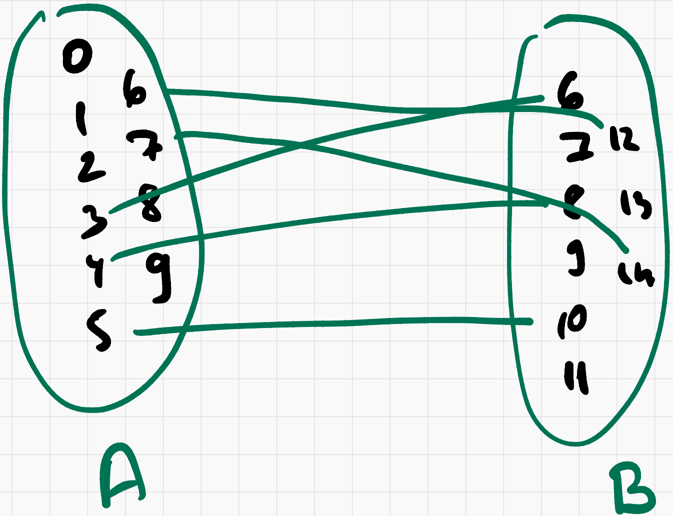
Example

$$A = \{i \in \mathbb{N} \mid i < 10\},\ B = \{i \in \mathbb{N} \mid 5 < i < 15\},\ R = \{((x, y) \in A \times B \mid y = 2x)\}$$



partial function

Building new relations from given ones

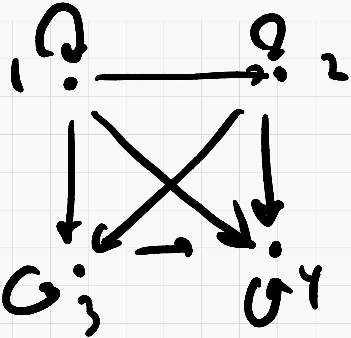
Inverse relation

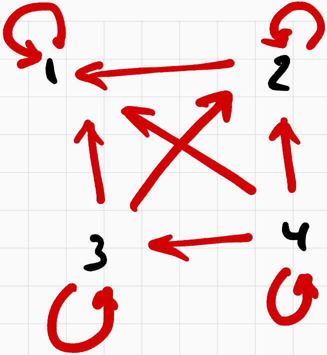
Definition Given a relation $R \subseteq A \times B$, we define the *inverse relation* $R^{-1} \subseteq B \times A$ by

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}.$$

Example: The inverse of the relation *is a parent of* on the set of people is the relation *is a child of*.

Example: $A = \{1, 2, 3, 4\}$, $R = \{(x, y) \mid x \leq y\}$





Composition of relations

$$\begin{tikzcd} A \arrow[r, dashed, "R"] \arrow[rr, bend right] & B \arrow[r, dashed, "S"] & C \end{tikzcd}$$

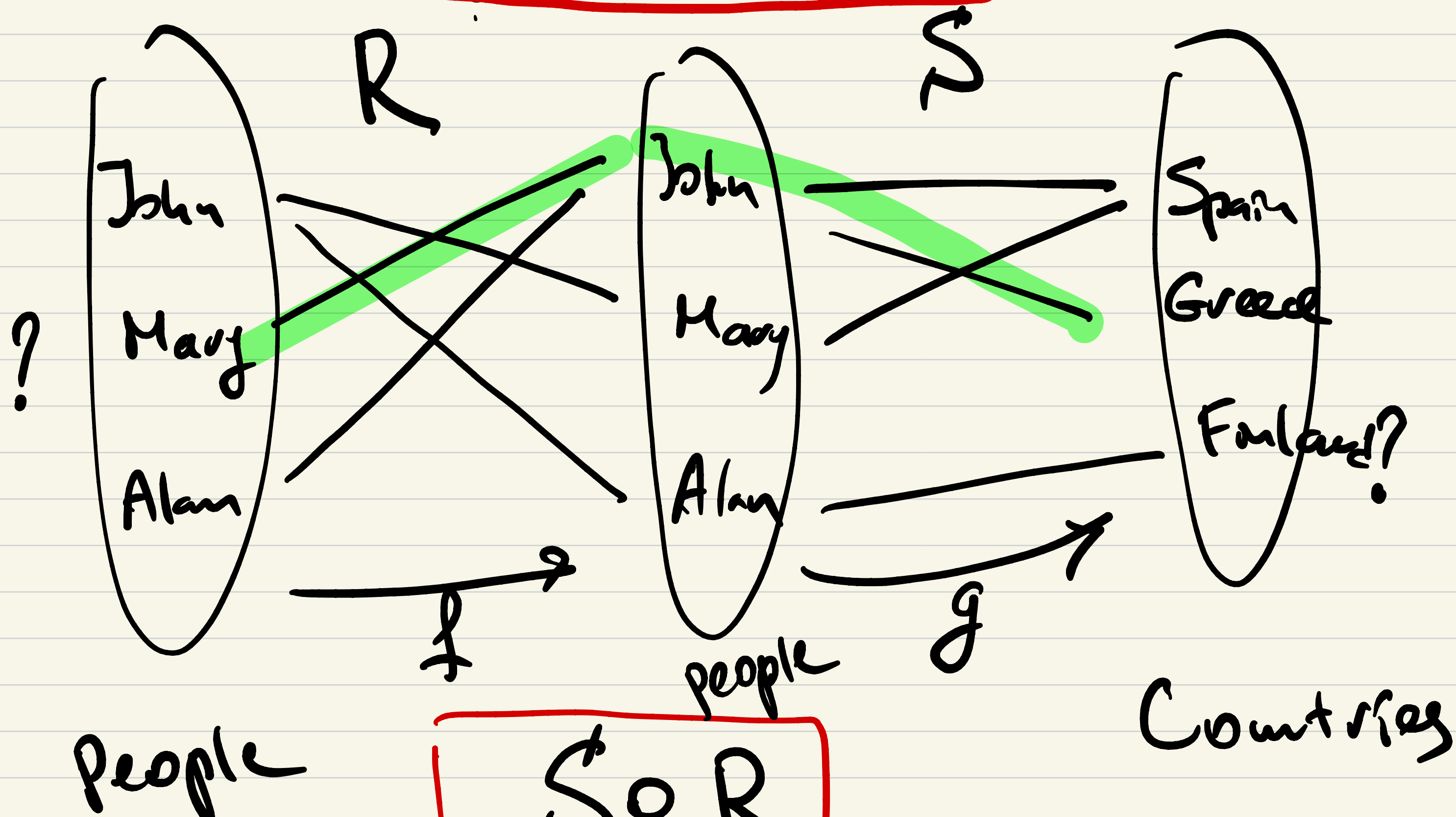
Definition Let $R \subseteq A \times B$ and $S \subseteq B \times C$. The (functional) composition of R and S, denoted by $S \circ R$, is the binary relation between A and C given by

$$S \circ R = \{(a, c) \mid \text{ exists } b \in B \text{ such that } aRb \text{ and } bSc\}.$$

Example: If R is the relation *is a sister of* and S is the relation *is a parent of*, then

- $S \circ R$ is the relation *is an aunt of*;
- $S \circ S$ is the relation *is a grandparent of*.

$$\boxed{g \circ f(x) = g(f(x))}$$



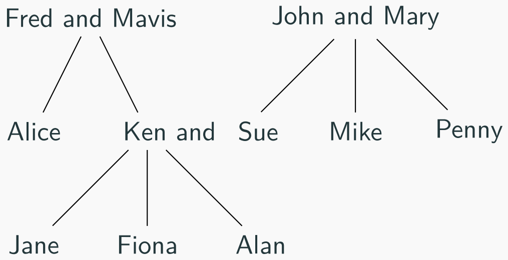
$$\boxed{S \circ R}$$

Example

$R:$ *is a sister of*

$S:$ *is a parent of*

$S \circ R = \{(a, c) \mid \text{ exists } b \in B \text{ such that } aRb \text{ and } bSc\}.$

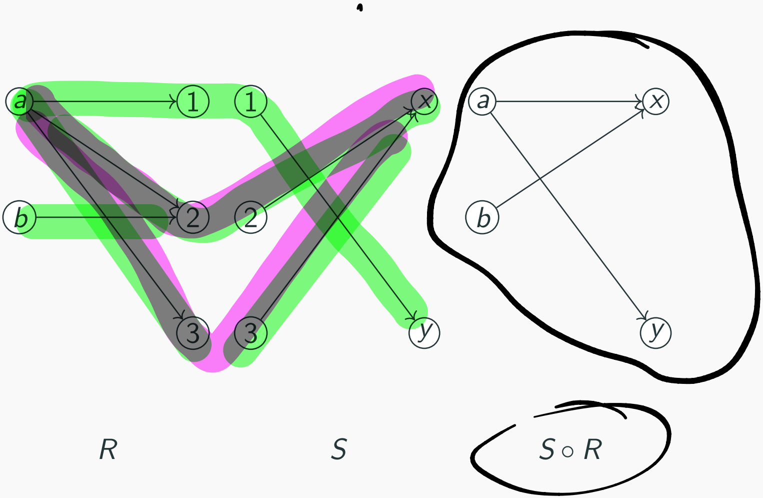


Alice R Ken and Ken S Alan so Alice $S \circ R$ Alan.

Penny R Sue and Sue S Jane so Penny $S \circ R$ Jane.

Fred S Ken and Ken S Fiona so Fred $S \circ S$ Fiona.

Digraph representation of compositions



Example

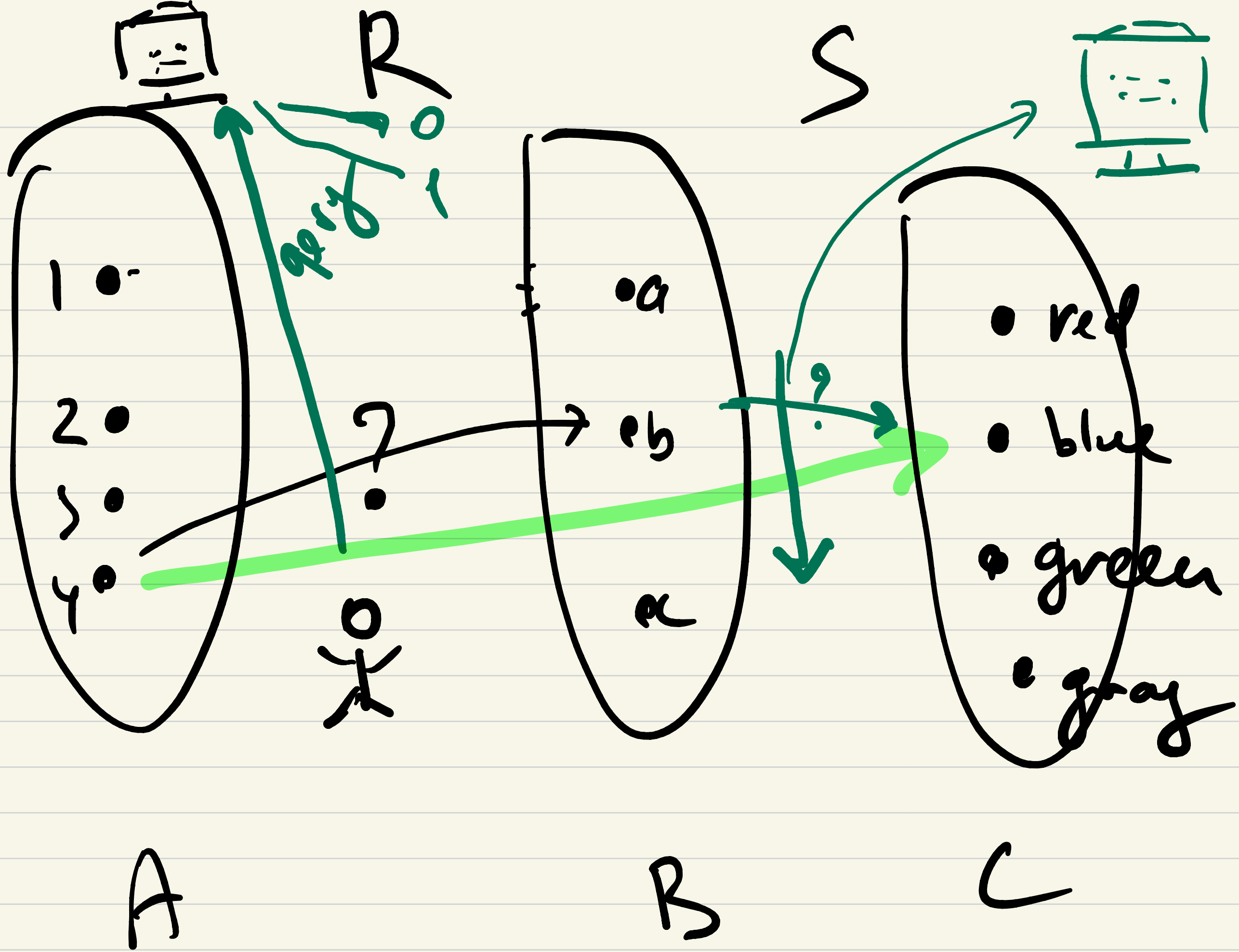
A – set of people, B – set of countries

$R \subseteq A \times A$, $R(x, y)$ represents x is a friend of y

$S \subseteq A \times B$, $S(u, v)$ represents u visited v

done

before


R
S
1
2
3
4
?
a
b
c
red
blue
green
gray
A
B
C

- Let $A = \{a_1, \dots, a_n\}$, $B = \{b_1, \dots, b_m\}$ and $R \subseteq A \times B$.
- We represent R by an array M of n rows and m columns. Such an array is called a *n by m matrix*.
- The entry in row i and column j of this matrix is given by $M(i, j)$ where

$$M(i,j) = \begin{cases} 1 & \text{if} \quad (a_i, b_j) \in R \\ 0 & \text{if} \quad (a_i, b_j) \notin R \end{cases}$$

Example 1

Let $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6\}$, and

$$U = \{(x, y) \in A \times B \mid x + y = 9\}$$

Assume an enumeration $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $a_4 = 7$ and $b_1 = 2$, $b_2 = 4$, $b_3 = 6$. Then M represents U, where

$$M = \begin{array}{cc} & \begin{array}{ccc} 2 & 4 & 6 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \\ 7 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

Example 2

Let $A = \{a, b, c, d\}$ and suppose that $R \subseteq A \times A$ has the following matrix representation:

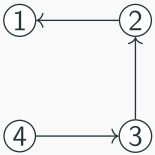
$$M = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

List the ordered pairs belonging to R.

$R = \{(a,b), (a,c), (b,c), (b,d),$
$(c,b), (d,a), (d,b), (d,d)\}$

Example

The binary relation R on $A = \{1, 2, 3, 4\}$ has the following digraph representation.



- The ordered pairs $R = \{(4,3), (3,2), (2,1)\}$
- The matrix

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & \cdot \\ 1 & \cdot & 0 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 0 & \cdot & 1 & \cdot \end{bmatrix} \end{array}$$

- In words: x, y are in the rel. R if x = y+1, x, y are elements of A