COMP229: Introduction to Data Science
Lecture 21: Orthogonal maps

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Lecture plan

- Bijections, linear and affine maps, isometries and their interconnections.
- Orthogonal maps.
- Constructing orthogonal maps and operating on them.

Reminder: isometries

An **isometry**

- preserves distances,
- decomposes into reflections and
- is an affine map.

Types of maps we have studied

Bijections are injective and surjective (on any sets).

Linear maps $\mathbb{R}^m \to \mathbb{R}^k$ have the form $\vec{v} \mapsto A\vec{v}$ for any $k \times m$ matrix A and a vector $\vec{v} \in \mathbb{R}^m$.

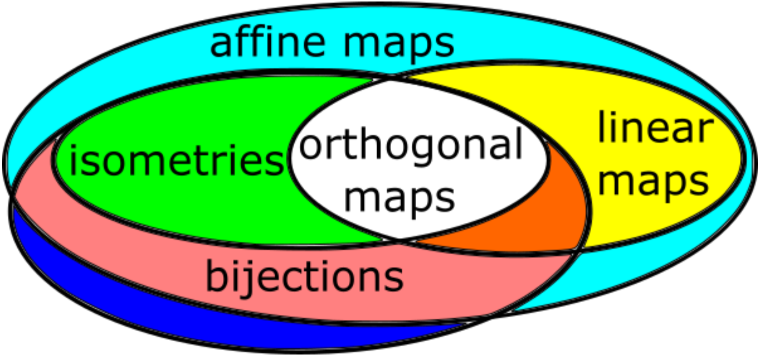
Affine maps $\mathbb{R}^m \to \mathbb{R}^k$ have the form $\vec{v} \mapsto A\vec{v} + \vec{u}$ with a translation by a fixed vector $\vec{u} \in \mathbb{R}^k$.

Isometries preserve distances, defined in any $\mathbb{R}^m$, the last two lectures solved exercises for $m = 2$.

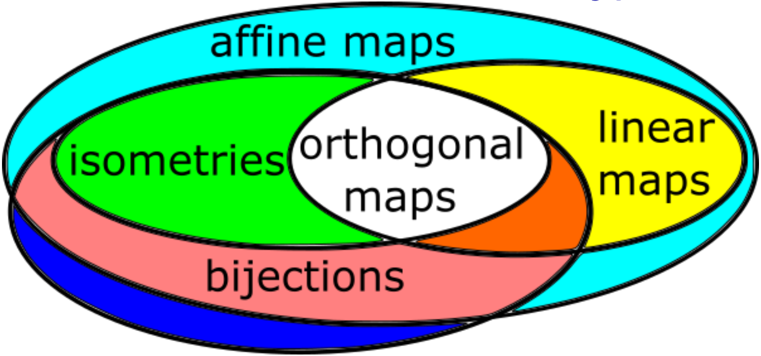
How are these four types of maps connected?

Types of maps in one diagram

Our aim is to understand this diagram, e.g. justify inclusions of types and find examples of maps in each intersection of types (of different maps).



Known inclusions of types



Any linear map $\vec{v} \mapsto A\vec{v}$ is affine by definition.

Theorem 20.3: any Euclidean isometry is bijective.

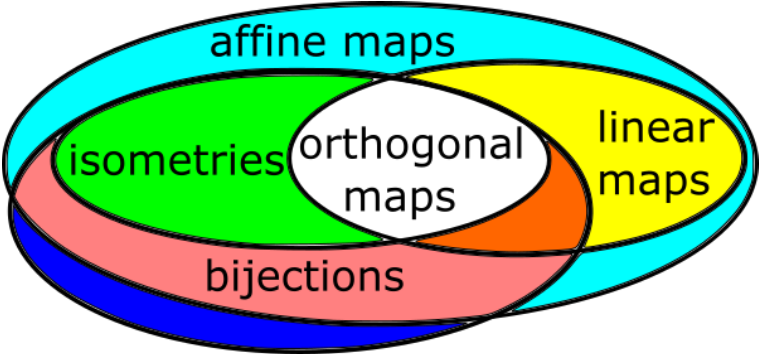
Theorem 20.11: any Euclidean isometry is affine.

Example maps: green and yellow

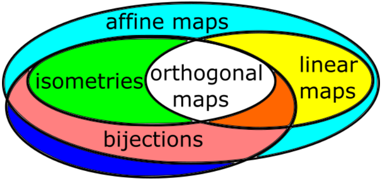
Problem 21.1. Find examples of maps $\mathbb{R} \to \mathbb{R}$ of following types

1) an isometry that is not linear (the green region)

2) a linear map that is not a bijection (yellow)



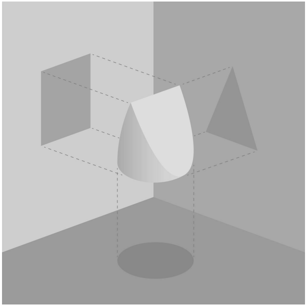
Example maps: green and yellow



Solution 21.1. 1) Green part: A translation by $\vec{u} \neq \vec{0}$ is an isometry, not a linear map (the green region).

2) Yellow part: $x \mapsto 0$ or a projection $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$, which is linear, not bijective (the yellow region).

The famous round square triangle object is a good example of non-bijectivity.

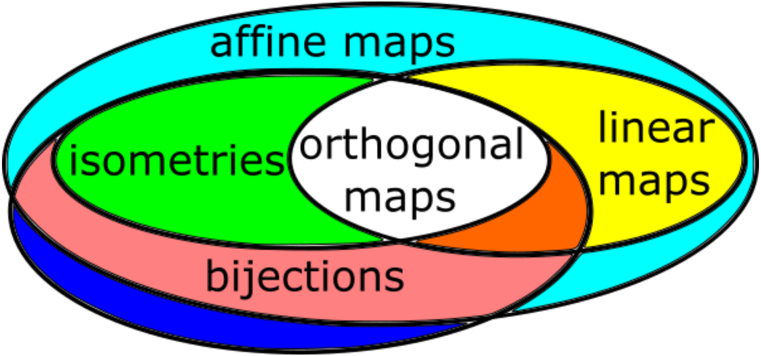


Example maps: cyan and orange

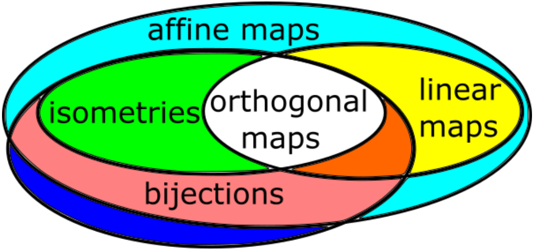
Problem 21.1. Find examples of maps $\mathbb{R} \to \mathbb{R}$.

3) an affine non-linear non-bijection (cyan)

4) a linear bijection that isn't an isometry (orange)



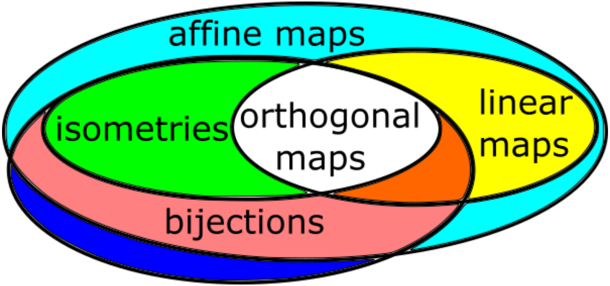
Example maps: cyan and orange



Solution 21.1. 3) Cyan: $x \mapsto 1$ or first project, then shift:
$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+1 \\ 0 \end{pmatrix}$, which is affine, not linear, not bijective.

4) Orange: $x \mapsto 2x$ linear, bijective, non-isometry.

Example maps: pink and dark blue

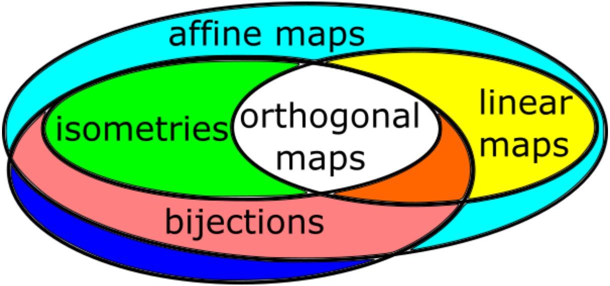


Problem 21.1. Find examples of maps $\mathbb{R} \to \mathbb{R}$.

5) an affine non-linear bijection that is not an isometry (pink)

6) a bijection that is not affine (dark blue)

Example maps: pink and dark blue



Solution 21.1. 5) Pink: $x \mapsto 2x + 1$ is affine, not linear, bijective, not an isometry.

6) Dark blue: $x \mapsto x^3$ is bijective, not affine.

An orthogonal map

Definition 21.2. A linear map $\mathbb{R}^m \to \mathbb{R}^m$, $\vec{v} \mapsto A\vec{v}$ for a square $m \times m$ matrix A, is called **orthogonal** if the scalar product is preserved: $\vec{u} \cdot \vec{v} = (A\vec{u}) \cdot (A\vec{v})$.

Then the matrix A is also called **orthogonal**.

Claim 21.3. Any orthogonal map is an isometry.

Proof. The distance between points $p, q \in \mathbb{R}^m$ is $|\vec{p} - \vec{q}|$. The squared length of any vector $\vec{v}$ is preserved :
$|A\vec{v}|^2 = (A\vec{v}) \cdot (A\vec{v}) = \vec{v} \cdot \vec{v} = |\vec{v}|^2$. $\square$

Linear isometries are orthogonal

Theorem 21.4. Any linear isometry is orthogonal.

Proof. For any $\vec{u}, \vec{v} \in \mathbb{R}^m$, since dinstances are preserved, the squared distances are also preserved:
$|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v})^2 = \vec{u}^2 + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}^2 = |\vec{u}|^2 + 2\vec{u}\cdot\vec{v} + |\vec{v}|^2$.

$|A(\vec{u}+\vec{v})|^2 = (A\vec{u}+A\vec{v})^2 = (A\vec{u})^2 + (A\vec{u})\cdot(A\vec{v}) + (A\vec{v})\cdot(A\vec{u}) + (A\vec{v})^2 = |A\vec{u}|^2 + 2(A\vec{u})\cdot(A\vec{v}) + |A\vec{v}|^2$.

Subtract the right hand sides.
Then use $|A\vec{u}|^2 = |\vec{u}|^2$, $|A\vec{v}|^2 = |\vec{v}|^2$ to get $(A\vec{u})\cdot(A\vec{v}) = \vec{u}\cdot\vec{v}$.
Hence the map $\vec{v} \mapsto A\vec{v}$ is orthogonal by Definition 20.2. $\square$

The transpose of a matrix

Definition 21.5. Any matrix A flipped over its diagonal gives the **transpose** A^T, i.e. $(A^T)_{ij} = a_{ji}$.

A matrix A is **symmetric** if $A^T = A$.
Any diagonal matrix with 0s outside the diagonal is symmetric.

Claim 21.6. The scalar product of vectors equals the matrix product of the vector-row by the vector-column: $\vec{u} \cdot \vec{v} = (\vec{u})^T \vec{v}$ for any $\vec{u}, \vec{v} \in \mathbb{R}^m$.

Proof. Both products are equal to $\sum_{j=1}^{m} u_j v_j$. $\square$

Orthogonal matrices

Lemma 21.7. $(A\vec{u})^T = \vec{u}^T A^T$ for any $k \times m$ matrix A and $\vec{u} \in \mathbb{R}^m$, then $(A\vec{u}) \cdot (A\vec{v}) = \vec{u}^T (A^T A)\vec{v}$.

Proof. The i-th coordinates are equal as follows:
$(A\vec{u})_i^T = \sum_{j=1}^{m} a_{ij} u_j = \sum_{j=1}^{m} u_j (A^T)_{ji} = (\vec{u}^T A^T)_i$. □

Theorem 21.8. A linear map $\vec{v} \mapsto A\vec{v}$ is orthogonal if and only if $A^T A = I$ is the identity matrix.

Proof. For the standard basis $\vec{e}_1, \dots, \vec{e}_m$, Lemma 21.7 implies:
$\vec{e}_i \cdot \vec{e}_j = (A\vec{e}_i) \cdot (A\vec{e}_j) = \vec{e}_i^T (A^T A)\vec{e}_j = (A^T A)_{ji} = 1$ if $i = j$, otherwise 0, hence $A^T A = I$. □

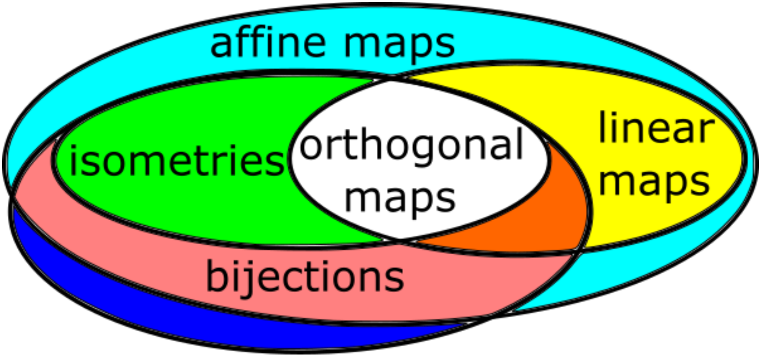
Orthogonality of columns

Claim 21.9. Columns $\vec{v}_1, \dots, \vec{v}_m$ of any orthogonal matrix A are orthogonal to each other and have length 1, i.e. briefly $\vec{v}_i \cdot \vec{v}_j = 1$ if $i = j$, otherwise 0.

Proof. $(A^T A)_{ij} = \sum_{k=1}^{m} A^T_{ik} A_{kj} = \sum_{k=1}^{m} A_{ki} A_{kj}$ is the scalar product of the i-th and j-th columns of A.

The columns of A are images of basis vectors under the map $\vec{v} \mapsto A\vec{v}$. If A is orthogonal, i.e. by Theorem 20.8 $A^T A = I$, then the scalar product of the i-th and j-th columns of A equals I_{ij}, i.e. 1 if $i = j$, otherwise 0. $\square$

Time to revise and ask questions



Problem 21.10. Is the matrix of a rotation about $0 \in \mathbb{R}^2$ orthogonal? $A = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$

References & links

- Composition of reflections.
- Orthogonal transformations with inner products.