Tutorial 9 (Wk10)
Bayesian Networks

This week's lecture covers Bayesian approaches for data analysis. One part of this is the concept of Bayesian Networks. In this tutorial we will look at the theory and in next week's lab we will implement some examples using Python. The solutions will be published on Wednesday.

Introduction to Bayesian Networks

Bayesian Networks (BNs), also known as *Belief Networks*, are probabilistic graphical models that represent relationships between random variables using a Directed Acyclic Graph (DAG). Each node in the graph corresponds to a random variable, and each directed edge represents a conditional dependency. The structure of the DAG allows Bayesian Networks to compactly represent complex joint probability distributions by capturing the conditional independence relationships between variables.

Conditional Independence

A key feature of Bayesian Networks is ***conditional independence***. This means that some variables become independent of others when specific conditions (or evidence) are known.

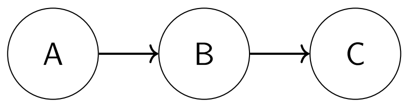
For example, consider three variables A, B, and C connected as $A \to B \to C$ in a Bayesian Network:

- The edge $A \to B$ implies that B is directly influenced by A.
- The edge $B \to C$ implies that C is directly influenced by B.

Conditional independence allows us to state that A and C are independent of each other given B. This is mathematically written as:

$$P(C \mid A, B) = P(C \mid B)$$

This means that once we know B, the probability of C no longer depends on A. This property simplifies computations and reduces the number of parameters needed to model the system.



Example

Imagine a weather prediction system.

- A: It is cloudy.
- B: It is raining.
- C: The ground is wet.

The relationship is modeled as $A \to B \to C$:
Clouds (A) cause rain (B). Rain (B) wets the ground (C).

Using conditional independence:

1. If we know it's raining ($B =$ true), the probability of the ground being wet (C) does not depend on whether it's cloudy (A).

2. This simplifies the computation of $P(A, B, C)$:

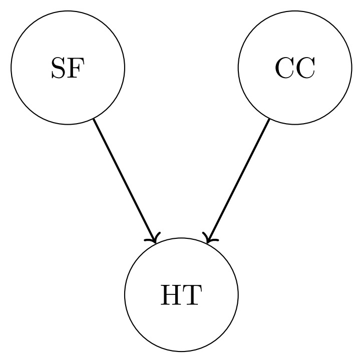
$$P(A, B, C) = P(A) \cdot P(B \mid A) \cdot P(C \mid B)$$

Example 2

Suppose we are building a system to diagnose diseases. A high temperature (HT) could be caused by:

- Seasonal Flu (SF).
- Contact with COVID-19 (CC).

Using Bayesian Networks, we model this as:



$$P(HT, SF, CC) = P(HT \mid SF, CC)P(SF)P(CC)$$

In this model:

- Seasonal flu (SF) and contact with COVID-19 (CC) are assumed to be independent.
- High temperature (HT) depends on SF and CC.

Conditional Probability Table (CPT)

CPTs quantify relationships in a Bayesian Network. The table lists probabilities of one variable given its parent(s). For example, for the node HT:

SF	CC	$P(HT = \text{true})$	$P(HT = \text{false})$
true	true	0.9	0.1
true	false	0.7	0.3
false	true	0.6	0.4
false	false	0.1	0.9

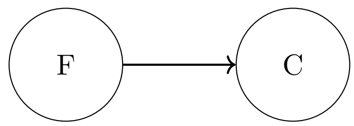
Conditional independence in this model implies that SF and CC are independent of each other but jointly influence HT. For example:

$$P(HT \mid SF, CC) = \text{value from CPT}$$

Exercises

Problem 1

Consider the following Bayesian network, where F represents "having the flu" and C represents "coughing."



CPTs:

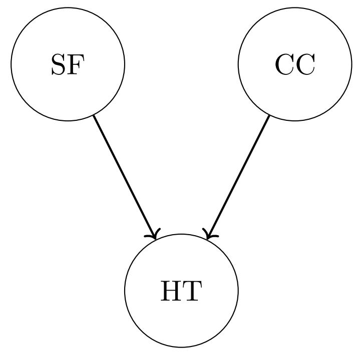
$$P(F) = 0.1, \quad P(C \mid F = \text{true}) = 0.8, \quad P(C \mid F = \text{false}) = 0.3$$

Questions

1. Write the joint probability table for this network. (i.e., list probabilities of all combinations of variables).
2. What is $P(C = \text{true})$?
3. Which nodes are conditionally independent?

Problem 2

Consider a situation where Bob has a high temperature (HT), which could be caused by Seasonal Flu (SF) or Contact with COVID-19 (CC).



Questions

1. Write the joint probability distribution for this network.

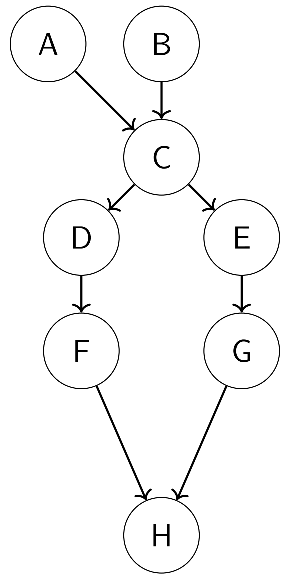
2. Compute $P(HT = \text{true})$ given the following:

   $$P(SF = \text{true}) = 0.1, \quad P(CC = \text{true}) = 0.05, \quad P(HT = \text{true} \mid SF, CC) = 0.9$$

3. Which nodes are conditionally independent?

Problem 3

Consider the following Bayesian network:



Questions

1. Are D and E necessarily independent given evidence about both A and B?

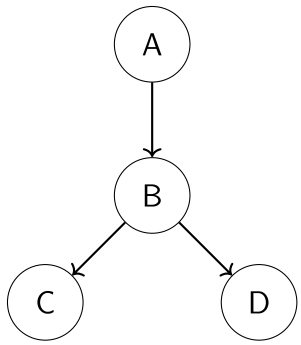
2. Are A and C necessarily independent given evidence about D?

3. Are A and H necessarily independent given evidence about C?

Problem 4

Using the Directed Acyclic Graph (DAG) and the CPTs below, calculate the probability:

$$P(A = \textbf{true}, B = \textbf{true}, C = \textbf{true}, D = \textbf{true})$$



The graph shows the following relationships:

- A influences B,
- B influences both C and D.

The Bayesian Network is defined as:

$$P(A, B, C, D) = P(A) \cdot P(B \mid A) \cdot P(C \mid B) \cdot P(D \mid B)$$

Conditional Probability Tables (CPTs):

1. $P(A)$: Probability of A

A	$P(A)$
false	0.6
true	0.4

2. $P(B \mid A)$: Probability of B given A

A	B	$P(B \mid A)$
false	false	0.01
false	true	0.99
true	false	0.7
true	true	0.3

3. $P(C \mid B)$**: Probability of** C **given** B

B	C	$P(C \mid B)$
false	false	0.4
false	true	0.6
true	false	0.9
true	true	0.1

4. $P(D \mid B)$**: Probability of** D **given** B

B	D	$P(D \mid B)$
false	false	0.02
false	true	0.98
true	false	0.05
true	true	0.95