# Distributed Systems
# COMP 212

Lecture 9

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# Leader Election

# Problem Statement

- *Elect a unique leader processor from among all the processors in the distributed system*
- Leader to be interpreted as:
  - coordinator
  - master processor
- Special case of consensus/agreement
- Processors should agree eventually on who they elect

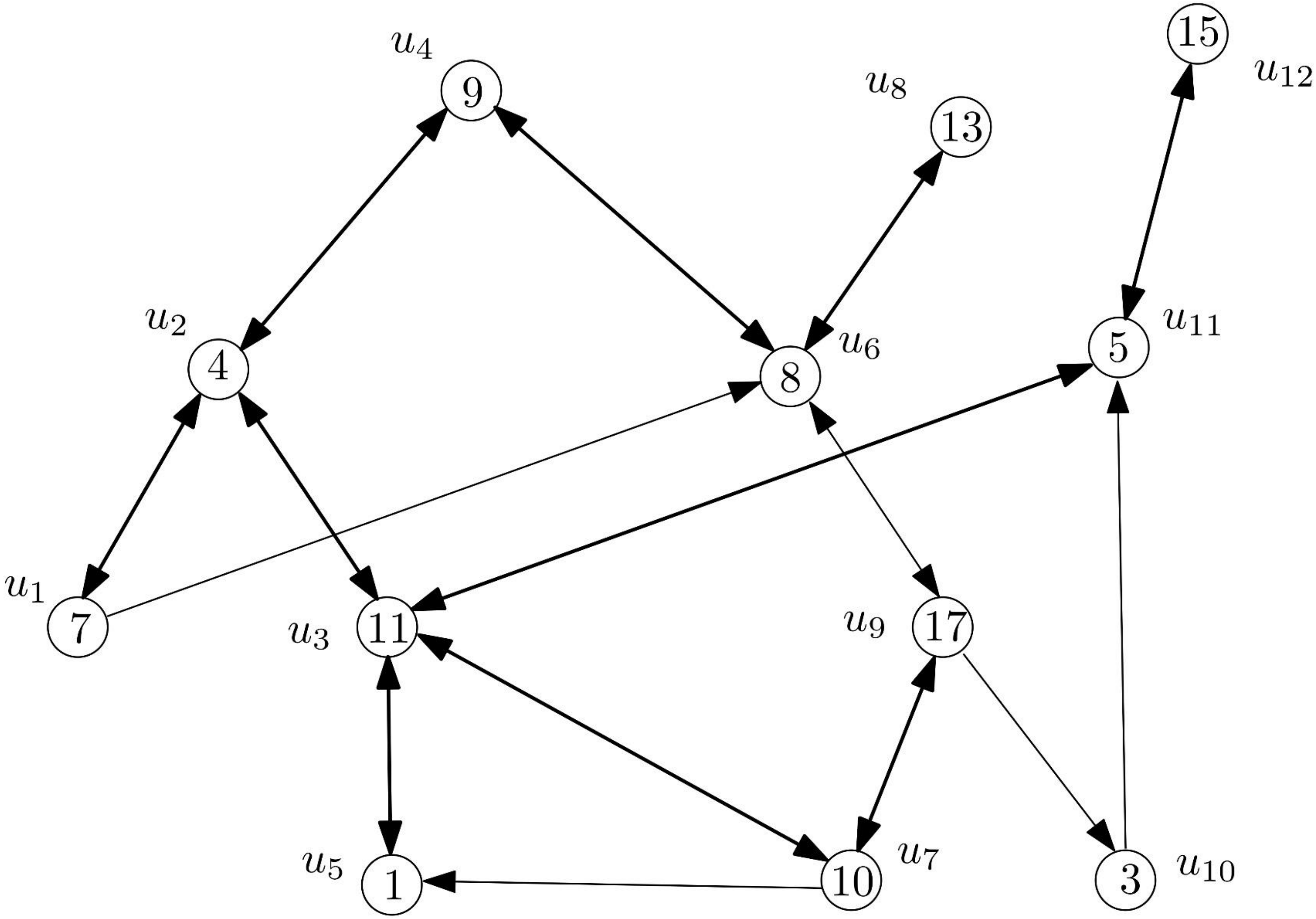
# Beyond Rings
# Leader Election in General Networks

# Leader Election in General Networks

- *Elect a unique leader processor from among all the processors in the distributed system*
- Now the network can be any strongly connected directed network
  - Strongly connected: For every processors u, v there is a path from u to v and a path from v to u
  - e.g., the directed ring is just a special case
  - *Why can't we use LCR in this case?*
- Processors have unique ids

# Leader Election in General Networks

- A strongly connected directed network



# A Simple Algorithm based on Flooding

- Processors have unique ids, do not know $n$ in advance, but do know the diameter $D$ of the network
  - Diameter:
    - the distance between two nodes is given by the shortest path between them
    - Then the diameter of the network is determined by the pair of nodes at maximum distance (and is equal to that distance)
    - In other words, it is the maximum shortest path in the network
- FloodMax algorithm: solves the problem
- Uses transmission, comparison, and storage of ids
- Main idea: Flood the maximum id
  - LCR also does something like this but does not require knowledge of $D$ and its termination condition works only for rings

# FloodMax: Informal description

- All processors know the diameter $D$ and their own id in advance
- All processors remember the greatest id that they have "heard" so far (initially their own)
- In every round all processors send the greatest known to all their out-neighbours
- After $D$ rounds compare the largest heard to your own
  - if *greatest heard = own id*, declare yourself the leader
  - otherwise, declare yourself non-leader
- Intuitively:
  - The maximum id will manage to reach the whole network
  - So everyone non-maximum will know that there is a greater id and $u_{max}$ can never receive a larger id

# FloodMax: Pseudocode

**Algorithm** FloodMax

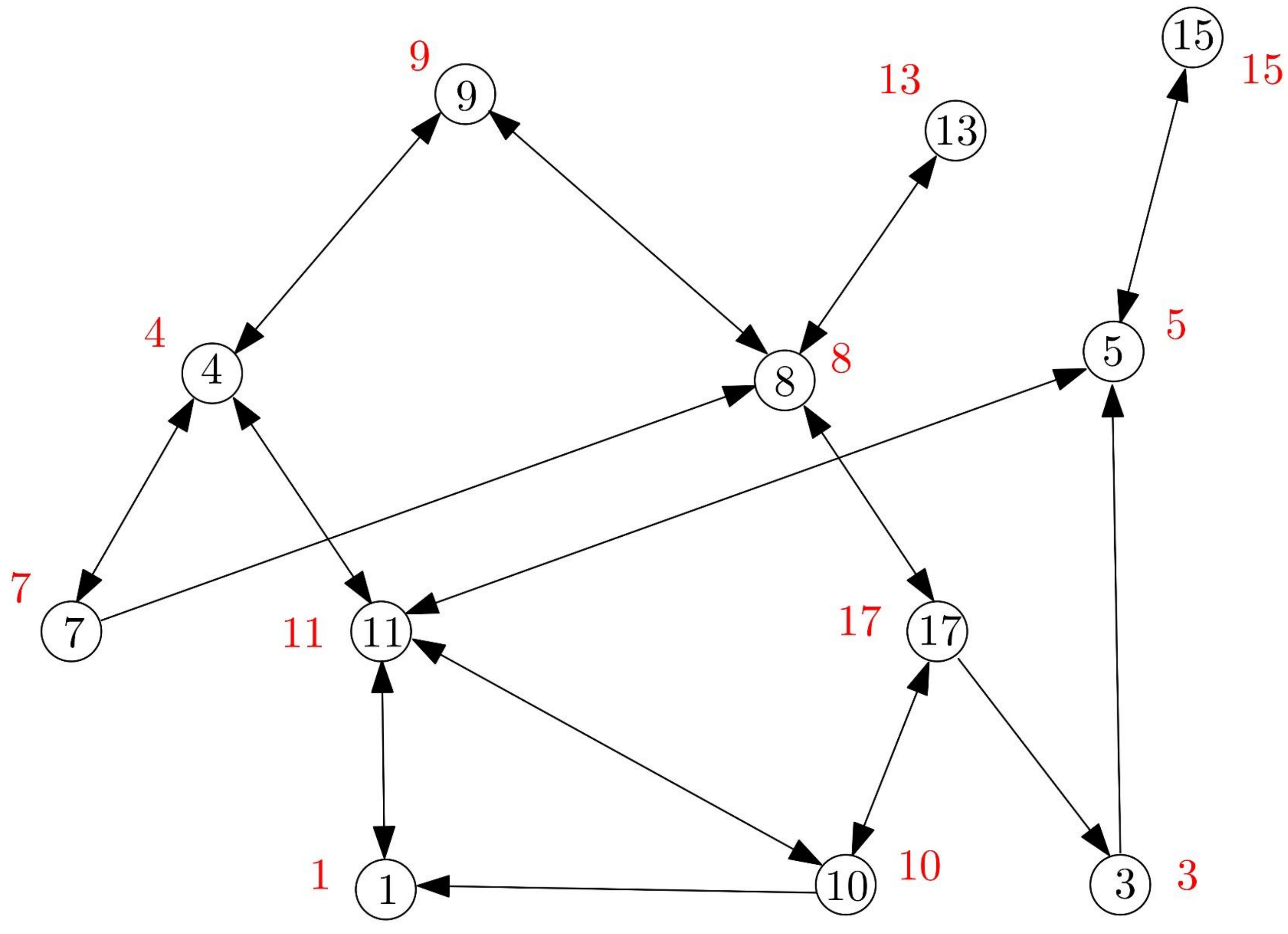
State of processor $u_i$ :

- $myID_i$: holds the processor's unique id
- $maxID_i$: holds the greatest id "heard" so far
- $status_i \in$ {"unknown", "leader", "non-leader"}: indicates whether $u_i$ has been elected ("leader"), not elected ("non-leader") or doesn' know yet ("unknown")

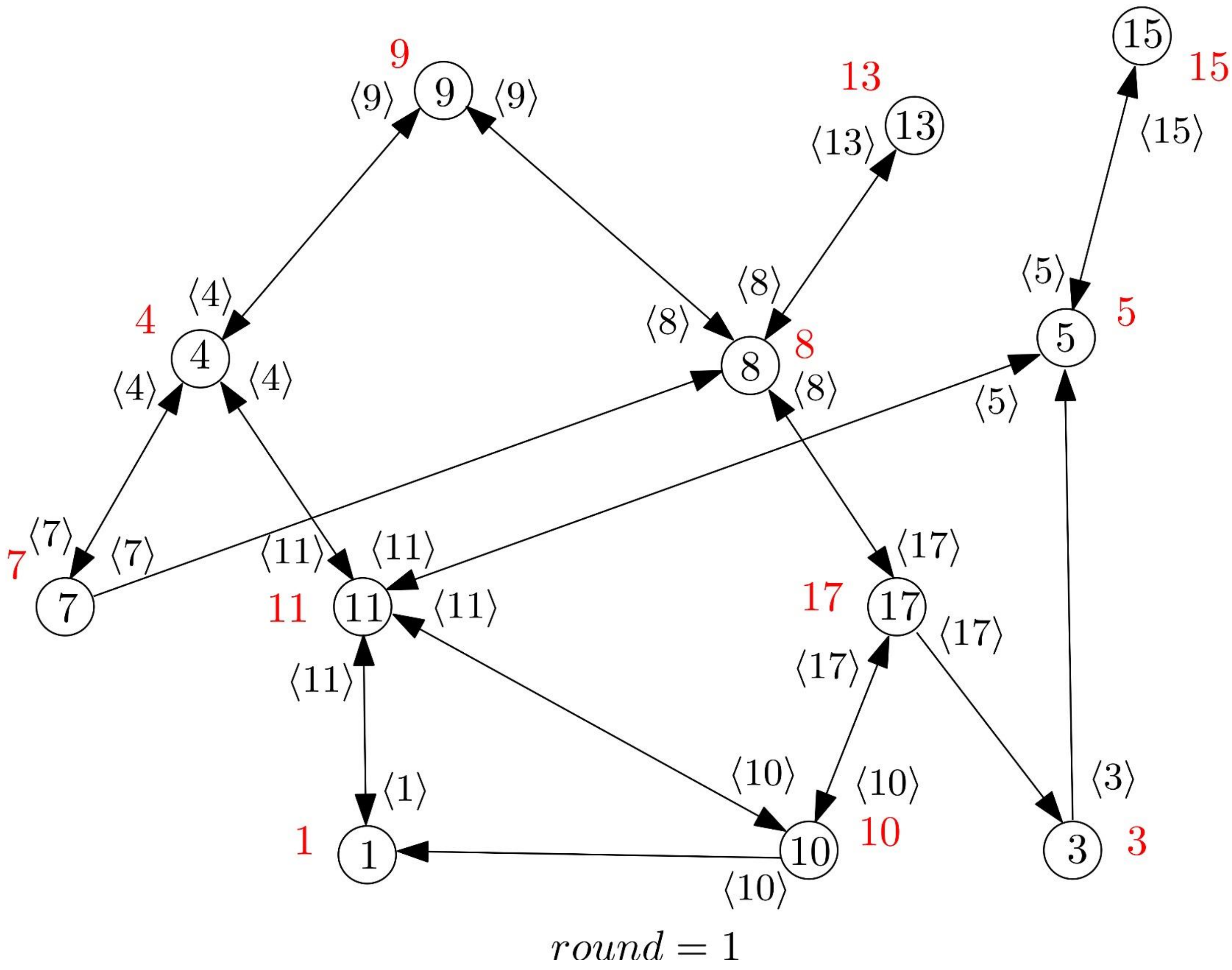
# FloodMax: Pseudocode

```
Algorithm FloodMax
Code for processor u_i , i ∈ {1, 2,… , n}:

Initially:
   u_i knows its own unique id stored in myID_i
   maxID_i := myID_i
   status_i := "unknown"
   Also has access to the current round and knows the diameter D

if round = 1 then
   send ⟨maxID_i⟩ to all out-neighbours
else
   upon receiving ⟨inIDs⟩ from in-neighbours       // one or more ids arriving from neighbours
   maxID_i := max({maxID_i} ∪ inIDs)               // remember only the maximum "heard" so far
   if round ≤ D then    // 1 < round ≤ D
      send ⟨maxID_i⟩ to all out-neighbours
   else    // round = D + 1
      if maxID_i = myID_i then        // if equal to your own, no greater id exists in the network
         status_i := "leader"         // therefore, elect yourself a leader
      else                            // greater than own
         status_i := "non-leader"     // therefore, declare yourself a non-leader
// observe that in the end all processors know the id of the elected leader, stored in their maxID_i
// variable
```

# Example Execution



# Example Execution



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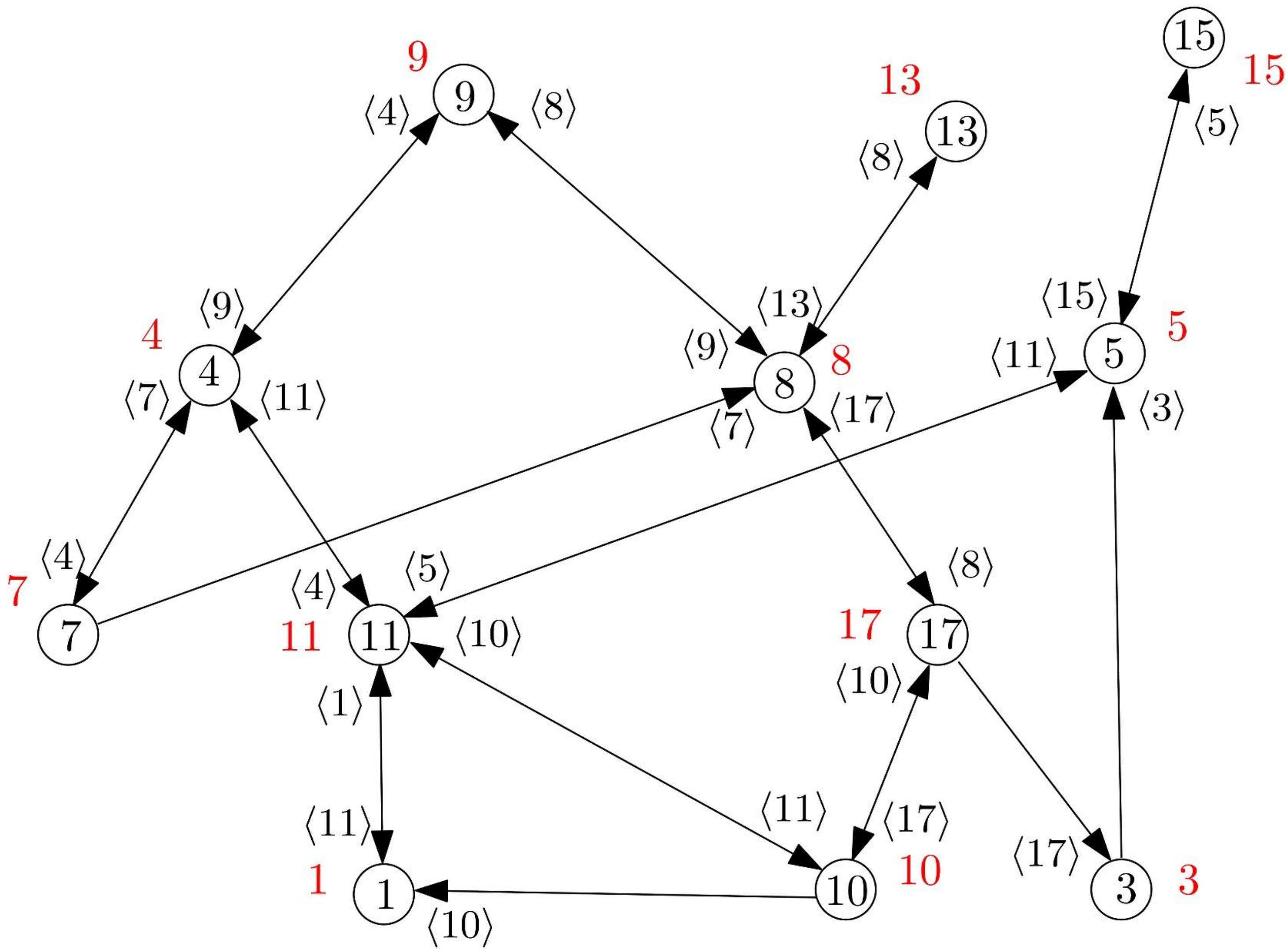
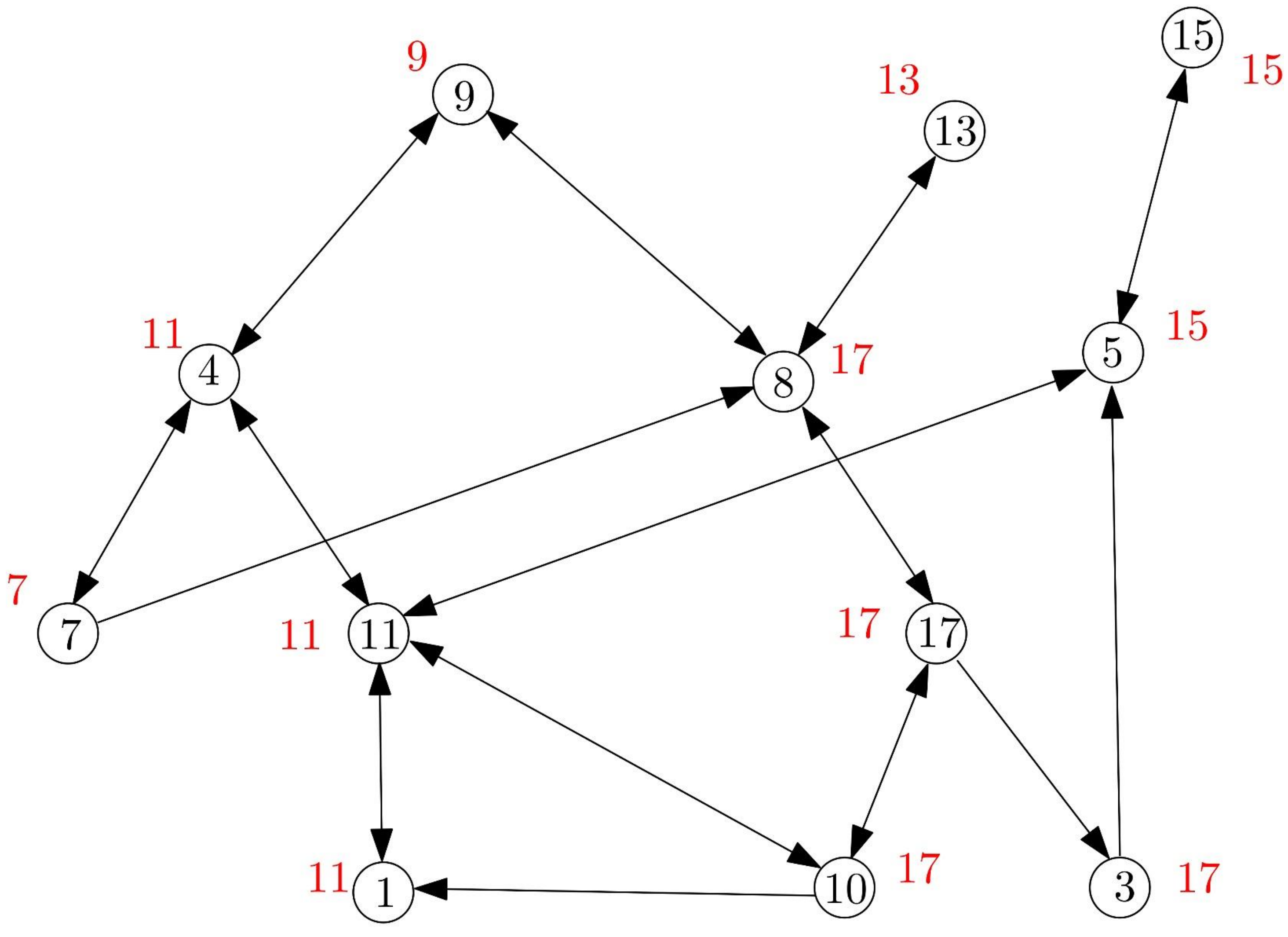


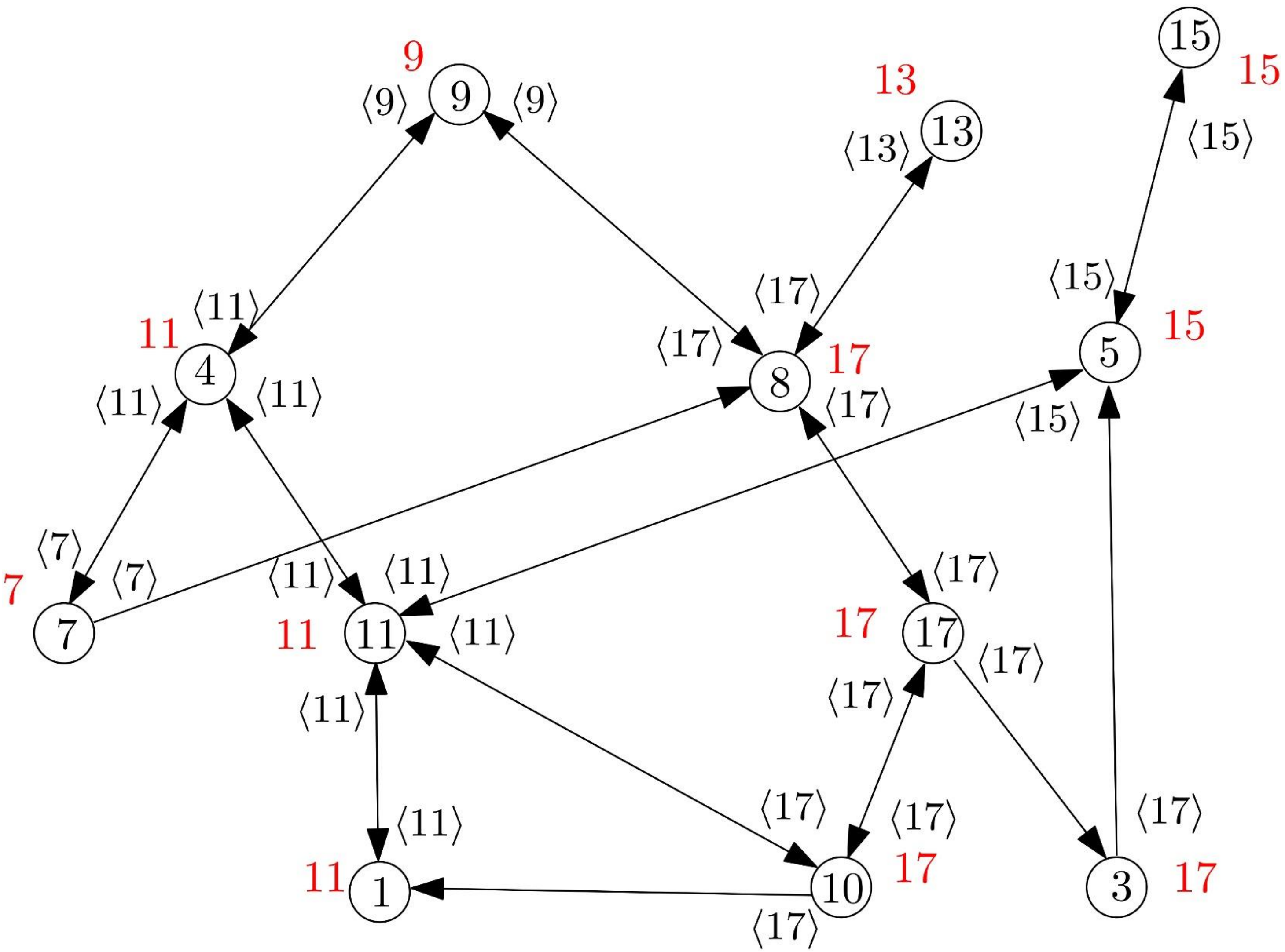

$round = 1$

# Example Execution




$round = 2$

# Example Execution




$round = 2$

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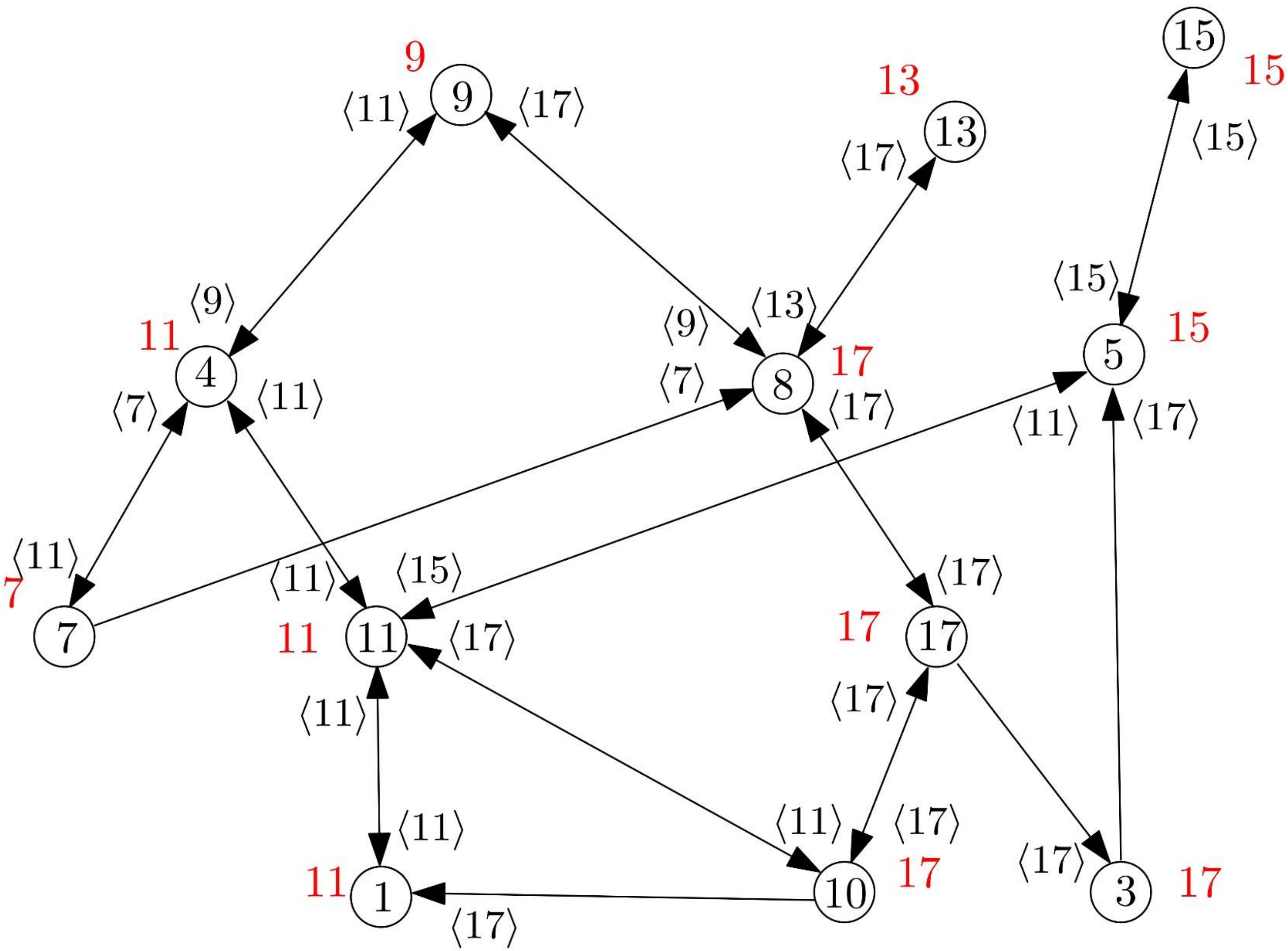
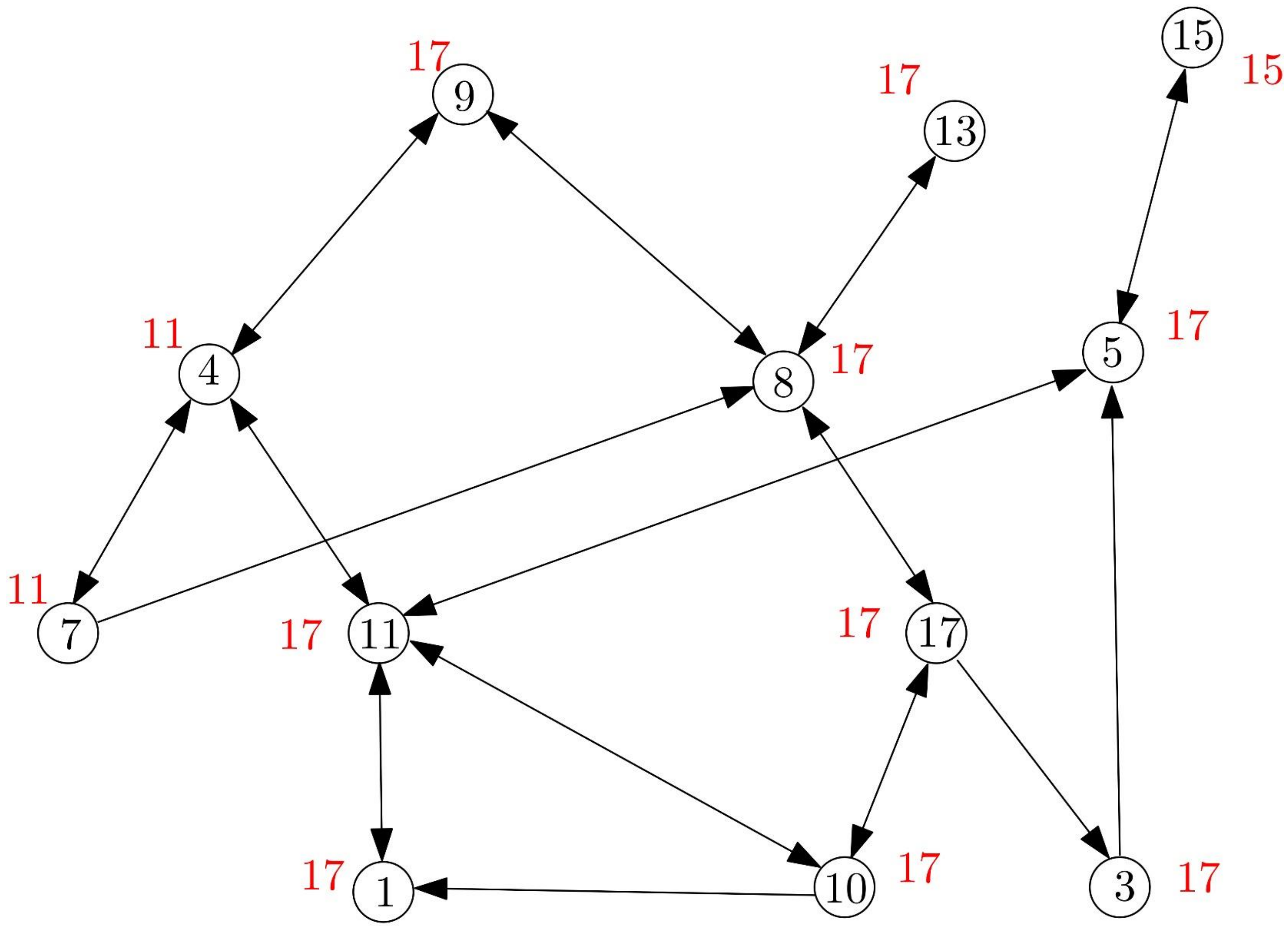


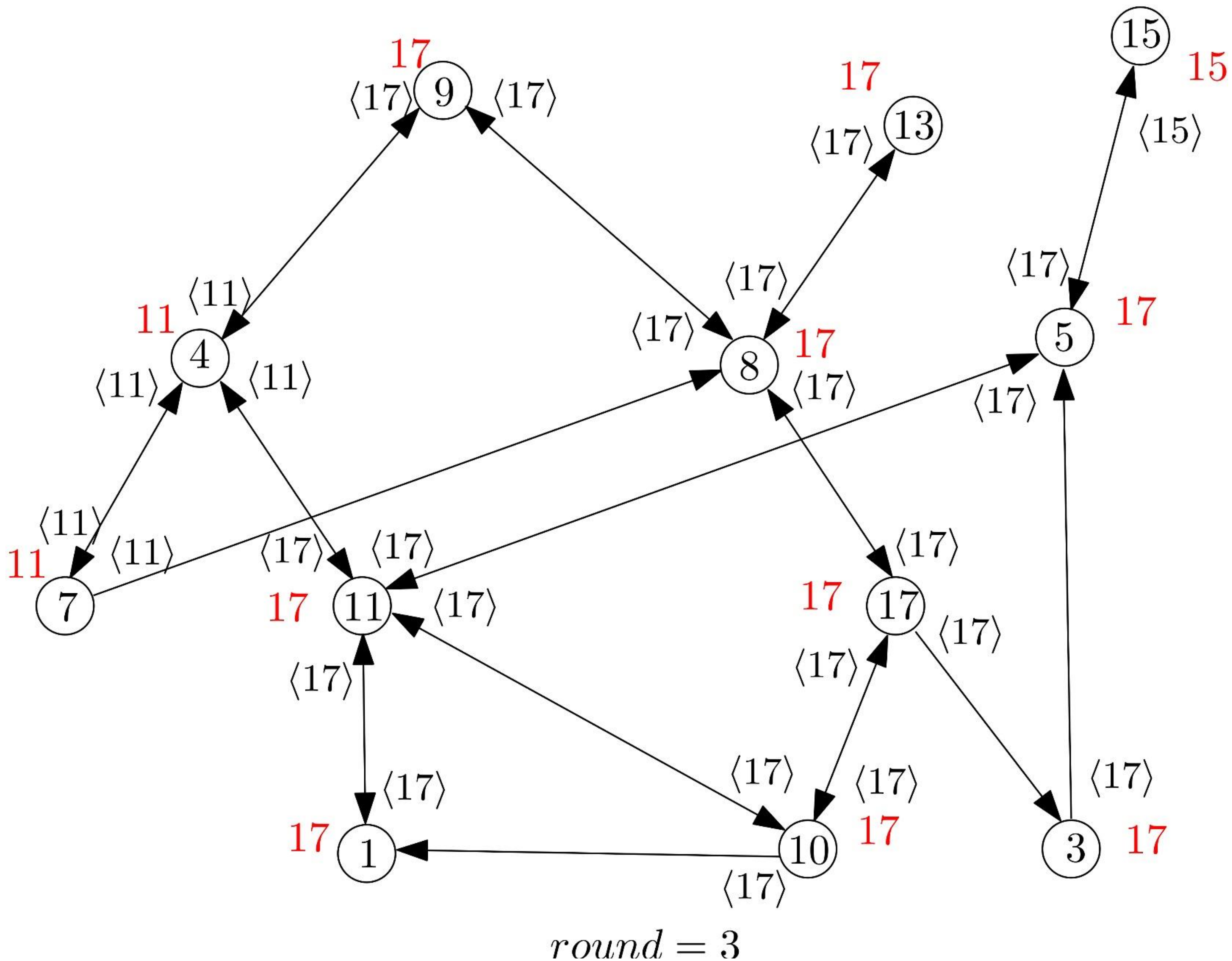

$round = 2$

# Example Execution




$round = 3$

# Example Execution



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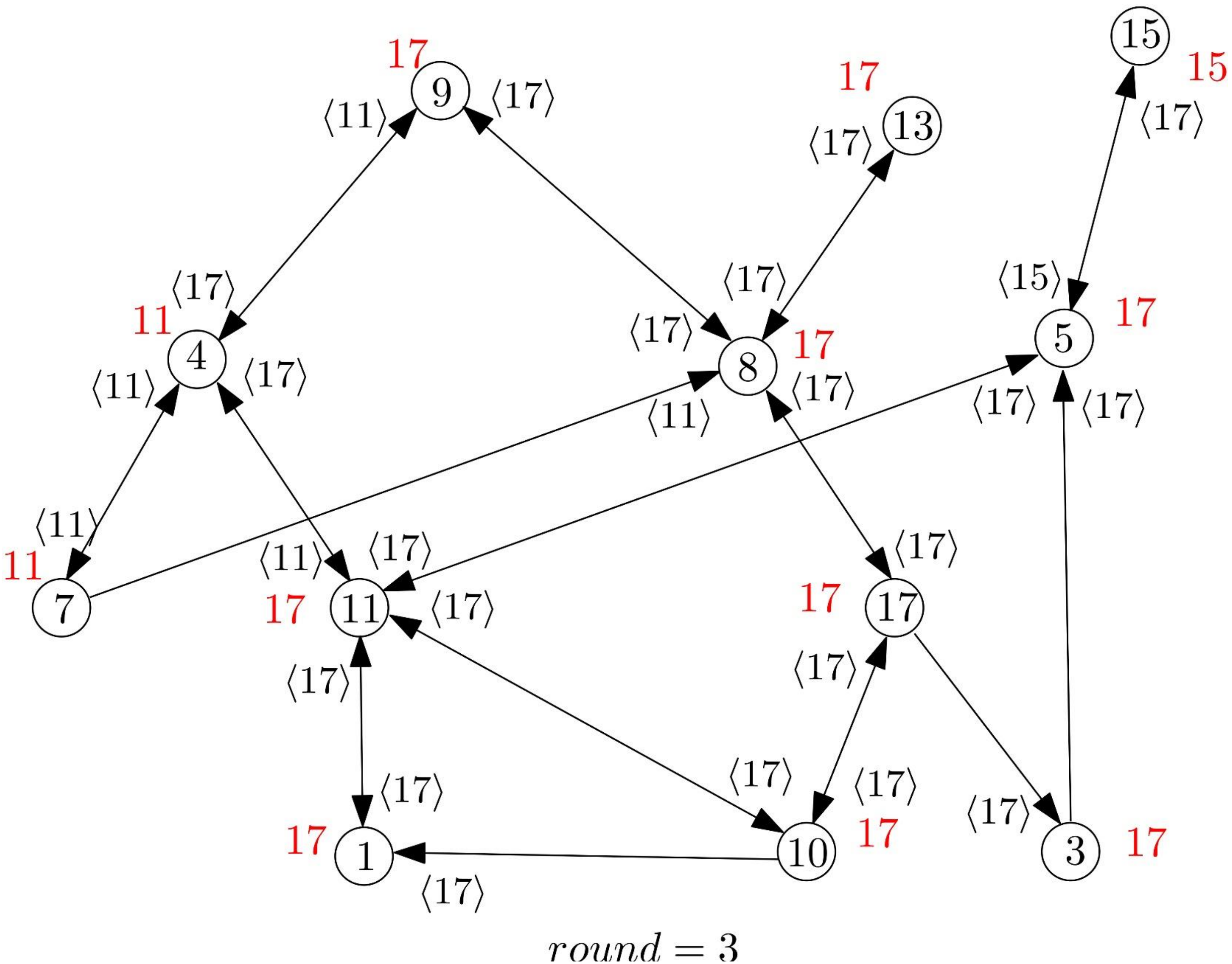
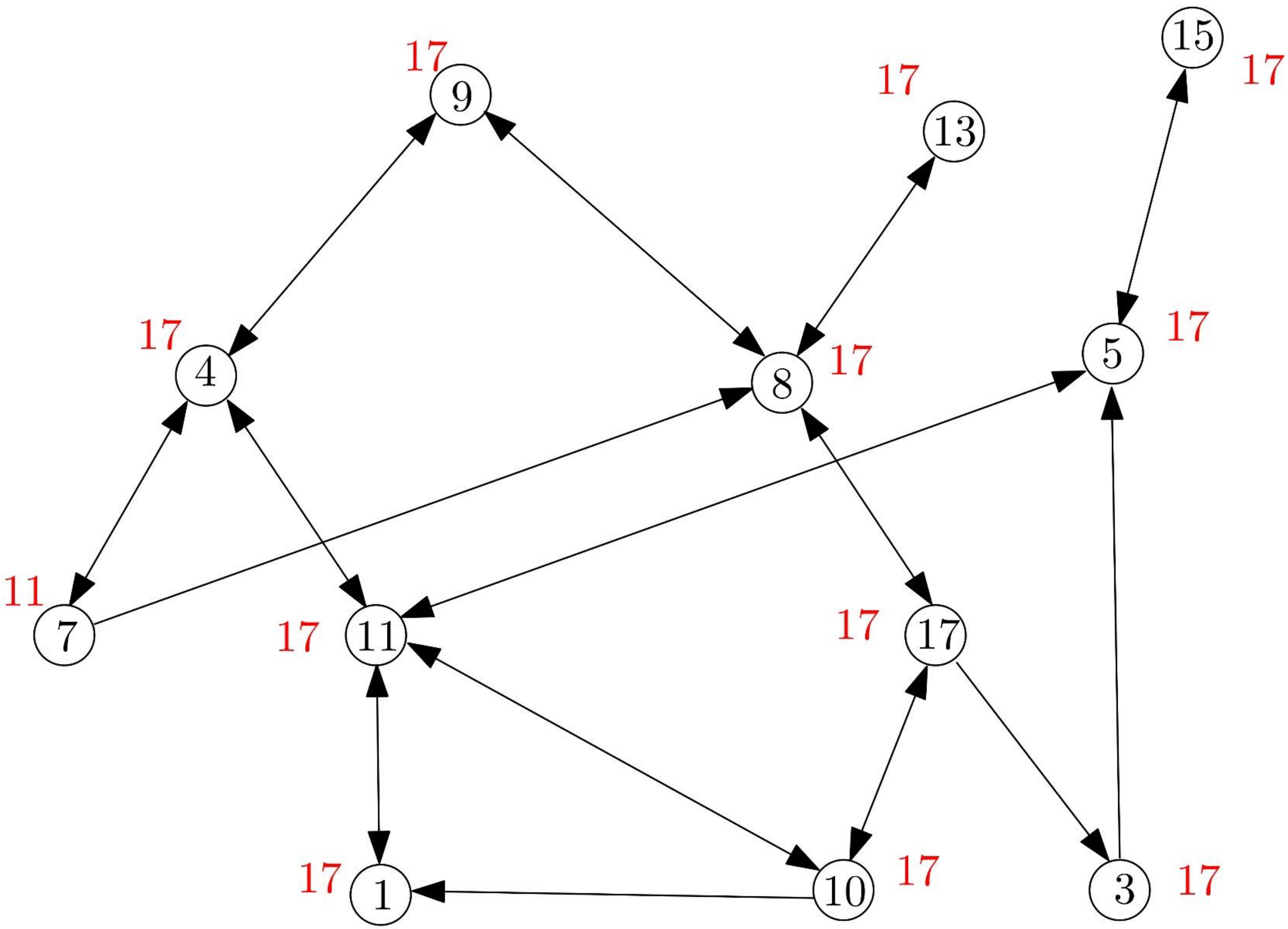
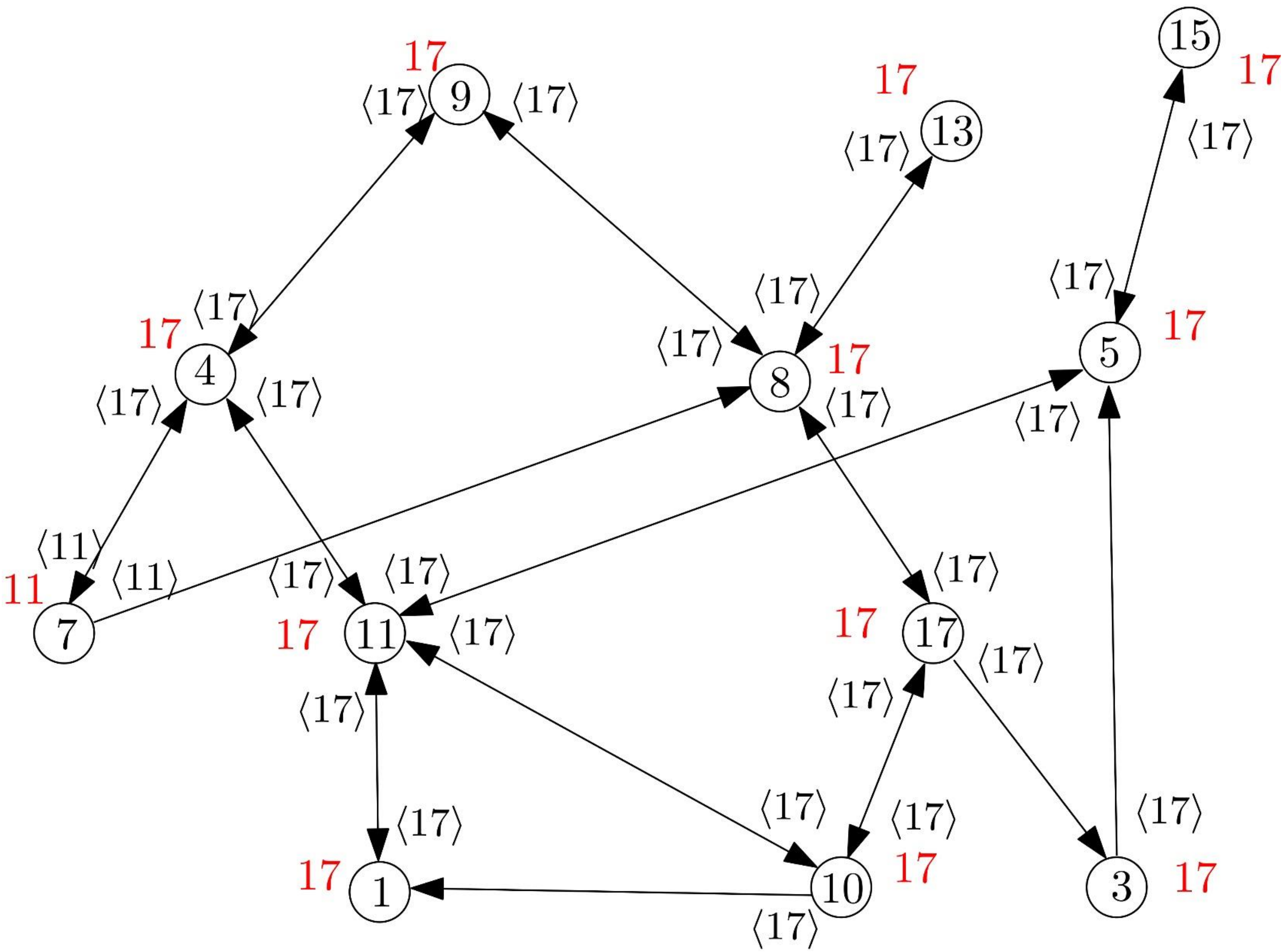


# Example Execution



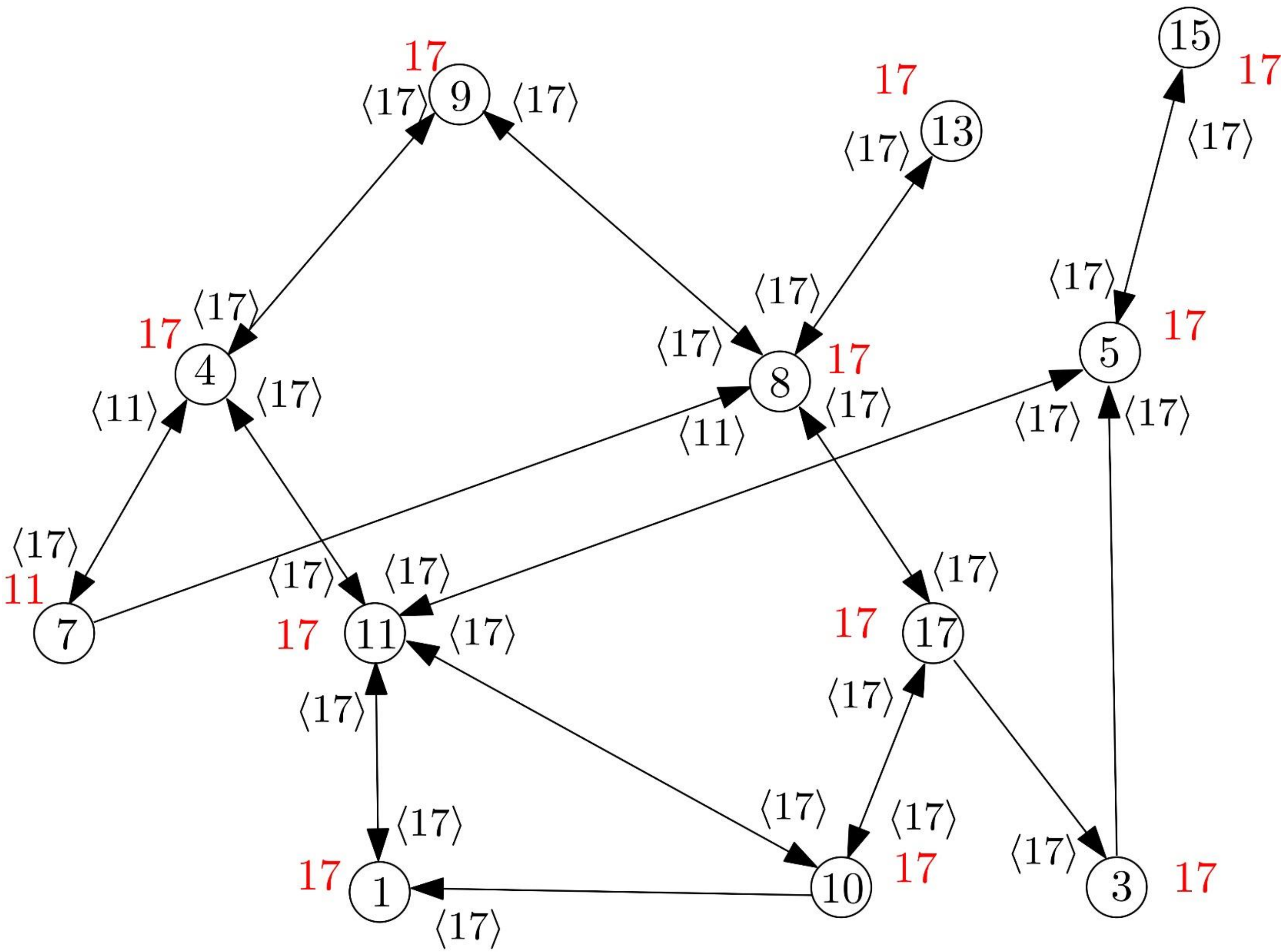

$round = 4$

# Example Execution



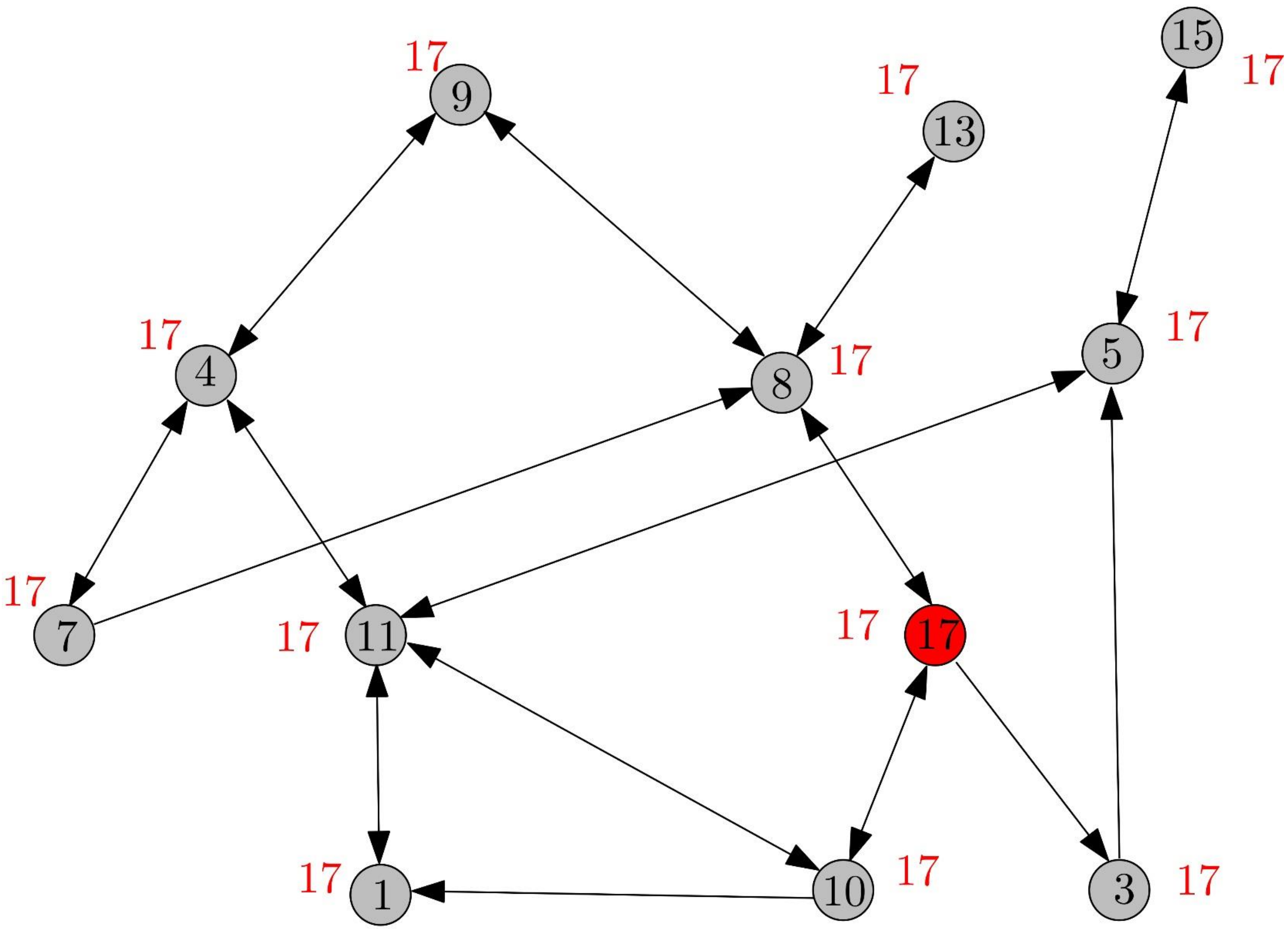

$round = 4$

# Example Execution




$round = 4$

# Example Execution




$round = 5$

# Correctness and Complexity

- Correctness:
  - exactly one processor is elected in the last round
- Time complexity:
  - $D + 1$ rounds (or $D$ depending on the round model)
- Communication complexity:
  - size of messages: encoding in bits of the maximum id
  - $D \cdot m$ messages always
    - $m$ is the number of directed links in the network

# FloodMax Correctness

- We have to show that:
  - Exactly one processor $u_i$ sets $status_i$ := "*leader*" in the last round
  - We know that the algorithm aims to elect the processor with the maximum id
    - Call it $u_{max}$
- Suffices to show that:
  - $u_{max}$ outputs "*leader*" in the last round
  - Every other $u_i$ outputs "*non-leader*" in the last round

# FloodMax Correctness

Lemma. $u_{max}$ outputs "*leader*" in round $D + 1$.

*Proof.* Trivial.

- We know that $id_{max}$ (i.e., the id of $u_{max}$) is the greatest id in the network
- Therefore, in every round it will hold at $u_{max}$ that
  - *maxID = myID*
  - as $u_{max}$ will never hear an id greater than its own
- So, this will also hold in round $D + 1$
  - Then *maxID = myID evaluates to "true" at* $u_{max}$
  - Therefore, $u_{max}$ outputs "*leader*" by setting *status* := "*leader*"

□

  - Essentially, by induction on the number of rounds *r, we show that maxID = myID holds for every r at* $u_{max}$

# FloodMax Correctness

Lemma. Every processor $u_i$ other than $u_{max}$ outputs "non-*leader*" in round $D + 1$.

*Proof.* It suffices to show that by the beginning of round $D + 1$ every processor $u_i$ has received $id_{max}$ (i.e., the id of $u_{max}$)

- Because then it must hold that in round $D + 1$
- $maxID_i = id_{max} > myID_i$
- and $u_i$ will set $status_i :=$ "*non-leader*"

- We will prove that: In round $r$, any $u_i$ at distance $r$ from $u_{max}$ receives $id_{max}$
  - By induction on $r$

# FloodMax Correctness

*Proof (continued).*

- $r = 1$: Trivially, as $u_{max}$ sends $id_{max}$ to all its neighbours
- Assume it holds for any round $r - 1 \geq 1$
  - that is, assume that all nodes at distance $r - 1$ from $u_{max}$ receive $id_{max}$ in round $r - 1$
- Then it must hold also for round r
  - Because all those nodes at distance $r - 1$, in round $r$ set $maxID_i := idmax$
  - Therefore send $id_{max}$ to all their neighbours
  - Implies that all nodes at distance $r$, receive $id_{max}$ in round $r$

□

# FloodMax Correctness

Theorem. The FloodMax algorithm solves the leader election problem in any strongly connected directed network (provided the availability of unique ids and knowledge of the diameter *D*).

- Observe that the diameter *D* concerns the maximum distance in the whole network
- *Can you think of a more precise parameter to replace D in this algorithm?*
  - In the worst case it will be equal to *D* but
  - In other cases it may be less

# FloodMax Correctness

- Observe that the diameter $D$ concerns the maximum distance in the whole network
- *Can you think of a more precise parameter to replace D in this algorithm?*
  - It is the maximum distance from $u_{max}$ to any other processor
    - known as the eccentricity of that node
  - Observe that it would be a bit artificial to assume that the algorithm knows this in advance
  - Knowing the diameter of the network as a whole is quite natural to assume

# FloodMax Time Complexity

- $D + 1$ rounds

- *Can you see why?*

# FloodMax Time Complexity

- $D + 1$ rounds
  - All processors perform a check in round $D + 1$
  - From that point on they do nothing
  - We could have explicitly added a halt (or terminate) command at that point
  - At that point one processor has been elected and all other know that they have not been elected
- Question: *Do they also know who the elected one is?*

# FloodMax Time Complexity

- $D + 1$ rounds
  - All processors perform a check in round $D + 1$
  - From that point on they do nothing
  - We could have explicitly added a halt (or terminate) command at that point
  - At that point 1 has been elected and all other know that they have not been elected
- Question: *Do they also know who the elected one is?*
  - *Yes*: In their *$maxID_i$ variable*

# FloodMax Communication Complexity

- $D \cdot m$ messages always
  - $m$ (also denoted $|E|$) is the number of directed links in the network

- *Can you see why?*

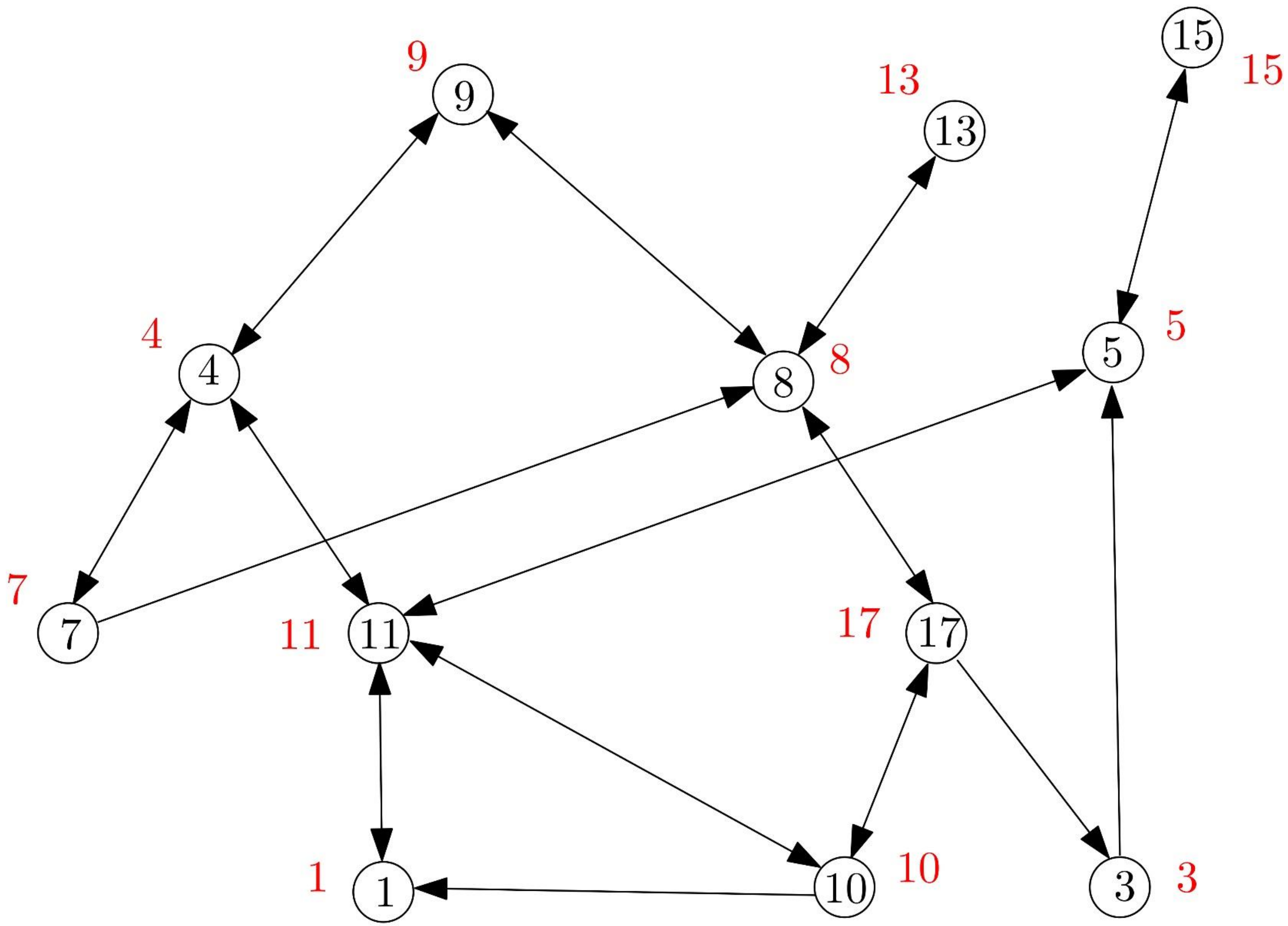
# FloodMax Communication Complexity

- $D \cdot m$ messages always
  - $m$ (also denoted $|E|$) is the number of directed links in the network
- In round $D + 1$ nothing is transmitted
  - Only local checks and termination decisions
- For the first $D$ rounds though:
  - Every processor $u_i$ sends $maxID_i$ to all its out-neighbours
  - Therefore, every link has one message in every round transmitted through it

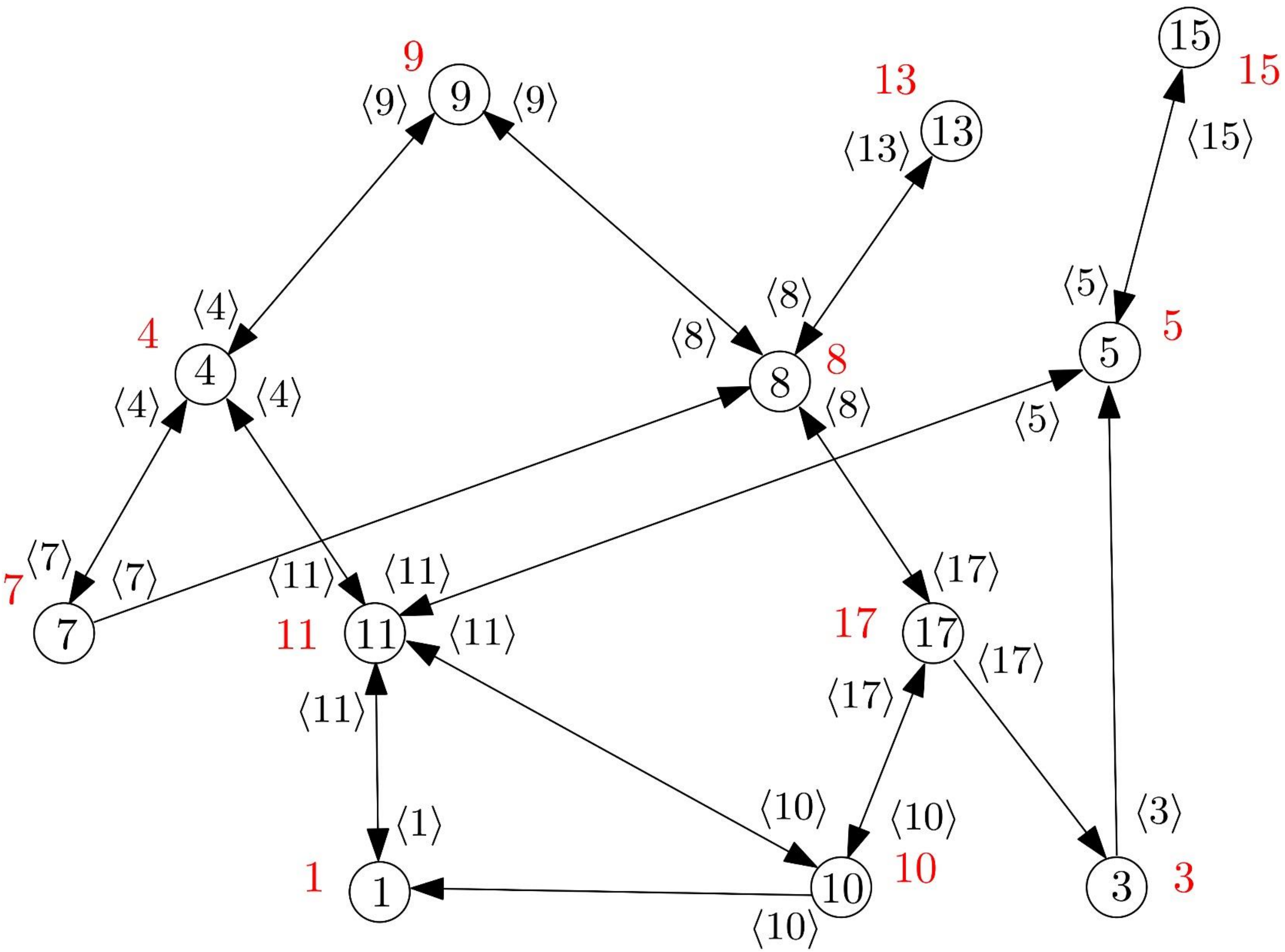
# FloodMax Communication Complexity

- *Is this the most message-efficient solution?*
- *Can you observe any “waste” of messages in FloodMax?*

# Example Execution

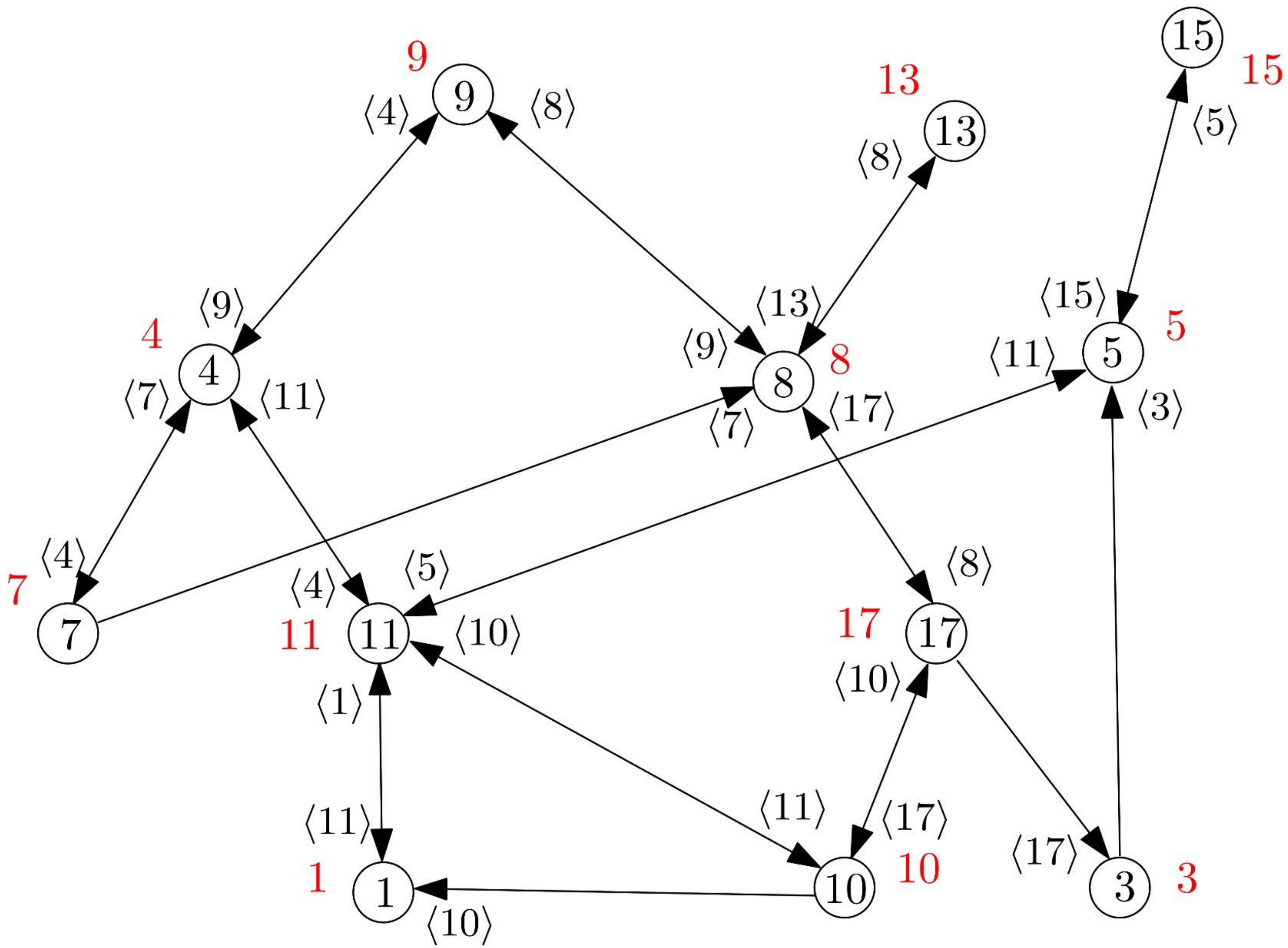


# Example Execution



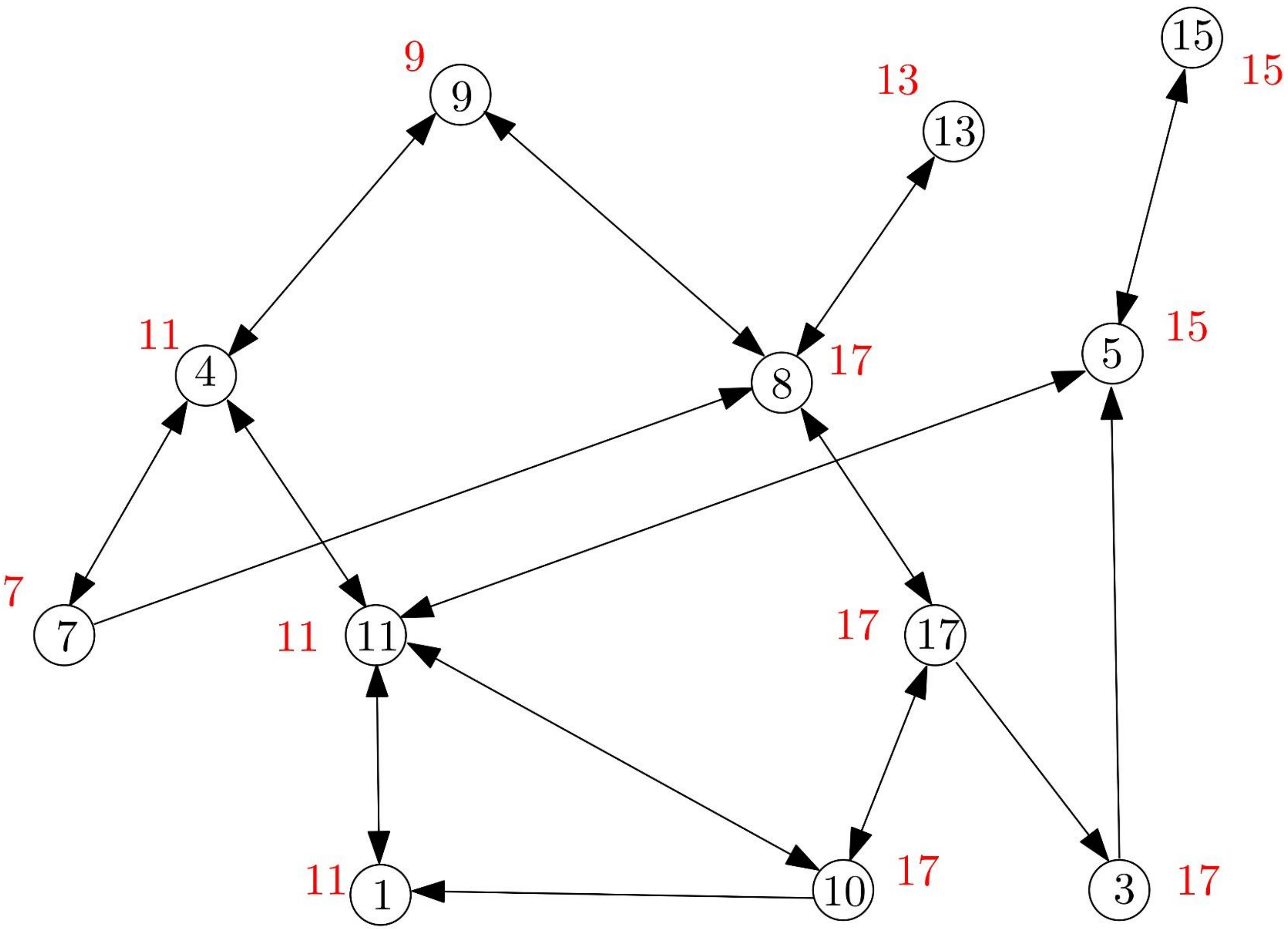

$round = 1$

# Example Execution



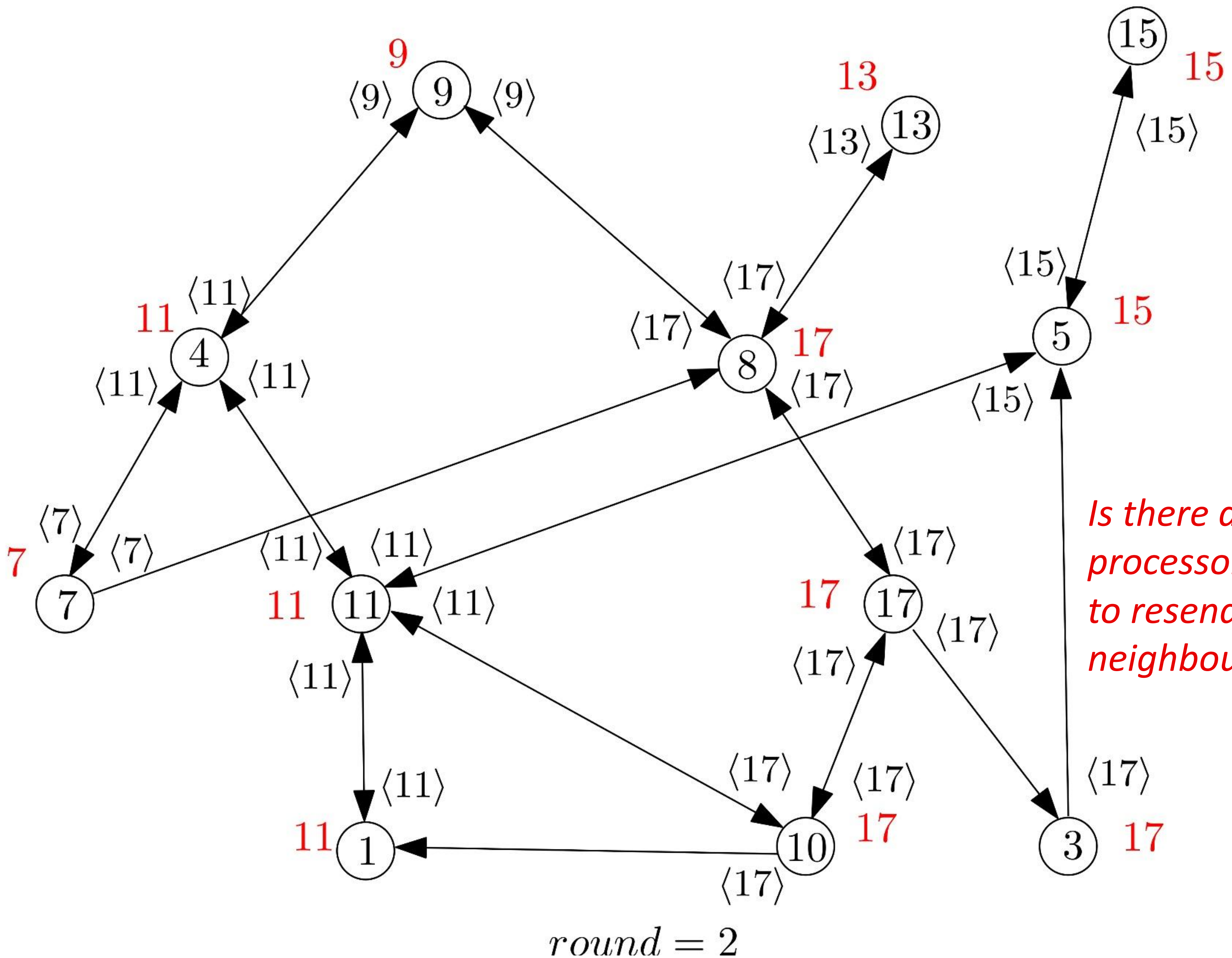

$round = 1$

# Example Execution



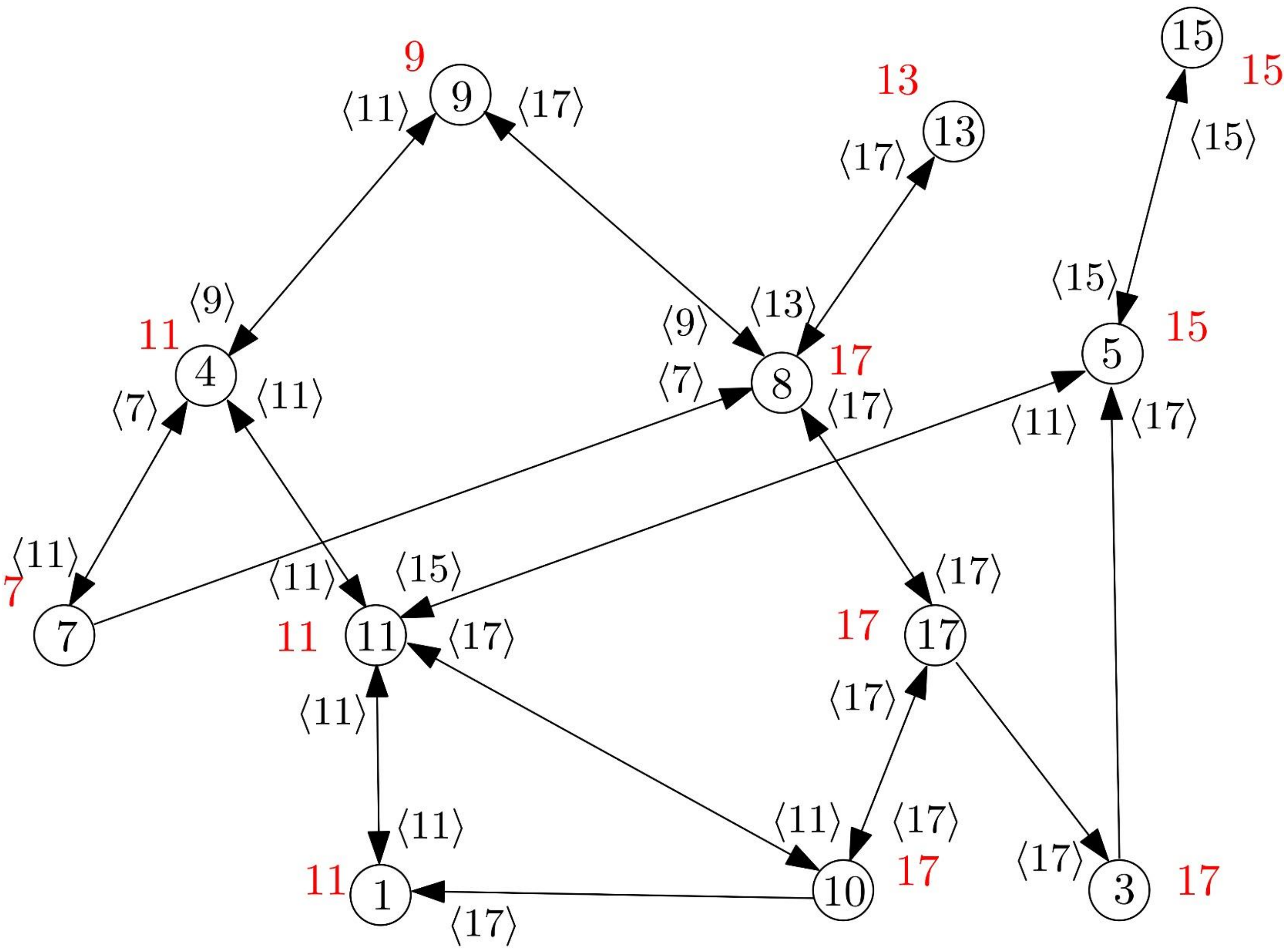

$round = 2$

# Example Execution



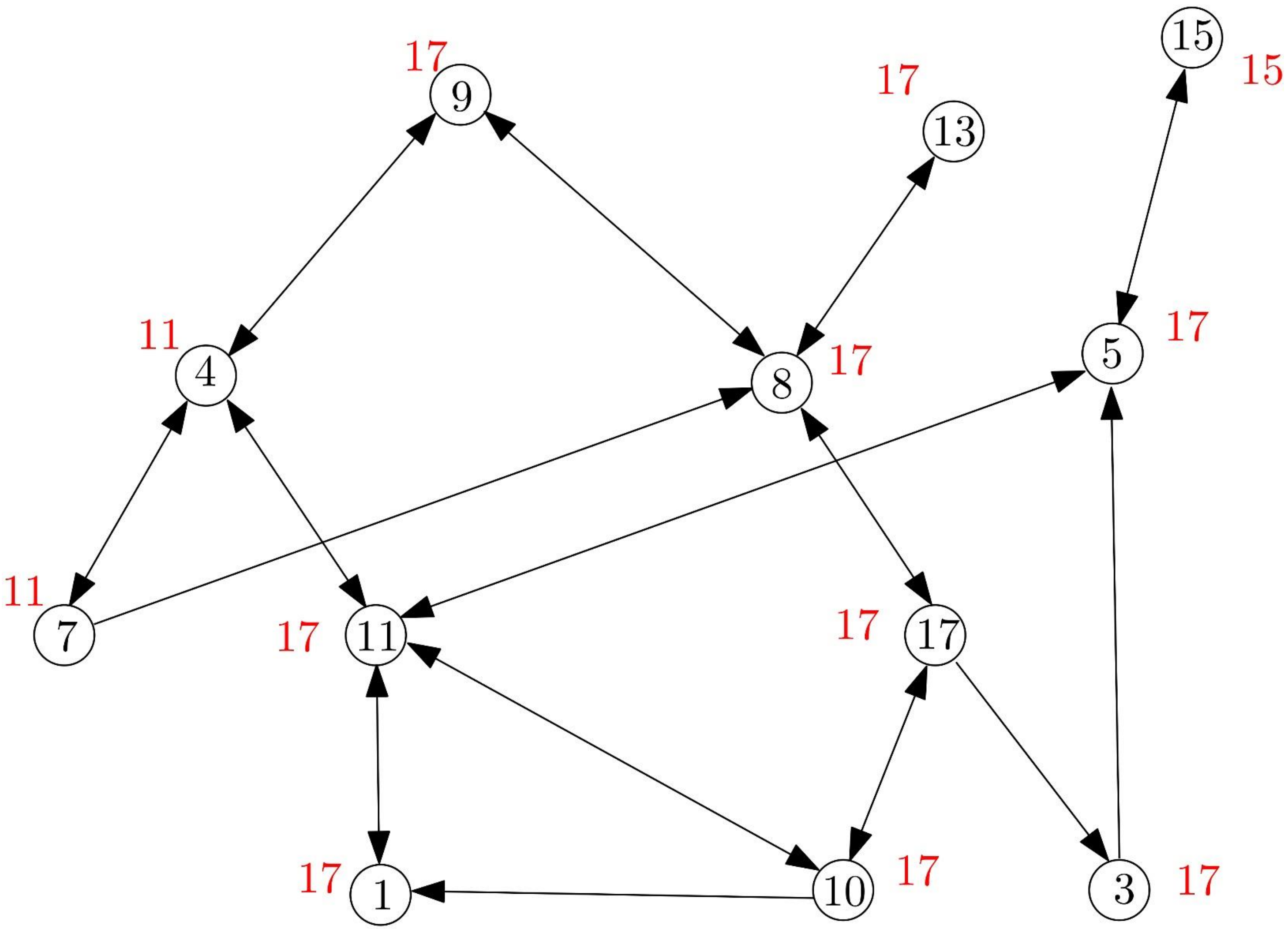

*Is there any need for processor with id = 17 to resend ⟨17⟩ to its neighbours?*

# Example Execution



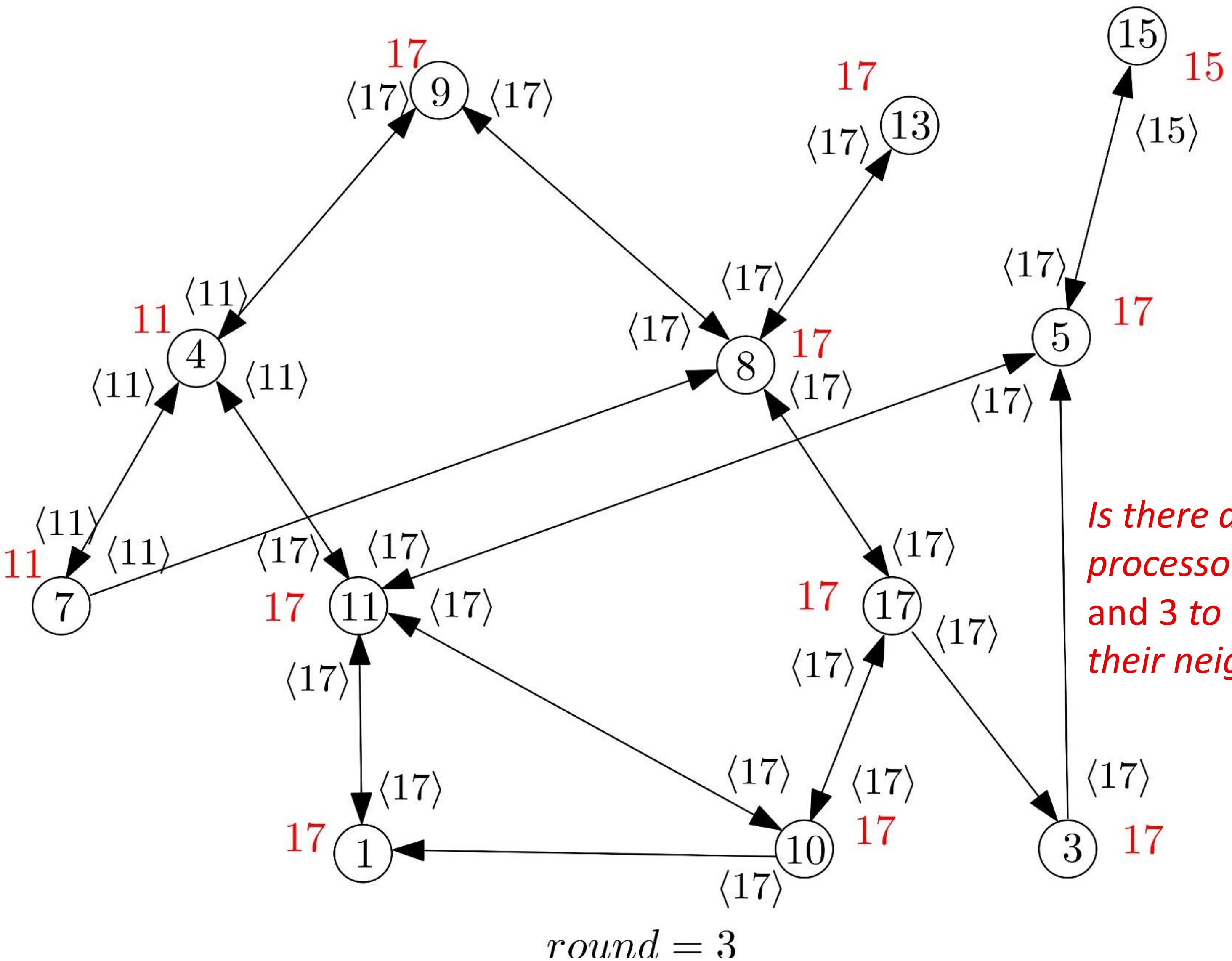

$round = 2$

# Example Execution




$round = 3$

# Example Execution




*Is there any need for processors* 17, 8, 10, and 3 *to resend* ⟨17⟩ *to their neighbours?*

# OptFloodMax: An Improvement of FloodMax

- Same as FloodMax but now
  - Processors do not send their $maxID_i$ in every round
  - They only send it whenever they hear a new maximum
  - That is, only in the rounds that they update their $maxID_i$

- Obvious to see that it reduces the #messages in various cases
- Not that obvious yet whether it improves the worst-case complexity

# OptFloodMax: Pseudocode

**Algorithm** OptFloodMax
Code for processor $u_i$ , $i \in \{1, 2, \ldots, n\}$:

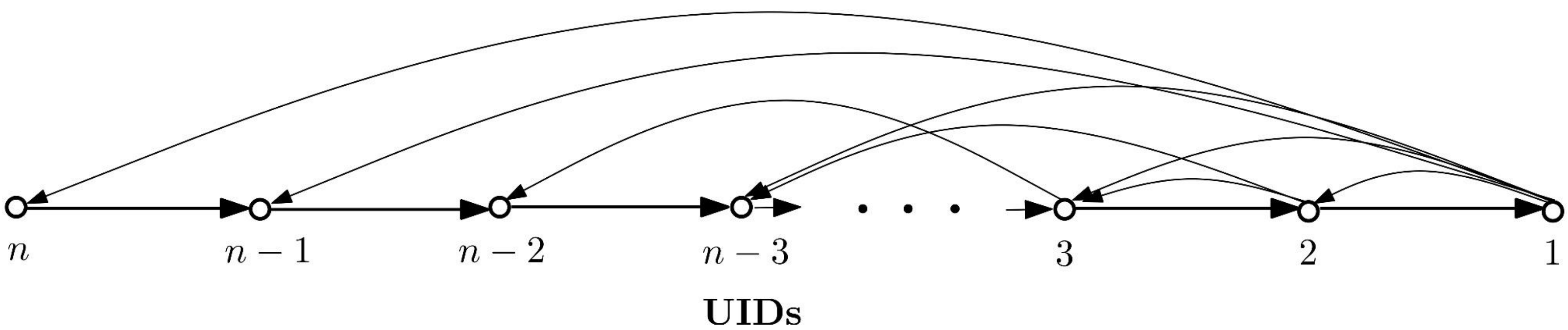
```
Initially:
   u_i knows its own unique id stored in myID_i
   maxID_i := myID_i
   status_i := "unknown"
   newInfo_i := true        // an additional Boolean variable                          ←
   Also has access to the current round and knows the diameter D

if round = 1 then
   send ⟨maxID_i⟩ to all out-neighbours
else
   upon receiving ⟨inIDs⟩ from in-neighbours        // one or more ids arriving from neighbours
   if max(inIDs) > maxID_i then                                                            ←
      maxID_i := max({maxID_i} ∪ inIDs)             // remember only the maximum "heard" so far
      newInfo_i := true                                                                    ←
   else                                                                                    ←
      newInfo_i := false                                                                   ←
   if round ≤ D and newInfo_i = true then   // 1 < round ≤ D                               ←
      send ⟨maxID_i⟩ to all out-neighbours
   else if round = D + 1 then                                                              ←
      if maxID_i = myID_i then          // if equal to your own, no greater id exists in the network
         status_i := "leader"           // therefore, elect yourself a leader
      else                              // greater than own
         status_i := "non-leader"       // therefore, declare yourself a non-leader
// observe that in the end all processors know the id of the elected leader, stored in their maxID_i
// variable
```

# OptFloodMax: Correctness and Complexity

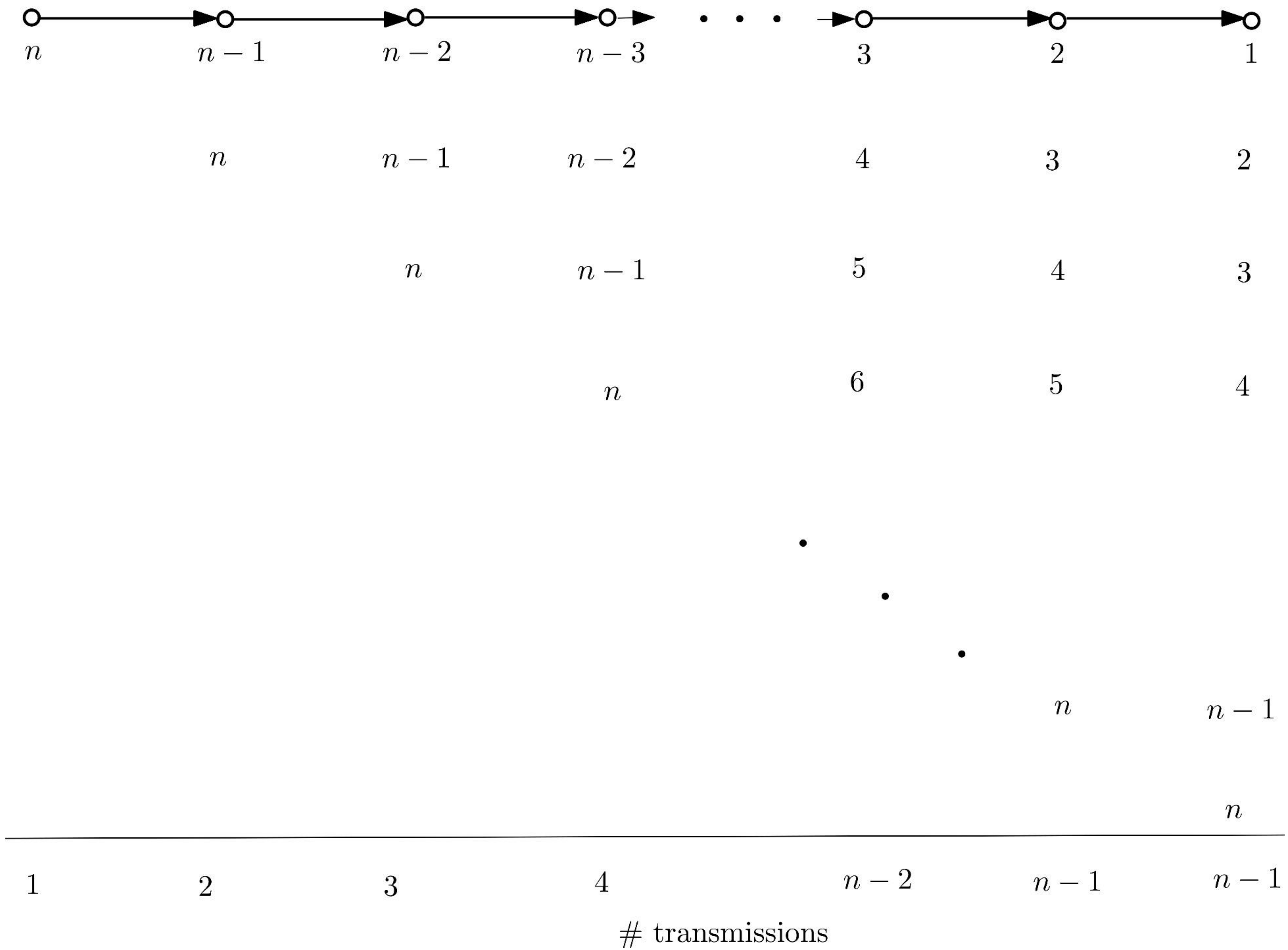
- Correctness:
  - remains correct (needs proof)
- Time complexity:
  - same as in FloodMax (immediate)
- Communication complexity:
  - *We can show that it does not improve the worst-case complexity compared to FloodMax*
  - FloodMax: $D \cdot m = O(n^3)$ messages
  - We can show that in some cases also OptFloodMax transmits $\Theta(n^3)$ messages

# OptFloodMax: Communication Complexity




- It happens that UIDs here are 1 through n (consecutive) - not necessary
  - But their order in the network (combined with the specific structure of this network) is important for this result
- Remark: The network has **all** inverse links (to the left), not only the ones shown here

# OptFloodMax: Communication Complexity



# OptFloodMax: Communication Complexity

$$
\begin{aligned}
\#messages &= (n-1)^2 + \sum_{i=1}^{n-1} (n-i)^2 \\
&= n^2 - 2n + 1 + \sum_{i=1}^{n-1} i^2 \\
&= \left[ \sum_{i=1}^{n} i^2 \right] - 2n + 1 \\
&= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} - 2n + 1 \\
&= \Theta(n^3)
\end{aligned}
$$

# (Opt)FloodMax Further Improvement

- *Can you think of any additional improvement?*
- *Any waste of messages that still remains?*

# (Opt)FloodMax Further Improvement

- *Can you think of any additional improvement?*
- *Any waste of messages that still remains?*

- Yes: No need to send to a processor that just send you the new maximum
  - Again won't improve the worst case complexity
  - Still, we gain something in many cases

# Summary

- Leader election is crucial for distributed systems
  - breaks symmetry
  - allows for coordination
- If all processors are initially identical then
  - impossible to elect a leader even in very simple networks
  - e.g., a ring
- Adding unique ids breaks this inconvenient initial symmetry
- The LCR algorithm elects a leader in any ring network
  - simple conceptually, assumes unique ids
  - $n$ rounds (or $2n$ for all to terminate), $O(n^2)$ messages
- The FloodMax algorithm elects a leader in any strongly connected network
  - like a generalisation of LCR
  - simple, assumes unique ids and knowledge of the diameter D
  - $D$ rounds, $D \cdot m$ messages
- The OptFloodMax algorithm is an improvement of FloodMax
  - Decreases the number of messages in many cases
  - Does not improve the worst-case complexity
  - Still such improvements may be very important for real systems and applications where we are not always faced with the worst cases