# COMP108
# Data Structures and Algorithms

## Dynamic Programming (Part II Assembly Line Scheduling)

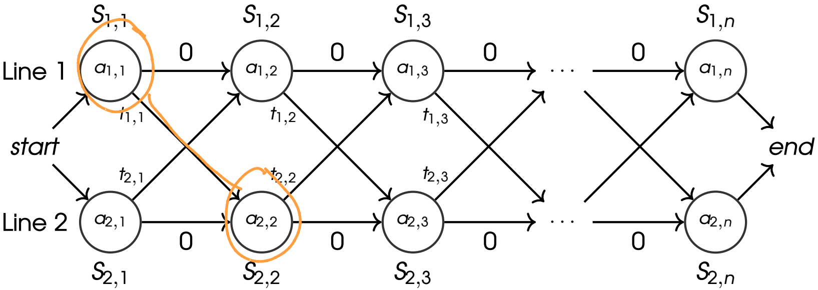
Professor Prudence Wong

pwong@liverpool.ac.uk

2022-23

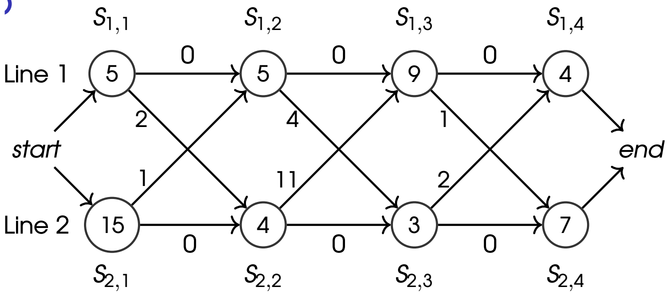
## Assembly Line Scheduling

- 2 assembly lines, each with $n$ stations ($S_{i,j}$: line $i$ station $j$)
- $S_{1,j}$ and $S_{2,j}$ perform same task but time taken is different
- $a_{i,j}$: assembly time at $S_{i,j}$
- $t_{i,j}$: transfer time from $S_{i,j}$

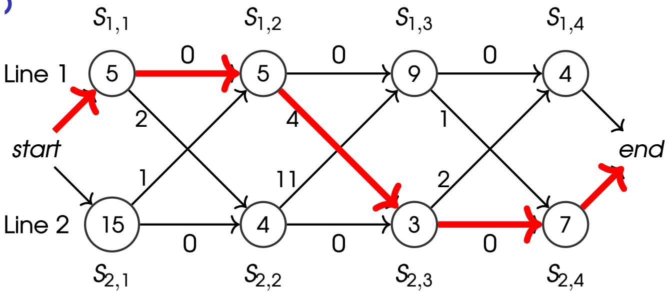



Problem: To determine which stations to go in order to **minimize** the total time through the n stations

## Example (1)

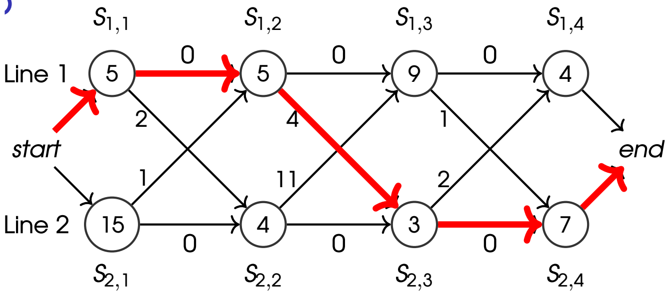

$S_{1,1}$
$S_{1,2}$
$S_{1,3}$
$S_{1,4}$
Line 1
5
5
9
4
0
0
0
2
4
1
start
end
1
11
2
Line 2
15
4
3
7
0
0
0
$S_{2,1}$
$S_{2,2}$
$S_{2,3}$
$S_{2,4}$

## Example (1)



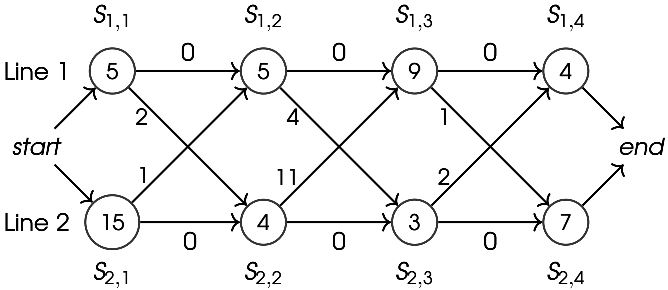

station chosen $S_{1,1}$ $S_{1,2}$ $S_{2,3}$ $S_{2,4}$

## Example (1)



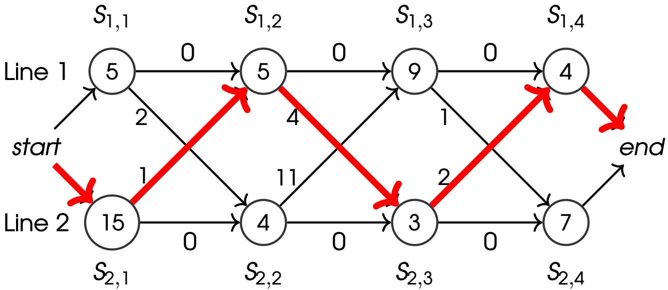

| | | | | | | |
|---|---|---|---|---|---|---|
| station chosen | $S_{1,1}$ | $S_{1,2}$ | | $S_{2,3}$ | $S_{2,4}$ | |
| time required | 5 | 5 | 4 | 3 | 7 | = 24 |

## Example (2)



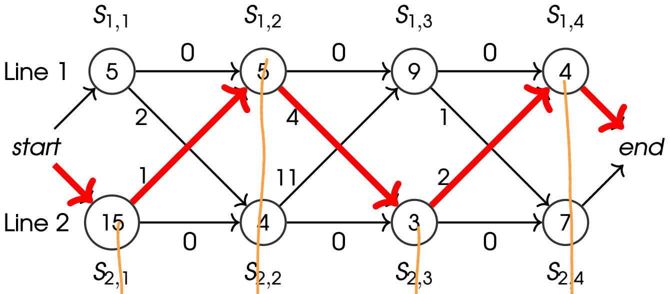
$S_{1,1}$
$S_{1,2}$
$S_{1,3}$
$S_{1,4}$
Line 1
5
5
9
4
0
0
0
2
4
1
start
end
1
11
2
Line 2
15
4
3
7
0
0
0
$S_{2,1}$
$S_{2,2}$
$S_{2,3}$
$S_{2,4}$

## Example (2)



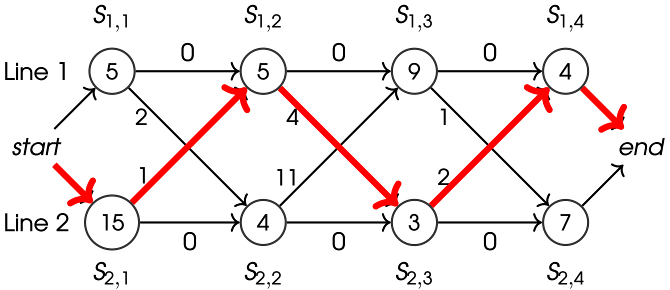

| | | | | |
|---|---|---|---|---|
| station chosen | $S_{2,1}$ | $S_{1,2}$ | $S_{2,3}$ | $S_{1,4}$ |

## Example (2)




| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| station chosen | $S_{2,1}$ | | $S_{1,2}$ | | $S_{2,3}$ | | $S_{1,4}$ | |
| time required | 15 | 1 | 5 | 4 | 3 | 2 | 4 | = 34 |

## Example (2)




| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| station chosen | $S_{2,1}$ | | $S_{1,2}$ | | $S_{2,3}$ | | $S_{1,4}$ | |
| time required | 15 | 1 | 5 | 4 | 3 | 2 | 4 | = 34 |

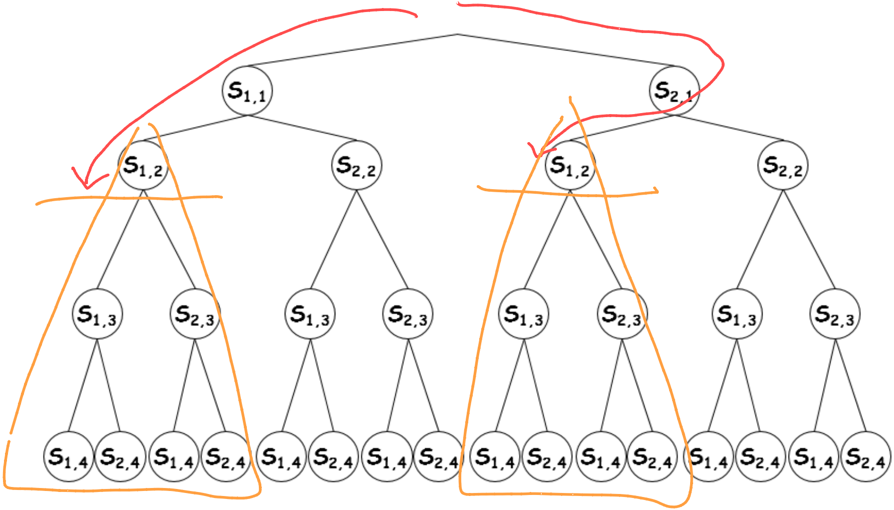
How to determine the best stations to go?

There are altogether $2^n$ choices of stations.

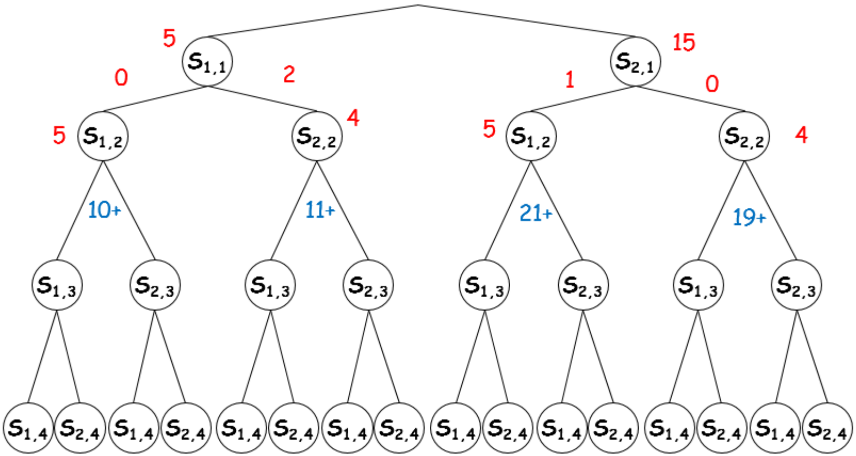
Should we try them all?

## All possible choices

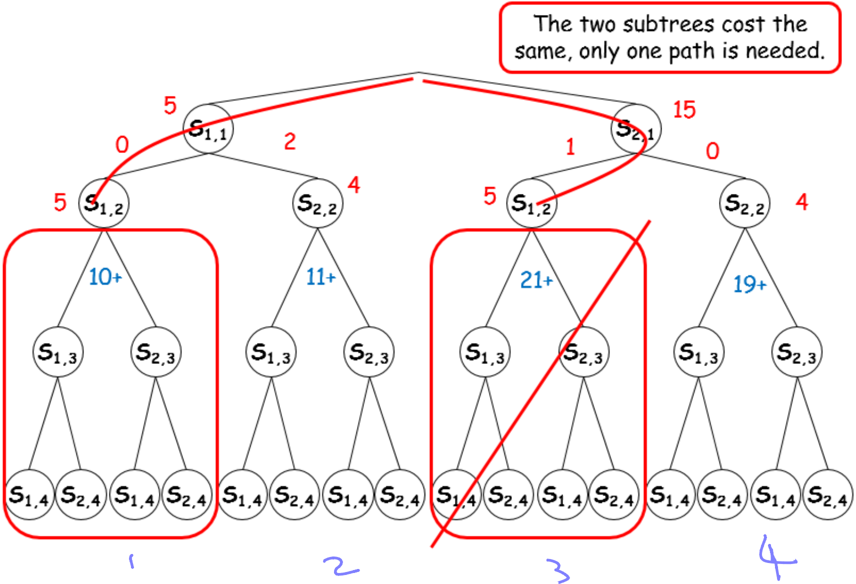
2^n possible paths, Q: can we reduce search space?



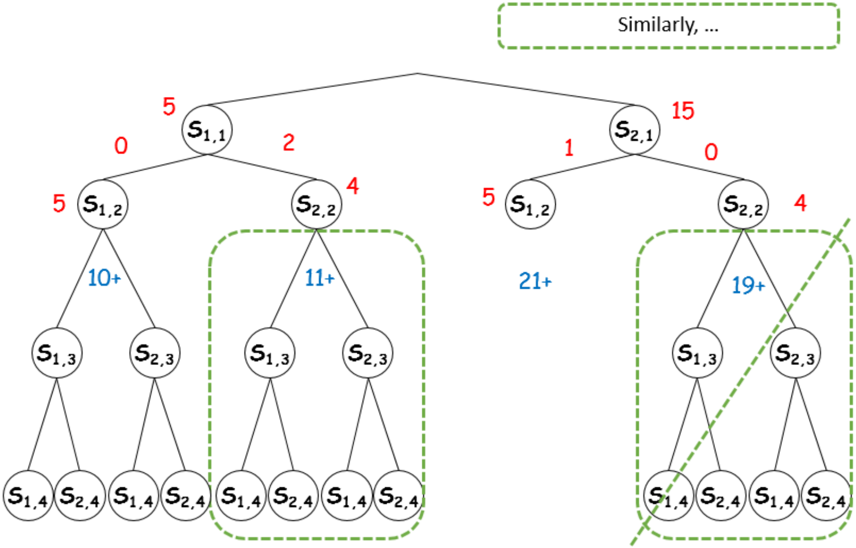
# All possible choices



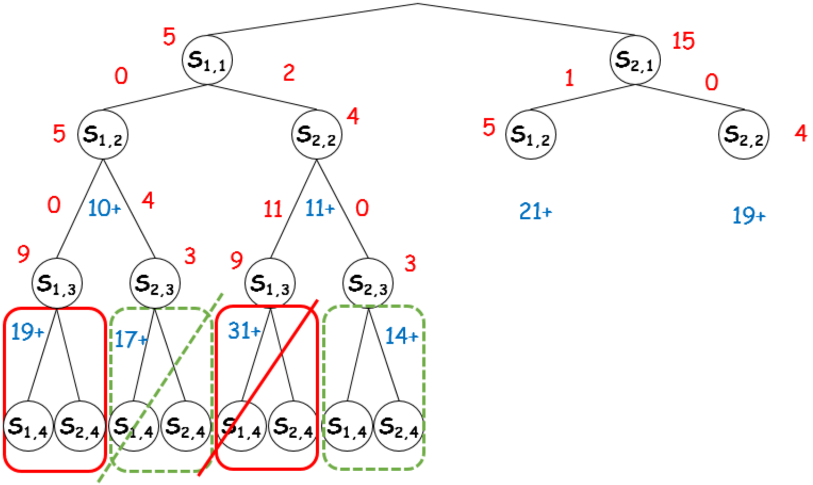
## All possible choices



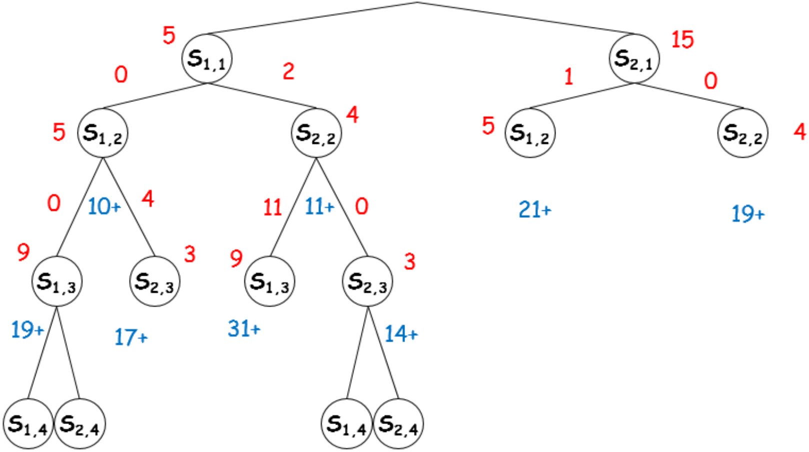
## All possible choices



## All possible choices



# All possible choices



## Good news: Dynamic Programming

- We don't need to try all possible choices.
- We can make use of **dynamic programming**:

## Good news: Dynamic Programming

- We don't need to try all possible choices.
- We can make use of **dynamic programming**:
  1. If we know the fastest ways to get thro' station $S_{1,n}$ and $S_{2,n}$

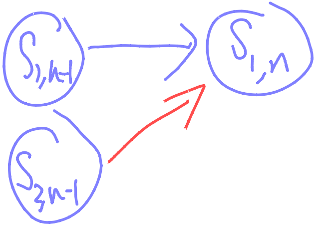
     $\Longrightarrow$ the faster of these two is overall fastest

$S_{1,n-1}$ $S_{1,n}$

$S_{2,n-1}$ $S_{2,n}$

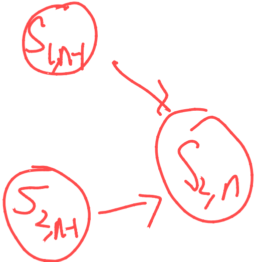
## Good news: Dynamic Programming

- We don′t need to try all possible choices.
- We can make use of **dynamic programming**:
  1. If we know the fastest ways to get thro′ station $S_{1,n}$ and $S_{2,n}$
     $\Longrightarrow$ the faster of these two is overall fastest
  2. Fastest ways to get thro′ $S_{1,n}$?
     need to know fastest way to get thro′ $S_{1,n-1}$ and $S_{2,n-1}$



## Good news: Dynamic Programming

- We don't need to try all possible choices.
- We can make use of **dynamic programming**:
  1. If we know the fastest ways to get thro' station $S_{1,n}$ and $S_{2,n}$
     $\Longrightarrow$ the faster of these two is overall fastest
  2. Fastest ways to get thro' $S_{1,n}$?
     need to know fastest way to get thro' $S_{1,n-1}$ and $S_{2,n-1}$
  3. Similarly for $S_{2,n}$



## Good news: Dynamic Programming

- We don't need to try all possible choices.
- We can make use of **dynamic programming**:
  1. If we know the fastest ways to get thro' station $S_{1,n}$ and $S_{2,n}$
     $\Longrightarrow$ the faster of these two is overall fastest
  2. Fastest ways to get thro' $S_{1,n}$?
     need to know fastest way to get thro' $S_{1,n-1}$ and $S_{2,n-1}$
  3. Similarly for $S_{2,n}$
  4. Generalsing, we want fastest way to get thro' $S_{1,j}$ and $S_{2,j}$, for all $j$.

# A dynamic programming solution - formalisation

What are the sub-problems?

- given $j$, what is the fastest way to get thro' $S_{1,j}$
- given $j$, what is the fastest way to get thro' $S_{2,j}$

## A dynamic programming solution - formalisation

What are the sub-problems?

- given $j$, what is the fastest way to get thro' $S_{1,j}$
- given $j$, what is the fastest way to get thro' $S_{2,j}$

Definitions:

- $f_1[j]$: the fastest time to get thro' $S_{1,j}$
- $f_2[j]$: the fastest time to get thro' $S_{2,j}$

## A dynamic programming solution - formalisation

What are the sub-problems?

- given $j$, what is the fastest way to get thro' $S_{1,j}$
- given $j$, what is the fastest way to get thro' $S_{2,j}$

Definitions:

- $f_1[j]$: the fastest time to get thro' $S_{1,j}$
- $f_2[j]$: the fastest time to get thro' $S_{2,j}$

The final solution equals to $\min\{f_1[n], f_2[n]\}$

## A dynamic programming solution - formalisation

What are the sub-problems?

- given $j$, what is the fastest way to get thro' $S_{1,j}$
- given $j$, what is the fastest way to get thro' $S_{2,j}$

Definitions:

- $f_1[j]$: the fastest time to get thro' $S_{1,j}$
- $f_2[j]$: the fastest time to get thro' $S_{2,j}$

The final solution equals to $\min\{f_1[n], f_2[n]\}$

Task:

- Starting from $f_1[1]$ and $f_2[1]$, compute $f_1[j]$ and $f_2[j]$ incrementally
- i.e., $f_1[2]\&f_2[2]$, $f_1[3]\&f_2[3]$, $\cdots$, $f_1[n]\&f_2[n]$

## Solving the sub-problems (1)

**Q1: What is the fastest time to get thro' $S_{1,j}$**

A: either

- the fastest way thro' $S_{1,j-1}$, then directly to $S_{1,j}$, or
- the fastest way thro' $S_{2,j-1}$, a transfer from line 2 to line 1, and then thro' $S_{1,j}$

## Solving the sub-problems (1)

**Q1: What is the fastest time to get thro' $S_{1,j}$**

$S_{1,j}$

A: either

- the fastest way thro' $S_{1,j-1}$, then directly to $S_{1,j}$, or
- the fastest way thro' $S_{2,j-1}$, a transfer from line 2 to line 1, and then thro' $S_{1,j}$

(i) $f_1[j-1] + a_{1,j}$

## Solving the sub-problems (1)

**Q1: What is the fastest time to get thro' $S_{1,j}$**

A: either

- the fastest way thro' $S_{1,j-1}$, then directly to $S_{1,j}$, or
- the fastest way thro' $S_{2,j-1}$, a transfer from line 2 to line 1, and then thro' $S_{1,j}$

(i) $f_1[j-1] + a_{1,j}$ (ii) $f_2[j-1] + t_{2,j-1} + a_{1,j}$

## Solving the sub-problems (1)

**Q1: What is the fastest time to get thro' $S_{1,j}$**

A: either

- the fastest way thro' $S_{1,j-1}$, then directly to $S_{1,j}$, or
- the fastest way thro' $S_{2,j-1}$, a transfer from line 2 to line 1, and then thro' $S_{1,j}$

$$\text{(i) } f_1[j-1] + a_{1,j} \qquad \text{(ii) } f_2[j-1] + t_{2,j-1} + a_{1,j}$$

$$\therefore f_1[j] = \min\{f_1[j-1] + a_{1,j}, \quad f_2[j-1] + t_{2,j-1} + a_{1,j}\}$$

## Solving the sub-problems (1)

**Q1: What is the fastest time to get thro' $S_{1,j}$**

A: either

- the fastest way thro' $S_{1,j-1}$, then directly to $S_{1,j}$, or
- the fastest way thro' $S_{2,j-1}$, a transfer from line 2 to line 1, and then thro' $S_{1,j}$

$$\text{(i) } f_1[j-1] + a_{1,j} \qquad \text{(ii) } f_2[j-1] + t_{2,j-1} + a_{1,j}$$

$$\therefore f_1[j] = \min\{f_1[j-1] + a_{1,j}, \quad f_2[j-1] + t_{2,j-1} + a_{1,j}\}$$

$$\text{Boundary case: } f_1[1] = a_{1,1}$$

## Solving the sub-problems (2)

**Q1: What is the fastest time to get thro' $S_{2,j}$**

A: either

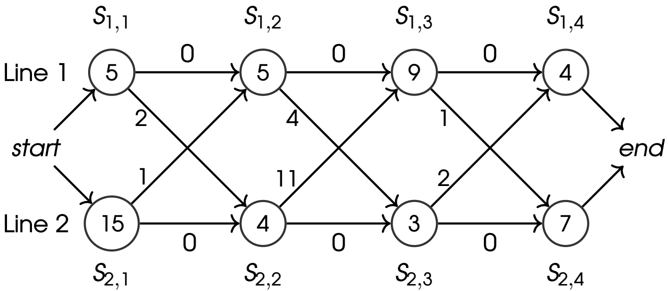
- the fastest way thro' $S_{1,j-1}$, a transfer from line 2 to line 1, and then thro' $S_{2,j}$, or
- the fastest way thro' $S_{2,j-1}$, then directly to $S_{2,j}$

$$\text{(i) } f_1[j-1] + t_{1,j-1} + a_{2,j} \qquad \text{(ii) } f_2[j-1] + a_{2,j}$$

$$\therefore\ f_2[j] = \min\{f_1[j-1] + t_{1,j-1} + a_{2,j},\quad f_2[j-1] + a_{2,j}\}$$

$$\text{Boundary case: } f_2[1] = a_{2,1}$$

## Example again

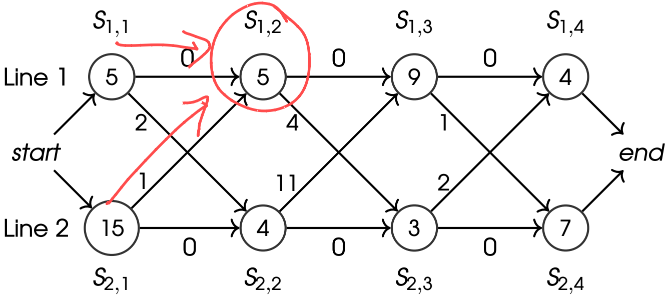



| $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
|---|---|---|---|
| 5 | | | |

minimum cost:

| $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
|---|---|---|---|
| 15 | | | |

## Example again



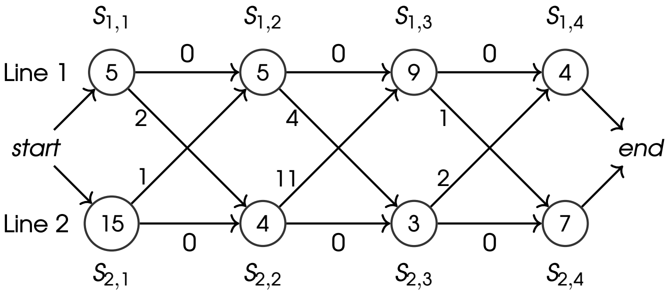

| $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
|---|---|---|---|
| 5 $\longrightarrow$ | 5+5=10 | | |

15+1+5=21

minimum cost: ____________

| $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
|---|---|---|---|
| 15 | | | |

# Example again

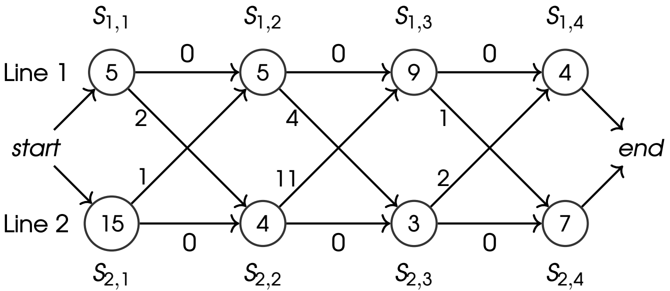



| $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
|---|---|---|---|
| 5 → | 5+5=10 | | |
| | 15+1+5 | | |

minimum cost:

| $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
|---|---|---|---|
| 15 | | | |

# Example again

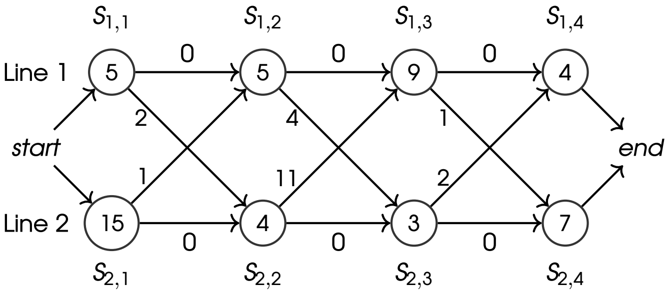



| $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
|---|---|---|---|
| 5 → | 5+5=10 | | |
| | ~~15+1+5~~ | | |

minimum cost:

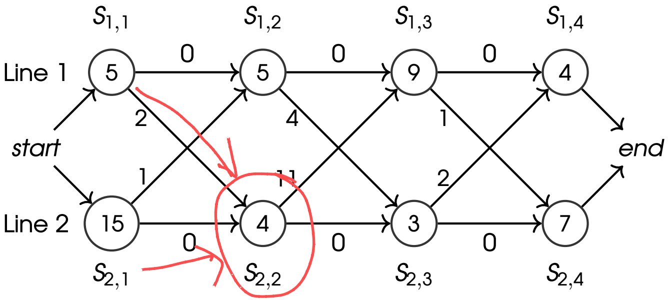
| $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
|---|---|---|---|
| 15 | | | |

# Example again




| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
|---|---|---|---|---|
| | 5 | 5+5=10 | | |
| | | ~~15+1+5~~ | | |
| minimum cost: | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
| | 15 | 5+2+4=11 | | |

# Example again

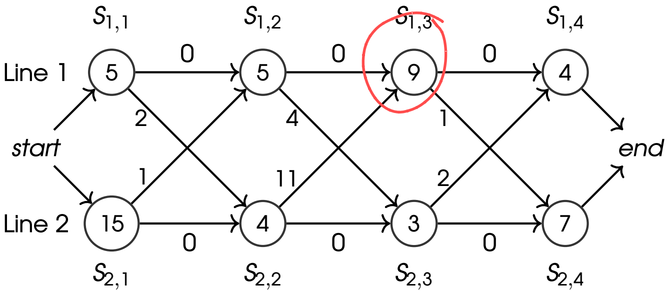



| | | | |
|---|---|---|---|
| $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
| 5 | 5+5=10 | | |
| | ~~15+1+5~~ | | |

minimum cost:

| | | | |
|---|---|---|---|
| $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
| 15 | 5+2+4=11 | | |
| | 15+4 | | |

# Example again

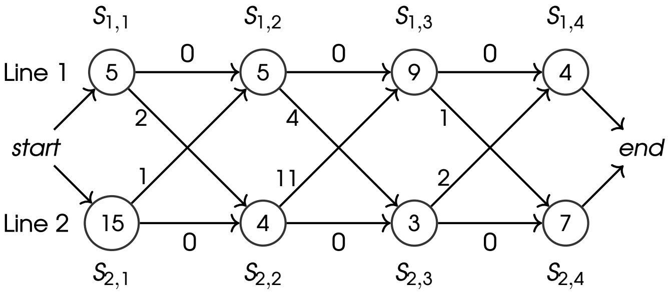



| $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
|---|---|---|---|
| 5 | 5+5=10 | | |
| | ~~15+1+5~~ | | |

minimum cost:

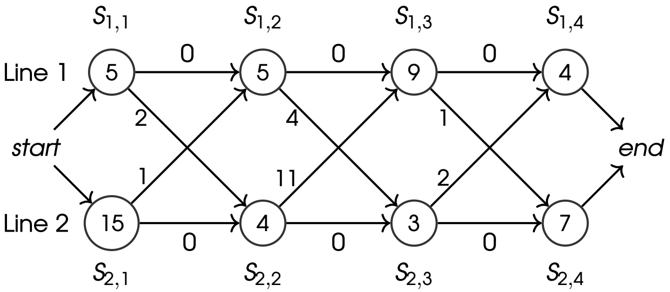
| $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
|---|---|---|---|
| 15 | 5+2+4=11 | | |
| | ~~15+4~~ | | |

## Example again



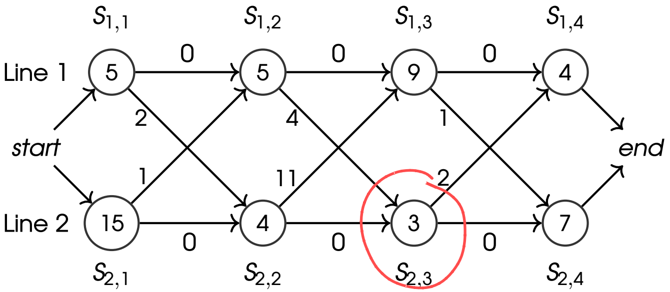

| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
|---|---|---|---|---|
| | 5 | → 5+5=10 → | 10+9=19 | |
| | | ~~15+1+5~~ | | |
| minimum cost: | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
| | 15 | 5+2+4=11 | | |
| | | ~~15+4~~ | | |

# Example again



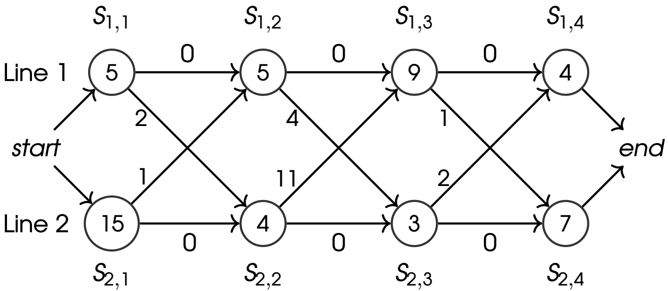

| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
|---|---|---|---|---|
| | 5 | 5+5=10 | 10+9=19 | |
| | | $\cancel{15+1+5}$ | 11+11+9 | |
| minimum cost: | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
| | 15 | 5+2+4=11 | | |
| | | $\cancel{15+4}$ | | |

# Example again



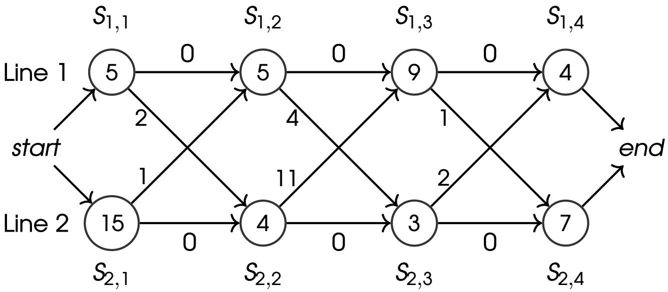

| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
|---|---|---|---|---|
| | 5 | 5+5=10 | 10+9=19 | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | |
| minimum cost: | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
| | 15 | 5+2+4=11 | | |
| | | ~~15+4~~ | | |

# Example again



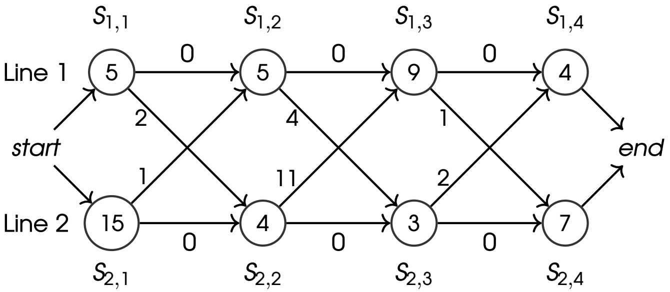

| | | | | |
|---|---|---|---|---|
| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
| | 5 | 5+5=10 | 10+9=19 | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | |
| minimum cost: | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
| | 15 | 5+2+4=11 | 10+4+3 | |
| | | ~~15+4~~ | | |

## Example again



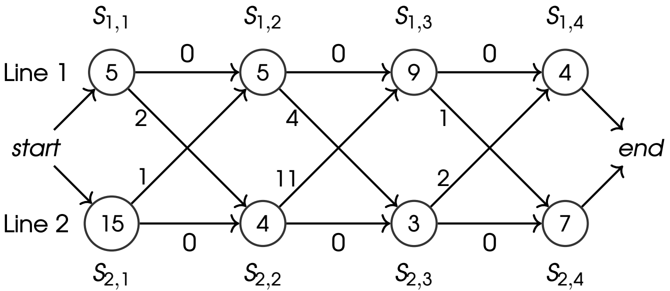

| | | | | |
|---|---|---|---|---|
| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
| | 5 | 5+5=10 | 10+9=19 | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | |
| minimum cost: | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
| | 15 | 5+2+4=11 | 10+4+3 | |
| | | ~~15+4~~ | 11+3=14 | |

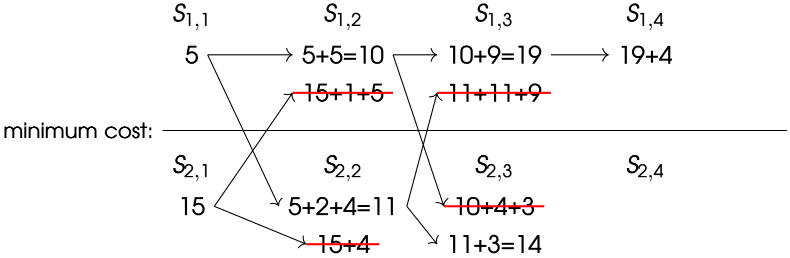
## Example again



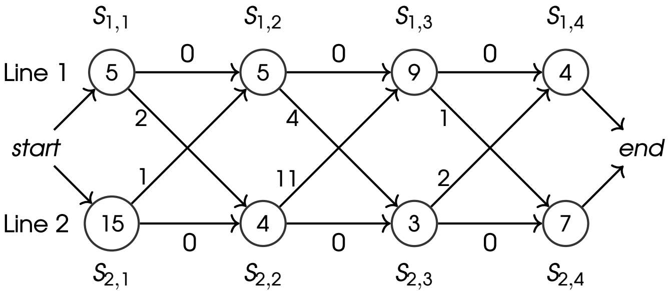

| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
|---|---|---|---|---|
| | 5 | 5+5=10 | 10+9=19 | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | |
| minimum cost: | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
| | 15 | 5+2+4=11 | ~~10+4+3~~ | |
| | | ~~15+4~~ | 11+3=14 | |

# Example again



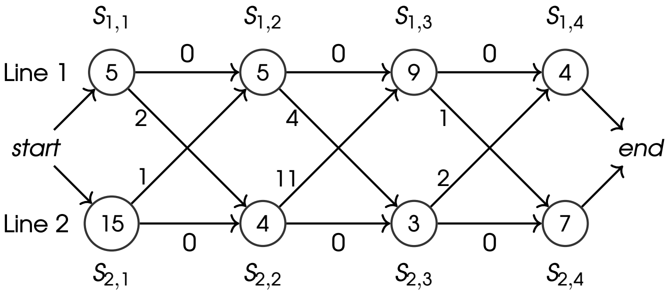



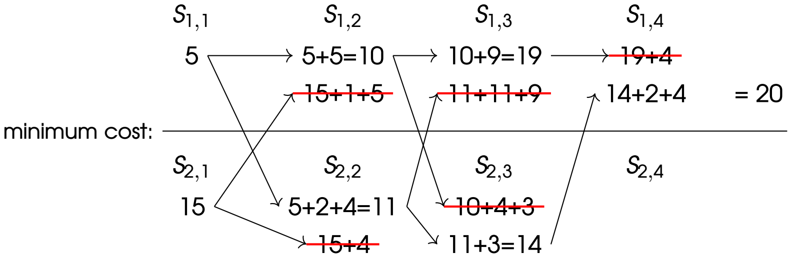
# Example again



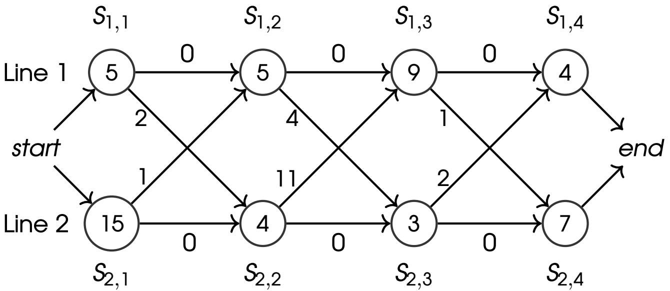

| | | | | | |
|---|---|---|---|---|---|
| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ | |
| | 5 | 5+5=10 | 10+9=19 | 19+4 | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | 14+2+4 | = 20 |
| minimum cost: | | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ | |
| | 15 | 5+2+4=11 | ~~10+4+3~~ | | |
| | | ~~15+4~~ | 11+3=14 | | |

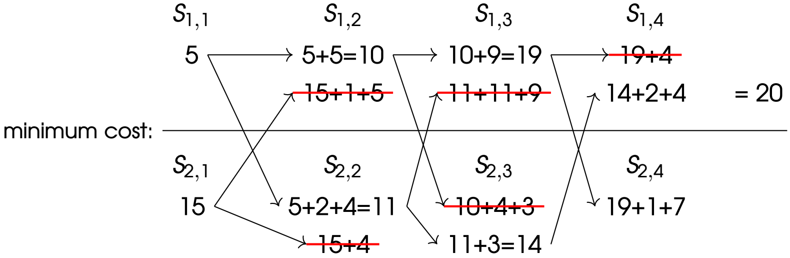
## Example again



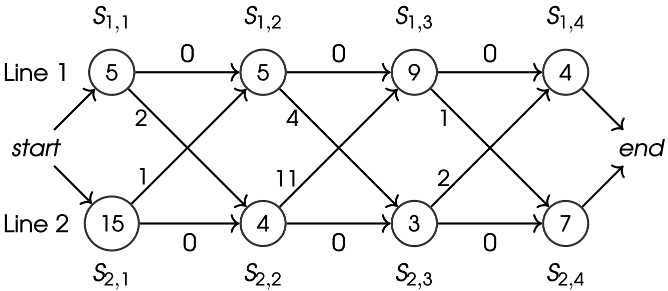



# Example again



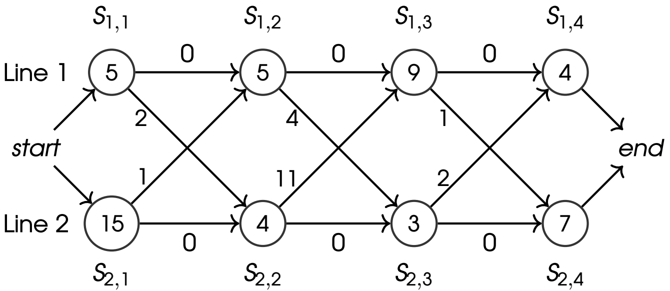



## Example again



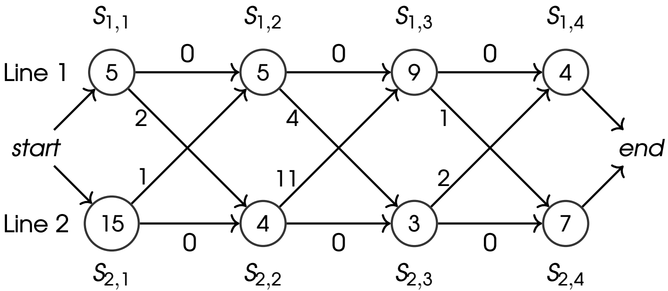

| | | | | | |
|---|---|---|---|---|---|
| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ | |
| | 5 | 5+5=10 | 10+9=19 | ~~19+4~~ | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | 14+2+4 | = 20 |
| minimum cost: | | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ | |
| | 15 | 5+2+4=11 | ~~10+4+3~~ | 19+1+7 | |
| | | ~~15+4~~ | 11+3=14 | 14+7 | |

# Example again



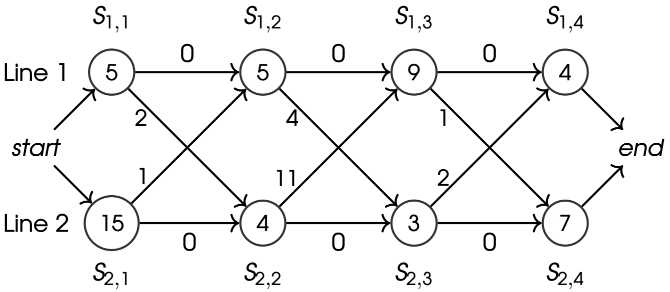

| | | | | | |
|---|---|---|---|---|---|
| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ | |
| | 5 | 5+5=10 | 10+9=19 | ~~19+4~~ | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | 14+2+4 | = 20 |
| minimum cost: | | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ | |
| | 15 | 5+2+4=11 | ~~10+4+3~~ | ~~19+1+7~~ | |
| | | ~~15+4~~ | 11+3=14 | 14+7 | = 21 |

# Example again



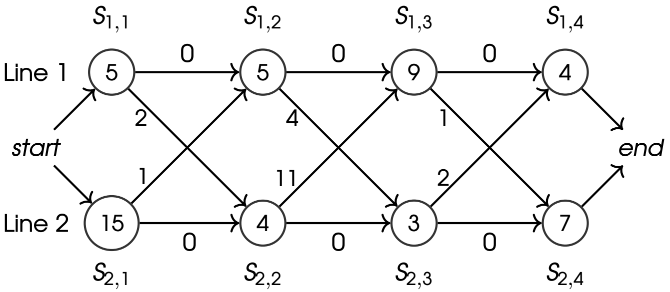

| | | | | | |
|---|---|---|---|---|---|
| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ | |
| | 5 | 5+5=10 | 10+9=19 | ~~19+4~~ | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | 14+2+4 | = 20 |
| minimum cost: | | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ | |
| | 15 | 5+2+4=11 | ~~10+4+3~~ | ~~19+1+7~~ | |
| | | ~~15+4~~ | 11+3=14 | 14+7 | ~~= 21~~ |

## Example again



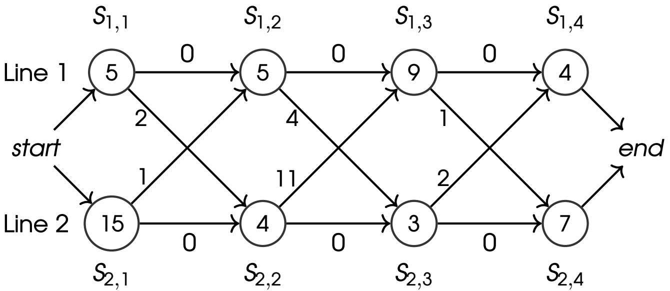

| | | | | | |
|---|---|---|---|---|---|
| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ | |
| | 5 | 5+5=10 | 10+9=19 | ~~19+4~~ | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | 14+2+4 | = 20 |
| minimum cost: | | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ | |
| | 15 | 5+2+4=11 | ~~10+4+3~~ | ~~19+1+7~~ | |
| | | ~~15+4~~ | 11+3=14 | 14+7 | ~~= 21~~ |

# Example again



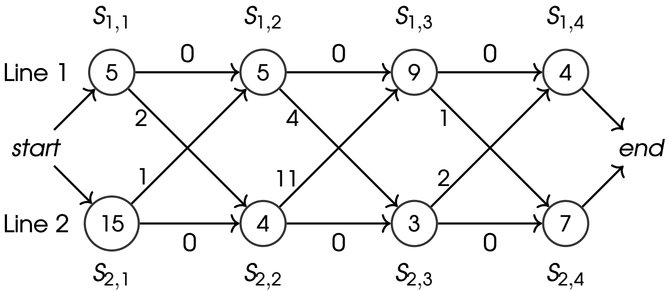

| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ | |
|---|---|---|---|---|---|
| | 5 | 5+5=10 | 10+9=19 | ~~19+4~~ | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | 14+2+4 | $\boxed{= 20}$ |
| minimum cost: | | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ | |
| | 15 | 5+2+4=11 | ~~10+4+3~~ | ~~19+1+7~~ | |
| | | ~~15+4~~ | 11+3=14 | 14+7 | ~~= 21~~ |

# Example again



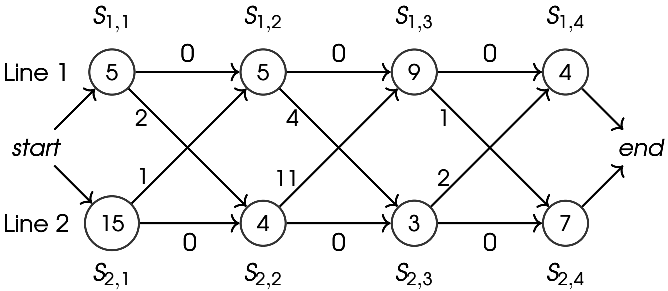

| | | | | | |
|---|---|---|---|---|---|
| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $\boxed{S_{1,4}}$ | |
| | 5 | 5+5=10 | 10+9=19 | ~~19+4~~ | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | 14+2+4 | $\boxed{= 20}$ |
| minimum cost: | | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ | |
| | 15 | 5+2+4=11 | ~~10+4+3~~ | ~~19+1+7~~ | |
| | | ~~15+4~~ | 11+3=14 | 14+7 | ~~= 21~~ |

# Example again



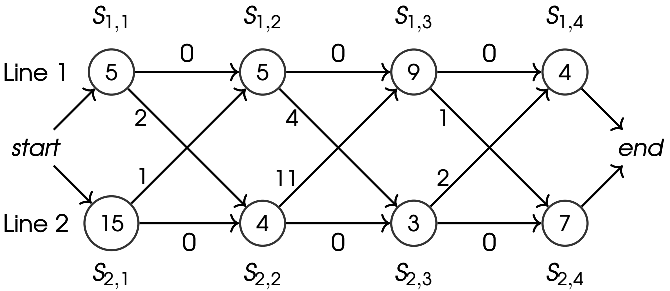

| | | | | | |
|---|---|---|---|---|---|
| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $\boxed{S_{1,4}}$ | |
| | 5 | 5+5=10 | 10+9=19 | ~~19+4~~ | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | 14+2+4 | $\boxed{= 20}$ |
| minimum cost: | | | | | |
| | $S_{2,1}$ | $S_{2,2}$ | $\boxed{S_{2,3}}$ | $S_{2,4}$ | |
| | 15 | 5+2+4=11 | ~~10+4+3~~ | ~~19+1+7~~ | |
| | | ~~15+4~~ | 11+3=14 | 14+7 | ~~= 21~~ |

## Example again



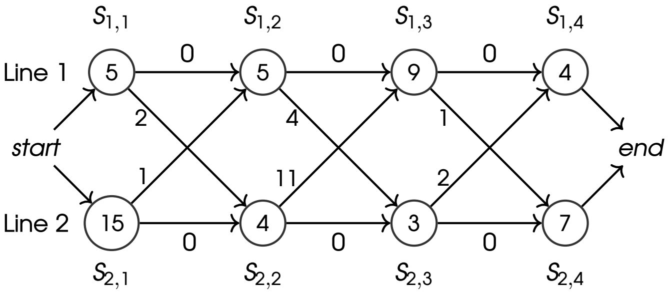

| | | | | | |
|---|---|---|---|---|---|
| | $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $\boxed{S_{1,4}}$ | |
| | 5 | 5+5=10 | 10+9=19 | ~~19+4~~ | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | 14+2+4 | $\boxed{= 20}$ |
| minimum cost: | | | | | |
| | $S_{2,1}$ | $\boxed{S_{2,2}}$ | $\boxed{S_{2,3}}$ | $S_{2,4}$ | |
| | 15 | 5+2+4=11 | ~~10+4+3~~ | ~~19+1+7~~ | |
| | | ~~15+4~~ | 11+3=14 | 14+7 | ~~= 21~~ |

# Example again



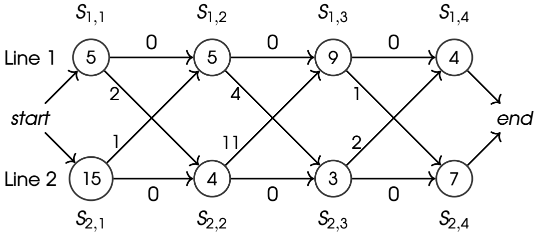

| | | | | | |
|---|---|---|---|---|---|
| | $\boxed{S_{1,1}}$ | $S_{1,2}$ | $S_{1,3}$ | $\boxed{S_{1,4}}$ | |
| | 5 | 5+5=10 | 10+9=19 | ~~19+4~~ | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | 14+2+4 | $\boxed{= 20}$ |
| minimum cost: | | | | | |
| | $S_{2,1}$ | $\boxed{S_{2,2}}$ | $\boxed{S_{2,3}}$ | $S_{2,4}$ | |
| | 15 | 5+2+4=11 | ~~10+4+3~~ | ~~19+1+7~~ | |
| | | ~~15+4~~ | 11+3=14 | 14+7 | ~~= 21~~ |

# Example again - $S_{1,1}$, $S_{2,2}$, $S_{2,3}$, $S_{1,4}$




| | $\boxed{S_{1,1}}$ | $S_{1,2}$ | $S_{1,3}$ | $\boxed{S_{1,4}}$ | |
|---|---|---|---|---|---|
| | 5 | 5+5=10 | 10+9=19 | ~~19+4~~ | |
| | | ~~15+1+5~~ | ~~11+11+9~~ | 14+2+4 | $\boxed{= 20}$ |
| minimum cost: | | | | | |
| | $S_{2,1}$ | $\boxed{S_{2,2}}$ | $\boxed{S_{2,3}}$ | $S_{2,4}$ | |
| | 15 | 5+2+4=11 | ~~10+4+3~~ | ~~19+1+7~~ | |
| | | ~~15+4~~ | 11+3=14 | 14+7 | ~~= 21~~ |

# Summary on the Example




| $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
|---|---|---|---|
| 5 | | | |

minimum cost: ______

| $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
|---|---|---|---|
| 15 | | | |

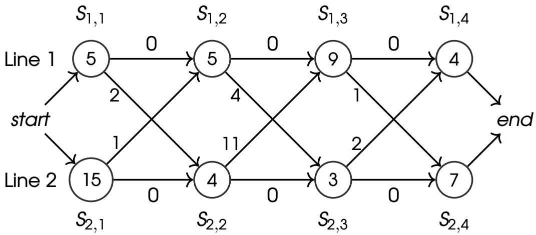
| $j$ | $f_1[j]$ | $f_2[j]$ |
|---|---|---|
| 1 | 5 | 15 |
| 2 | | |
| 3 | | |
| 4 | | |

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j = 1 \\ \min\{f_1[j-1] + a_{1,j},\ f_2[j-1] + t_{2,j-1} + a_{1,j}\} & \text{if } j > 1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j = 1 \\ \min\{\min\{f_1[j-1] + t_{1,j-1} + a_{2,j},\ f_2[j-1] + a_{2,j}\} & \text{if } j > 1 \end{cases}$$

$$f^* = \min\{f_1[n], f_2[n]\}$$

# Summary on the Example




| $S_{1,1}$ | $S_{1,2}$ | $S_{1,3}$ | $S_{1,4}$ |
|---|---|---|---|
| 5 | 5+5=10 | | |
| | ~~15+1+5~~ | | |

minimum cost:

| $S_{2,1}$ | $S_{2,2}$ | $S_{2,3}$ | $S_{2,4}$ |
|---|---|---|---|
| 15 | 5+2+4=11 | | |
| | ~~15+4~~ | | |

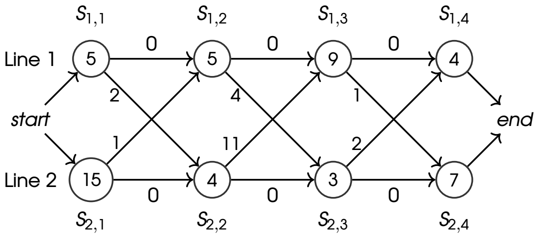
| $j$ | $f_1[j]$ | $f_2[j]$ |
|---|---|---|
| 1 | 5 | 15 |
| 2 | 10 | 11 |
| 3 | | |
| 4 | | |

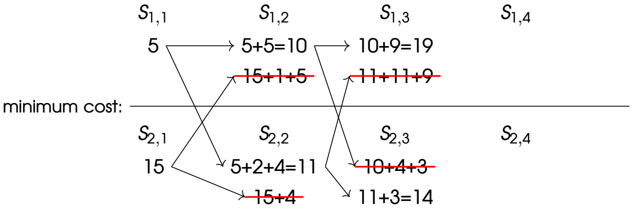
$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j = 1 \\ \min\{f_1[j-1] + a_{1,j}, f_2[j-1] + t_{2,j-1} + a_{1,j}\} & \text{if } j > 1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j = 1 \\ \min\{\min\{f_1[j-1] + t_{1,j-1} + a_{2,j}, f_2[j-1] + a_{2,j}\} & \text{if } j > 1 \end{cases}$$

$$f^* = \min\{f_1[n], f_2[n]\}$$

## Summary on the Example



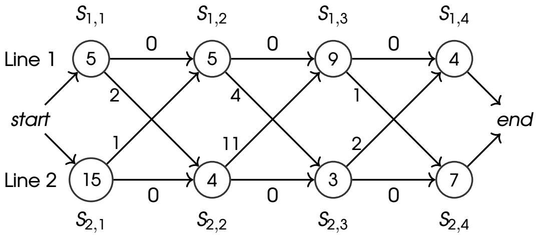

| $j$ | $f_1[j]$ | $f_2[j]$ |
|---|---|---|
| 1 | 5 | 15 |
| 2 | 10 | 11 |
| 3 | 19 | 14 |
| 4 | | |

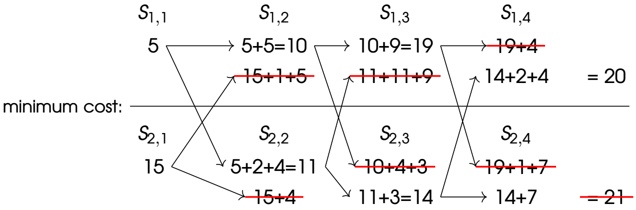
$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j = 1 \\ \min\{f_1[j-1] + a_{1,j}, f_2[j-1] + t_{2,j-1} + a_{1,j}\} & \text{if } j > 1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j = 1 \\ \min\{\min\{f_1[j-1] + t_{1,j-1} + a_{2,j}, f_2[j-1] + a_{2,j}\} & \text{if } j > 1 \end{cases}$$

$$f^* = \min\{f_1[n], f_2[n]\}$$

# Summary on the Example




| $j$ | $f_1[j]$ | $f_2[j]$ |
|---|---|---|
| 1 | 5 | 15 |
| 2 | 10 | 11 |
| 3 | 19 | 14 |
| 4 | 20 | 21 |

$$f_1[j] = \begin{cases} a_{1,1} & \text{if } j = 1 \\ \min\{f_1[j-1] + a_{1,j}, f_2[j-1] + t_{2,j-1} + a_{1,j}\} & \text{if } j > 1 \end{cases}$$

$$f_2[j] = \begin{cases} a_{2,1} & \text{if } j = 1 \\ \min\{\min\{f_1[j-1] + t_{1,j-1} + a_{2,j}, f_2[j-1] + a_{2,j}\} & \text{if } j > 1 \end{cases}$$

$$f^* = \min\{f_1[n], f_2[n]\}$$

## Pseudo code

$O(n)$

$f_1[1] \leftarrow a_{1,1}$

$f_2[1] \leftarrow a_{2,1}$

for $j \leftarrow 2$ to $n$ do $\leftarrow O(n)$

begin

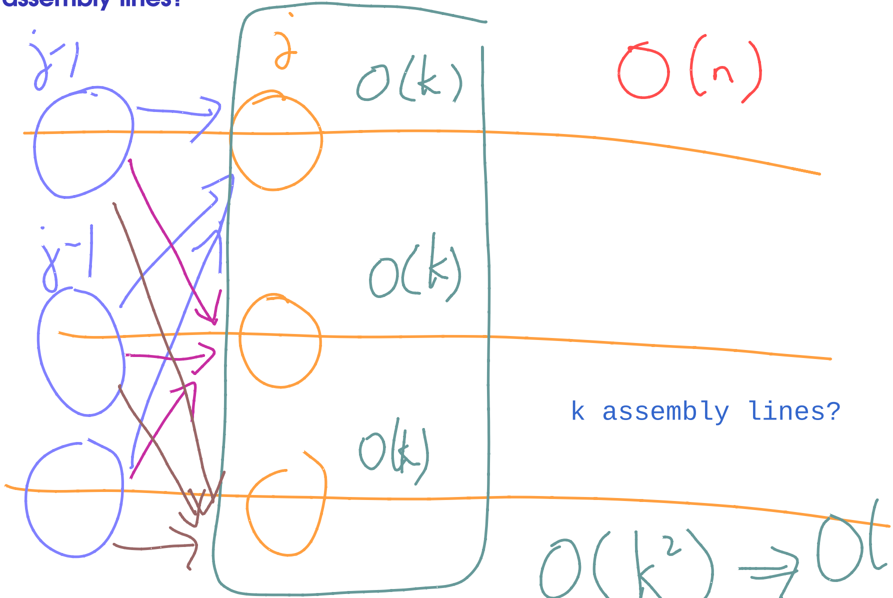
$\quad f_1[j] \leftarrow \min\{f_1[j-1] + a_{1,j}, f_2[j-1] + t_{2,j-1} + a_{1,j}\}$ $O(1)$

$\quad f_2[j] \leftarrow \min\{f_2[j-1] + a_{2,j}, f_1[j-1] + t_{1,j-1} + a_{2,j}\}$ $O(1)$

end

$f^* \leftarrow \min\{f_1[n], f_2[n]\}$

## What about 3 assembly lines?




$O(n)$

k assembly lines?

$O(k^2) \rightarrow O(k^2 n)$

## Summary

Summary: Dynamic Programming for Assembly Line Scheduling

Next week: Revision Lecture

## For note taking