COMP108
Data Structures and Algorithms

Divide-and-Conquer Algorithms (Part I)

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2022-23

Outline

Divide-and-Conquer algorithms

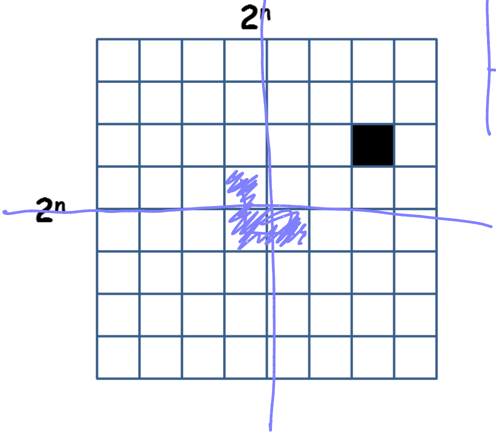
- Basic idea
- Learn a few examples

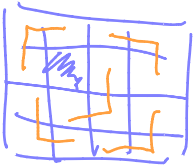
Learning outcomes:

- Able to describe the principle of divide-and-conquer algorithms
- Able to design divide-and-conquer algorithm for some simple problems

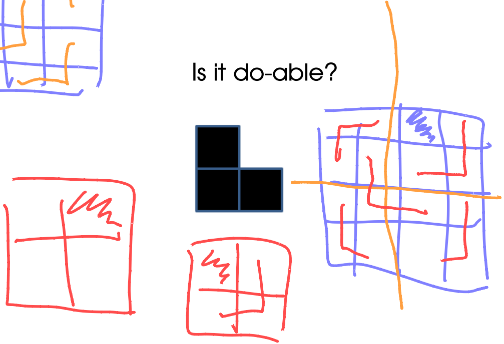
Triomino Puzzle

- **Input:** **2^n-by-2^n** chessboard with **one** missing square & many **L-shaped** tiles of **3** adjacent squares
- **Question:** **Cover** the chessboard with L-shaped tiles without overlapping





Is it do-able?



Divide-and-conquer algorithms

One of the best-known algorithm design techniques

Idea:

- A problem instance is divided into several smaller instances of the same problem, ideally of about same size
- The smaller instances are solved, typically recursively
- The solutions for the smaller instances are combined to get a solution to the large instance

Some problems we have seen before

Finding the sum of all numbers in an array

- Suppose we have 8 numbers:
  4 6 3 2 8 7 5 1

Iterative version:

sum $\leftarrow$ 0
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while i $\leq$ n do
begin
  sum $\leftarrow$ sum + $A[i]$
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output sum

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- Suppose we have 8 numbers:
  4 6 3 2 8 7 5 1
- Divide: let's divide them into two halves
  $\underbrace{4\ 6\ 3\ 2}$ AND $\underbrace{8\ 7\ 5\ 1}$

  Sum1 + Sum2

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- Recursively: How to find the sum of each half? *Apply* divide-and-conquer on each half
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Divide-and-conquer to find the sum

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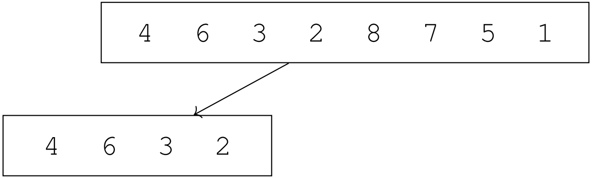

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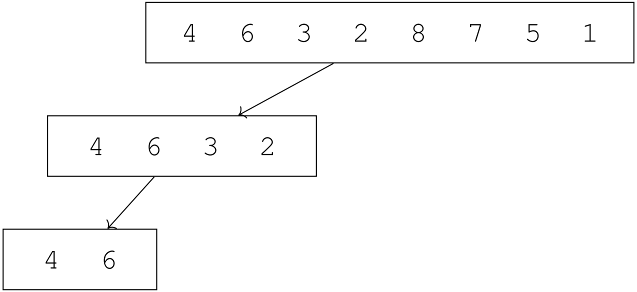

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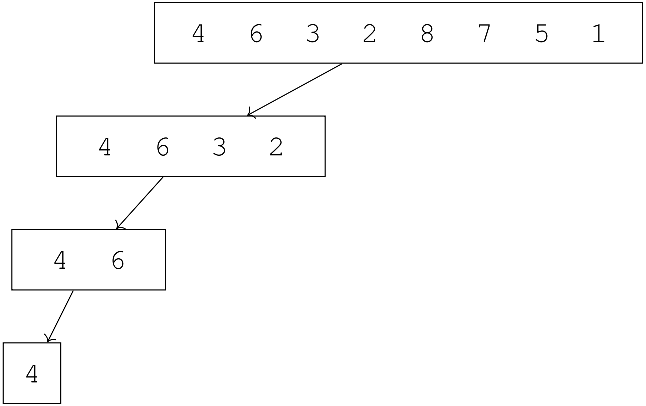

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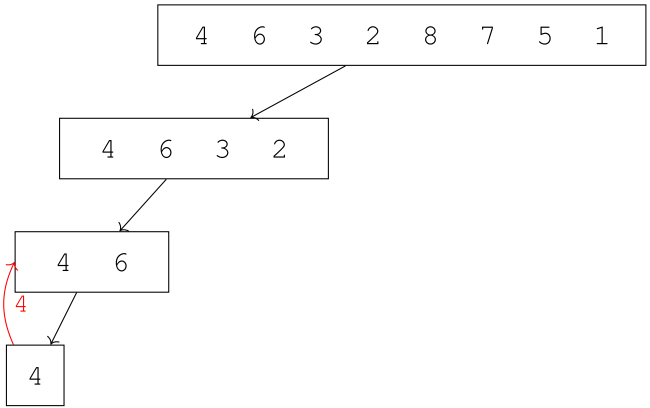

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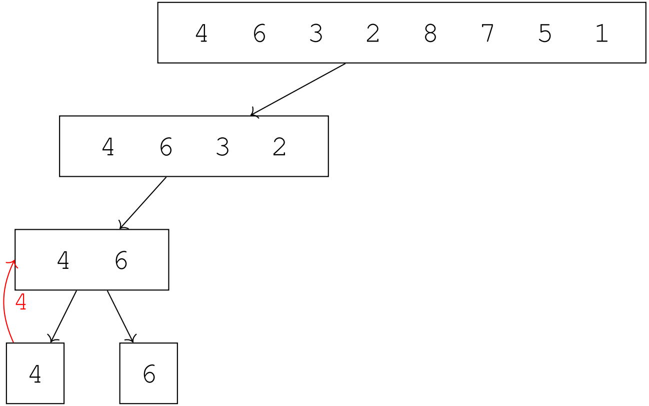

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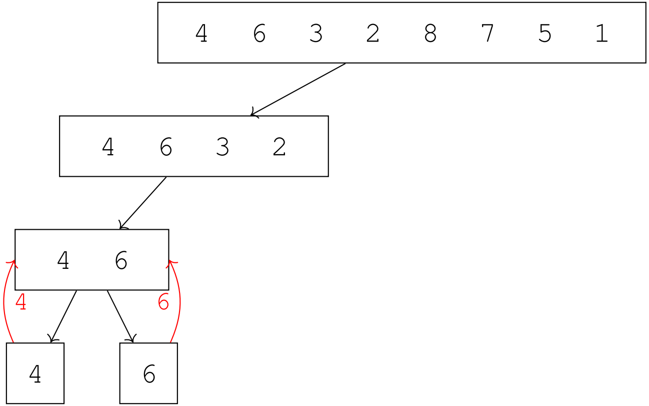

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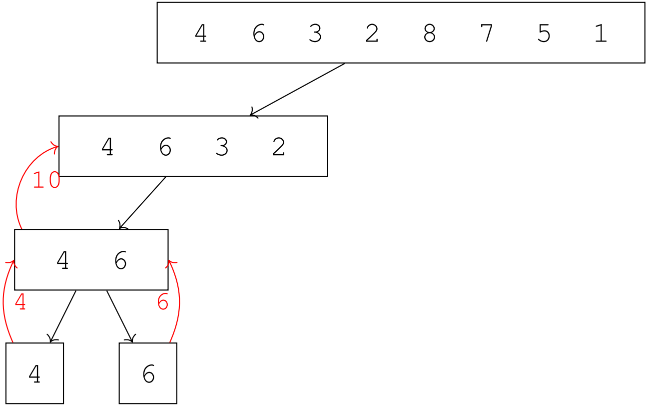

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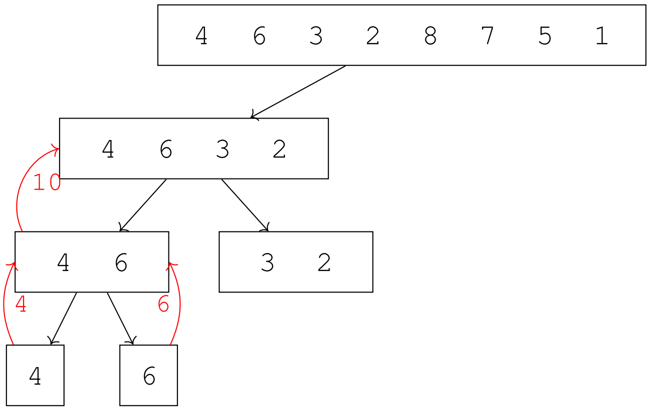

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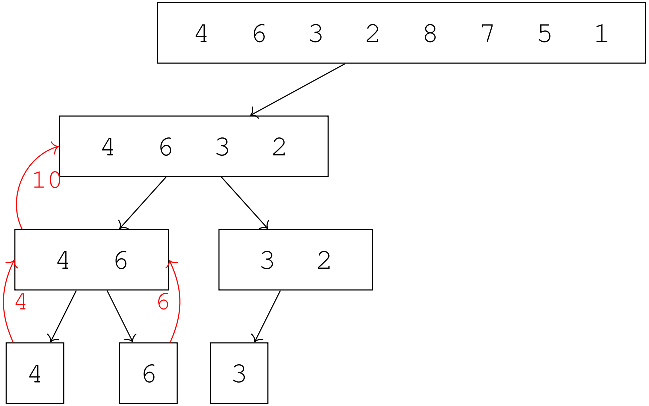

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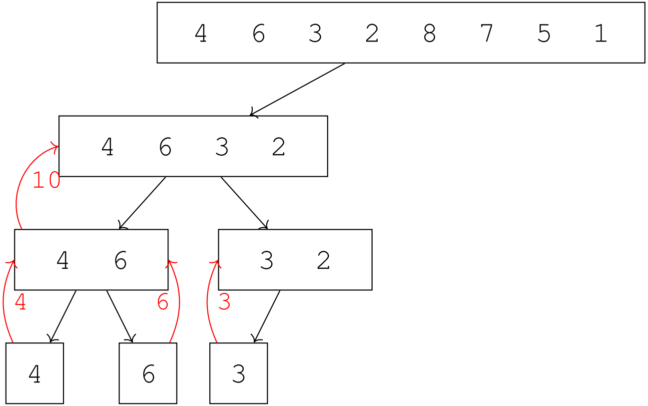

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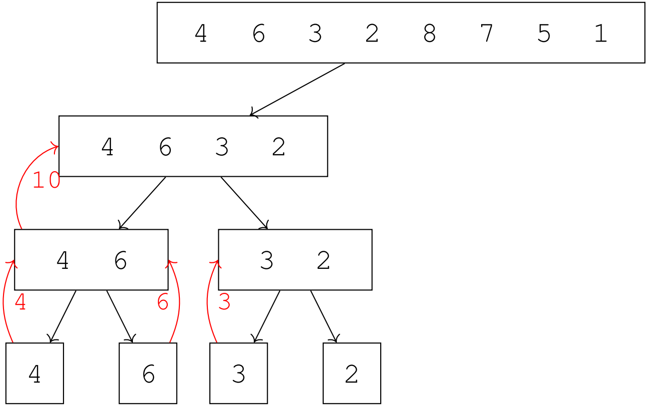

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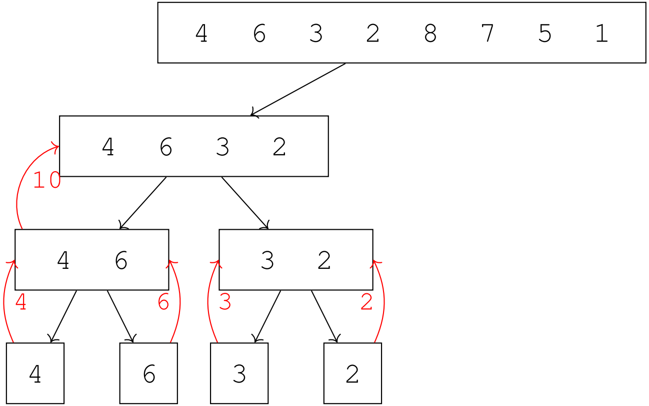

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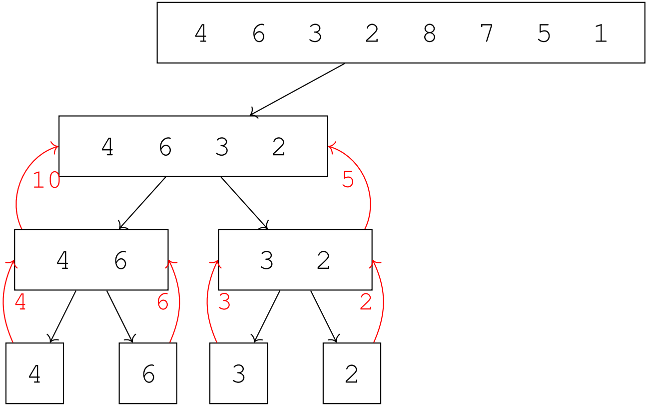

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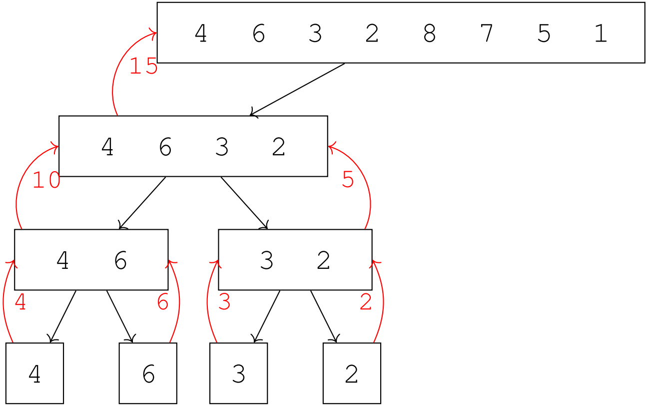

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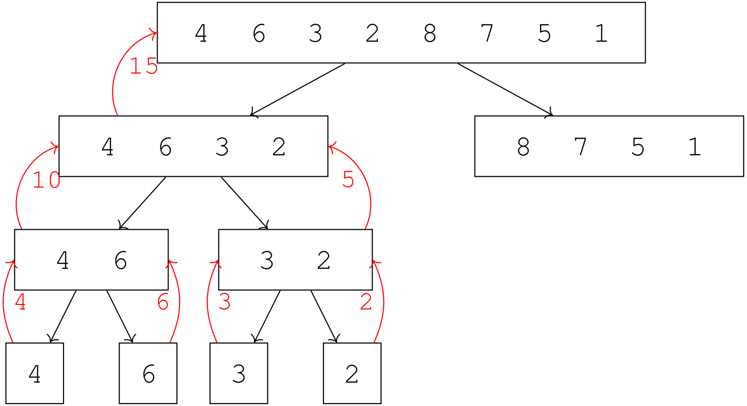

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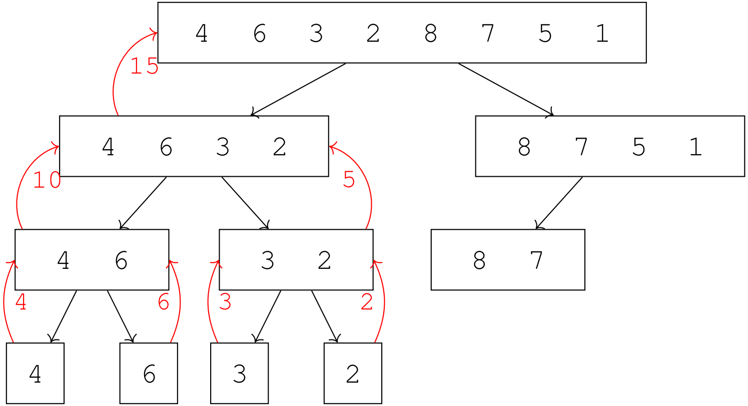

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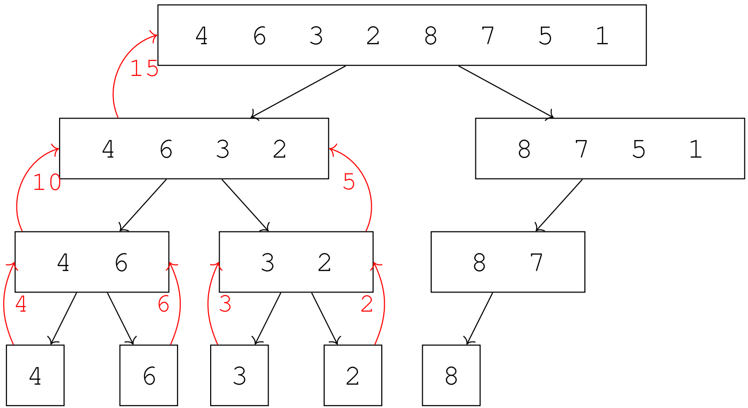

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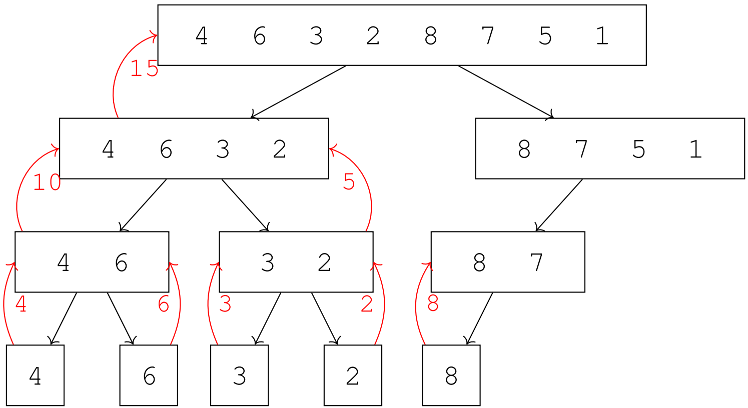

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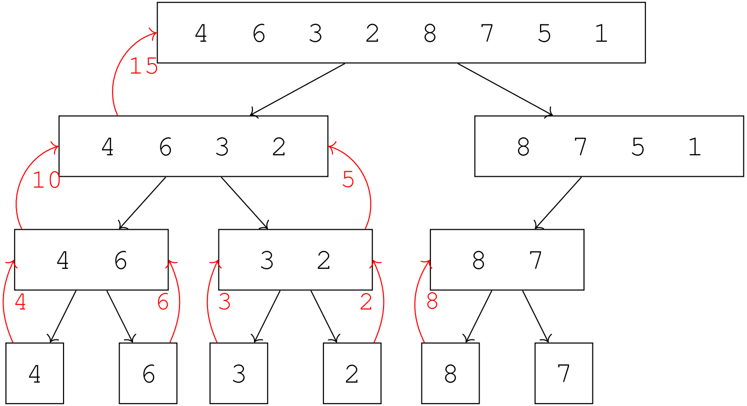

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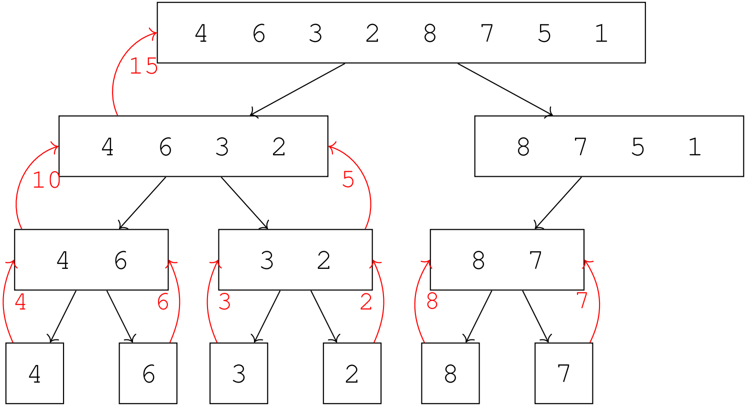

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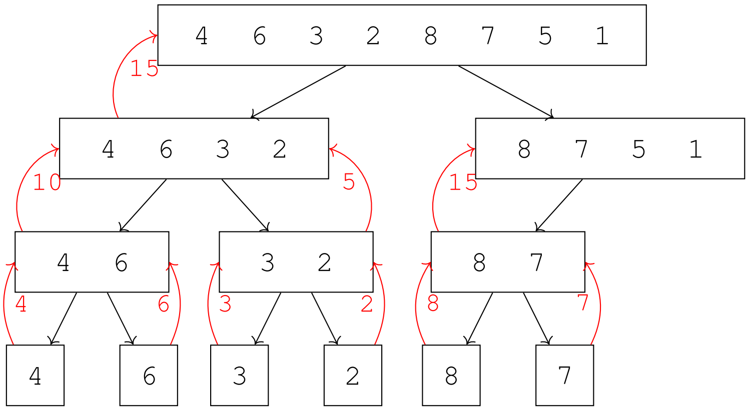

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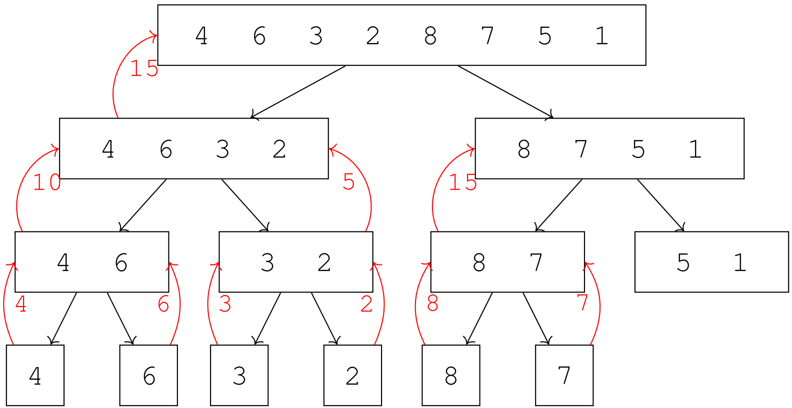

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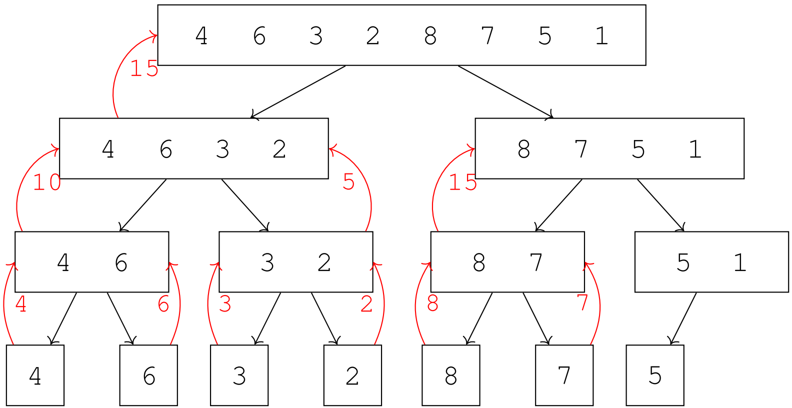

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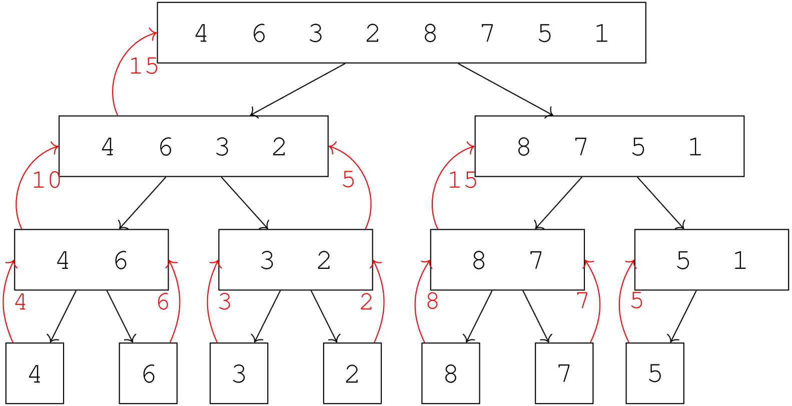

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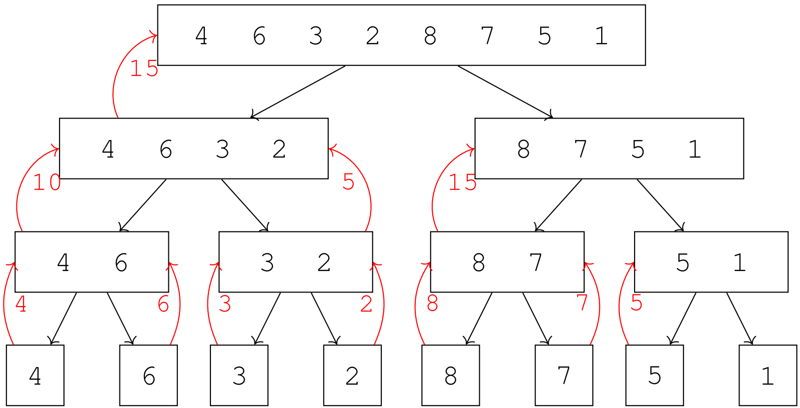

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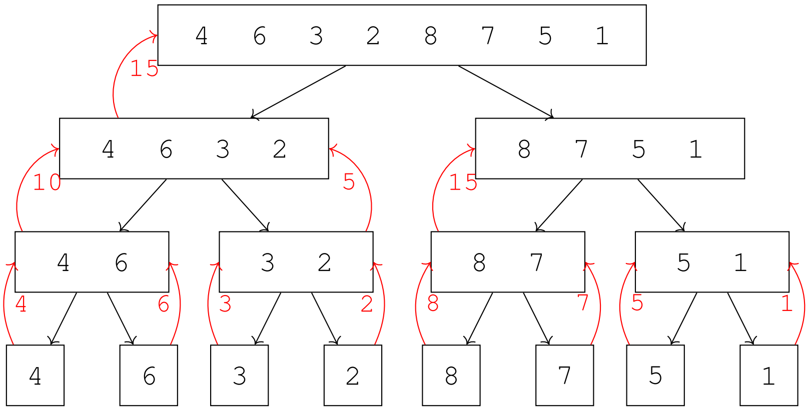

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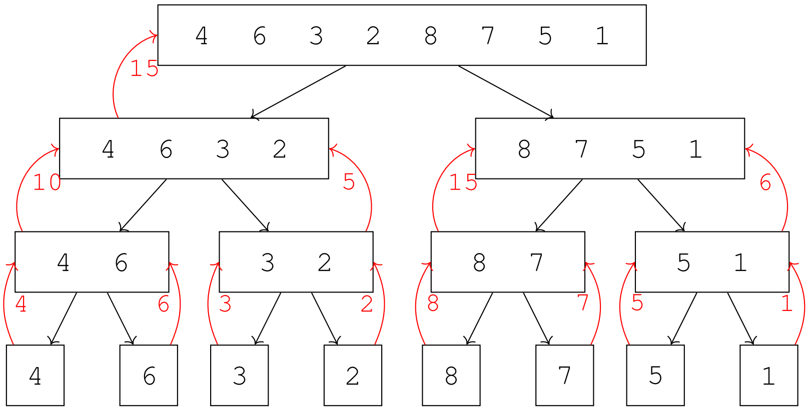

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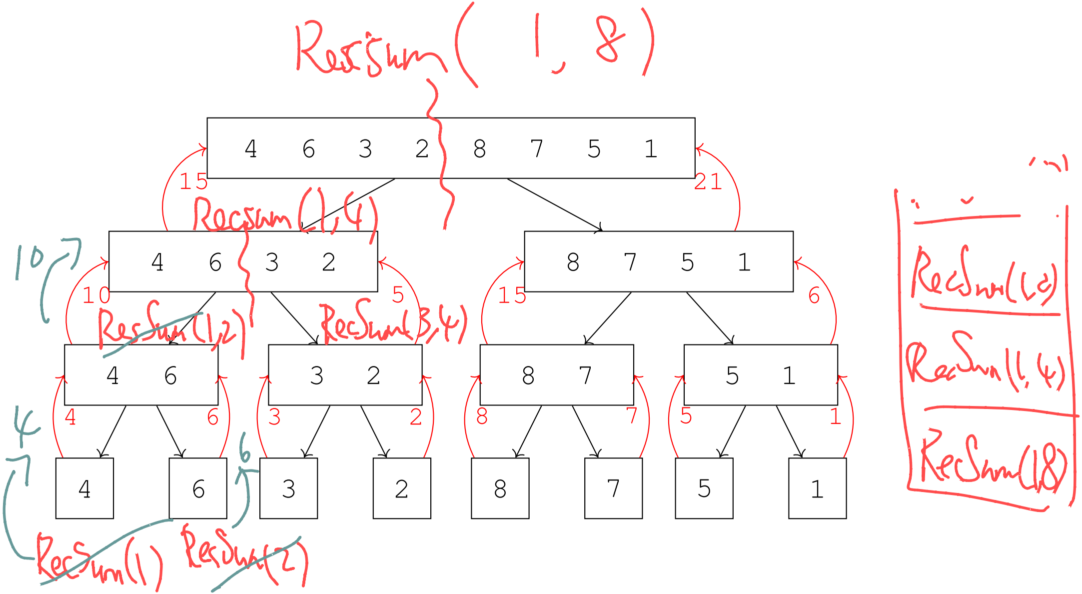

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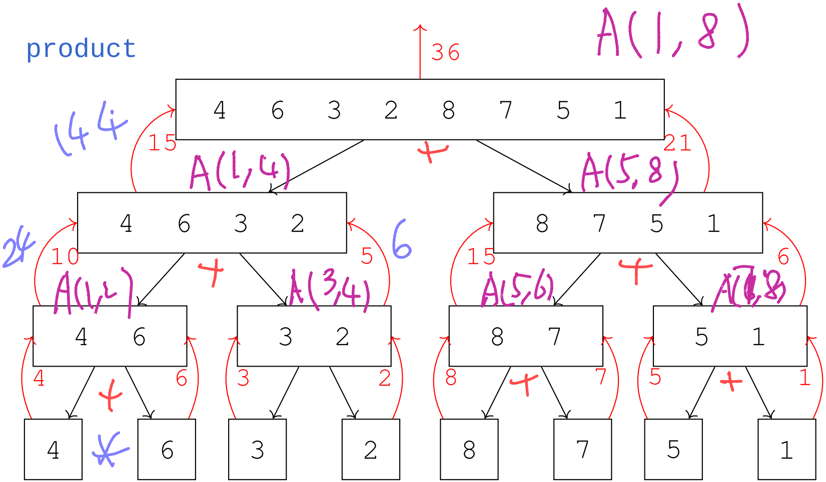

Divide-and-conquer to find the sum

For simplicity, assume n is a power of 2

We can call the following algorithm by RecSum(A, 1, n)

Algorithm RecSum(A[ ], p, q)

```
p is index of entry of first element in the
current call
q … last ...
```

Iterative version:

sum $\leftarrow$ 0
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while i $\leq$ n do
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Divide-and-conquer to find the sum

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```
Algorithm RecSum(A[ ], p, q)
    if p == q then
        return A[p]
    else
    begin



    end
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Iterative version:

```
sum ← 0
i ← 1
while i ≤ n do
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Divide-and-conquer to find the sum

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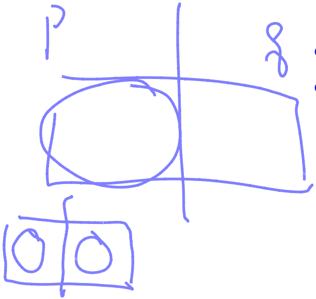
Algorithm RecSum($A[\,]$, p, q)
    if $p == q$ then
        return $A[p]$
    else
    begin
        sum1 $\leftarrow$ RecSum(A, p, $\frac{p+q-1}{2}$)
        sum2 $\leftarrow$ RecSum(A, $\frac{p+q+1}{2}$, q)
        return sum1 + sum2
    end

*



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Another problem we have seen before

Finding the minimum over all numbers in an array

- Suppose we have 8 numbers:
  4 6 3 2 8 7 5 1

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min ← A[1]
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Divide-and-conquer to find the minimum

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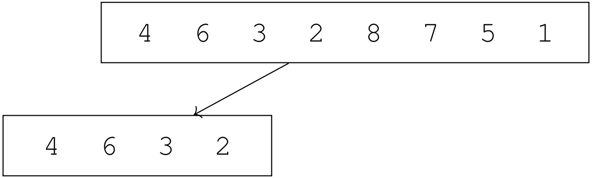
Divide-and-conquer to find the minimum



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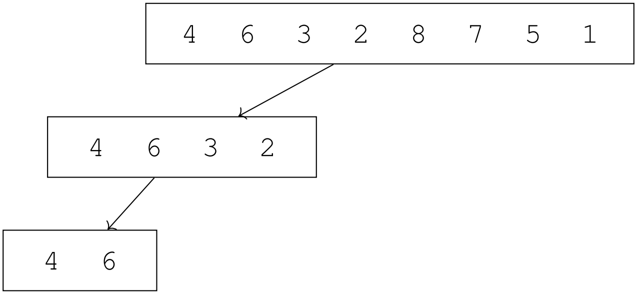

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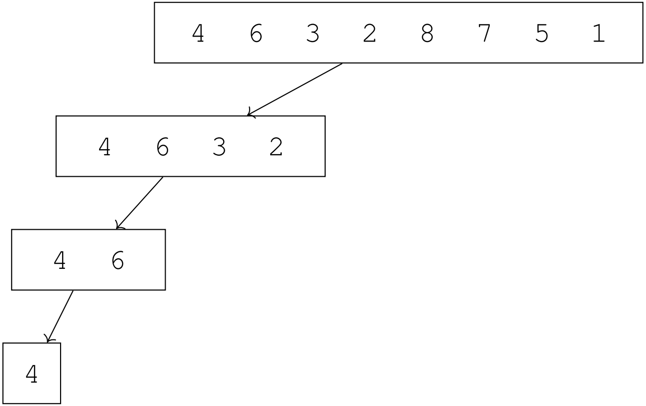

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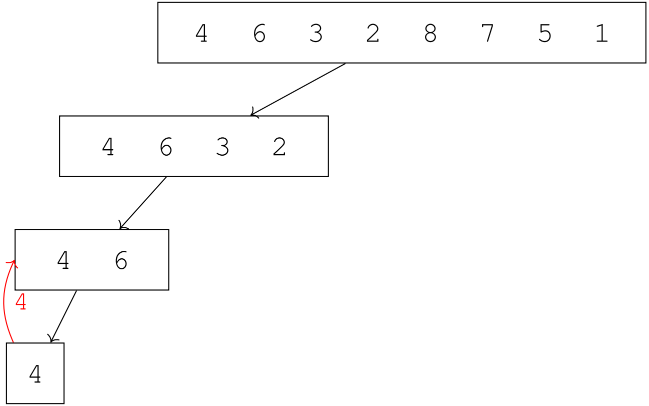

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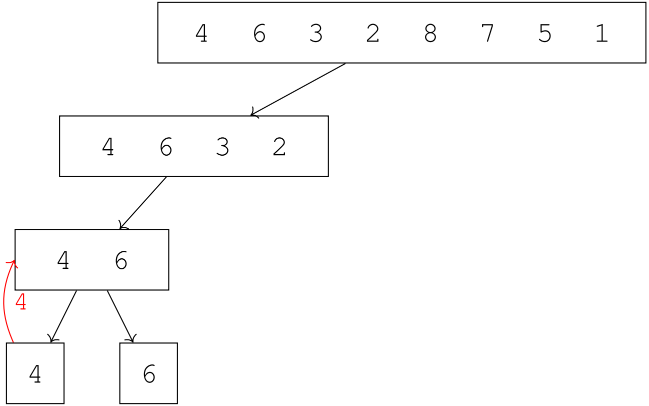

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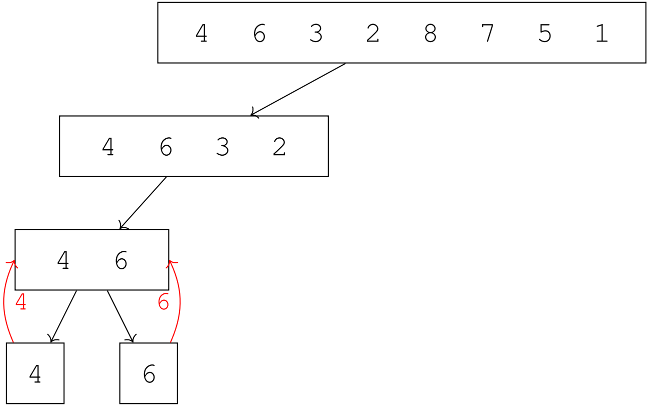

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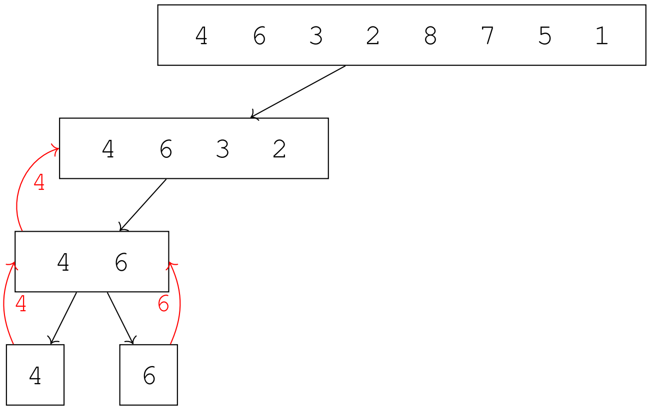

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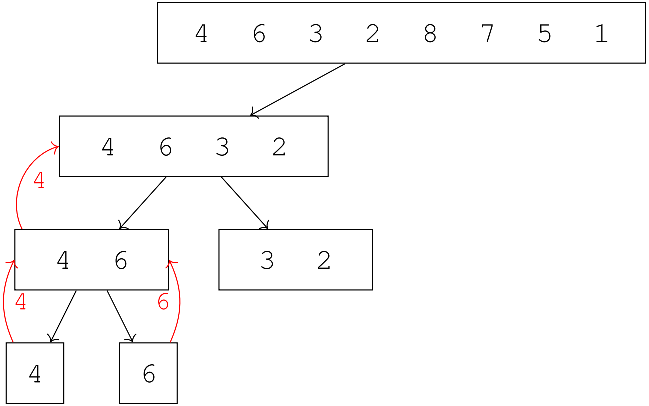

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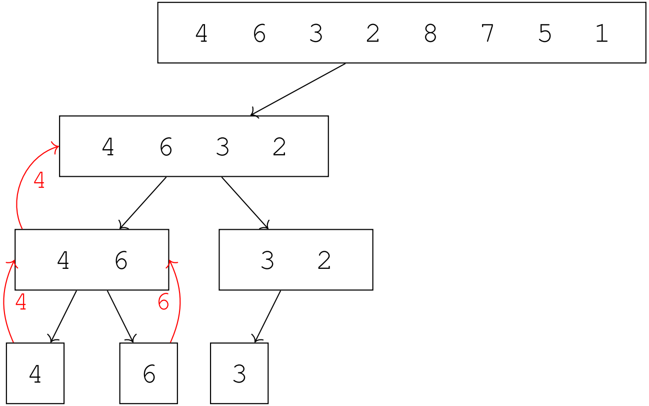

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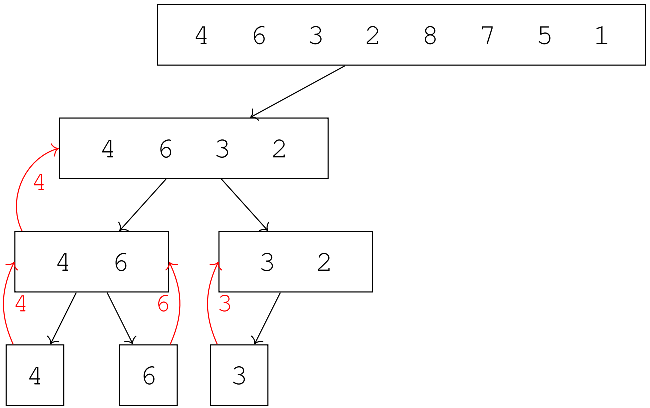

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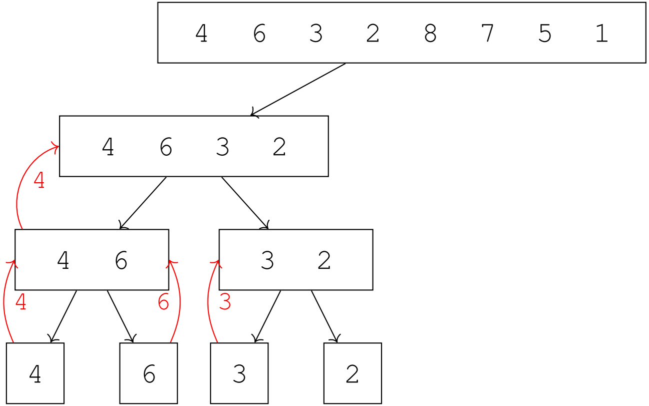

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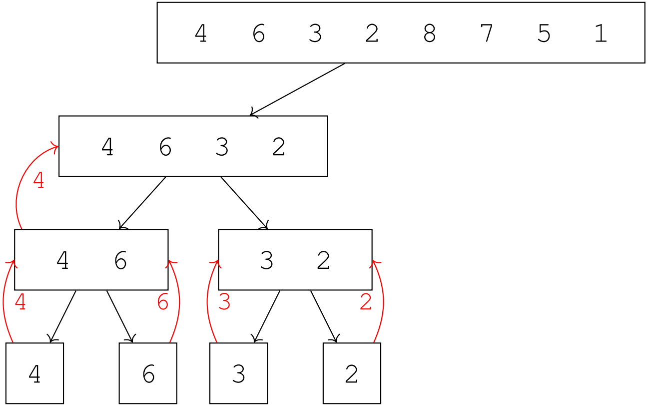

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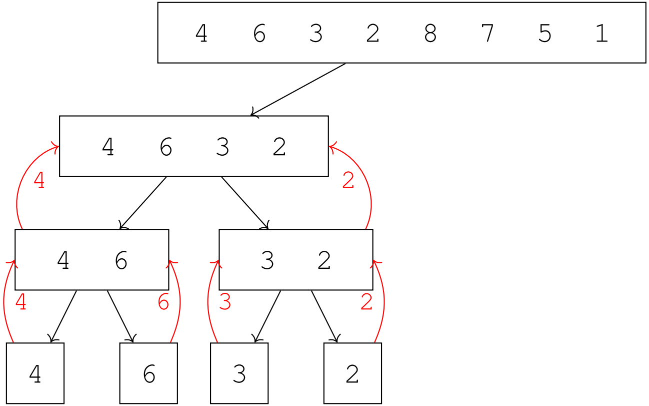

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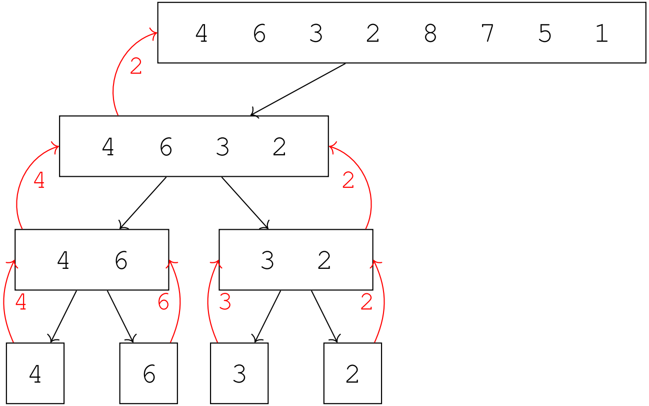

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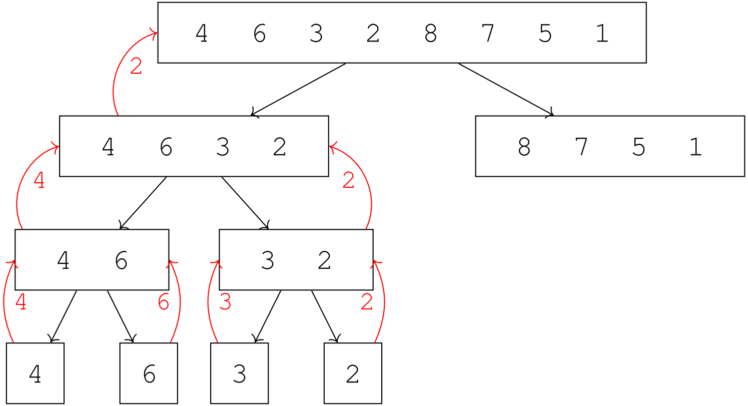

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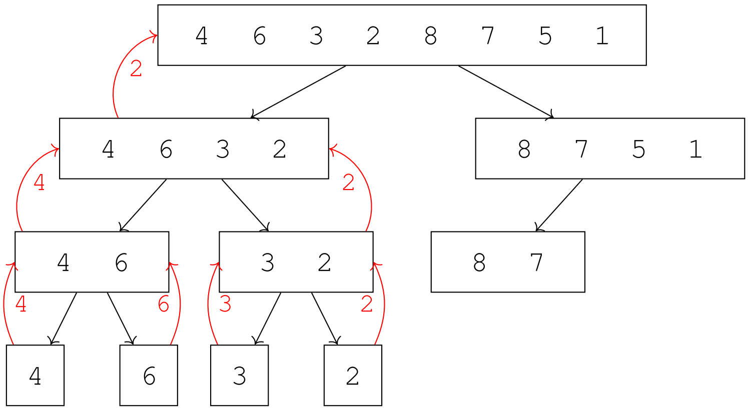
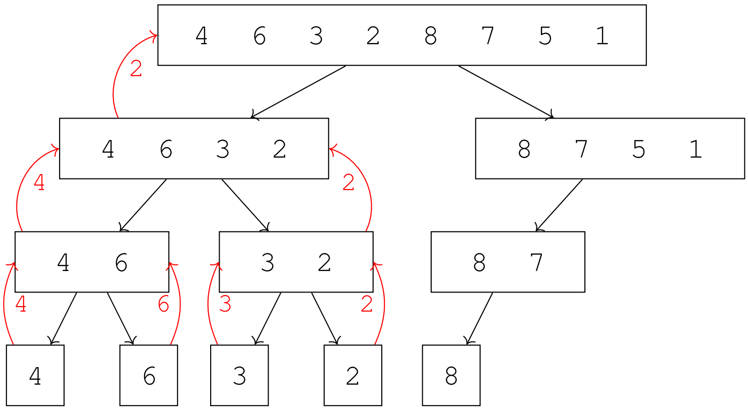
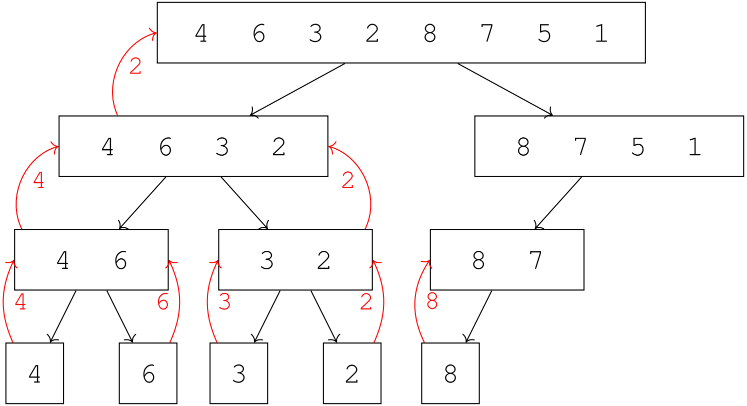


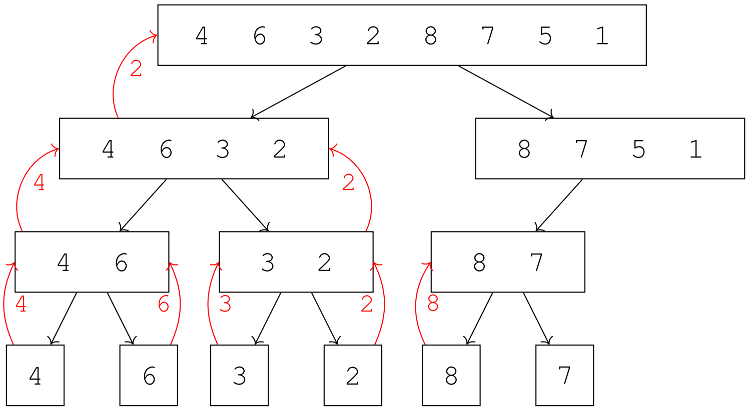
Divide-and-conquer to find the minimum



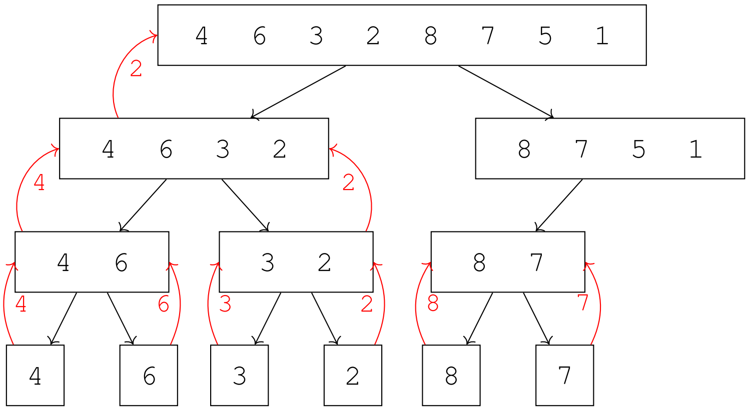
Divide-and-conquer to find the minimum



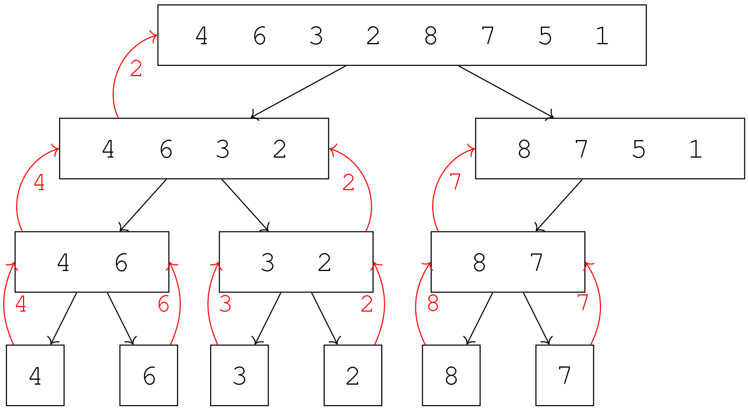
Divide-and-conquer to find the minimum



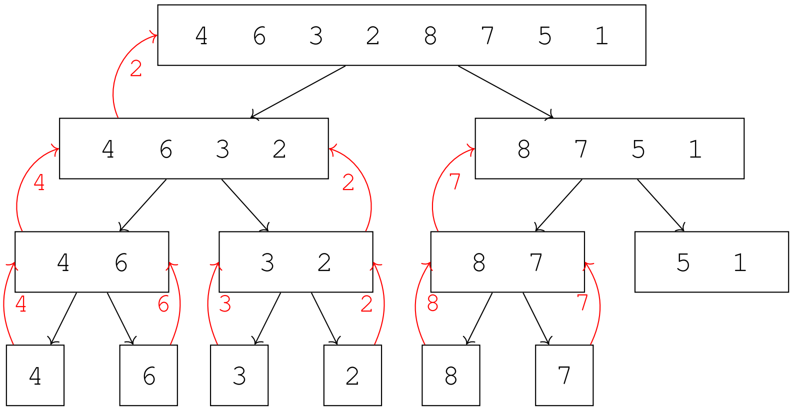
Divide-and-conquer to find the **minimum**



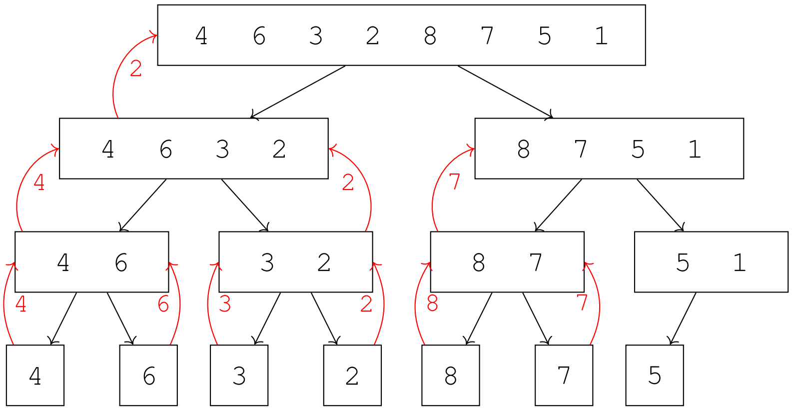
Divide-and-conquer to find the minimum



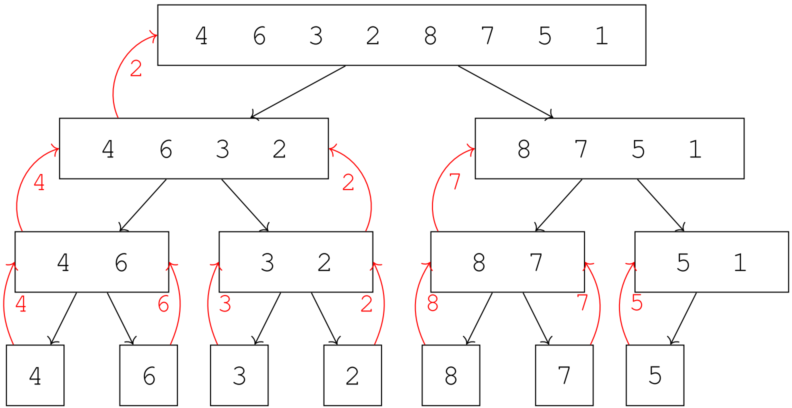
Divide-and-conquer to find the minimum



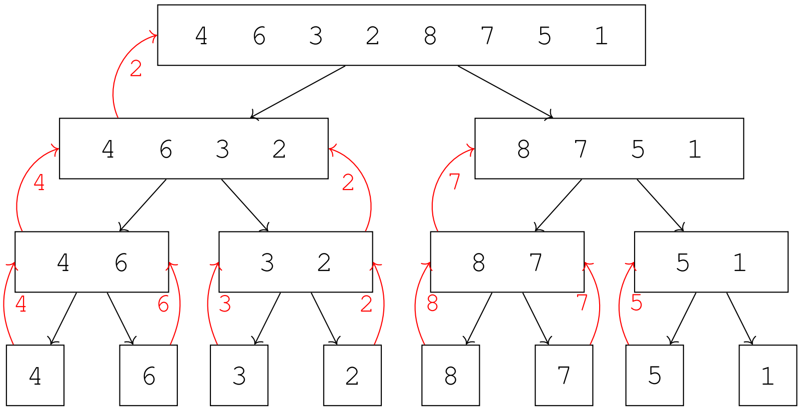
Divide-and-conquer to find the minimum



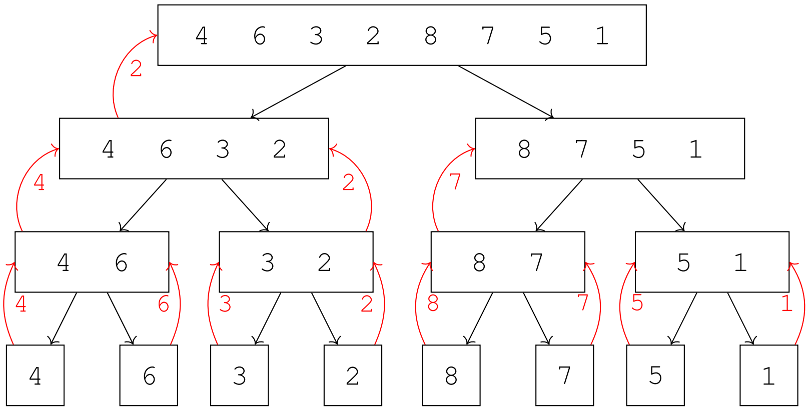
Divide-and-conquer to find the minimum



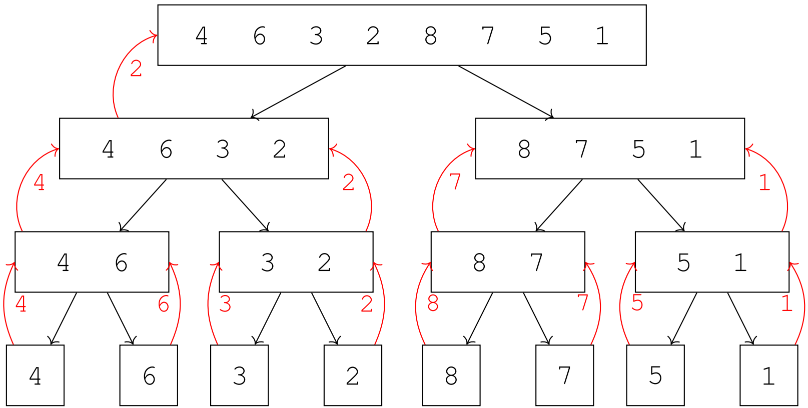
Divide-and-conquer to find the minimum



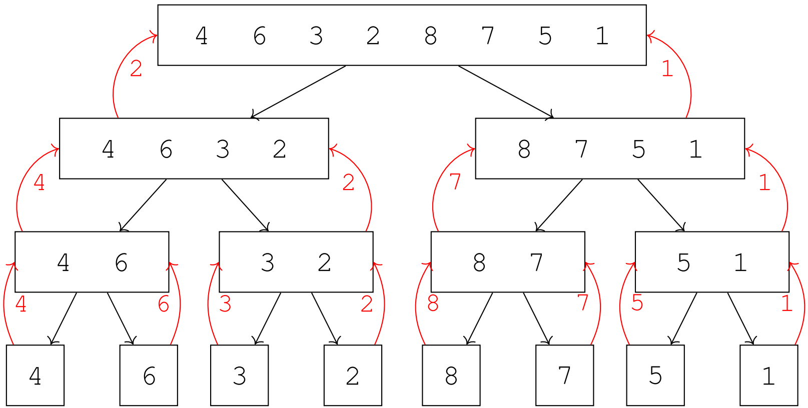
Divide-and-conquer to find the minimum



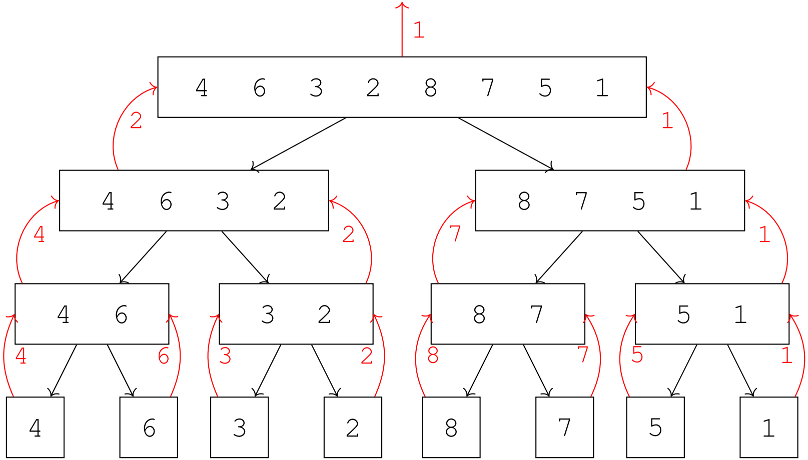
Divide-and-conquer to find the minimum



Divide-and-conquer to find the minimum

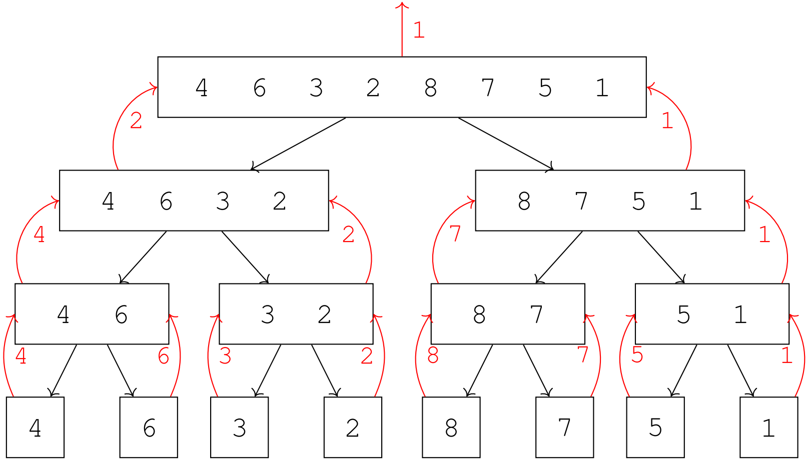


Divide-and-conquer to find the minimum



Divide-and-conquer to find the minimum

What about the maximum?



Divide-and-conquer to find the minimum

For simplicity, assume n is a power of 2

We can call the following algorithm by RecMin(A, 1, n)

Algorithm RecMin($A[\,]$, p, q)

Iterative version:

min $\leftarrow A[1]$
i $\leftarrow$ 2
while i $\leq$ n do
begin
  if $A[i] \leq$ min then
    min $\leftarrow A[i]$
  i $\leftarrow$ i + 1
end
output min

Divide-and-conquer to find the minimum

For simplicity, assume n is a power of 2

We can call the following algorithm by RecMin(A, 1, n)

Algorithm RecMin($A[\,]$, p, q)
    if $p == q$ then
        return $A[p]$
    else begin

    end

Iterative version:
min $\leftarrow A[1]$
i $\leftarrow$ 2
while i $\leq$ n do
begin
    if $A[i] \leq$ min then
        min $\leftarrow A[i]$
    i $\leftarrow$ i + 1
end
output min

Divide-and-conquer to find the minimum

For simplicity, assume n is a power of 2

We can call the following algorithm by RecMin(A, 1, n)

Algorithm RecMin($A[\,]$, p, q)
  if $p == q$ then
    return $A[p]$
  else begin
    answer1 $\leftarrow$ RecMin(A, p, $\frac{p+q-1}{2}$)
    answer2 $\leftarrow$ RecMin(A, $\frac{p+q+1}{2}$, q)

  end

Iterative version:
min $\leftarrow A[1]$
i $\leftarrow$ 2
while i $\leq$ n do
begin
  if $A[i] \leq$ min then
    min $\leftarrow A[i]$
  i $\leftarrow$ i + 1
end
output min

Divide-and-conquer to find the minimum

For simplicity, assume n is a power of 2

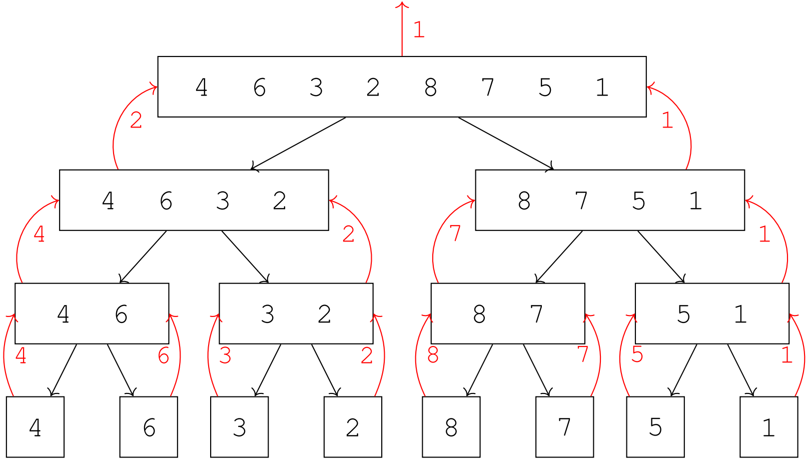
We can call the following algorithm by RecMin(A, 1, n)

```
Algorithm RecMin(A[ ], p, q)
    if p == q then
        return A[p]
    else begin
        answer1 ← RecMin(A, p, (p+q-1)/2)
        answer2 ← RecMin(A, (p+q+1)/2, q)
        if answer1 ≤ answer2 then
            return answer1
        else return answer2
    end
```

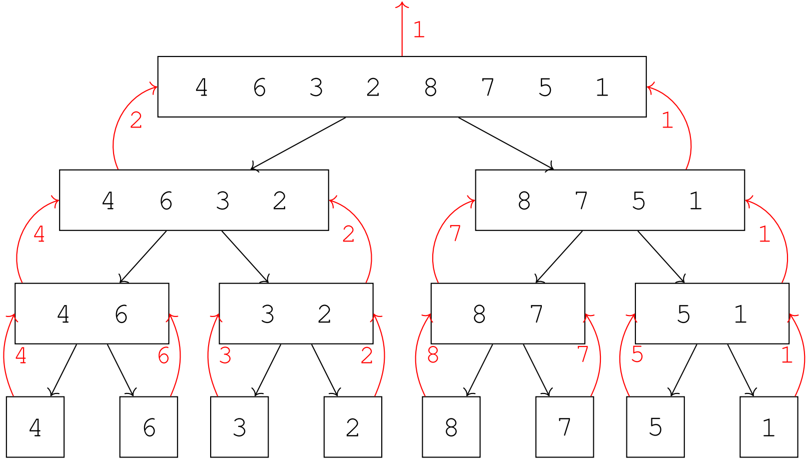
Iterative version:

```
min ← A[1]
i ← 2
while i ≤ n do
begin
    if A[i] ≤ min then
        min ← A[i]
    i ← i + 1
end
output min
```

Time complexity analysis



Time complexity analysis



$O(n)$ Why?

Summary

Summary: Basic Divide-and-Conquer algorithms

Next: Merge Sort Algorithm

For note taking