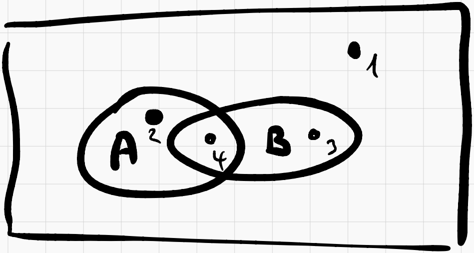
Proving identities: $A \Delta B = (A \cup B) - (A \cap B)$

Prove for $\forall A, B$ if A and B are sets then $A \Delta B = (A \cup B) - (A \cap B)$

Suppose that x is a particular but arbitrarily chosen



<u>Case 1</u> $x \notin A$, $x \notin B$

By def. of $A \Delta B$, $x \notin A \Delta B$

By def. of $\cup, \cap$ we have $x \notin A \cup B$, $x \notin A \cap B$

Hence $x \notin (A \cup B) - (A \cap B)$

Proof continues

Case 2 $\quad x \in A,\ x \notin B$

$x \in A \Delta B$

By def of union, $\quad x \in A \cup B$

By def. of intersection, $\quad x \notin A \cap B$

So $\quad x \in (A \cup B) - (A \cap B)$

Case 3 is similar

Case 4 $\quad x \in A, \quad x \in B$

By def. of Δ, $\quad x \notin A \Delta B$

$x \in A \cup B$

$x \in A \cap B$

$x \notin (A \cup B) - (A \cap B)$

The algebra of sets

The algebra of sets (1)

Suppose that A, B, C, U are sets with $A \subseteq U$, $B \subseteq U$, $C \subseteq U$

Commutative laws (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.

Associative laws (a) $A \cup (B \cup C) = (A \cup B) \cup C$ and
(b) $A \cap (B \cap C) = (A \cap B) \cap C$.

Distributive laws (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and
(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Identity laws (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$

Complement laws (a) $A \cup \sim A = U$ and (b) $A \cap \sim A = \emptyset$.

The algebra of sets (2)

Double complement law	$\sim (\sim A) = A$.
Idempotent laws	(a) $A \cup A = A$ and (b) $A \cap A = A$.
Universal bound laws	(a) $A \cup U = U$ and (b) $A \cap \emptyset = \emptyset$.
De Morgan's law	(a) $\sim (A \cup B) = \sim A \cap \sim B$ and (b) $\sim (A \cap B) = \sim A \cup \sim B$
Absorption laws	(a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$
Complement of U and $\emptyset$	(a) $\sim U = \emptyset$ and (b) $\sim \emptyset = U$
Set difference law	$A - B = A \cap \sim B$

Proving the commutative law $A \cup B = B \cup A$

Definition: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$ $B \cup A = \{x \mid x \in B \text{ or } x \in A\}$.

These are the same set. To see this, check all possible cases.

Case 1: Suppose $x \in A$ and $x \in B$. Since $x \in A$, the definitions above show that x is in both $A \cup B$ and $B \cup A$.

Case 2: Suppose $x \in A$ and $x \notin B$. Since $x \in A$, the definitions above show that x is in both $A \cup B$ and $B \cup A$.

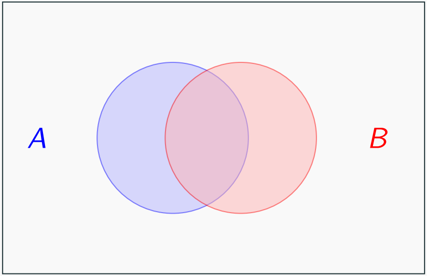
Case 3: Suppose $x \notin A$ and $x \in B$. Since $x \in B$, the definitions above show that x is in both $A \cup B$ and $B \cup A$.

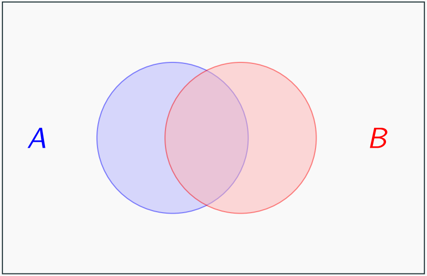
Case 4: Suppose $x \notin A$ and $x \notin B$. The definitions above show that x is not in $A \cup B$ and x is not in $B \cup A$.

So, for all possible x, either x is in both $A \cup B$ and $B \cup A$, or it is in neither. We conclude that the sets $A \cup B$ and $B \cup A$ are the same.

De Morgan's laws

$$\sim (A \cap B) = \sim A \cup \sim B.$$





A proof of De Morgan's law $\sim(A \cap B) = \sim A \cup \sim B$

Case 1: Suppose $x \in A$ and $x \in B$. From the definition of $\cap$, $x \in A \cap B$. So from the definition of $\sim$, $x \notin \sim(A \cap B)$. From the definition of $\sim$, $x \notin \sim A$ and also $x \notin \sim B$. So from the definition of $\cup$, $x \notin \sim A \cup \sim B$.

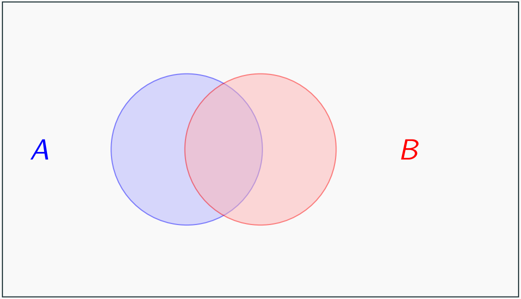
Case 2: Suppose $x \in A$ and $x \notin B$. From the definition of $\cap$, $x \notin A \cap B$. So from the definition of $\sim$, $x \in \sim(A \cap B)$. From the definition of $\sim$, $x \notin \sim A$ but $x \in \sim B$. So from the definition of $\cup$, $x \in \sim A \cup \sim B$.

Case 3: Suppose $x \notin A$ and $x \in B$. From the definition of $\cap$, $x \notin A \cap B$. So from the definition of $\sim$, $x \in \sim(A \cap B)$. From the definition of $\sim$, $x \in \sim A$ but $x \notin \sim B$. So from the definition of $\cup$, $x \in \sim A \cup \sim B$.

Case 4: Suppose $x \notin A$ and $x \notin B$. From the definition of $\cap$, $x \notin A \cap B$. So from the definition of $\sim$, $x \in \sim(A \cap B)$. From the definition of $\sim$, $x \in \sim A$ and $x \in \sim B$. So from the definition of $\cup$, $x \in \sim A \cup \sim B$.

Using the algebra of sets

Prove that $\boxed{(A \cap \sim B) \cup (B \cap \sim A) = (A \cup B) \cap \sim (A \cap B).}$



Algebraic proof

$$
\begin{aligned}
\underline{(A \cup B) \cap \sim (A \cap B)} &= (A \cup B) \cap (\sim A \cup \sim B) \text{ De Morgan} \\
&= ((A \cup B) \cap \sim A) \cup ((A \cup B) \cap \sim B) \text{ distributive} \\
&= (\sim A \cap (A \cup B)) \cup (\sim B \cap (A \cup B)) \text{ commutative} \\
&= ((\sim A \cap A) \cup (\sim A \cap B)) \cup ((\sim B \cap A) \cup (\sim B \cap B)) \text{ distributive} \\
&= ((A \cap \sim A) \cup (B \cap \sim A)) \cup ((A \cap \sim B) \cup (B \cap \sim B)) \text{ commutative} \\
&= (\emptyset \cup (B \cap \sim A)) \cup ((A \cap \sim B) \cup \emptyset) \text{ complement} \\
&= \underline{(A \cap \sim B) \cup (B \cap \sim A)} \text{ commutative and identity}
\end{aligned}
$$

Cardinality of sets

Cardinality of sets

Definition The cardinality of a *finite* set A is the number of distinct elements in A, and is denoted by $|A|$.

$\#A$ $\quad\quad$ $\|A\|$

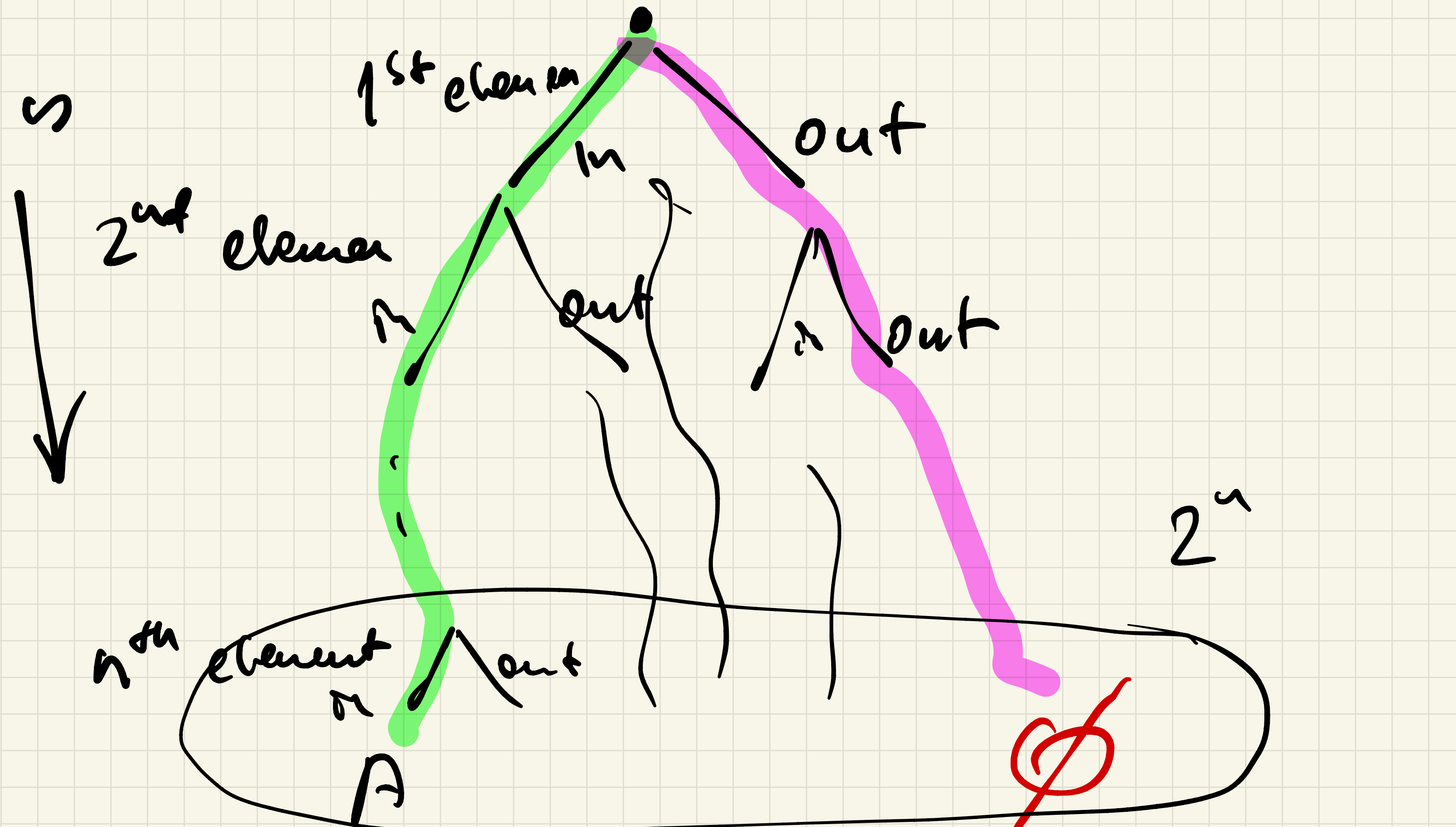
$$|\{1, 2, 5\}| = 3$$

$$|\{1, 1, 2, 2, 3\}| = 3$$

$$A = \{1, 2, 3\}$$

$$\{1\},\ \{2\},\ \{3\},\ \{1,2,3\},$$
$$\{1,2\},\ \{1,3\},\ \{2,1\},\ \emptyset$$

$$|Pow(A)| = 2^{|A|} \qquad |2^A| = 2^{|A|}$$



1^{st} element
in
out
2^{nd} element
in
out
in
out
n^{th} element
in
out
2^n
A
$\emptyset$

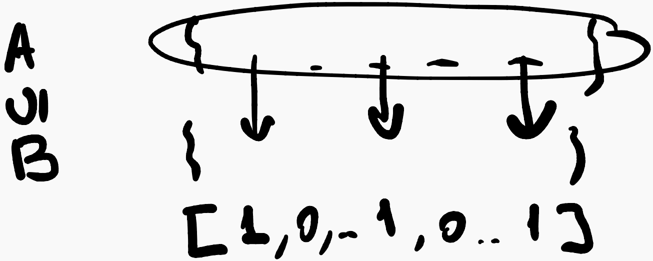
Example: The cardinality of the set of all subsets

Definition The **power set** $Pow(A)$ of a set A is the set of all subsets of A. In other words,

$$Pow(A) = \{C \mid C \subseteq A\}.$$

For all $n \in \mathbb{Z}^+$ and all sets A: if $|A| = n$, then $|Pow(A)| = 2^n$.

Power set and bit vectors



We use the correspondence between bit vectors and subsets: $|Pow(A)|$ is the number of bit vectors of length n.

The number of n-bit vectors is 2^n

We prove the statement by induction.

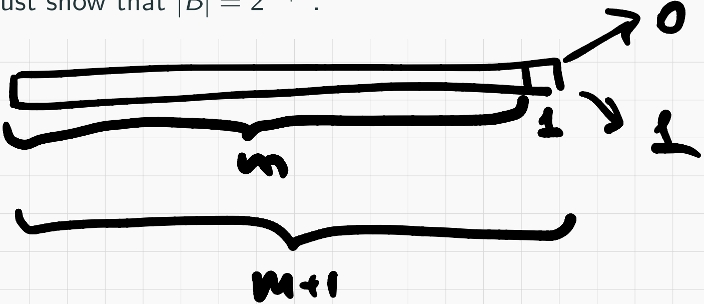
Base Case: $n = 1$

$[1]$
$[0]$

$2^1 = 2$

The number of n-bit vectors is 2^n

Inductive Step: Assume that the property holds for $n = m$, so the number of m-bit vectors is 2^m. Now consider the set B of all $(m+1)$-bit vectors. We must show that $|B| = 2^{m+1}$.

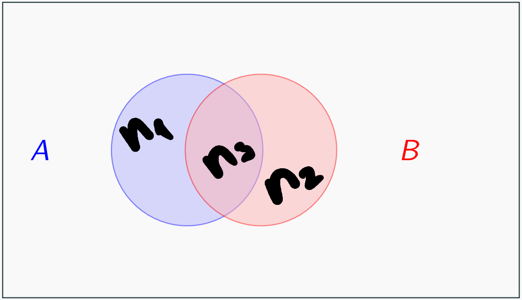


$$|B| = 2 \cdot 2^m = 2^{m+1}$$

If A and B are sets then

n_3

$$|A \cup B| = |A| + |B| - \underbrace{|A \cap B|}_{}.$$



$|A| = n_1 + n_3$

$|B| = n_2 + n_3$

$|A \cup B| = n_1 + n_2 + n_3$

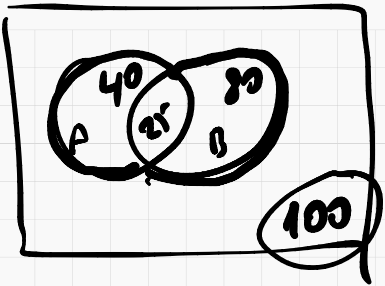
$|A| + |B| = n_1 + n_3 + n_2 + n_3$

Example

Suppose there are 100 third-year students. 40 of them take the module "Sequential Algorithms" and 80 of them take the module "Multi-Agent Systems". 25 of them took both modules. How many students took neither modules?

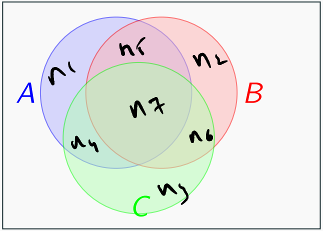
$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 40 + 80 - 25 = 95$$



Computing the cardinality of a union of three sets

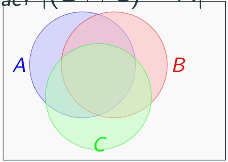
$$|A \cup B \cup C| = \underline{|A| + |B| + |C|} - \underline{|A \cap B| - |A \cap C| - |B \cap C|} + \underline{|A \cap B \cap C|}$$



Proof (optional)

We need lots of notation.

- $|A - (B \cup C)| = n_a$, $|B - (A \cup C)| = n_b$, $|C - (A \cup B)| = n_c$,
- $|(A \cap B) - C| = n_{ab}$, $|(A \cap C) - B| = n_{ac}$, $|(B \cap C) - A| = n_{bc}$,
- $|A \cap B \cap C| = n_{abc}$.



Then

$$\begin{aligned}
|A \cup B \cup C| &= n_a + n_b + n_c + n_{ab} + n_{ac} + n_{bc} + n_{abc} \\
&= (n_a + n_{ab} + n_{ac} + n_{abc}) + (n_b + n_{ab} + n_{bc} + n_{abc}) \\
&+ (n_c + n_{ac} + n_{bc} + n_{abc}) - (n_{ab} + n_{abc}) \\
&- (n_{ac} + n_{abc}) - (n_{bc} + n_{abc}) + n_{abc}
\end{aligned}$$

These are special cases of the principle of inclusion and exclusion

Principle of inclusion and exclusion

Let $A_1, A_2, \ldots, A_n$ be sets.

Then

$$\begin{aligned} |A_1 \cup \cdots \cup A_n| &= \sum_{1 \le k \le n} |A_i| \\ &- \sum_{1 \le j < k \le n} |A_j \cap A_k| \\ &+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| \\ &- \cdots \\ &+ (-1)^{n-1} |A_1 \cap \cdots \cap A_n|. \end{aligned}$$