Foundations of Computer Science Comp109

University of Liverpool

Boris Konev

konev@liverpool.ac.uk

Part 6. Combinatorics

Comp109 Foundations of Computer Science

Reading

- Discrete Mathematics with Applications, S. Epp, Chapter 9.
- Discrete Mathematics and Its Applications, K. H. Rosen, Sections 6.1, 6.3, 6.4

Contents

- Basics of counting
- Notation for sums and products. The factorial function.
- Counting permutations and combinations.
- Binomial coefficients.

Developing ideas (1)

All chairs in a room are labelled with a single digit followed by a lower-case letter. What is the largest number of differently numbered chairs?

Developing ideas (2)

How many different bit strings of length 8 are there?

- How many different bytes are there?

$$0000\,0000,\ \ 0000\,0001,\ \ 0000\,0010,\ \ 0000\,0011, \ldots$$

$$\underbrace{2}\times\underbrace{2}\times\underbrace{2}\times\underbrace{2}\times\underbrace{2}\times\underbrace{2}\times\underbrace{2}\times\underbrace{2} = 2^8$$

Developing ideas (3)

How many ways there are to select 3 students for a prospectus photograph (order matters) from a group of 5?

$$\underline{5} \times \underline{4} \times \underline{3} = 60$$

The product rule

If there is a sequence of k events with $n_1, \ldots, n_k$ possible outcomes, then the total number of outcomes for the sequence of k events is

$$n_1 \times n_2 \times \cdots \times n_k.$$

Example

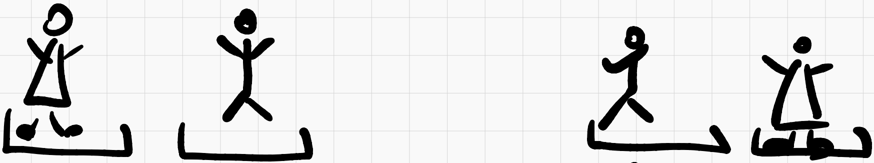
How many distinct car licence plates are there consisting of six characters, the first three of which are letters and the last three of which are digits?

$$\boxed{\underbrace{A..Z}_{26} \times \underbrace{A..Z}_{26} \times \underbrace{A..Z}_{26} \times \underbrace{0..9}_{10} \times \underbrace{0..9}_{10} \times \underbrace{0..9}_{10}}$$

$$26^3 \times \underbrace{1000}_{10^3}$$

Developing ideas (4)

How many ways there are to select a male and a female student for a prospectus photograph (order matters) from a group of 2 male and 3 female students?



$3 \times 2 + 2 \times 3$

Disjoint events

Two events are said to be disjoint (or "mutually exclusive") if they can't occur simultaneously.

Example: If you have 3 pairs of blue jeans and 2 pairs of black jeans, then there are $3 + 2 = 5$ different pairs of jeans which are blue or black which you could wear.

The sum rule

If A and B are disjoint events and there are n_1 possible outcomes for event A and n_2 possible outcomes for event B then there are $n_1 + n_2$ possible outcomes for the event "either A or B".

Example

How many three-digit numbers begin with 3 or 4?

$\underbrace{3/4}\ \underbrace{0..9}\ \underbrace{0..9}$

$2 \times \underbrace{10 \times 10}$

2×100

$3 \sqcup \sqcup \quad 100$

$4 \sqcup \sqcup \quad 100$

$= 100 + 100$

Example

I wish to take two pieces of fruit with me for lunch. I have three bananas, four apples and two pears. How many ways can I select two pieces of fruit of different type?

A B P

4 3 2

AB AP BP

$4 \times 3 + 4 \times 2 + 3 \times 2 = ?$

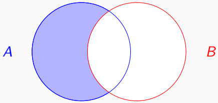
Set-theoretic interpretation

- If A and B are disjoint sets (that is, $A \cap B = \emptyset$) then $|A \cup B| = |A| + |B|$.
- Any sequence of k events can be regarded as an element of the Cartesian product $A_1 \times \cdots \times A_k$. This set has size $|A_1| \times \cdots \times |A_k|$.

Developing ideas (5)

A computer password is a string of 8 characters, where each character is an uppercase letter or a digit. Each password must contain ~~at least one digit~~.

How many different passwords are there?



A..Z

0..9

36

$$36^8 - 26^8$$

Answer

$$2,612,282,842,880$$

Note: lazy users

How many different 8-character passwords can be obtained by combining 3-letter word, a 4-letter word and a digit?

(According to `http://www.scrabblefinder.com` there are 1015 3-letter and 4030 4-letter English words.)

$\underline{3L}\ \underline{4L}\ \underline{D}$ $\quad$ $\underline{3L}\ \underline{D}\ \underline{4L}$ $\quad$ $\underline{4L}\ \underline{3L}\ \underline{D}$ $\quad$ $\underline{4L}\ \underline{D}\ \underline{3L}$

$\underline{D}\ \underline{3L}\ \underline{4L}$ $\quad$ $\underline{D}\ \underline{4L}\ \underline{3L}$

$6 \times 1015 \times 4030 \times 10$

Answer

$$245,427,000 \text{ (about } 0.009\%)$$

Beware of passwords like HOT4FUZZ

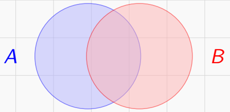
Developing ideas (6)

How many bit strings of length 8 start with 1 or finish with 00?

1

00

1 00



$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$2^8 - 2^7 - 2^6 + 2^5$$

The subtraction rule

If there are n_1 possible outcomes for event A, n_2 possible outcomes for event B and n_3 of these outcomes are shared between A and B then there are

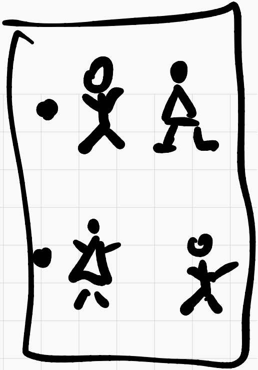
$$n_1 + n_2 - n_3$$

possible outcomes for the event "A or B".

Developing ideas (7)

How many ways there are to select 2 representatives from a group of 5 students?

$$\frac{5 \times 4}{2}$$



The division rule

Given n possible outcomes, if

- some of the n outcomes are the same
- every group of indistinguishable outcomes contains exactly d elements

there are n/d different outcomes.

Mini summary

Counting problems can be hard

Four decomposition rules:

- The product rule
- The sum rule
- The subtraction rule
- The division rule

To move further we need some mathematical notation