Solutions for Problem Set 1
Mathematical Preliminaries

Excercise 1

Given $\overline{X} = (1,2,3)^T$ and $\overline{Y} = (3,2,1)^T$ find

1. $\overline{X} + \overline{Y}$
2. $\overline{X}^T\overline{Y}$
3. $\overline{Y}\overline{X}^T$

Solution

1. $\overline{X} + \overline{Y} = (1+3, 2+2, 3+1) = (4,4,4)$
2. $\overline{X}^T\overline{Y} = 1\cdot 3 + 2\cdot 2 + 3\cdot 1 = 10$
3. $\overline{Y}\overline{X}^T = \begin{pmatrix} 3 & 6 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix}$

Excercise 2

Given two matrices $\overline{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and $\overline{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}$

1. Compute $\overline{A} + \overline{B}$
2. Compute $\overline{B} + \overline{A}$. Is it equal to $\overline{A} + \overline{B}$? Is it always the case?
3. Compute $\overline{A} \cdot \overline{B}$
4. Compute $\overline{B} \cdot \overline{A}$. Is it equal to $\overline{A} \cdot \overline{B}$?

Solution

1. $\overline{A} + \overline{B} = \overline{B} + \overline{A} = \begin{pmatrix} 1 & 3 & 3 \\ 5 & 7 & 9 \\ 6 & 8 & 10 \end{pmatrix}$

2. It is always the case that $\overline{A} + \overline{B} = \overline{B} + \overline{A}$.

3. $\overline{A} \cdot \overline{B} = \begin{pmatrix} -1 & 5 & 9 \\ -1 & 14 & 21 \\ -1 & 23 & 33 \end{pmatrix}$

4. $\overline{B} \cdot \overline{A} = \begin{pmatrix} 4 & 5 & 6 \\ 30 & 36 & 42 \\ 6 & 6 & 6 \end{pmatrix}$. Comparing the two products we conclude that $\overline{A} \cdot \overline{B} \neq \overline{B} \cdot \overline{A}$

Excercise 3

Compute the inverse of the following matrix $\overline{A} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, if one exsits. Verify that the matrix product of $\overline{A}$ and its inverse is the 2x2 identity matrix.

Solution

$\overline{A}^{-1} = \frac{1}{|A|} * Adj(A)$

$|A| = (1 * 1) - (2 * -2) = 5$

$Adj(A) = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}^T$

$\overline{A}^{-1} = \frac{1}{5} \cdot \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

Excercise 4

Show that the vectors $\overline{A} = (1, 2, -3, 4)^T$, $\overline{B} = (1, 1, 0, 2)^T$, and $\overline{C} = (-1, -2, 1, 1)^T$ are linearly independent.

Solution

By definition, $\overline{A}, \overline{B}, \overline{C}$ are linearly independent if $\lambda_1 \overline{A} + \lambda_2 \overline{B} + \lambda_3 \overline{C} = \overline{0}$ implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

The vector equation $\lambda_1 \overline{A} + \lambda_2 \overline{B} + \lambda_3 \overline{C} = \overline{0}$ is equivalent to the following system of equations:

$$\begin{cases} \lambda_1 + \lambda_2 - \lambda_3 = 0 \\ 2\lambda_1 + \lambda_2 - 2\lambda_3 = 0 \\ -3\lambda_1 + \lambda_3 = 0 \\ 4\lambda_1 + 2\lambda_2 + \lambda_3 = 0 \end{cases}$$

By solving this system, one can see that it has a unique solution $\lambda_1 = \lambda_2 = \lambda_3 = 0$, and hence the vectors $\overline{A}, \overline{B}, \overline{C}$ are linearly independent.

Excercise 5

Find the ranks of the following matrices $\overline{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $\overline{B} = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{pmatrix}$.

Solution

1. The rank of $\overline{A}$ is less than three because the three rows of the matrix are linearly dependent ($0 \cdot \overline{R_1}^T + 0 \cdot \overline{R_2}^T + \overline{R_3}^T = \overline{0}$, where $\overline{R_i}^T$ is the i-th row of the matrix). On the other hand, the rank is at least 2 because the first two rows are linearly independent. Hence the rank of $\overline{A}$ is 2.

2. The rank of $\overline{B}$ is less than three because the three rows of the matrix are linearly dependent ($\overline{R_1}^T - \overline{R_2}^T - \overline{R_3}^T = \overline{0}$, where $\overline{R_i}^T$ is the i-th row of the matrix). On the other hand, the rank is at least 2 because the first two rows are linearly independent. Hence the rank of $\overline{B}$ is 2.

Excercise 6

Find the eigenvalues and the corresponding eigenvectors of $\overline{A} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$

Solution

Let $\overline{X} = (x_1, x_2)^T$ be an eigenvector of $\overline{A}$. Then, by definition, $\overline{AX} = \lambda \overline{X}$.

Then we can write $(\overline{A} - \lambda)\overline{X}$=0, $(\overline{A} - \lambda I)\overline{X}$=0. Now $\overline{X} \neq 0$. Hence $(\overline{A} - \lambda I)$=0. From this we get the following systen of equations

$$(4 - \lambda)(3 - \lambda) - 2 = 0$$

Solving this system with respect to λ we find two solutions (which are the eigenvalues of $\overline{A}$): $\lambda = 2$ and $\lambda = 5$.

From the first equation we find that $x_2 = x_1 \frac{\lambda - 4}{2}$, i.e. an eigenvector corresponding to an eigenvalue λ is $(1, (\lambda - 4)/2)^T$.

Hence, the eigenvector corresponding to the eigenvalue $\lambda_1 = 2$ is $(1, -1)^T$ and the eigenvector corresponding to the eigenvalue $\lambda_2 = 5$ is $(1, 0.5)^T$.

Excercise 7

Given $f(x) = \log(x)$ (where log denotes the natural logarithm) and $g(x) = 2x + 1$, compute

1. $f'(x)$

2. $g'(x)$

3. $(f(x) + g(x))'$
4. $(f(x)g(x))'$
5. $\left(\frac{f(x)}{g(x)}\right)'$
6. $(g(f(x)))'$

Solution

1. $f'(x) = \frac{1}{x}$
2. $g'(x) = 2$
3. $(f(x) + g(x))' = \frac{1}{x} + 2$
4. $(f(x)g(x))' = \frac{2x+1}{x} + 2\log x$
5. $\left(\frac{f(x)}{g(x)}\right)' = \frac{(2x+1)/x - 2\log x}{(2x+1)^2}$
6. $(g(f(x)))' = \frac{2}{x}$

Excercise 8

Given $f(x, y) = (x + 2y^3)^2$ compute

1. $\frac{\partial f}{\partial x}$
2. $\frac{\partial f}{\partial y}$
3. $\nabla_{(x,y)} f$

Solution

1. $\frac{\partial f}{\partial x} = 2(x + 2y^3)$
2. $\frac{\partial f}{\partial y} = 2(x + 2y^3) \cdot 6y^2$
3. $\nabla_{(x,y)} f = (2(x + 2y^3), 2(x + 2y^3) \cdot 6y^2)^T$