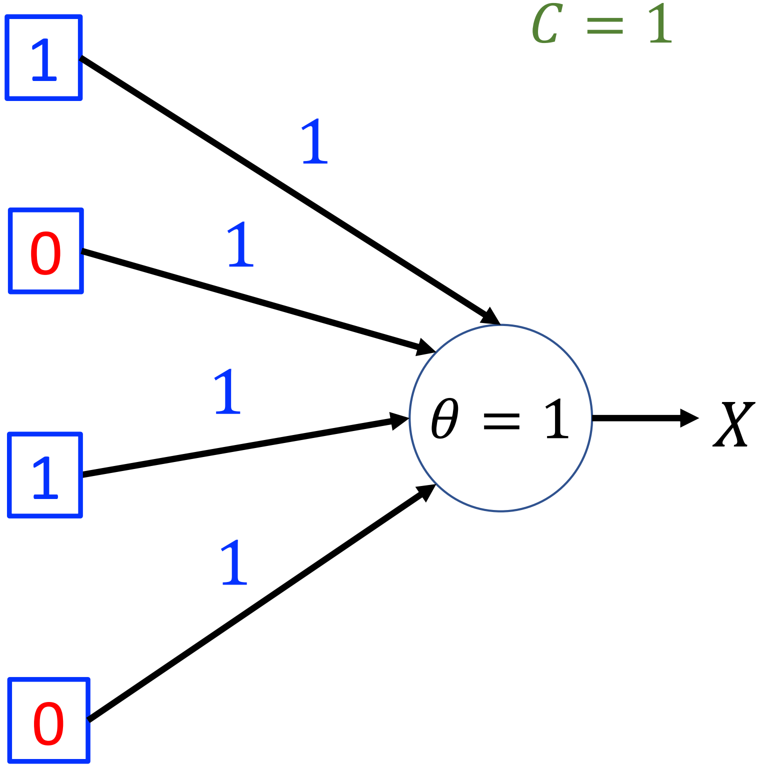


$C = 1$ $t = 0$

a_1^0	a_2^0	a_3^0	a_4^0
1	0	1	0

w_1^0	w_2^0	w_3^0	w_4^0
1	1	1	1

A Running Example



$C = 1$ $t = 0$

a_1^0	a_2^0	a_3^0	a_4^0
1	0	1	0

w_1^0	w_2^0	w_3^0	w_4^0
1	1	1	1

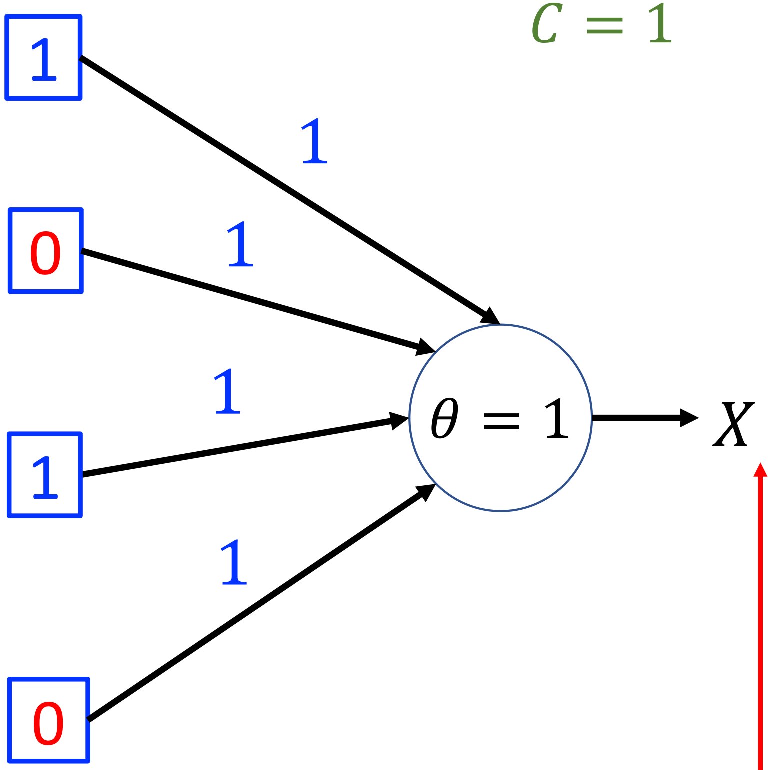
$$S^0 = \sum_{i=1}^{4} w_i^0 a_i^0$$

$$= w_1^0 \times a_1^0 + w_2^0 \times a_2^0 + w_3^0 \times a_3^0 + w_4^0 \times a_4^0$$

$$= 1 \times 1 + 1 \times 0 + 1 \times 1 + 1 \times 0 = 2 \geq \theta$$

$$X^1 = 1$$

A Running Example



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1	0	1	0

w_1^0	w_2^0	w_3^0	w_4^0
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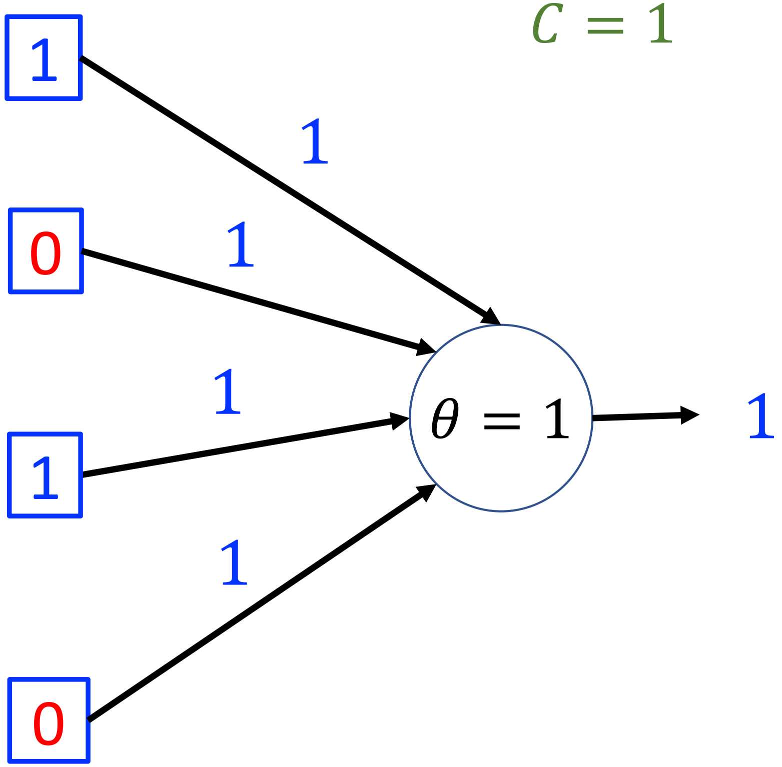
$$= 1 \times 1 + 1 \times 0 + 1 \times 1 + 1 \times 0 = 2 \geq \theta$$

$$\Downarrow$$

$$X^1 = 1$$

A Running Example



$C = 1$ $t = 0$

a_1^0	a_2^0	a_3^0	a_4^0
1	0	1	0

w_1^0	w_2^0	w_3^0	w_4^0
1	1	1	1

$$S^0 = \sum_{i=1}^{4} w_i^0 a_i^0$$
$$= w_1^0 \times a_1^0 + w_2^0 \times a_2^0 + w_3^0 \times a_3^0 + w_4^0 \times a_4^0$$
$$= 1\times 1 + 1\times 0 + 1\times 1 + 1\times 0 = 2 \geq \theta$$

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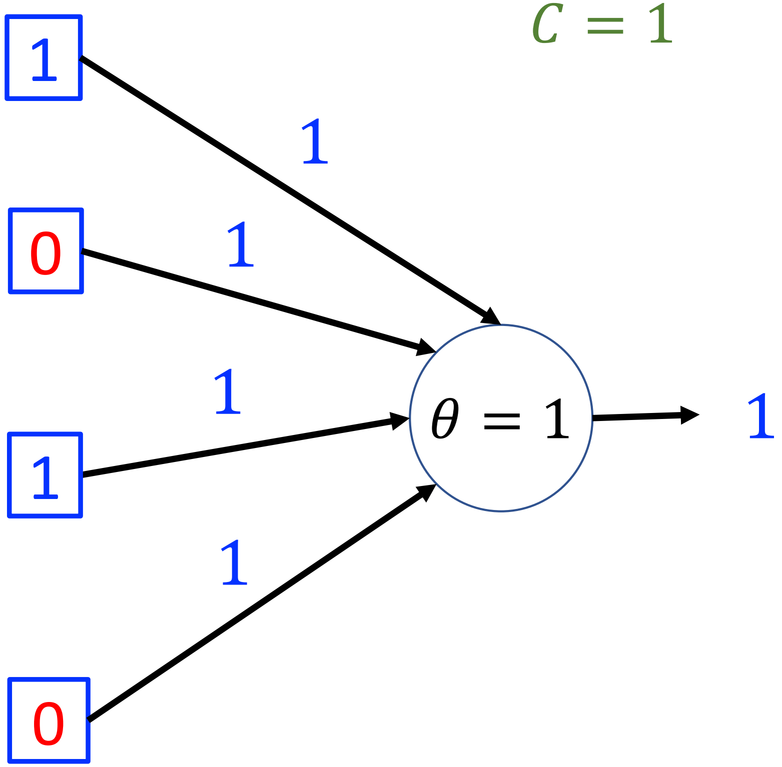
A Running Example



$C = 1$ $t = 0$

a_1^0	a_2^0	a_3^0	a_4^0
1	0	1	0

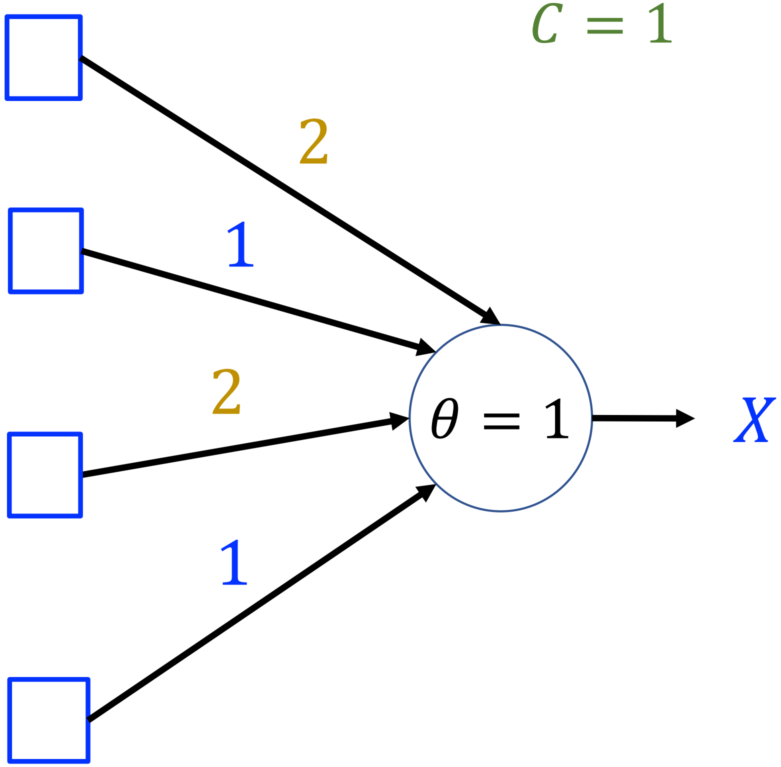
w_1^0	w_2^0	w_3^0	w_4^0
1	1	1	1

$$\Delta w_i^0 = C a_i^0 X^1 \qquad w_i^1 = w_i^0 + \Delta w_i^0$$

$$\Delta w_1^0 = 1\times1\times1 = 1, \quad w_1^1 = w_i^0 + \Delta w_i^0 = 1 + 1 = 2;$$
$$\Delta w_2^0 = 1\times0\times1 = 0, \quad w_2^1 = w_2^0 + \Delta w_2^0 = 1 + 0 = 1;$$
$$\Delta w_3^0 = 1\times1\times1 = 1, \quad w_3^1 = w_3^0 + \Delta w_3^0 = 1 + 1 = 2;$$
$$\Delta w_4^0 = 1\times0\times1 = 0, \quad w_4^1 = w_4^0 + \Delta w_4^0 = 1 + 0 = 1;$$

A Running Example

$C = 1$ $t = 1$



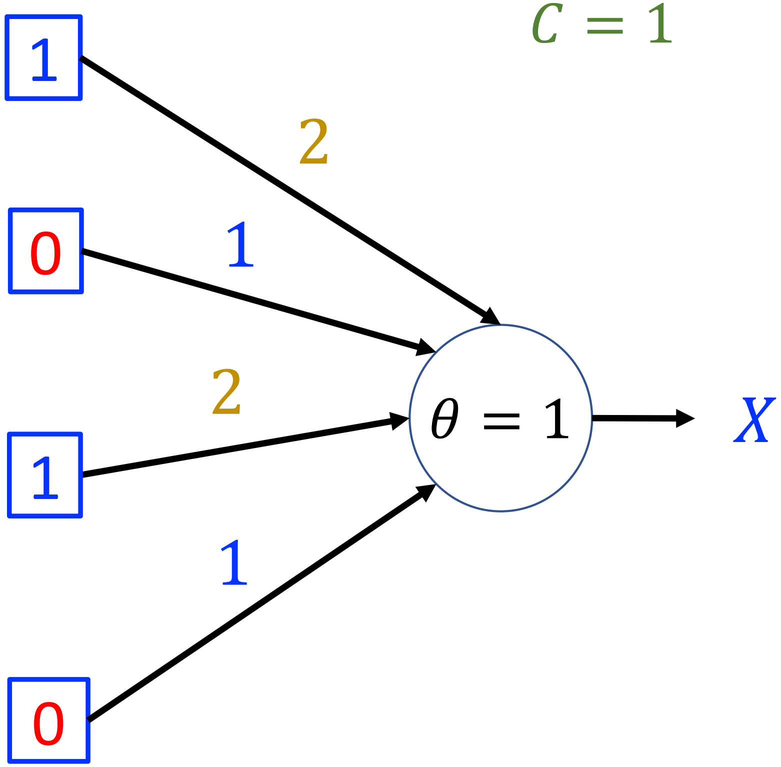
a_1^0	a_2^0	a_3^0	a_4^0
1	0	1	0

w_1^1	w_2^1	w_3^1	w_4^1
2	1	2	1

$$\Delta w_i^0 = C a_i^0 X^1 \qquad w_i^1 = w_i^0 + \Delta w_i^0$$

$$\Delta w_1^0 = 1\times1\times1 = 1,\ \ w_1^1 = w_i^0 + \Delta w_i^0 = 1 + 1 = 2;$$
$$\Delta w_2^0 = 1\times0\times1 = 0,\ \ w_2^1 = w_2^0 + \Delta w_2^0 = 1 + 0 = 1;$$
$$\Delta w_3^0 = 1\times1\times1 = 1,\ \ w_3^1 = w_3^0 + \Delta w_3^0 = 1 + 1 = 2;$$
$$\Delta w_4^0 = 1\times0\times1 = 1,\ \ w_4^1 = w_4^0 + \Delta w_4^0 = 1 + 0 = 1;$$

A Running Example



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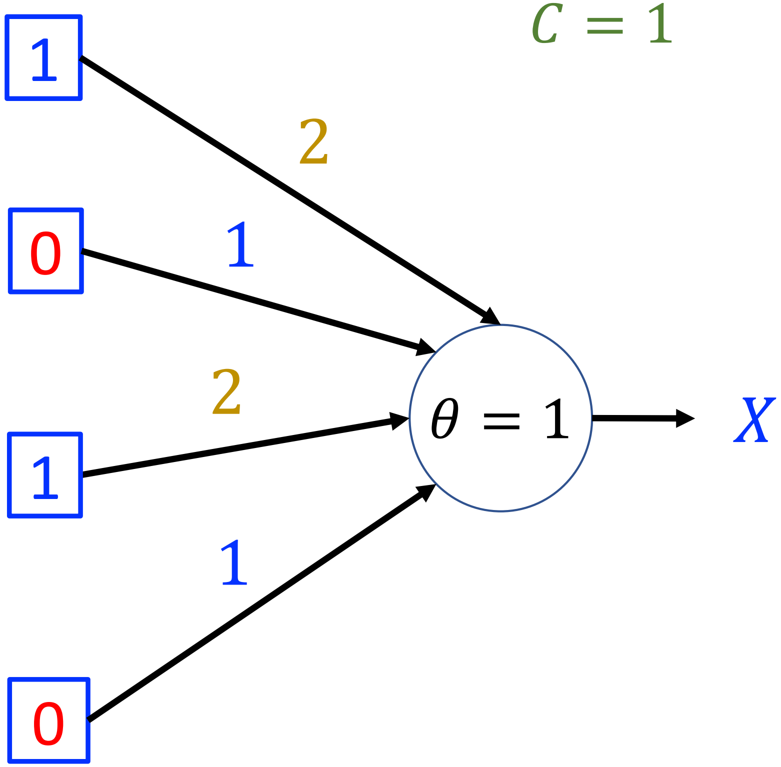
a_1^1	a_2^1	a_3^1	a_4^1
1	0	1	0

w_1^1	w_2^1	w_3^1	w_4^1
2	1	2	1

A Running Example



$C = 1$ $t = 1$

a_1^1	a_2^1	a_3^1	a_4^1
1	0	1	0

w_1^1	w_2^1	w_3^1	w_4^1
2	1	2	1

$$S^1 = \sum_{i=1}^{4} w_i^1 a_i^1$$

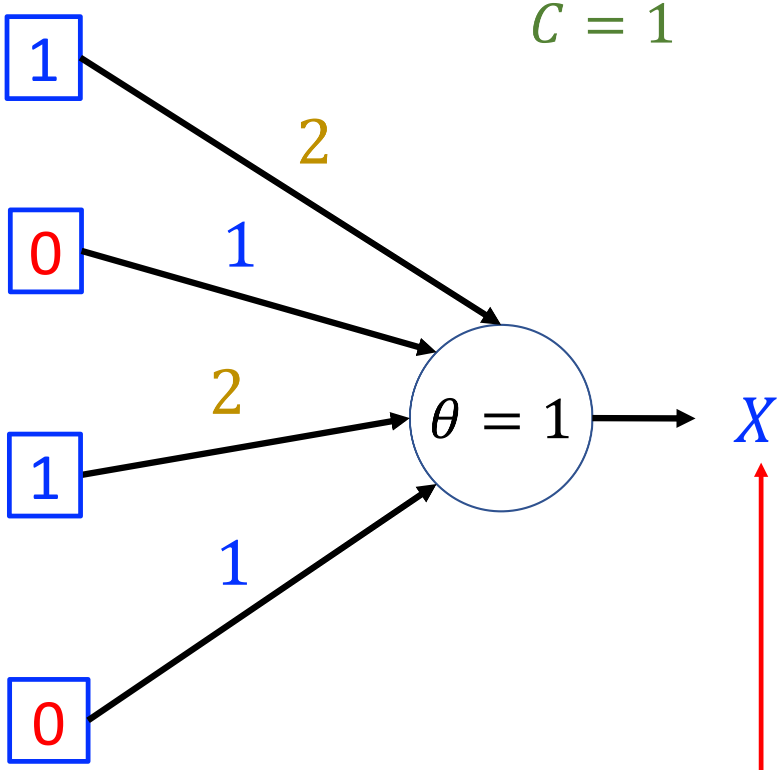
$$= w_1^1 \times a_1^1 + w_2^1 \times a_2^1 + w_3^1 \times a_3^1 + w_4^1 \times a_4^1$$

$$= 2\times1 + 1\times0 + 2\times1 + 1\times0 = 4 \geq \theta$$

$$X^2 = 1$$

A Running Example

$C = 1$ $\quad$ $t = 1$



a_1^1	a_2^1	a_3^1	a_4^1
1	0	1	0

w_1^1	w_2^1	w_3^1	w_4^1
2	1	2	1

$$S^1 = \sum_{i=1}^{4} w_i^1 a_i^1$$

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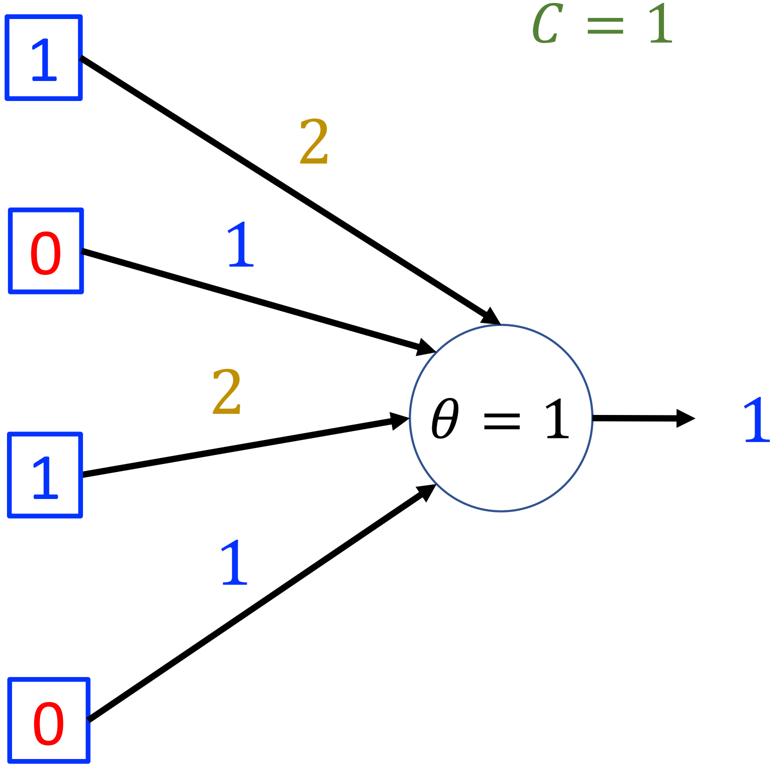
A Running Example



$C = 1$ $t = 1$

a_1^1	a_2^1	a_3^1	a_4^1
1	0	1	0

w_1^1	w_2^1	w_3^1	w_4^1
2	1	2	1

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$$= w_1^1 \times a_1^1 + w_2^1 \times a_2^1 + w_3^1 \times a_3^1 + w_4^1 \times a_4^1$$

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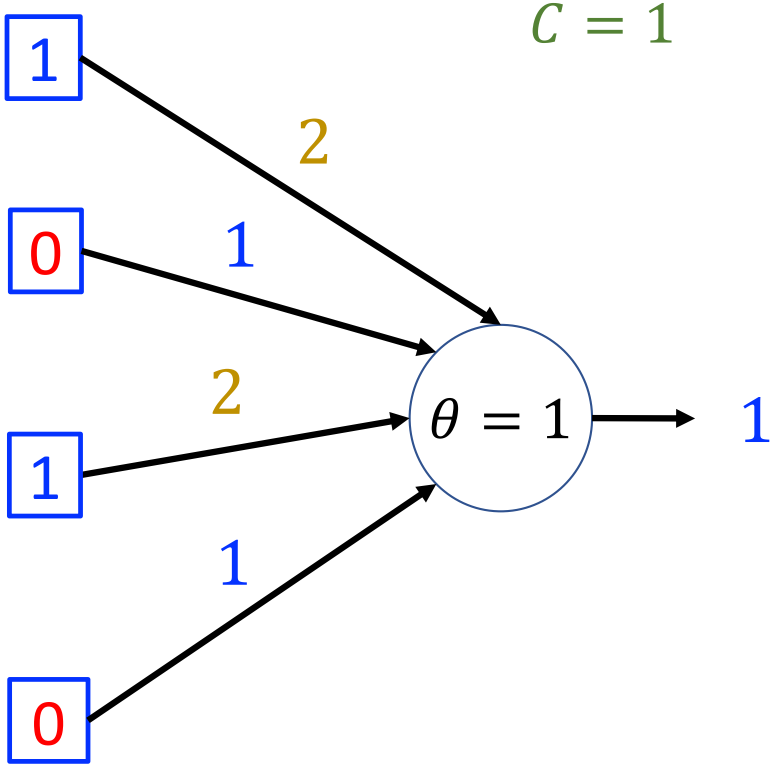
A Running Example



$C = 1$ $t = 1$

a_1^1	a_2^1	a_3^1	a_4^1
1	0	1	0

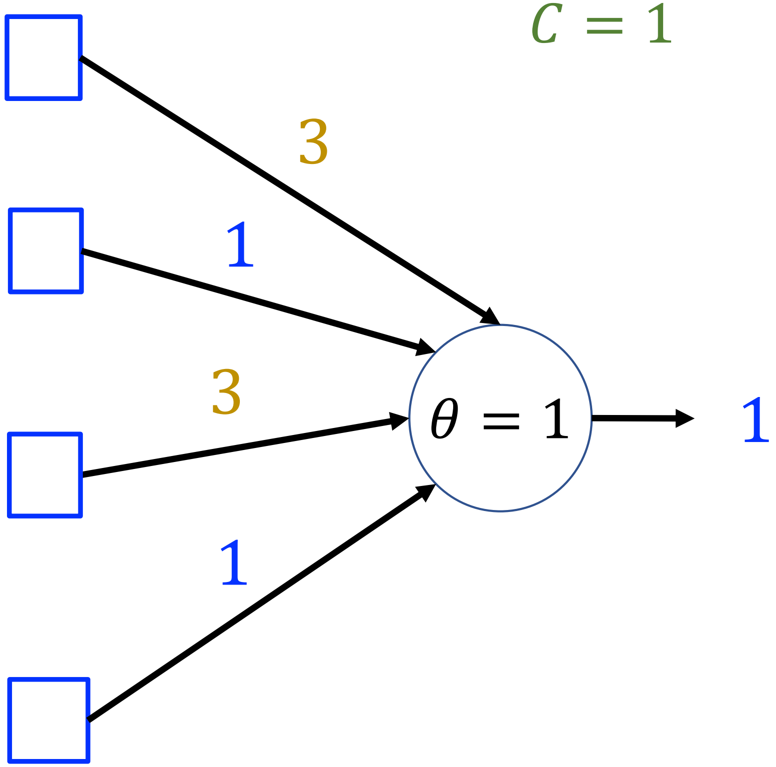
w_1^1	w_2^1	w_3^1	w_4^1
2	1	2	1

$$\Delta w_i^1 = C a_i^1 X^2 \qquad w_i^2 = w_i^1 + \Delta w_i^1$$

$$\Delta w_1^1 = 1\times1\times1 = 1, \quad w_1^2 = w_i^1 + \Delta w_i^1 = 2 + 1 = 3;$$
$$\Delta w_2^1 = 1\times0\times1 = 0, \quad w_2^2 = w_2^1 + \Delta w_2^1 = 1 + 0 = 1;$$
$$\Delta w_3^1 = 1\times1\times1 = 1, \quad w_3^2 = w_3^1 + \Delta w_3^1 = 2 + 1 = 3;$$
$$\Delta w_4^1 = 1\times0\times1 = 0, \quad w_4^2 = w_4^1 + \Delta w_4^1 = 1 + 0 = 1;$$

A Running Example

$C = 1$ $\quad$ $t = 2$



a_1^1	a_2^1	a_3^1	a_4^1
1	0	1	0

w_1^2	w_2^2	w_3^2	w_4^2
3	1	3	1

$$\Delta w_i^1 = C a_i^1 X^2 \qquad w_i^2 = w_i^1 + \Delta w_i^1$$

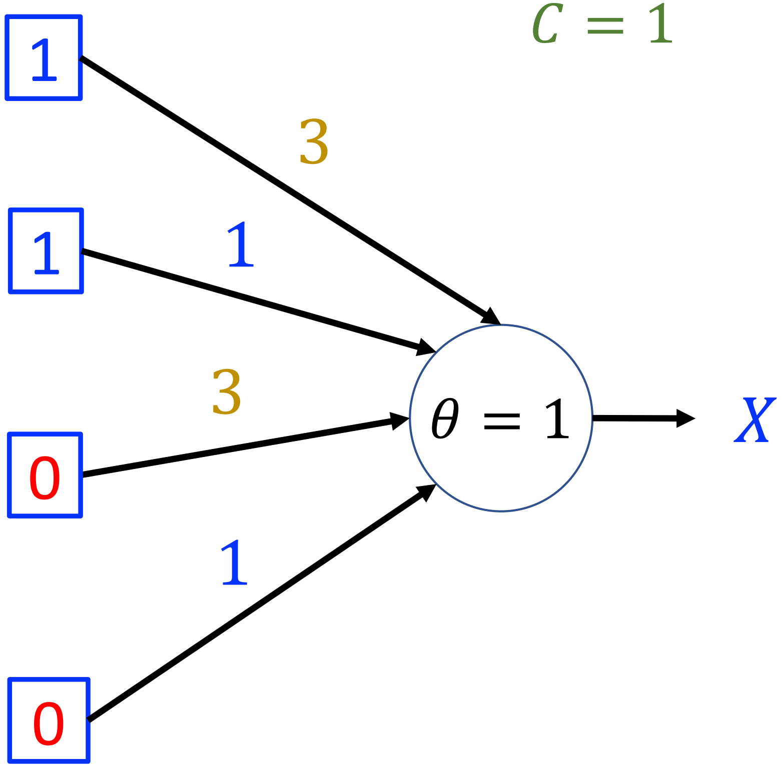
$$\Delta w_1^1 = 1\times1\times1 = 1,\ \ w_1^2 = w_i^1 + \Delta w_i^1 = 2 + 1 = 3;$$

$$\Delta w_2^1 = 1\times0\times1 = 0,\ \ w_2^2 = w_2^1 + \Delta w_2^1 = 1 + 0 = 1;$$

$$\Delta w_3^1 = 1\times1\times1 = 1,\ \ w_3^2 = w_3^1 + \Delta w_3^1 = 2 + 1 = 3;$$

$$\Delta w_4^1 = 1\times0\times1 = 1,\ \ w_4^2 = w_4^1 + \Delta w_4^1 = 1 + 0 = 1;$$

A Running Example



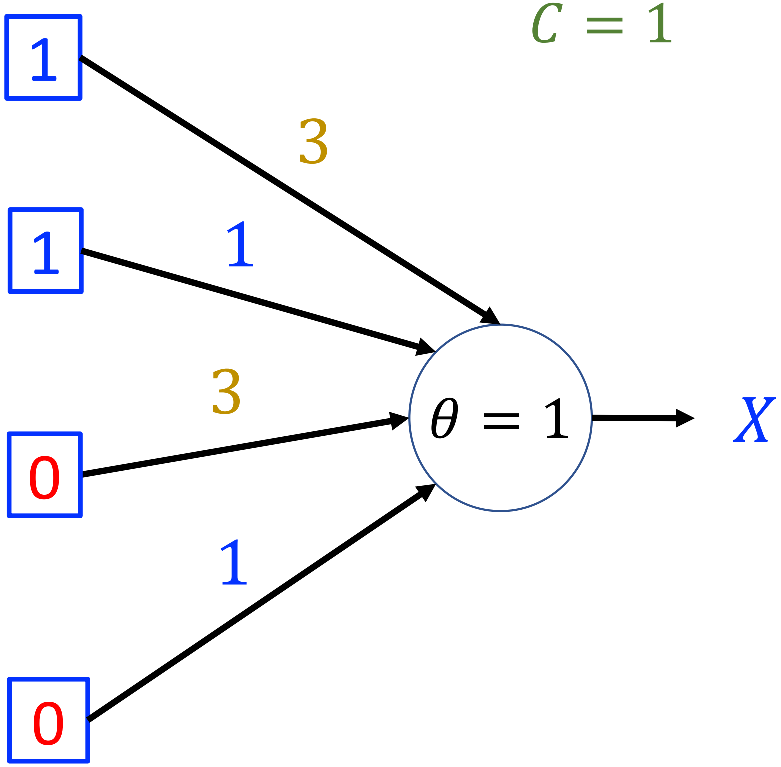
$C = 1$ $t = 2$

a_1^2	a_2^2	a_3^2	a_4^2
1	1	0	0

w_1^2	w_2^2	w_3^2	w_4^2
3	1	3	1

A Running Example

$C = 1$ $t = 2$



a_1^2	a_2^2	a_3^2	a_4^2
1	1	0	0

w_1^2	w_2^2	w_3^2	w_4^2
3	1	3	1

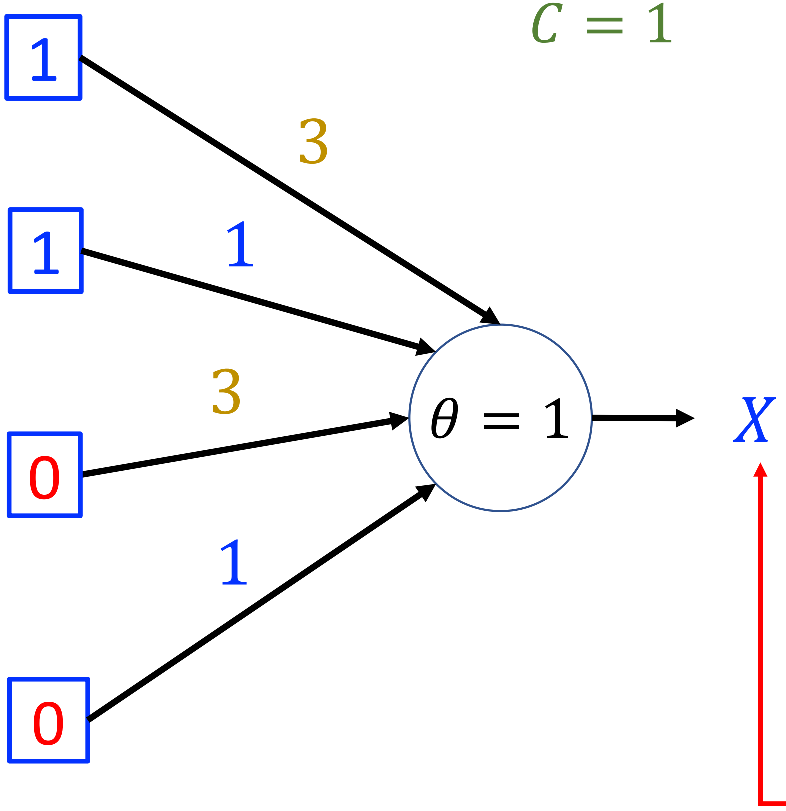
$$S^2 = \sum_{i=1}^{4} w_i^2 a_i^2$$

$$= w_1^2 \times a_1^2 + w_2^2 \times a_2^2 + w_3^2 \times a_3^2 + w_4^2 \times a_4^2$$

$$= 3 \times 1 + 1 \times 1 + 2 \times 0 + 1 \times 0 = 4 \geq \theta$$

$$X^3 = 1$$

A Running Example



$C = 1$ $t = 2$

a_1^2	a_2^2	a_3^2	a_4^2
1	1	0	0

w_1^2	w_2^2	w_3^2	w_4^2
3	1	3	1

$$S^2 = \sum_{i=1}^{4} w_i^2 a_i^2$$

$$= w_1^2 \times a_1^2 + w_2^2 \times a_2^2 + w_3^2 \times a_3^2 + w_4^2 \times a_4^2$$

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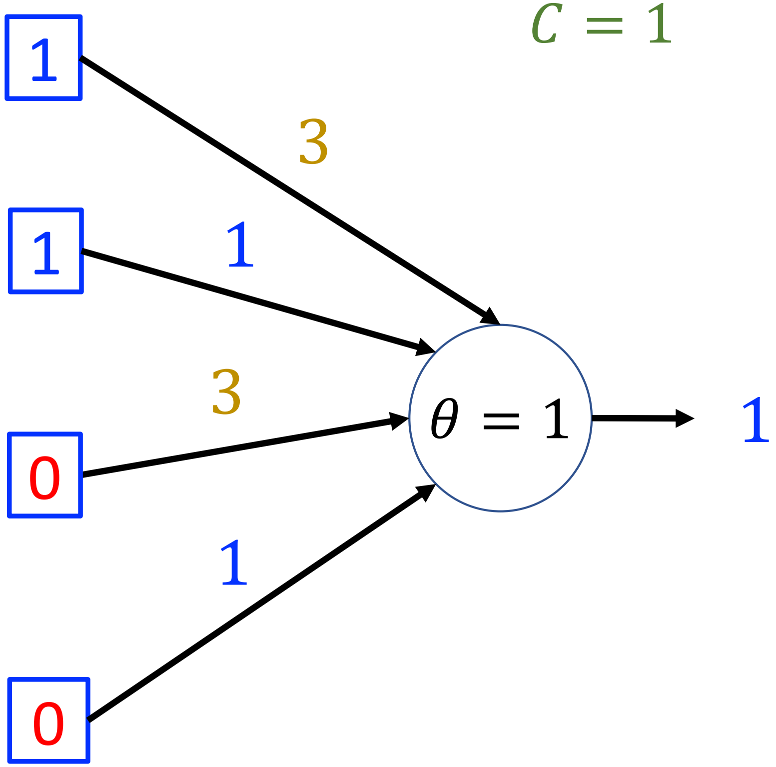
A Running Example



$C = 1$ $t = 2$

a_1^2	a_2^2	a_3^2	a_4^2
1	1	0	0

w_1^2	w_2^2	w_3^2	w_4^2
3	1	3	1

$$S^2 = \sum_{i=1}^{4} w_i^2 a_i^2$$

$$= w_1^2 \times a_1^2 + w_2^2 \times a_2^2 + w_3^2 \times a_3^2 + w_4^2 \times a_4^2$$

$$= 3 \times 1 + 1 \times 1 + 2 \times 0 + 1 \times 0 = 4 \geq \theta$$

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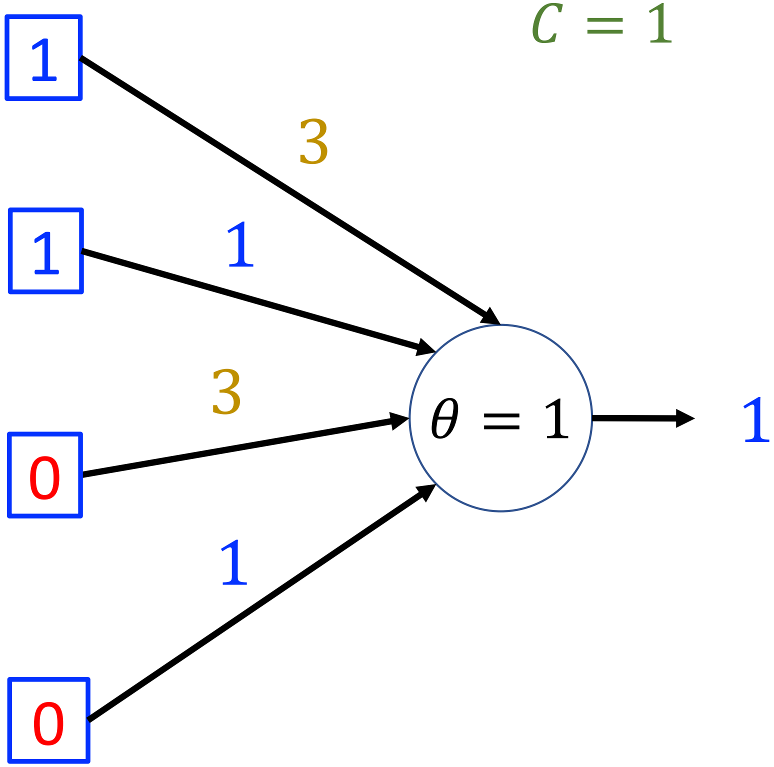
A Running Example



$C = 1$ $\quad$ $t = 2$

a_1^2	a_2^2	a_3^2	a_4^2
1	1	0	0

w_1^2	w_2^2	w_3^2	w_4^2
3	1	3	1

$$\Delta w_i^2 = C a_i^2 X^3 \qquad w_i^3 = w_i^2 + \Delta w_i^2$$

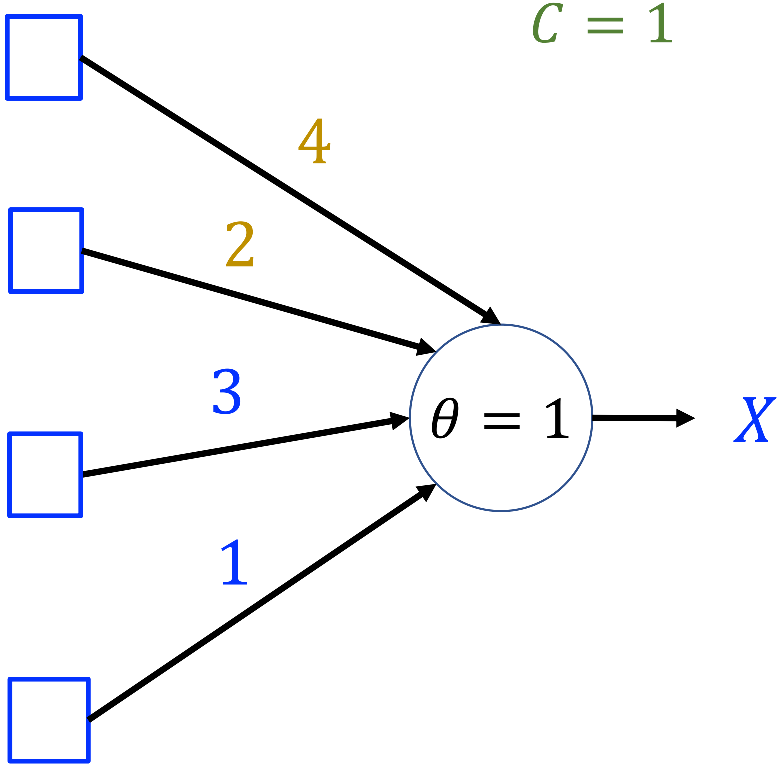
$$\Delta w_1^2 = 1\times1\times1 = 1,\ \ w_1^3 = w_i^2 + \Delta w_i^2 = 3 + 1 = 4;$$

$$\Delta w_2^2 = 1\times1\times1 = 1,\ \ w_2^3 = w_2^2 + \Delta w_2^2 = 1 + 1 = 2;$$

$$\Delta w_3^2 = 1\times0\times1 = 0,\ \ w_3^3 = w_3^2 + \Delta w_3^2 = 3 + 0 = 3;$$

$$\Delta w_4^2 = 1\times0\times1 = 0,\ \ w_4^3 = w_4^2 + \Delta w_4^2 = 1 + 0 = 1;$$

A Running Example



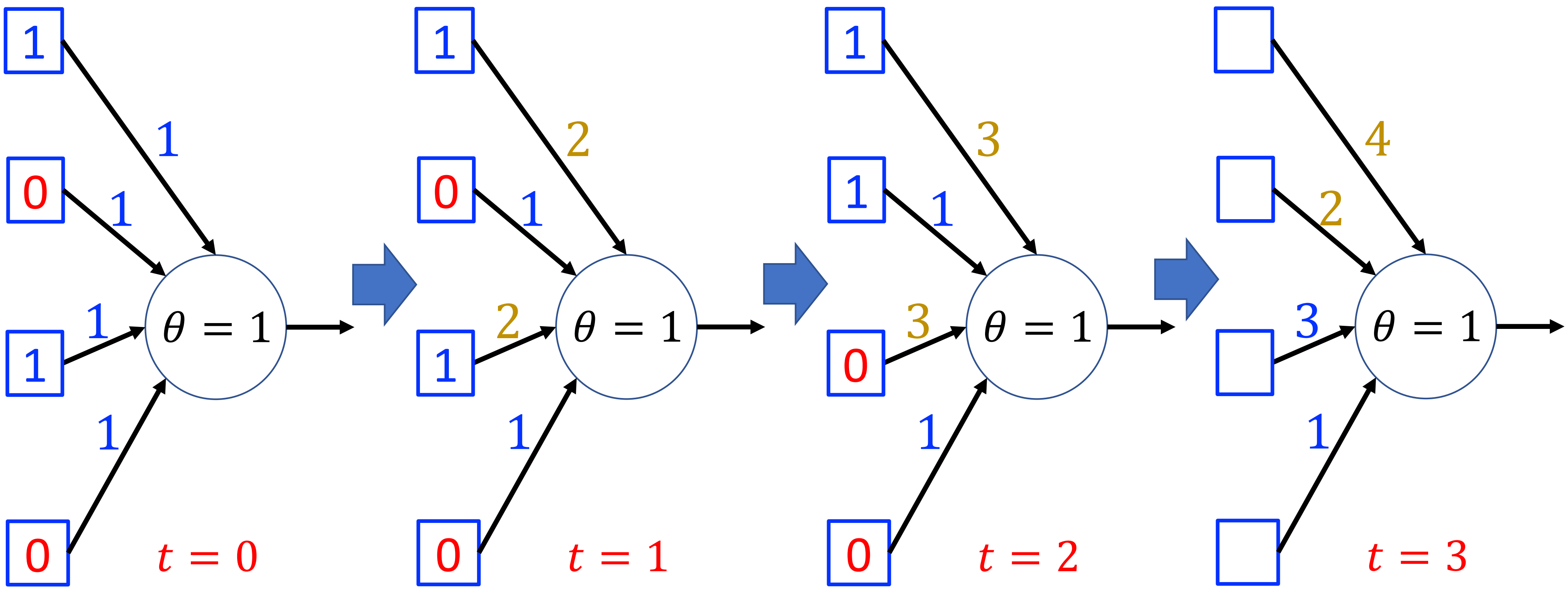
$C = 1$ $t = 3$

a_1^3	a_2^3	a_3^3	a_4^3

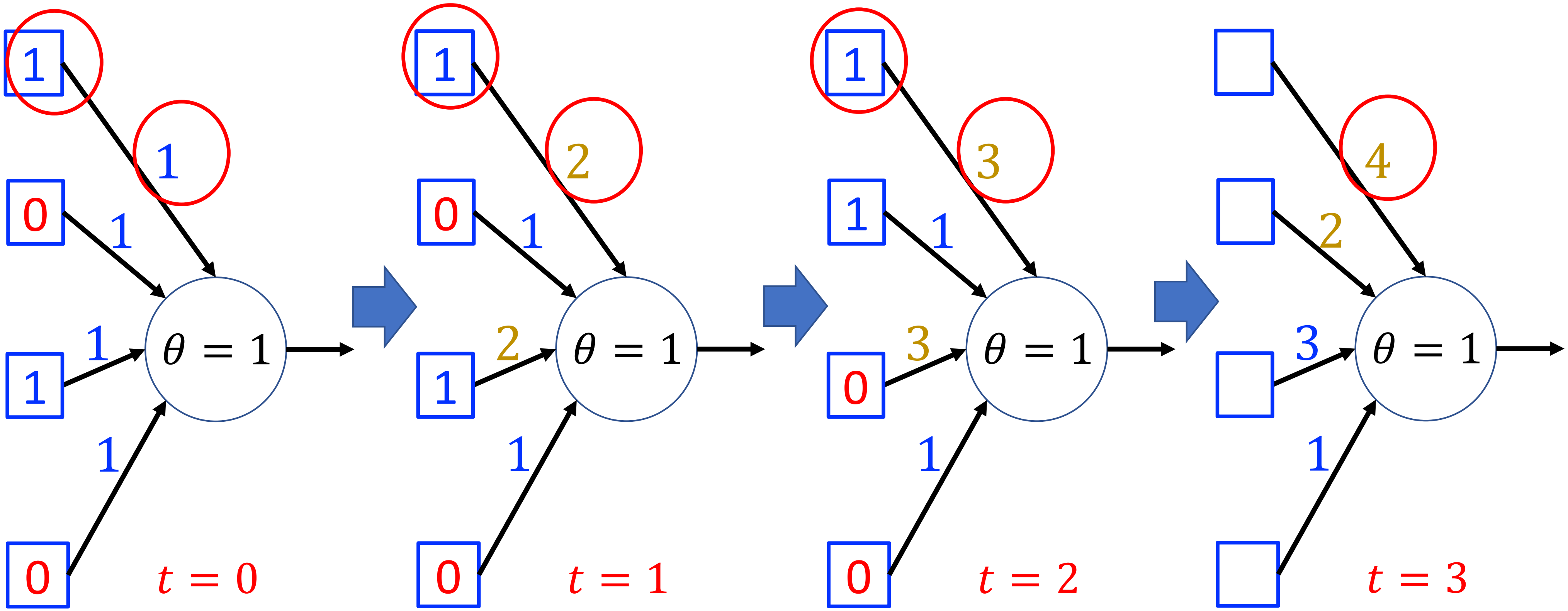
w_1^3	w_2^3	w_3^3	w_4^3
4	2	3	1

Wait for the inputs at $t = 3$ ...

Meaning behind Hebb's Rule



Meaning behind Hebb's Rule



Intuition: If two adjacent neurons always fire together, they should have strong relation (large weight).

Theory behind Hebb's Rule

Under some assumptions:

Assumption 1: Simplified activation function

$$X^{t+1} = g(S^t) = H(S^t - \theta) = \begin{cases} 1, & S^t \geq \theta; \\ 0, & S^t < \theta. \end{cases} \Rightarrow X^{t+1} = g(S^t) = \sum_i w_i^t a_i^t$$

Assumption 2: All the inputs a^t are from a given finite data set D with the cardinality N. In each epoch, we train the neuron with all the inputs within the data set. We have infinite epochs.

Theory behind Hebb's Rule

From the discrete $\Delta w_i^t = C a_i^t X^{t+1}$ to the continuous form.

$$\frac{dw_i}{dt} = C a_i X = C a_i \sum_i w_i a_i \quad \Rightarrow \quad \frac{dw}{dt} = C a X = \underline{C a a^T} w$$ **(Matrix form)**

Where $a = [a_1 \quad \cdots \quad a_n]^T, w = [w_1 \quad \cdots \quad w_n]^T$

In each epoch, all the N inputs within the data set are considered, thus we can use the average $\bar{a}$ over the data set as the increase rate, represented as $\bar{a}\bar{a}^T$.

$$\frac{dw}{dt} = \underline{C\bar{a}\bar{a}^T} w = \alpha w \qquad \text{, with } \alpha = C\bar{a}\bar{a}^T$$

Theory behind Hebb's Rule

From the discrete $\Delta w_i^t = Ca_i^t X^{t+1}$ to the continuous form.

$$\frac{dw_i}{dt} = Ca_i X = Ca_i \sum_i w_i a_i \Rightarrow \frac{dw}{dt} = CaX = \underline{Caa^T w}$$ **(Matrix form)**

Where $a = [a_1 \quad \cdots \quad a_n]^T, w = [w_1 \quad \cdots \quad w_n]^T$

In each epoch, all the N inputs within the data set are considered, thus we can use the average $\bar{a}$ over the data set as the increase rate, represented as $\bar{a}\bar{a}^T$.

$$\frac{dw}{dt} = \underline{C\bar{a}\bar{a}^T} w = \alpha w \qquad \text{, with } \alpha = C\bar{a}\bar{a}^T$$

Positive-definite

The average $\bar{a}$ always exists, but we don't know what it is. The "**average**" is the feature of the data set D that is learnt by the Hebb's rule.

Theory behind Hebb's Rule

$$\frac{dw}{dt} = C\bar{a}\bar{a}^T W = \alpha W \quad \text{, with } \alpha = C\ \bar{a}\bar{a}^T$$

Positive-definite

We can obtain the (**unique**) solution of this ordinary differential equation (ODE).

$$W(t) = k_1 e^{\alpha_1 t}\delta_1 + k_2 e^{\alpha_2 t}\delta_2 + \cdots + k_m e^{\alpha_m t}\delta_m$$

where

Power expressions with the same base e: the exponent matters

$\boldsymbol{k_i}$: coefficients determined by initial value,
$\boldsymbol{\delta_i}$: the i-th eigenvector of α, $\boldsymbol{\alpha_i}$: the i-th eigenvalue of α

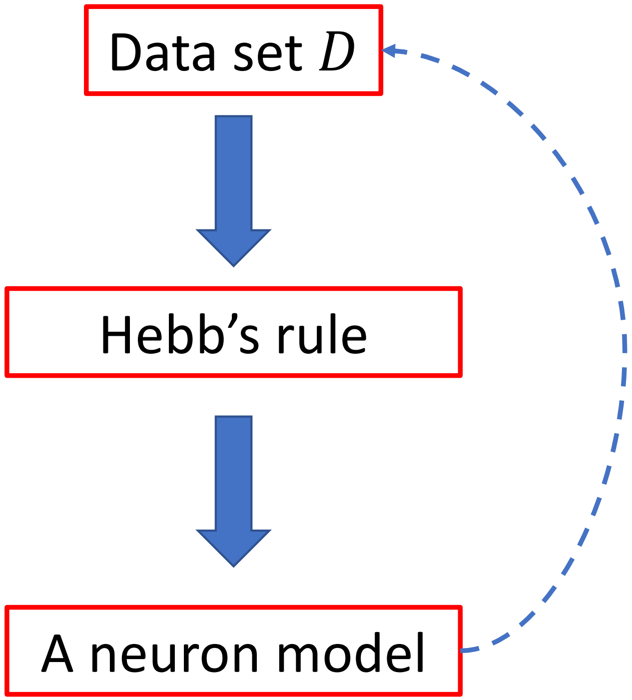
PCA
Principal components analysis

When the sufficient time has passed, that is, t is large enough,

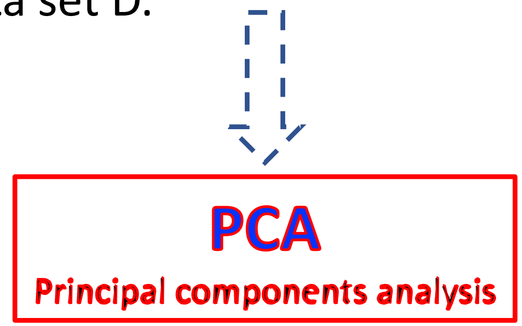
$$W(t) \approx k^* e^{\alpha^* t}\delta^* \qquad \Longrightarrow \qquad X(t) \approx k^* e^{\alpha^* t}\delta^* a$$

Largest eigenvalue: Dominated by the term of largest eigenvalue

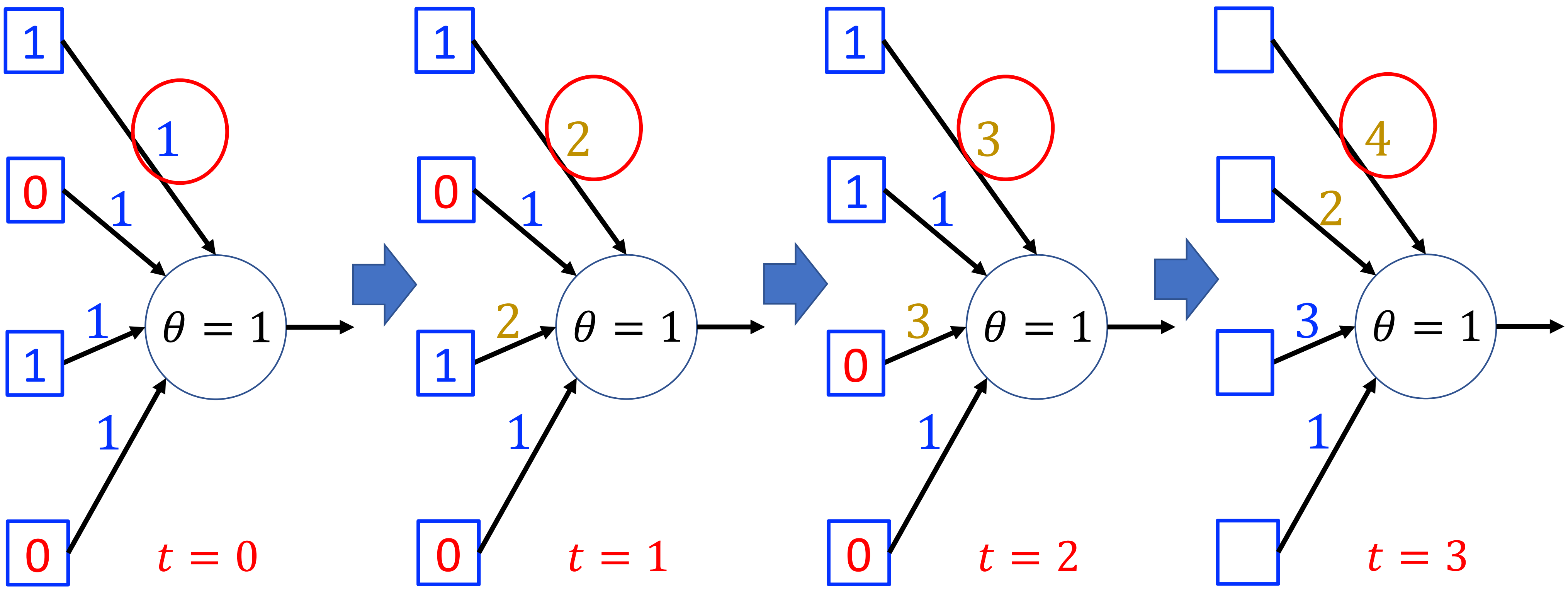
Theory behind Hebb's Rule



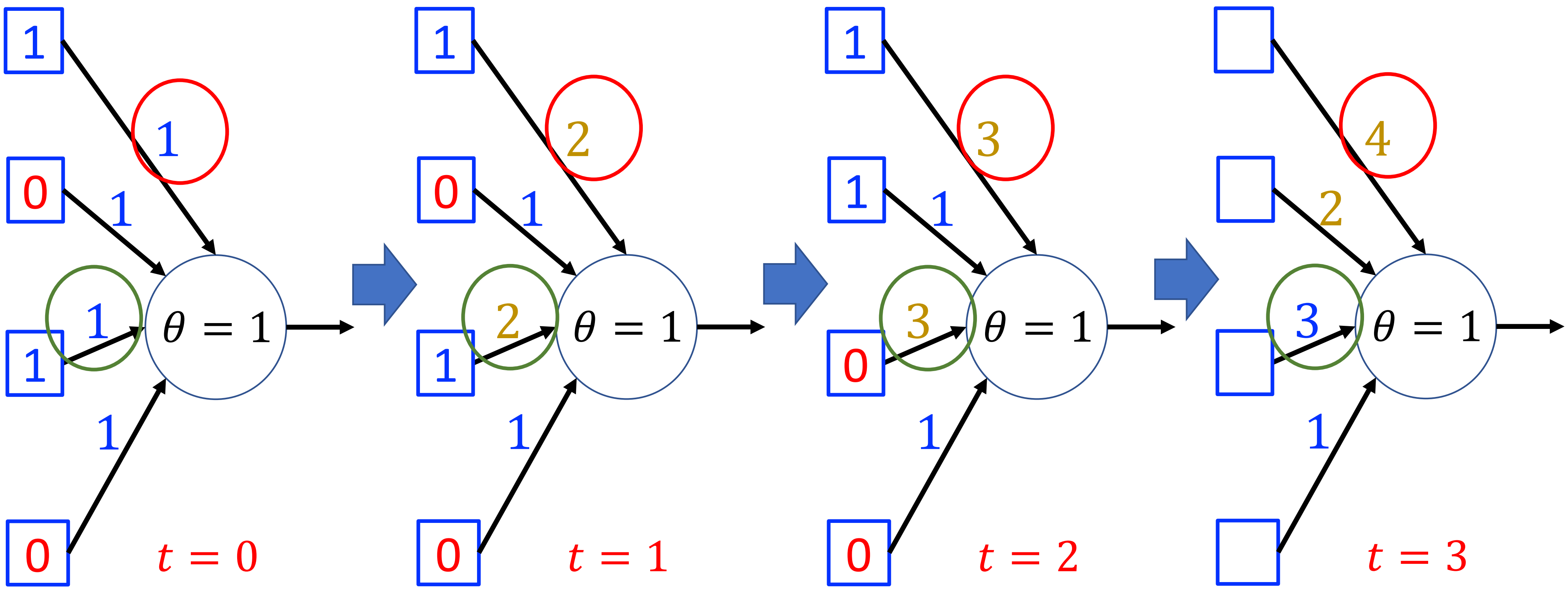
The **weights** of the neuron model after Hebbian learning, reflects **the most important feature of the average $\overline{a}$** of the data set D.



Example: w_1 and w_3



Example: w_1 and w_3



All the weights increase **monotonously**. Finally, each weight will become large enough such that any activated input can fire the neuron alone.