Comp305

Biocomputation

Lecturer: Yi Dong

Comp305 Part I.

Artificial Neural Networks

Topic 2.

The McCulloch-Pitts Neuron (1943)

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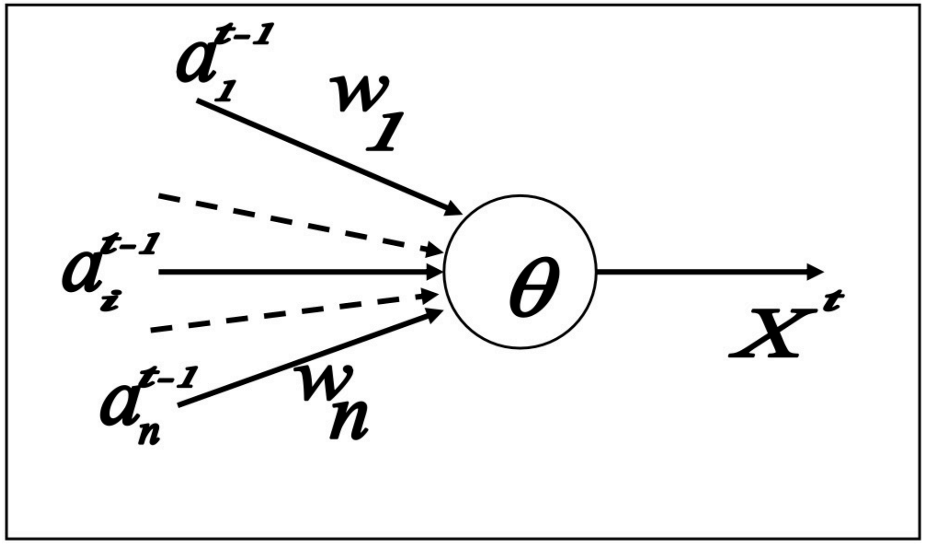
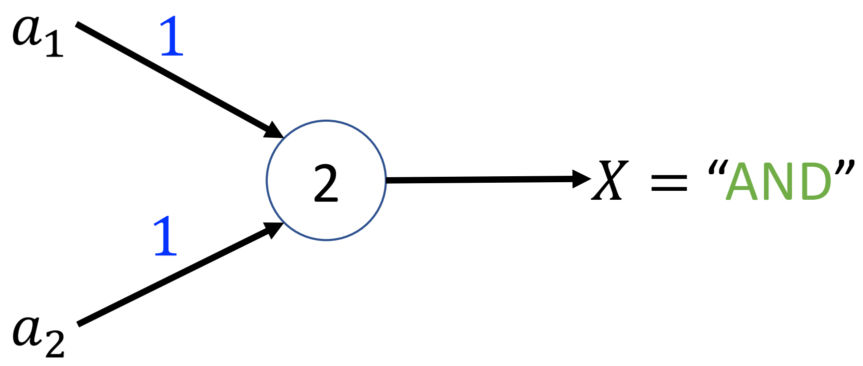
McCulloch and Pitts modelled the neuron as

- a ***binary,*** ***discrete-time*** input
- with ***excitatory*** ***and*** ***inhibitory*** ***connections and an*** ***excitation threshold***.



MP-Neuron Logic: Two Inputs

a_1	a_2	"AND"
1	1	1
0	1	0
1	0	0
0	0	0



"AND" – the output fires if a_1 and a_2 both fire.

MP-Neuron Logic: Two Inputs

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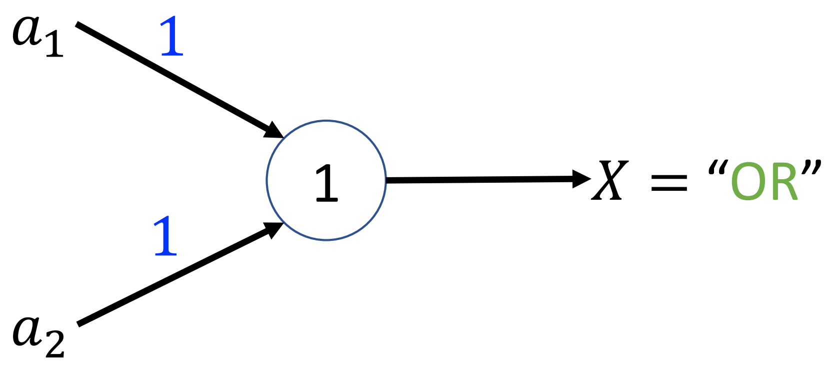
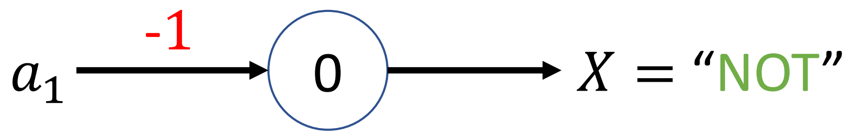


"OR" – the output fires if a_1 fires or a_2 fires or both fire.

MP-Neuron Logic: Two Inputs

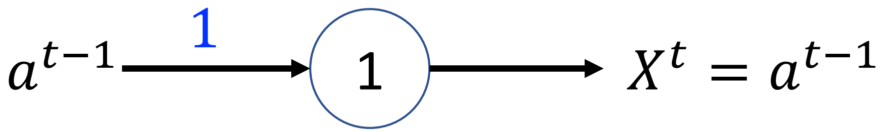
a_1	**"NOT"**
1	0
0	1



"NOT" – the output fires if a_1 does **NOT** fire and vice versa.

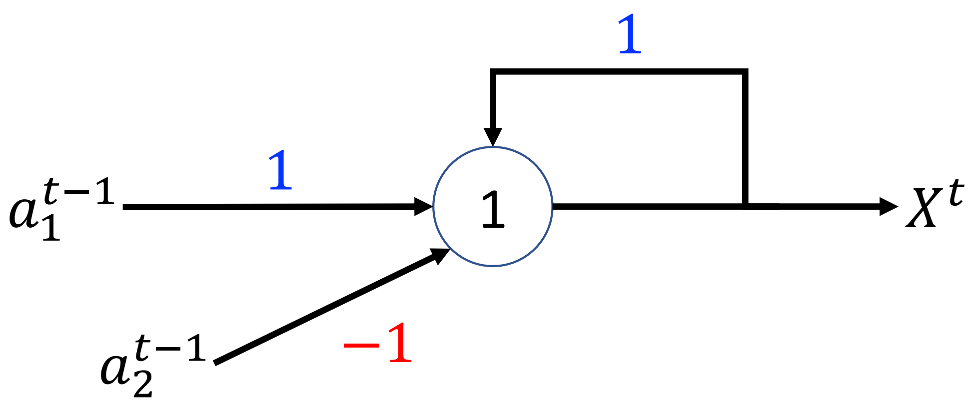
MP-Neuron as a Register Cell

a_1	"Reg"
1	1
0	0



Such a single neuron thus behaves as a single register cell able to retain the input for one period elapsing between two instant.

(Extended) MP-Neuron as a Memory Cell



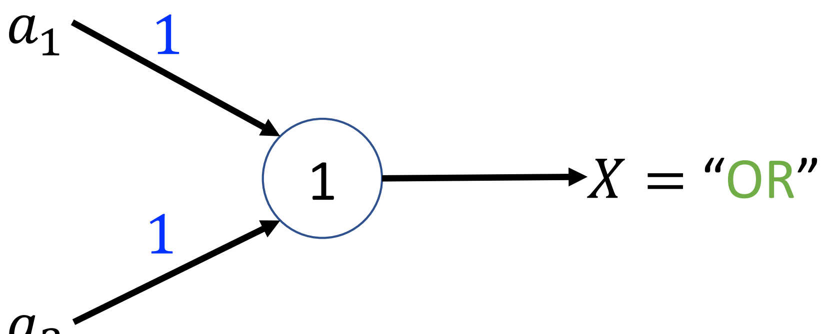
With a feedback loop closed around the neuron, as it is shown above, we obtain a memory cell. Note that it is not a single MP-neuron with the classical definition.

Topic of Today's Lecture

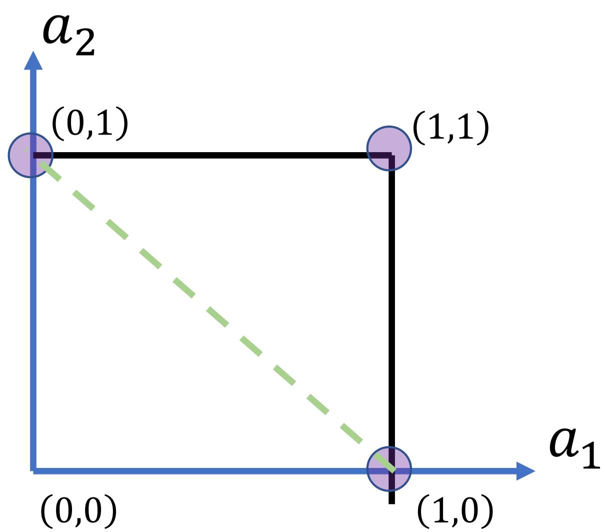
What kind of propositions can be represented by a single MP neuron (without time)?

Representation Power

Geometric Interpretation: 2D “OR”

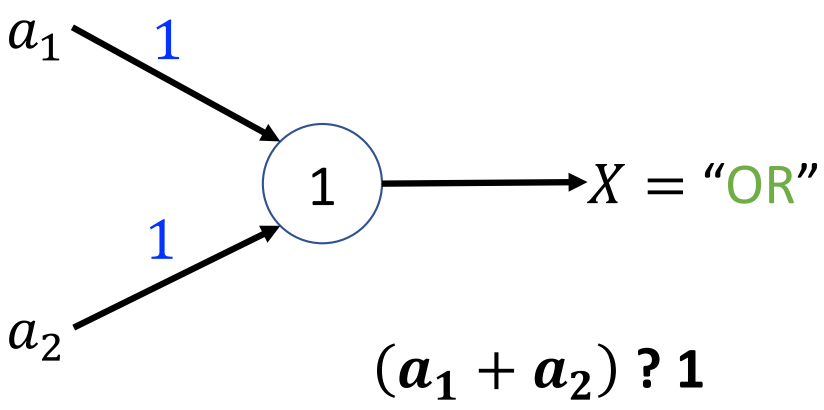


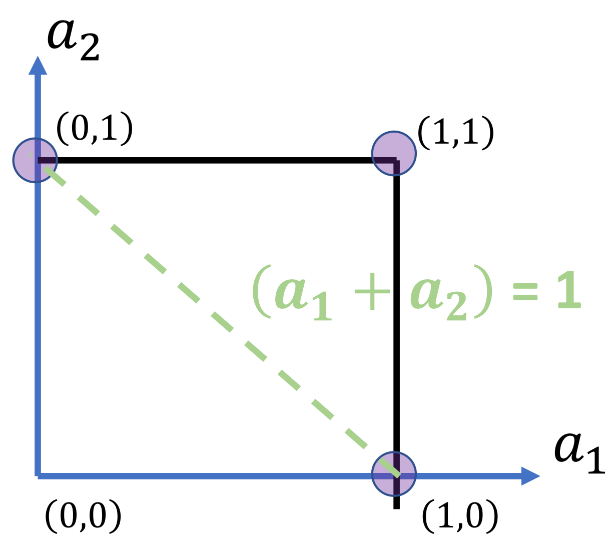
$$(\boldsymbol{a_1} + \boldsymbol{a_2})\ \boldsymbol{?}\ \boldsymbol{1}$$



- A single MP neuron splits the input space (4 points in the case of 2 inputs) into two halves

Geometric Interpretation: 2D "OR"





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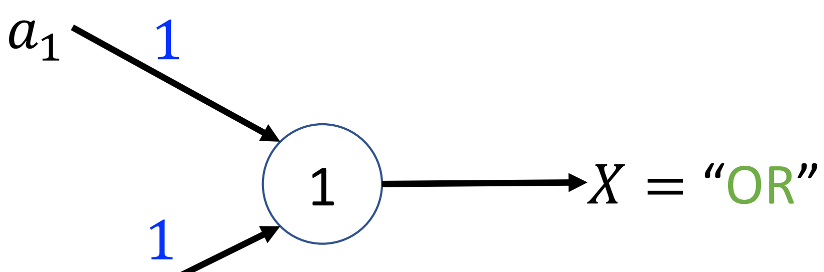
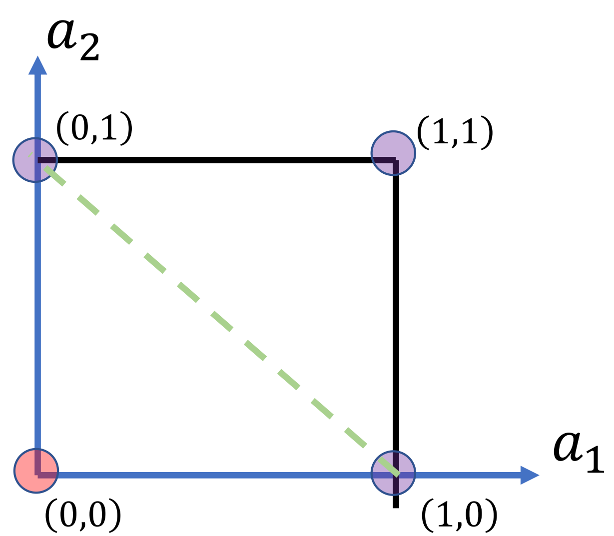
Geometric Interpretation: 2D "OR"

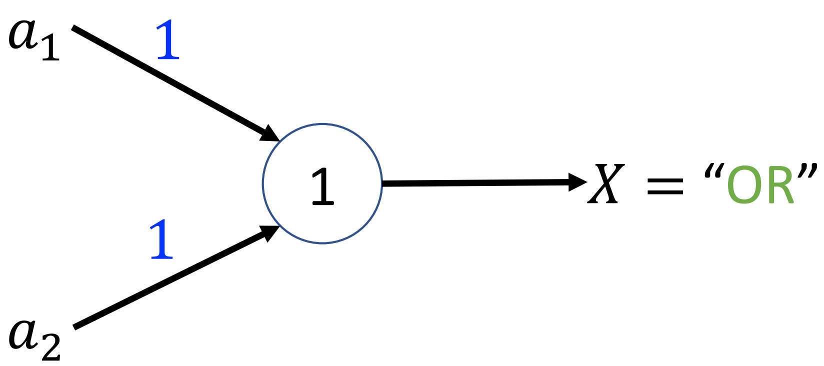


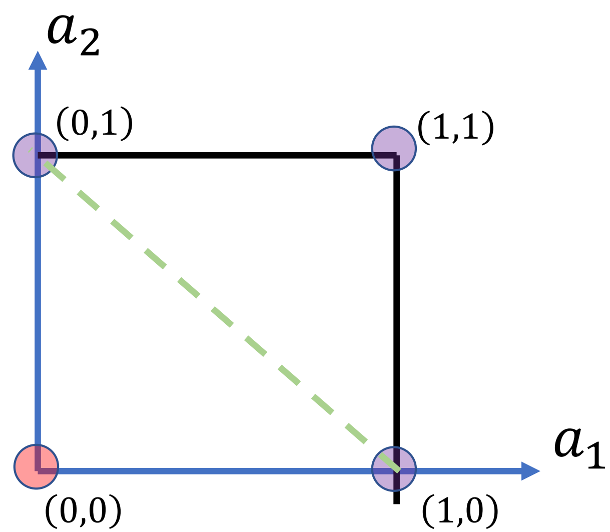
$$(\boldsymbol{a_1} + \boldsymbol{a_2})\ \mathbf{?}\ \mathbf{1}$$



- Two categories: Points lying on or above the line $\sum_{i=1}^{2} a_i - 1 = 0$ and points lying below this line

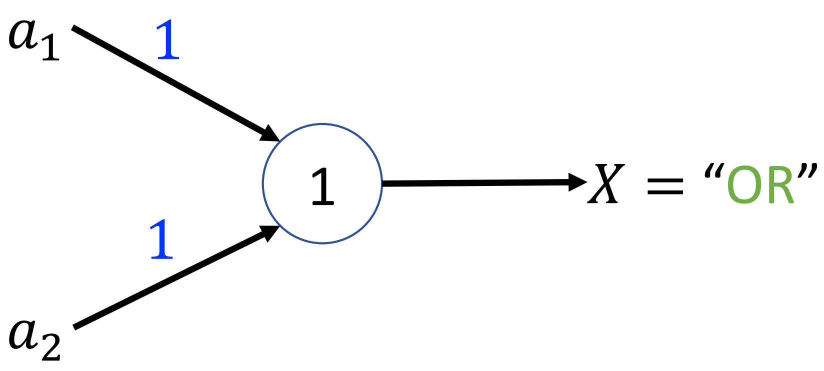
Geometric Interpretation: 2D "OR"

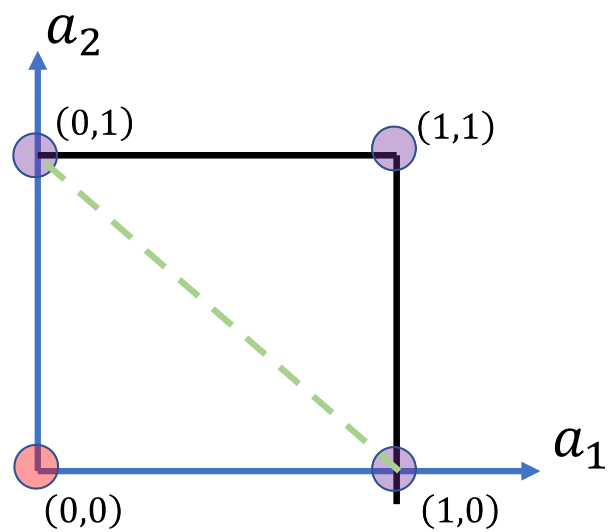




- In other words, all inputs firing the neuron will be one side of the line, while all inputs inhibiting the neuron lie on the other side.

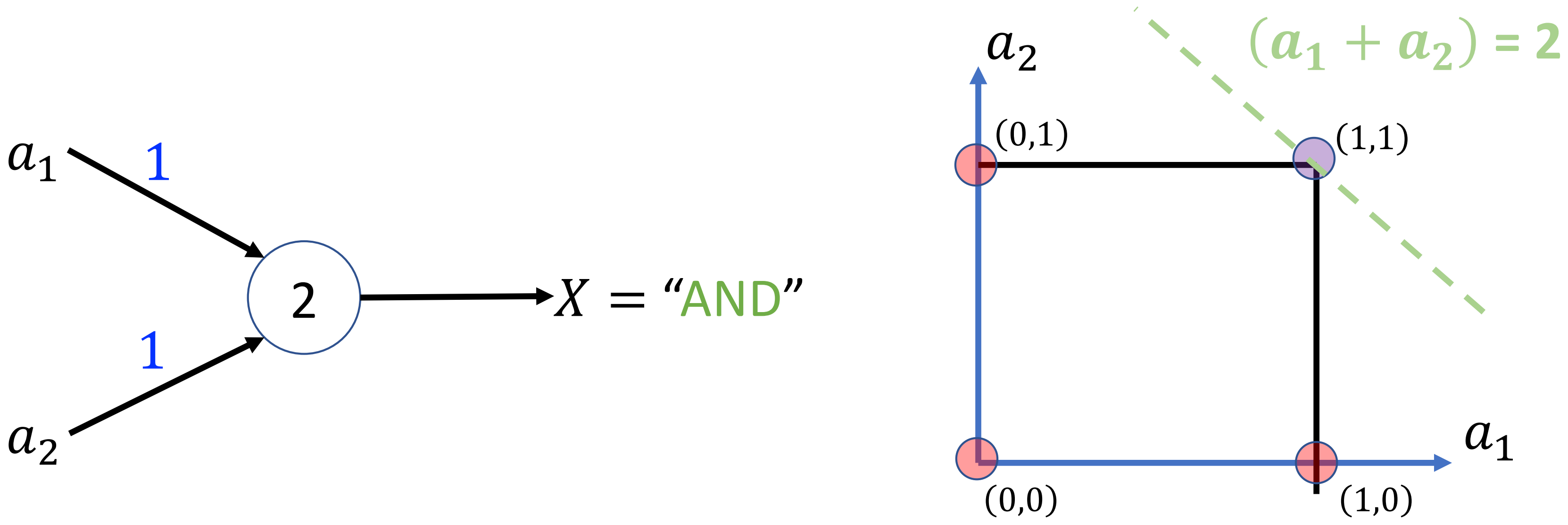
Geometric Interpretation: 2D "OR"





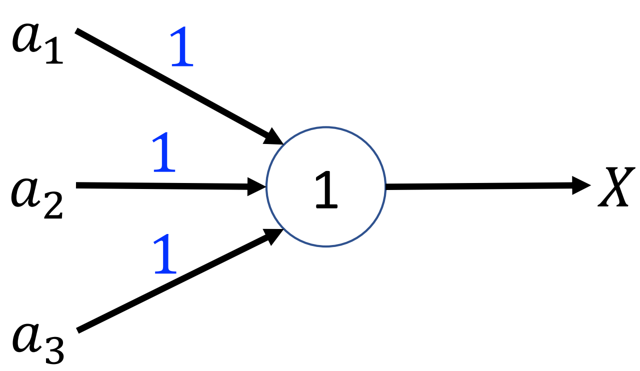
- Does a MP neuron implement a boundary that linearly separates the input space? Let us see more examples…

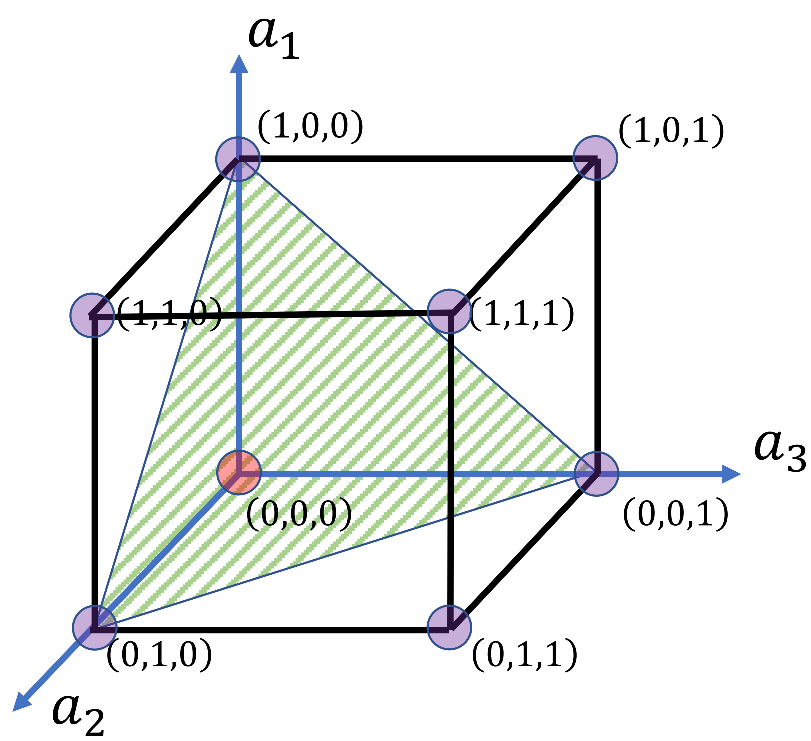
Geometric Interpretation: 2D “AND”



- For “AND” operation, the MP neuron above represents a decision maker separating the input space linearly with $\sum_{i=1}^{2} a_i - 2 = 0$

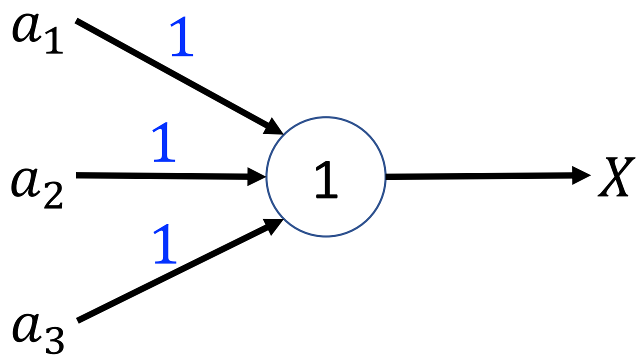
Geometric Interpretation: Multiple Inputs Case

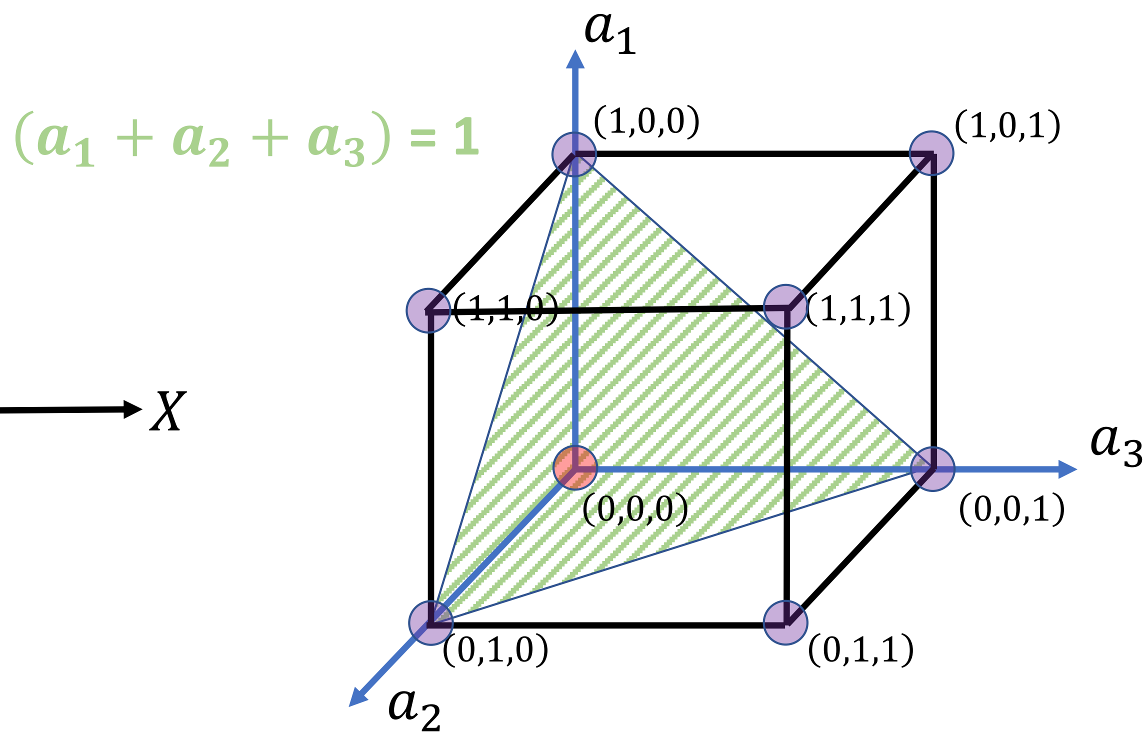




- What if we have more than 2 inputs?

Geometric Interpretation: Multiple Inputs Case





- What if we have more than 2 inputs?
- Instead of a line, we have a plane.

Representation Power of a single MP Neuron

- A single MP neuron can be used to represent <u>some</u> Boolean functions which are linearly separable.

- Linear separability (for Boolean functions): There exists a line (plane) such that all inputs which produce a 1 for the function lie on one side of the line (plane) and all inputs which produce a 0 lie on other side of the line (plane).

Representation Power of a single MP Neuron

- A single MP neuron can be used to represent some Boolean functions which are linearly separable.

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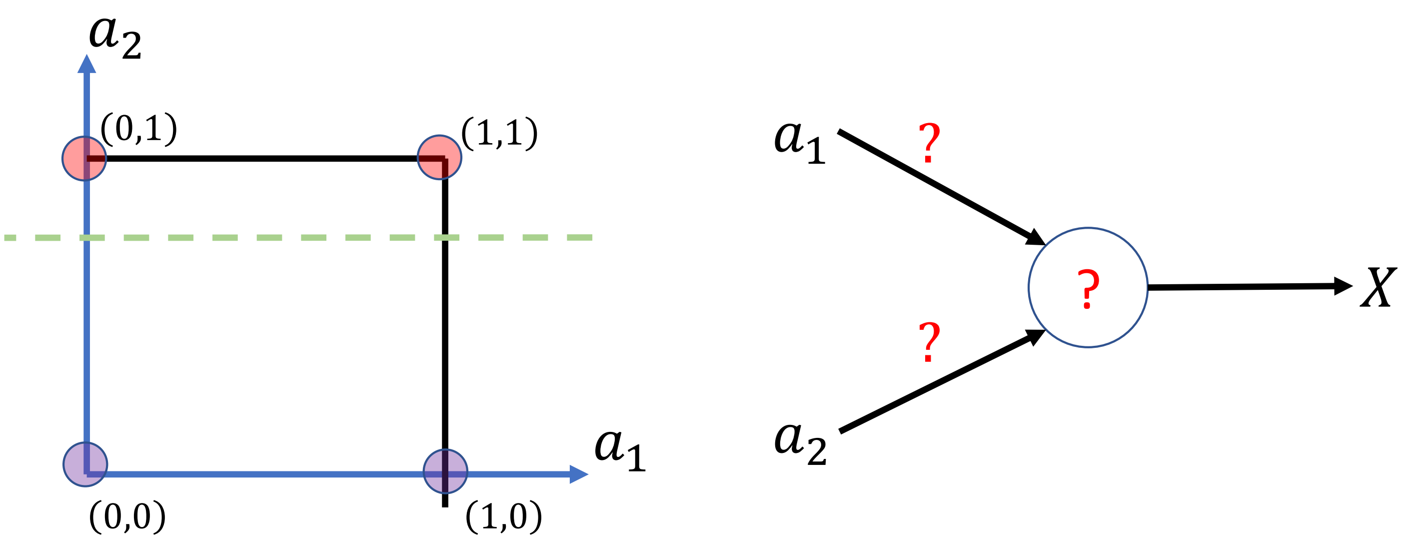
- Completeness: Can each linearly separable function be represented by a single MP neuron?

A Counterexample



- Try to prove there is no single MP neuron implementation for the linear separable function indicated in the figure.

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- Completeness: Can each linearly separable function be represented by a single MP neuron? <u>**No**</u>!

Representation Power of a single MP Neuron

- A single MP neuron can be used to represent some Boolean functions which are linearly separable.

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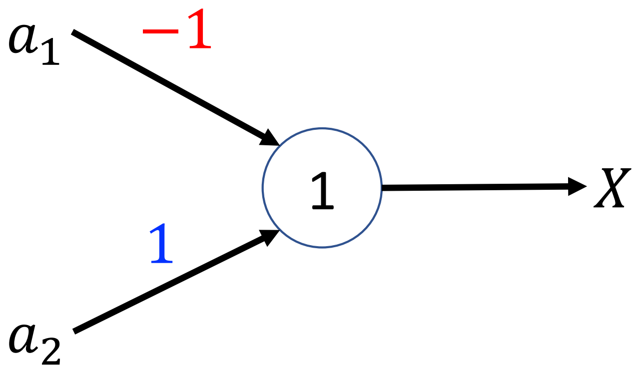
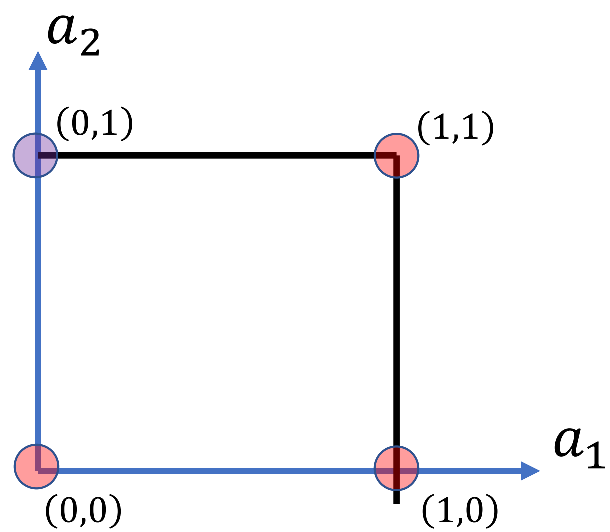
- Completeness: Can each linearly separable function be represented by a single MP neuron? **No**!

- Is there any other function that can be represented by an MP neuron?

What if we have inhibitory connections?





- Enumerating all the cases, it is easy to obtain that only $(0,1)$ fires the neuron. Does the neuron act as a linear boundary in this case?

Revisit definition

Recall the definition of MP neuron.

$$X^t = 1 \text{ if and only if } S^{t-1} = \sum_{i=1}^{n} w_i a_i^{t-1} \geq \theta, \text{ and } w_i > 0, \forall a_i^{t-1} > 0.$$

Assume that $w_1 = -1$,

$$S^{t-1} = \begin{cases} -1, & a_1 = 1 \\ \sum_{i=2}^{n} w_i a_i^{t-1}, & \boxed{a_1 = 0} \end{cases}$$

A plain as a linear boundary.

Revisit definition

Recall the definition of MP neuron.

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$$S^{t-1} = \begin{cases} -1, & a_1 = 1 \\ \sum_{i=2}^{n} 1 \times a_i^{t-1}, & \boxed{a_1 = 0} \end{cases}$$

A plain as a linear boundary.

Revisit definition

Recall the definition of MP neuron.

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Assume that $w_1 = -1$,

$$S^{t-1} = \begin{cases} -1, \\ \sum_{i=2}^{n} a_i^{t-1}, \end{cases}$$

$\boxed{a_1 = 1}$ The face of $a_1 = 1$.

$\boxed{a_1 = 0}$ A plain as a linear boundary.

After obtaining a boundary, remove the face of $a_1 = 1$ from the positive side.

Revisit definition

Recall the definition of MP neuron.

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Assume that $w_1 = -1$,

$$S^{t-1} = \begin{cases} -1, \\ \sum_{i=2}^{n} a_i^{t-1}, \end{cases} \quad \boxed{a_1 = 1} \quad \boxed{a_1 = 0}$$

The face of $a_1 = 1$.

Q: $= \sum_{i=1}^{n} a_i^{t-1}$?

After obtaining a boundary, remove the face of $a_1 = 1$ from the positive side.

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The face of $a_1 = 1$.

For a plane $\sum_{i=1}^{n} a_i^{t-1} = \theta$, remove the face of $a_1 = 1$ from the positive side.

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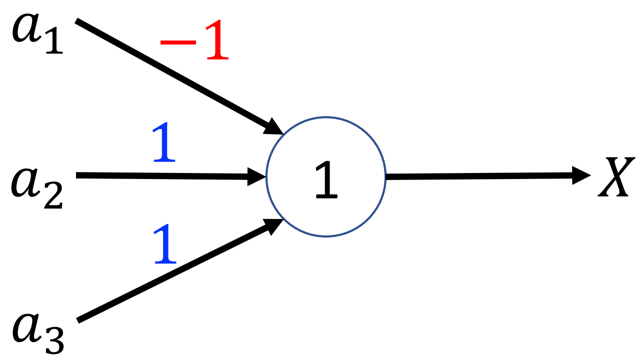
$$S^{t-1} = \begin{cases} -1, & \boxed{a_1 = 1} \\ \sum_{i=1}^{n} a_i^{t-1}, & \boxed{a_1 = 0} \end{cases}$$

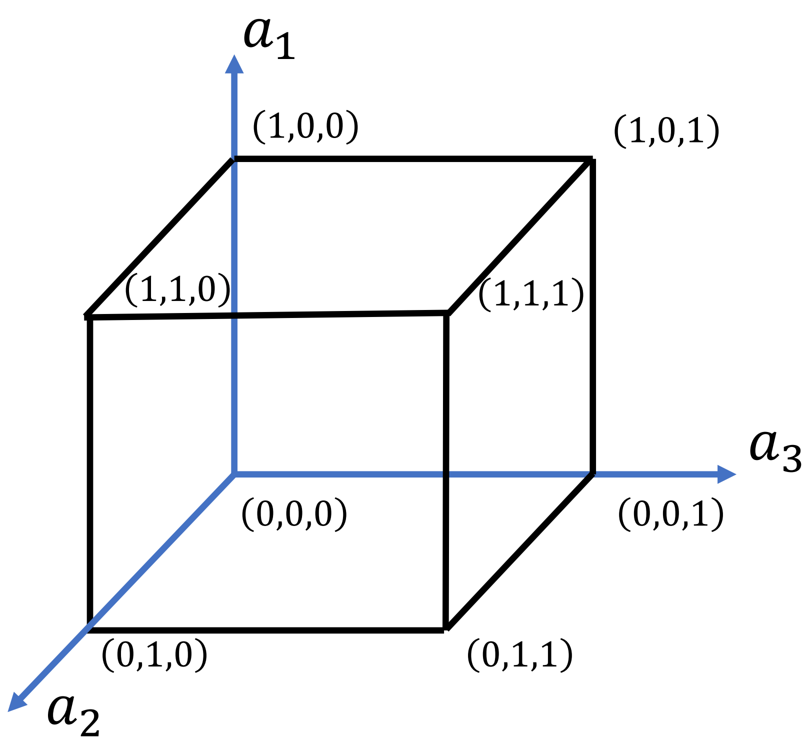
The face of $a_1 = 1$.

For a plane $\sum_{i=1}^{n} a_i^{t-1} = \theta$, remove the face of $a_1 = 1$ from the positive side.

Iteratively do the above, until all the inhibitory connections are considered.

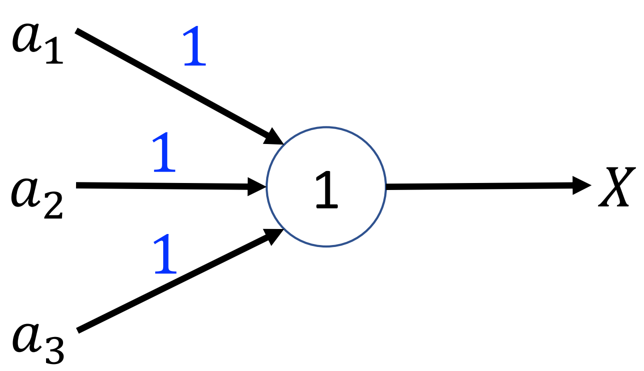
Clearer in Multiple Inputs Case

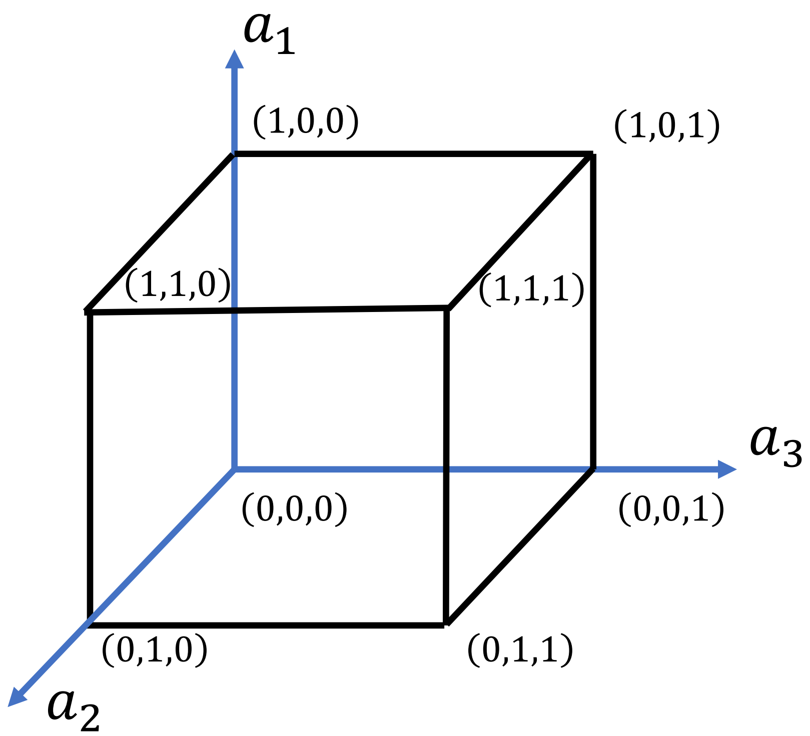




- The state space is consisted of all the vertexes of a cube .

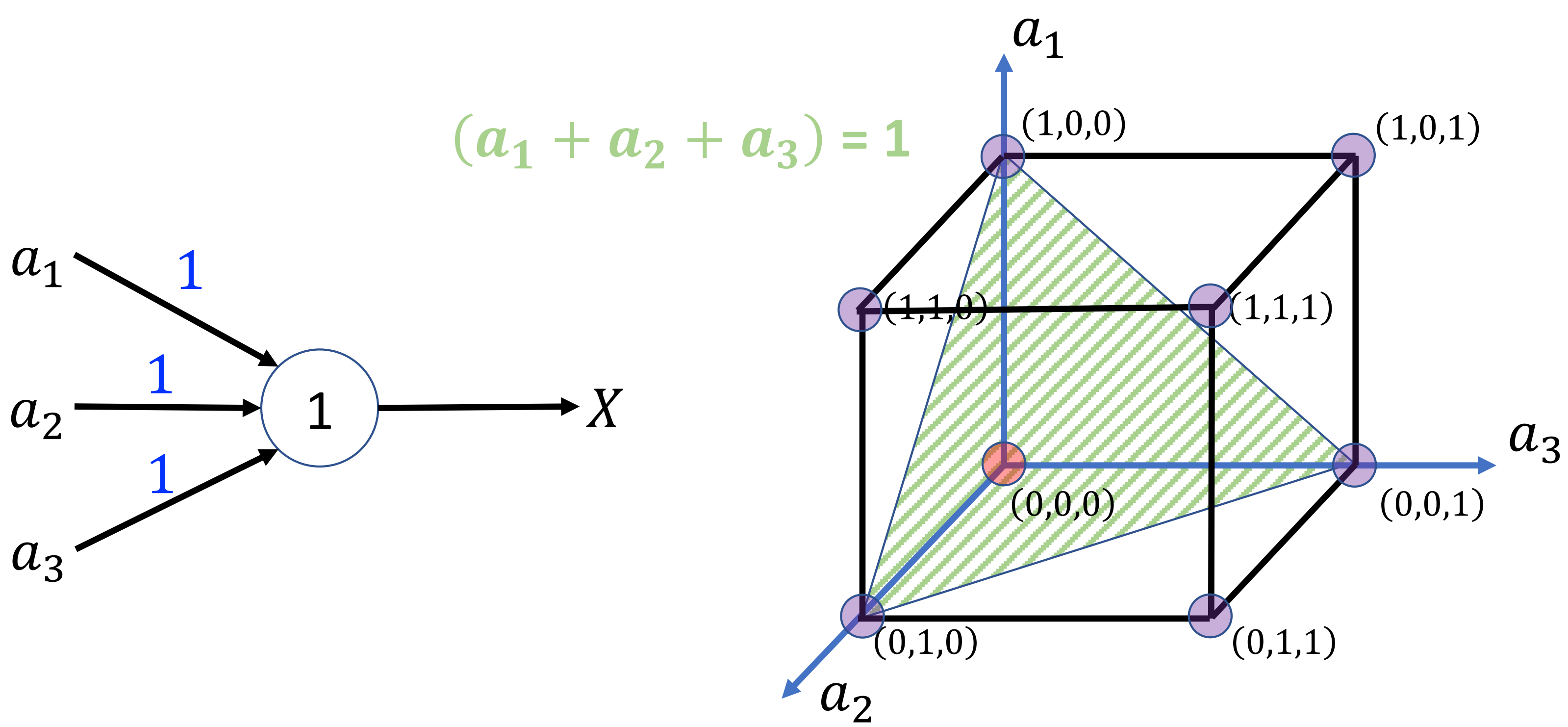
Clearer in Multiple Inputs Case





- We first consider the revised neuron as above.

Clearer in Multiple Inputs Case



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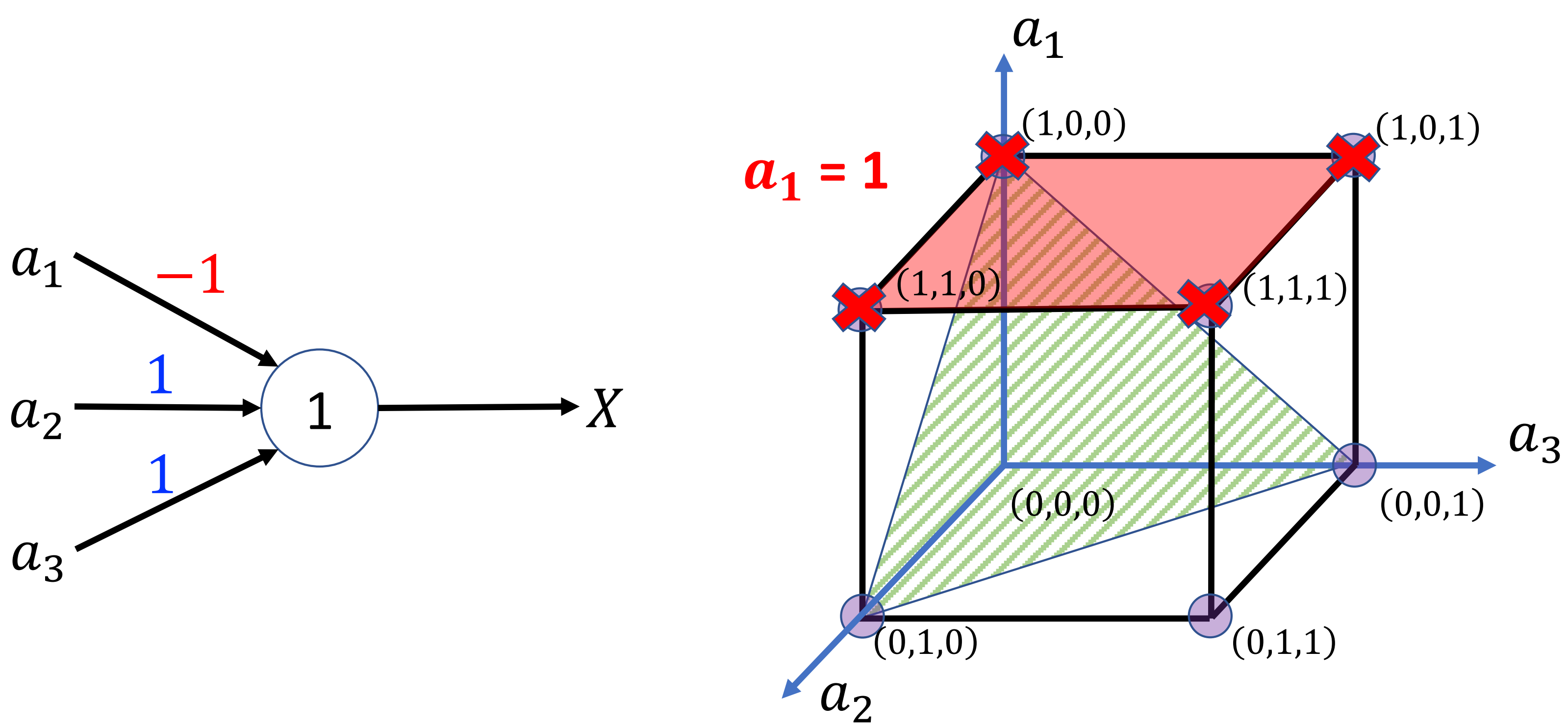
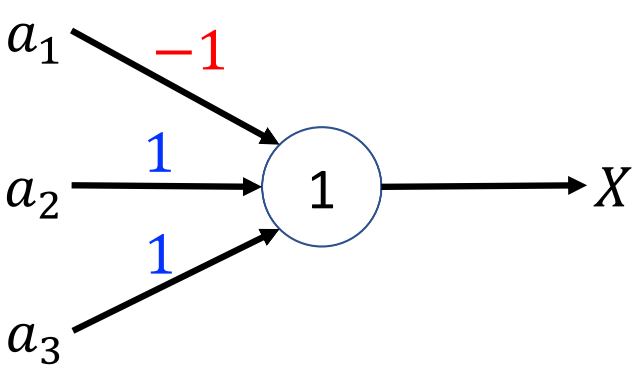
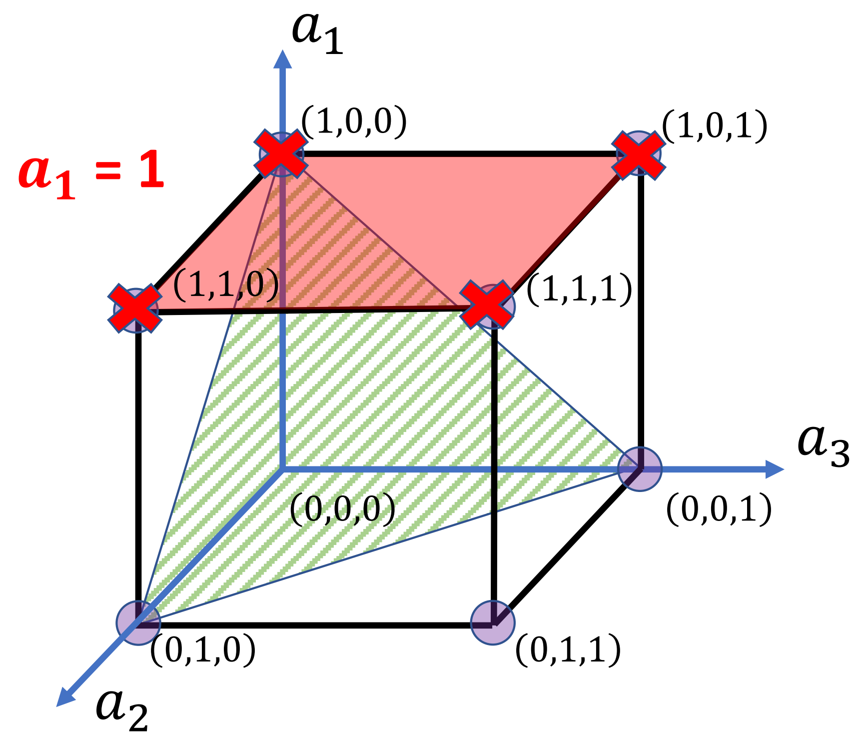
Clearer in Multiple Inputs Case



- The we remove the face of $a_1 = 1$ from the positive side.

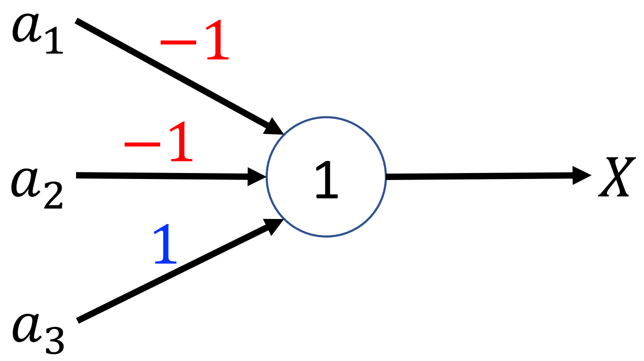
Clearer in Multiple Inputs Case





- The inputs that fire the neuron are (0,1,1), (0,0,1), (0,1,0).

Another example



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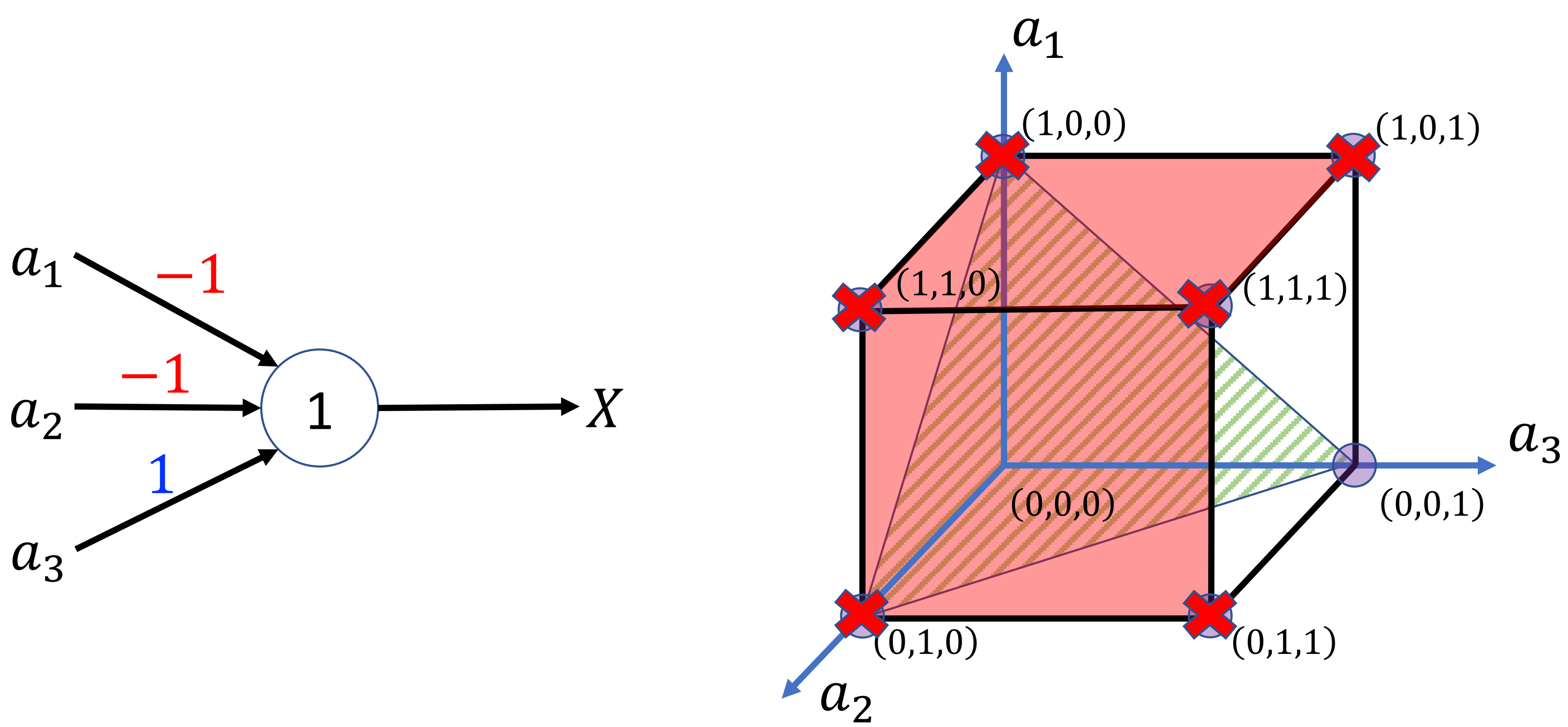

- The only input that fires the neuron is (0,0,1).

Representation Power of a single MP Neuron

- A single MP neuron can be used to represent some Boolean functions which are linearly separable.

- Linear separability (for Boolean functions): There exists a line (plane) such that all inputs which produce a 1 for the function lie on one side of the line (plane) and all inputs which produce a 0 lie on other side of the line (plane).

- Completeness: Can each linear separable function be represented by a single MP neuron? **No**!

- A single MP neuron describes a specific linear boundary that is only determined by the threshold in the hyper-cube state space, removing the faces corresponding to inhibitory connections.