2nd test is moved to

NOV 24

Example

Which of the following define a relation that is reflexive, symmetric, antisymmetric or transitive?

- x divides y on the set $\mathbb{Z}^+$ of positive integers;
- $x \neq y$ on the set $\mathbb{Z}$ of integers;
- x *has the same age as* y on the set of people.

Digraph representation

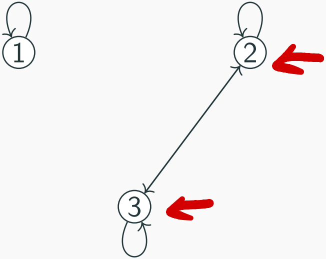
In the directed graph representation, R is

- *reflexive* if there is always an arrow from every vertex to itself;
- *symmetric* if whenever there is an arrow from x to y there is also an arrow from y to x;
- *antisymmetric* if whenever there is an arrow from x to y and $x \neq y$, then there is no arrow from y to x;
- *transitive* if whenever there is an arrow from x to y and from y to z there is also an arrow from x to z.

Example 1

- *reflexive* $\forall x : xRx$
- *symmetric* $\forall x, y : xRy \implies yRx$
- *antisymmetric* $\forall x, y : xRy,\ yRx \implies x = y$
- *transitive* $\forall x, y, z : xRy,\ yRz \implies xRz$

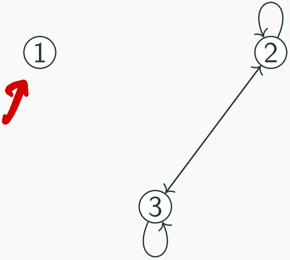
Let $A = \{1, 2, 3\},\ R_1 = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$



Example 2

- *reflexive* $\forall x : xRx$
- *symmetric* $\forall x, y : xRy \implies yRx$
- *antisymmetric* $\forall x, y : xRy,\ yRx \implies x = y$
- *transitive* $\forall x, y, z : xRy,\ yRz \implies xRz$

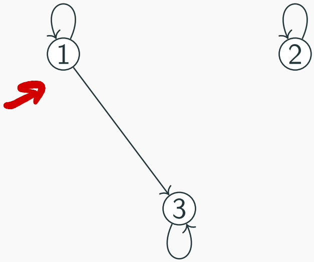
Let $A = \{1, 2, 3\},\ R_2 = \{(2, 2), (2, 3), (3, 2), (3, 3)\}$



Example 3

- *reflexive* $\forall x : xRx$
- *symmetric* $\forall x, y : xRy \implies yRx$
- *antisymmetric* $\forall x, y : xRy,\ yRx \implies x = y$
- *transitive* $\forall x, y, z : xRy,\ yRz \implies xRz$

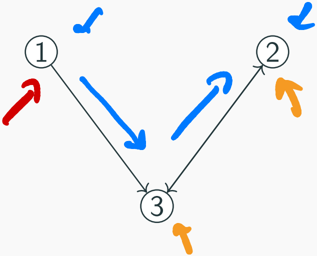
Let $A = \{1, 2, 3\}$, $R_3 = \{(1,1), (2,2), (3,3), (1,3)\}$



Example 4

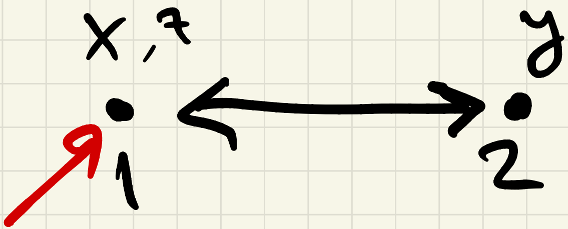
- *reflexive* $\forall x : xRx$
- *symmetric* $\forall x, y : xRy \implies yRx$
- *antisymmetric* $\forall x, y : xRy,\ yRx \implies x = y$
- *transitive* $\forall x, y, z : xRy,\ yRz \implies xRz$

Let $A = \{1, 2, 3\},\ R_4 = \{(1, 3), (3, 2), (2, 3)\}$

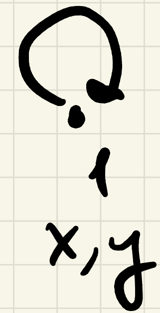


$A = \{1, 2\}$

- reflexive ✗
- symmetric ✓
- anti-symmetric ✓
- transitive ✓

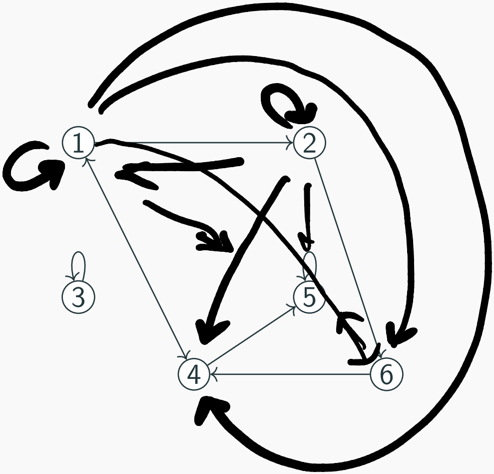


- reflexivity ✗
- symmetry ✓
- anti-symmetry ✗
- transitivity ✗



anti-sym
refl.
symm
trans

Example: Reachability relation



Transitive closure

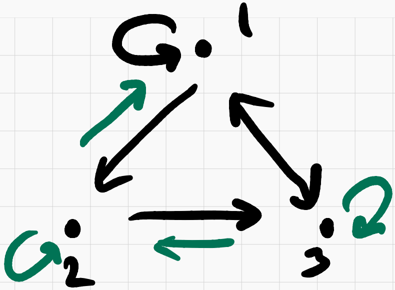
Given a binary relation R on a set A, the *transitive closure* R^* of R is the (uniquely determined) relation on A with the following properties:

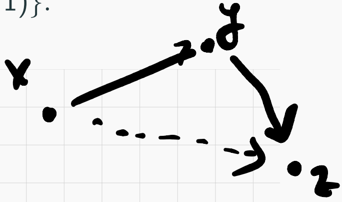
- R^* is transitive;
- $R \subseteq R^*$;
- If S is a transitive relation on A and $R \subseteq S$, then $R^* \subseteq S$.

Example

Let $A = \{1, 2, 3\}$. Find the transitive closure of

$$R = \{(1,1), (1,2), (1,3), (2,3), (3,1)\}.$$





$R^* = \{(1,1), (2,2), (3,3), (1,2), (1,3) \ldots \}$

Transitivity and composition

A relation S is transitive if and only if $S \circ S \subseteq S$.

This is because

$$S \circ S = \{(a, c) \mid \text{ exists } b \text{ such that } aSb \text{ and } bSc\}.$$

Let S be a relation. Set $S^1 = S$, $S^2 = \underline{S \circ S,}$ $S^3 = \underline{S \circ S \circ S,}$ and so on.

Theorem Denote by S^* the transitive closure of S. Then xS^*y if and only if there exists $n > 0$ such that xS^ny.

Transitive closure in matrix form

The relation R on the set $A = \{1, 2, 3, 4, 5\}$ is represented by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Determine the matrix $R \circ R$ and hence explain why R is not transitive.

Computation

$$
\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}
=
\begin{bmatrix} . & . & . & . & . \\ . & . & . & 1 & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix}
$$

$$R \circ R = \{(a, c) \mid \text{ exists } b \in A \text{ such that } aRb \text{ and } bRc\}.$$

Note (in red) that there are pairs (a, c) that are in $R \circ R$ but not in R. Hence, R is not transitive.

Detour: Warshall's algorithm

```
def warshall(a):
    n = len(a)
    for k in range(n):
        for i in range(n):
            for j in range(n):
                a[i][j] = (a[i][j] or
                           (a[i][k] and a[k][j]))
    return a

print warshall([[1,0,0,1,0],
                [0,1,0,0,1],
                [0,0,1,0,0],
                [1,0,1,0,0],
                [0,1,0,1,0]])
```

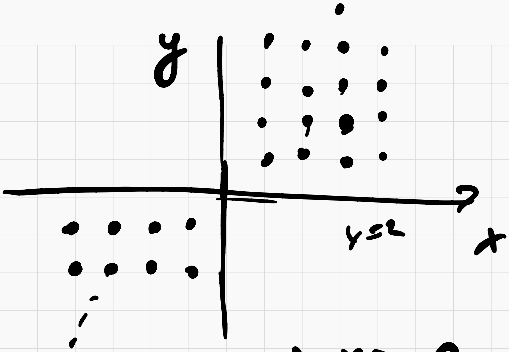
Important relations

Important relations: Equivalence relations

Definition A binary relation R on a set A is called an *equivalence relation* if it is reflexive, transitive, and symmetric.

Example

The relation R on the non-zero integers given by xRy if $xy > 0$;



$$x \cdot x > 0$$

$$\left.\begin{matrix} x \cdot y > 0 \\ y \cdot z > 0 \end{matrix}\right\} \Rightarrow x \cdot z > 0$$

Equiv.
rel.

Example

- The relation *has the same age* on the set of people.

- reflexive
- symmetric
- transitivity

- Same length on the set of cars.
- Same tax band on the set of salaries.

Functions and equivalence relations

Let $f\colon A \to B$ be a function. Define a relation R on A by

$$a_1 R a_2 \Leftrightarrow f(a_1) = f(a_2).$$

A is a set of cars, B is the set of real numbers, and f assigns to any car in A its length. Then $a_1 R a_2$ if and only if a_1 and a_2 are of the same length.

Cars