Partition of a set

A *partition* of a set A is a collection of non-empty subsets $A_1, \ldots, A_n$ of A satisfying:

- $A = A_1 \cup A_2 \cup \cdots \cup A_n$;
- $A_i \cap A_j = \emptyset$ for $i \neq j$.

The A_i are called the blocks of the partition.

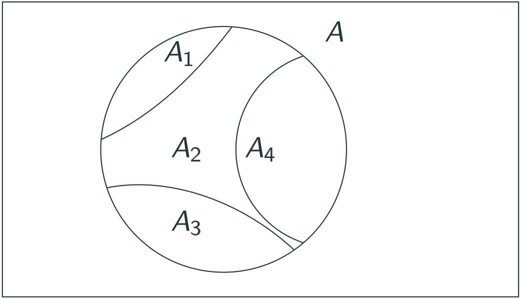
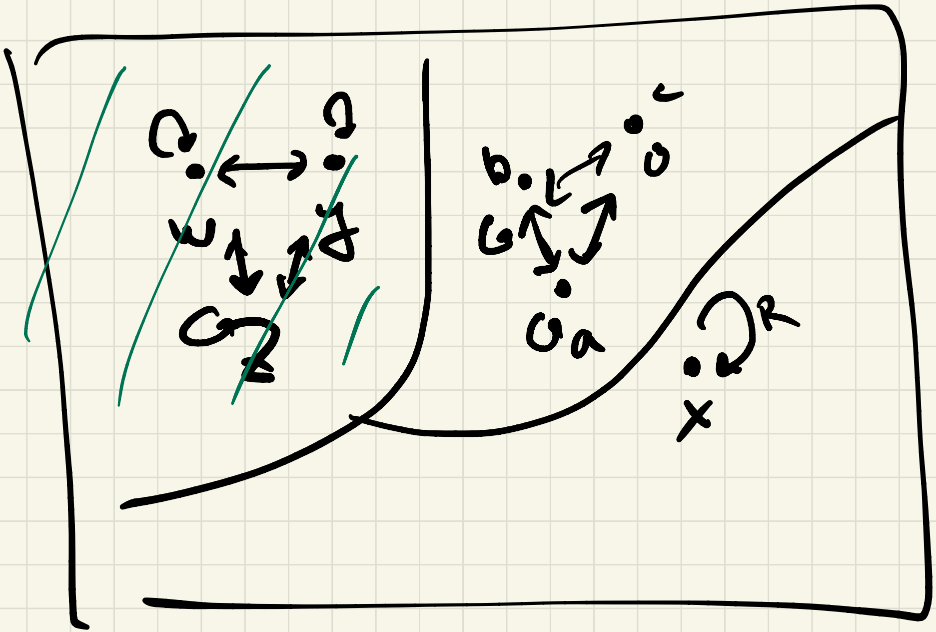


Figure 3: Partition of A

A , R - an equivalence relation on A

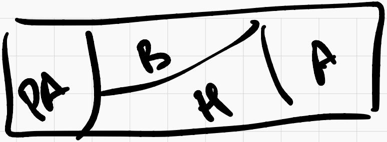


Equivalence classes

Definition The *equivalence class* E_x of any $x \in A$ is defined by

$$E_x = \cancel{\{y \mid yRx\}}. \quad \{y \in A \mid y R x\}$$

Example: equivalence classes for the relation 'same tax band' on the set of salaries



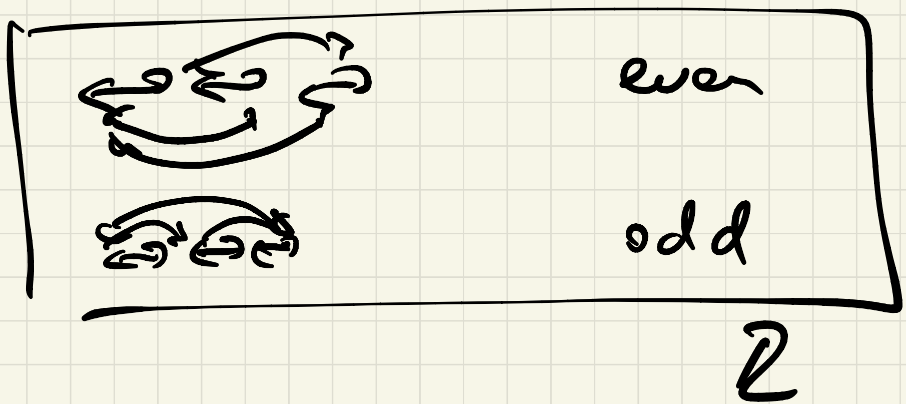
$S \subseteq \mathbb{Z} \times \mathbb{Z}$ $\qquad (i,j) \in S$ if

$i \bmod 2 = j \bmod 2$

$E_1 =$ set of odd int. $= E_3 = E_{239}$

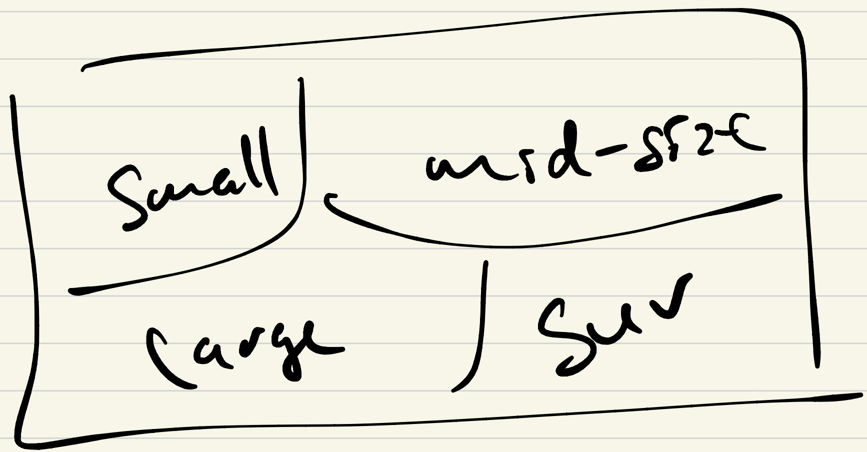
E_2 = set of even int $= E_4 = E_{-26} = \dots$

$= E_0 \dots$



A = set of cars

T - being of the same size



Connecting partitions and equivalence relations

Theorem Let R be an equivalence relation on a non-empty set A. Then the equivalence classes $\{E_x \mid x \in A\}$ form a partition of A.

Proof (Optional)

The proof is in four parts:

(1) We show that the equivalence classes $E_x = \{y \mid yRx\}$, $x \in A$, are non-empty subsets of A: by definition, each E_x is a subset of A. Since R is reflexive, xRx. Therefore $x \in E_x$ and so E_x is non-empty.

(2) We show that A is the union of the equivalence classes $E_x, x \in A$: We know that $E_x \subseteq A$, for all E_x, $x \in A$. Therefore the union of the equivalence classes is a subset of A. Conversely, suppose $x \in A$. Then $x \in E_x$. So, A is a subset of the union of the equivalence classes.

(Optional) Proof (continued)

The purpose of the last two parts is to show that distinct equivalence classes are disjoint, satisfying (ii) in the definition of partition.

(3) We show that if xRy then $E_x = E_y$: Suppose that xRy and let $z \in E_x$. Then, zRx and xRy. Since R is a transitive relation, zRy. Therefore, $z \in E_y$. We have shown that $E_x \subseteq E_y$. An analogous argument shows that $E_y \subseteq E_x$. So, $E_x = E_y$.

(4) We show that any two distinct equivalence classes are disjoint: To this end we show that if two equivalence classes are not disjoint then they are identical. Suppose $E_x \cap E_y \neq \emptyset$. Take a $z \in E_x \cap E_y$. Then, zRx and zRy. Since R is symmetric, xRz and zRy. But then, by transitivity of R, xRy. Therefore, by (3), $E_x = E_y$.

Connecting partitions and equivalence relations

Theorem Suppose that $A_1, \ldots, A_n$ is a partition of A. Define a relation R on A by setting: xRy if and only if there exists i such that $1 \leq i \leq n$ and $x, y \in A_i$. Then R is an equivalence relation.

Proof (Optional)

- Reflexivity: if $x \in A$, then $x \in A_i$ for some i. Therefore xRx.
- Transitivity: if xRy and yRz, then there exists A_i and A_j such that $x, y \in A_i$ and $y, z \in A_j$. $y \in A_i \cap A_j$ implies $i = j$. Therefore $x, z \in A_i$ which implies xRz.
- Symmetry: if xRy, then there exists A_i such that $x, y \in A_i$. Therefore yRx.

Application: Rational numbers

Recall: r is rational if $r = \frac{k}{l}$, where k, l are integers and $l \neq 0$.

Evidently, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \dots$

Consider the set $A = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \neq 0\}$ and relation R on A defined as:

$$\underline{(a, b)R(c, d)} \Leftrightarrow \underline{\underline{ad = bc}}$$

R is and equivalence relation on A and the set of all equivalence classes of R is the set of rationals

$\mathbb{Q}$

$$\begin{array}{c|cccc}
\hline
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \cdots & \\
\frac{2}{1} & \frac{2}{2} & & & \\
\frac{3}{1} & & \frac{3}{3} & & \\
& & & \frac{4}{4} & \\
\vdots & & & &
\end{array}$$

Important relations: Partial orders

Definition A binary relation R on a set A which is reflexive, transitive and antisymmetric is called a partial order.

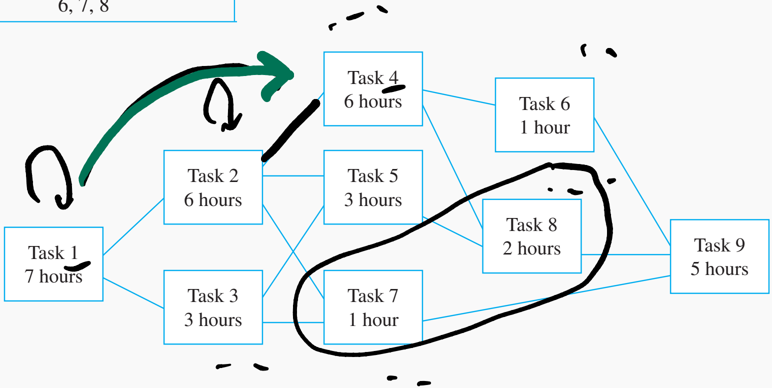
Partial orders are important in situations where we wish to characterise precedence.

Examples:

- the relation $\leq$ on the the set $\mathbb{R}$ of real numbers;
- the relation $\subseteq$ on $Pow(A)$;
- "*is a divisor of*" on the set $\mathbb{Z}^+$ of positive integers.

Example: Job scheduling

Task	Immediately Preceding Tasks
1	
2	1
3	1
4	2
5	2, 3
6	4
7	2, 3
8	4, 5
9	6, 7, 8



Predecessors in partial orders

If R is a partial order on a set A and xRy, $x \neq y$ we call x a predecessor of y.

If x is a predecessor of y and there is no $z \notin \{x, y\}$ for which xRz and zRy, we call x an immediate predecessor of y.

Hasse Diagram

The Hasse Diagram of a partial order is a digraph. The vertices of the digraph are the elements of the partial order, and the edges of the digraph are given by the "immediate predecessor" relation.

It is typical to *assume* that the arrows pointing upwards.

$S = \langle a, b, c \rangle$

$\{a, b, c\}$

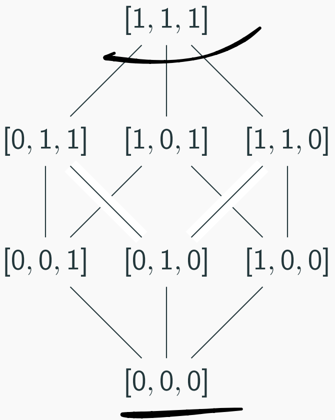
$\{b, c\} \quad \{a, c\} \quad \{a, b\}$

$\cup| \quad \subseteq \quad \cup| \ldots$

$\{c\} \quad \{b\} \quad \{a\}$

$\cup| \quad \subseteq \quad \subseteq$

$\emptyset$



characterize vectors of subsets of a set A

$|A| = 3$

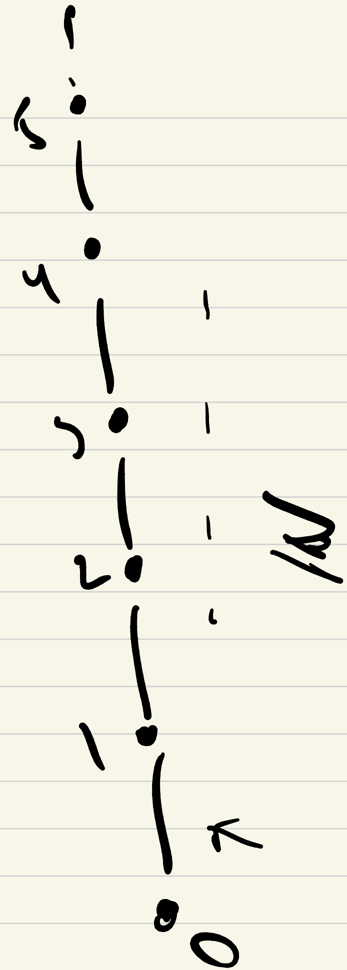
Important relations: Total orders

Definition A binary relation R on a set A is a total order if it is a partial order such that for any $x, y \in A$, xRy or yRx.

The Hasse diagram of a total order is a chain.

Examples

- the relation $\leq$ on the set $\mathbb{R}$ of real numbers;
- the usual lexicographical ordering on the words in a dictionary;
- the relation "is a divisor of" is *not* a total order.



n-ary relations

The Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ of sets $A_1, A_2, \ldots, A_n$ is defined by

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, \ldots, a_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}.$$

Here $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ if and only if $a_i = b_i$ for all $1 \leq i \leq n$.

An n-ary relation is a subset of $A_1 \times \ldots \ldots A_n$

Databases and relations

A database table $\approx$ relation

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Students $= \{$ (A, 231455 . . .), ( ) . . $\}$

Unary relations

Unary relations are just subsets of a set.

Example: The unary relation `EvenPositiveIntegers` on the set $\mathbb{Z}^+$ of positive integers is

$$\{x \in \mathbb{Z}^+ \mid x \text{ is even}\}.$$