COMP229: Introduction to Data Science
Lecture 12: Equivalence relations and Vector spaces

Olga Anosova, O.Anosova@liverpool.ac.uk
Autumn 2023, Computer Science department
University of Liverpool, United Kingdom

Lecture plan

- Functions vs relations
- Equivalence relations
- Free vs fixed vectors
- Equivalent vectors

Last lecture recap

- Not everything is Normal; **ratios** and **power law** distributions are very different.
- Rules of NHST:
  1. Check the assumptions!!
  2. Avoid statements about the truth of H_0.
  3. Report the whole process, including underlying definitions and data ~~torture~~ exploration.
  4. Avoid statements about effect of absence of effect.
  5. Avoid statements about the effect size.
  6. Use multiple types of analysis and **measures**.
- Clearly state all definitions, including what we mean by "equal".

A real-valued function

For simplicity we consider real-valued functions below, however the real line $\mathbb{R}$ will be later replaced by a higher dimensional space $\mathbb{R}^n$.

Definition 12.1. A **function** f can be considered as a set of ordered pairs $(x, f(x))$ such that for any argument (input) x there *is* a *unique* value (output) $f(x) \in \mathbb{R}$.

If $f(x)$ is defined for any $x \in \mathbb{R}$, the notation $f : \mathbb{R} \to \mathbb{R}$ means that each real x is mapped to a real value, but possibly the same value like in $f(x) = 0$.

A relation between data objects

The unit circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ doesn't define a function $y(x)$ on $[-1, 1]$, because any $x \in (0, 1)$ corresponds to 2 values $y = \pm\sqrt{1 - x^2}$.

Formula $\{(x, y) \in \mathbb{R}^2 : y = \frac{1}{x}\}$ doesn't define a function on $\mathbb{R}$, because of no output for $x = 0$. But it is a function on $\mathbb{R}\backslash\{0\}$.

Definition 12.2. Any collection of pairs (x, y), where x belongs to one fixed set (say, $\mathbb{R}$) and y is from another set (possibly, $\mathbb{R}$) is called a **relation**.

For example, (x, y) from the unit circle and all pairs (student X, module taken by X) form relations.

Surjective functions

The notation $f : \mathbb{R} \to \mathbb{R}$ doesn't mean that *any* real value is attained, i.e. the *image* of the function f $f(\mathbb{R}) = \{f(x) : x \in \mathbb{R}\}$ can be a small subset of $\mathbb{R}$.

Definition 12.3. A function f is called **surjective** if $f(\mathbb{R}) = \mathbb{R}$, i.e. for any $y \in \mathbb{R}$ there exist $x \in \mathbb{R}$ (not necessarily unique) such that $f(x) = y$.

Problem 12.4. Is $\sin(x) : \mathbb{R} \to \mathbb{R}$ surjective?

Solution 12.4. No, because $-1 \leqslant \sin x \leqslant 1$.

Injective and bijective functions

Definition 12.5. $f : \mathbb{R} \to \mathbb{R}$ is **injective** if f takes *different* values $f(x) \neq f(y)$ on any different arguments $x \neq y$, i.e. if $f(x) = f(y)$ then $x = y$.

f is called **bijective** (or one-to-one: 1-1) if f is simultaneously surjective and injective.

Problem 12.6. For an integer $n \geqslant 1$, is the function $f(x) = x^n : \mathbb{R} \to \mathbb{R}$ surjective, bijective?

Solution 12.6. If n is even, $x^n \geqslant 0$, not surjective.
If n is odd, then for any $y \in \mathbb{R}$ we find $x = \sqrt[n]{y}$, hence $y = x^n$, bijective.

An equivalence relation

Definition 12.7. If an object A is related to B, we write an ordered pair (A, B). This relation is called an **equivalence** if and only if the following axioms hold

reflexivity : any A is equivalent to A, i.e. (A, A);

symmetry : if A is equivalent to B, then B is equivalent to A, i.e. $(A, B) \Rightarrow (B, A)$;

transitivity : if A is equivalent to B and B is equivalent to C, then A is equivalent to C, i.e. (A, B) and $(B, C) \Rightarrow (A, C)$.

Examples

Problem 12.8. Are the following relations on pairs (x, y) of real numbers equivalence relations?

(1) $x < y$;

(2) $x \leqslant y$;

(3) distance $|x - y| \leqslant 1$.

Solution 12.8. (1) fails reflexivity: $x < x$ is false.

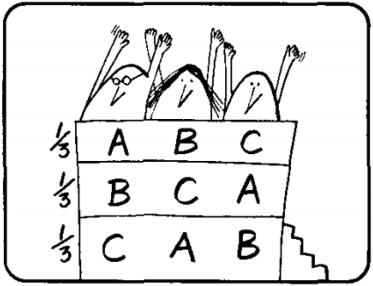
(2) fails the axiom symmetry: if $x \leqslant y$ then $y \leqslant x$ holds only for $x = y$, not for all $x, y \in \mathbb{R}$.

(3) fails the transitivity axiom: the Euclidean distance $|\pm 1 - 0| = 1$, but $|-1 - 1| = 2 > 1$. So, we can't just set a distance threshold under which all points are equivalent.

(Hello again, the *p*-value!)

Probability and ranking

Suppose we need to rank 3 items A,B,C by voters preferences. A poll shows that 2/3 prefer A to B and 2/3 prefer B to C. That means that most voters prefer A to C, right?



Not really: if each preference row is supported by 1/3 of voters, then C is preferred to A by 2/3 of voters.

This is known as Arrow's impossibility (voting) theorem.

Transitivity

Transitivity of a relation R:
aRb and bRc means that aRc, but only for *transitive* relations.

Transitivity is crucial in understanding of causality, aka the famous chicken or the egg problem.

Intransitivity can be implemented by Efron's dice.

Intransitivity contradicts our spacial intuition, as demonstrated by the Penrose staircase.

Ascending & Descending by M.Escher

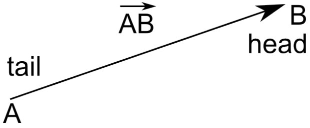
Equivalence classes

Claim 12.9. For any equivalence relation, all objects can be classified (split) into well-defined *classes* consisting of all objects that are pairwise equivalent to each other.

Proof. Take any object A and form its own class
$[A] = \{\text{all } B \text{ equivalent to } A\}$.
Take any $C \notin [A]$ and form its class
$[C] = \{\text{all } B \text{ equivalent to } C\}$ and so on until all objects are classified.
If classes overlap, they should coincide by the transitivity axiom. So all objects split into well-defined classes.

Fixed vectors are directed segments

The notation $\mathbb{R}^n$ means a product of n copies of $\mathbb{R}$, i.e. any point $A \in \mathbb{R}^n$ has n ordered real coordinates written as $(a_1, \ldots, a_n)$.



Definition 12.10. For points $A, B \in \mathbb{R}^n$, the *fixed vector* $\overrightarrow{AB}$ is the directed segment from A to B.

Fixed vectors are **equivalent** (or **equal**), $\overrightarrow{AB} = \overrightarrow{CD}$), if $b_i - a_i = d_i - c_i$ for each i-th coordinate.

Free vectors are classes of vectors

Claim 12.11. The equality $\overrightarrow{AB} = \overrightarrow{CD}$ defines an equivalence relations on all fixed vectors (i.e. with fixed endpoints).

Definition 12.12. Any *class of equal* fixed vectors is a called **free** vector (without fixed endpoints) and has n coordinates $\vec{v} = (b_1 - a_1, \dots, b_n - a_n)$.

A free vector $\vec{v} = (v_1, \dots, v_n)$ is often represented by using the origin $A = (0, \dots, 0)$ as the tail point, then the head point is $B = (v_1, \dots, v_n)$.

The length of a vector

Definition 12.13. The **Euclidean distance** between points $A = (a_1, \dots, a_n), B = (b_1, \dots, b_n) \in \mathbb{R}^n$ is $|AB| = \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}$.

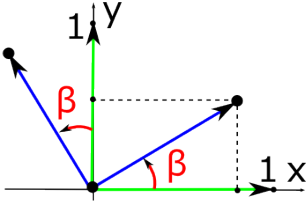
The **length** (or **magnitude**) of a free vector $\vec{v} = (v_1, \dots, v_n)$ is $|\vec{v}| = \sqrt{v_1^2 + \cdots + v_n^2} = \sqrt{\sum_{i=1}^{n} v_i^2}$.

Problem 12.14. Is it an equivalence relation on free vectors: $\vec{u} \sim \vec{v}$ if $|\vec{u}| = |\vec{v}|$?

Solution 12.14. Yes, all axioms hold.

The angle between vectors

Definition 12.15. The **angle** $\angle(\vec{u}, \vec{v})$ between free vectors $\vec{u}, \vec{v}$ is measured after identifying their tail points, say at the origin in the plane spanned by $\vec{u}, \vec{v}$.



$\angle(\vec{u}, \vec{v}) \in [0, 2\pi)$ is defined as the fraction of the anticlockwise turn from $\vec{u}$ to $\vec{v}$ assuming that the angle of the full turn is 2π or 360°.

The **opposite** angle is $\angle(\vec{v}, \vec{u}) = 2\pi - \angle(\vec{u}, \vec{v})$.

Another possible range of angles is $(-180^\circ, 180^\circ]$.

Parallel or collinear vectors

Definition 12.16. Free vectors are called **parallel** (or **collinear**) if the angle between them is 0 or π.

Claim 12.17. The collinearity is an equivalence.

Proof. All parallel vectors with a common tail point lie in the same straight line, all axioms hold.

Problem 12.18. Do all pairs (α, β) such that $\alpha - \beta$ is divisible by 2π form an equivalence relation?

Solution 12.18. Yes, axioms are satisfied. The *classes of equivalence* can be considered as angles between vectors.

Time to revise and ask questions

- A *bijection* is both injective ($f(x) = f(y) \Rightarrow x = y$) and surjective (for any y there is x with $f(x) = y$).
- An *equivalence*: reflexive, symmetric, transitive.
- A *free* vector is a class of all equal fixed vectors.
- The *length* of a free vector is $|\vec{v}| = \sqrt{\sum_{i=1}^{n} v_i^2}$
- The *angle* $\angle(\vec{u}, \vec{v})$ is measured counter-clockwise from $\vec{u}$ to $\vec{v}$.

Problem 12.19. Friendship relations (on Facebook or in real life): we say (A, B), if a person A knows a person B. Is it an equivalence relation on people?