Inviscid Burger's Equation in 2D

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1 Introduction

Burger's Equation, discovered by Johannes (Jan) Burger, is part of a group of quasi/non-linear hyperbolic partial differential equations which describe fluid dynamics which form the Navier-Stokes equation. Under simplified conditions, Navier-Stokes reduces to Burger's equation in 1D which can then be used as a toy model to describe phenomena such as gas dynamics, nonlinear acoustics and traffic flow.

A conservation equation, Burger's equation describes the flux of a fluid element as it moves along a path. Burger's equation differs from the advection equation in that the velocity of the fluid is dependent on the equation of motion which governs the fluid. It should be noted that this velocity dependence leads to the development of a shock wave or discontinuity as the fluid propagates. Although important, the numerical treatment of the shock is outside the scope of this project.

Burger's equation has two forms, a viscid and inviscid form which describe the motion of the fluid with or without the viscosity of the fluid. In 1D conservative form,

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$
$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u)}{\partial x} = v\frac{\partial^2 u}{\partial x^2}$$

where u is the velocity of the fluid, f is the flux and in the second equation, v is the viscosity. In 2D, the inviscid form of Burger's equation is modified to retain both x and y components of the velocity which are related by the direction of travel.

$$\frac{\partial F(x,y,t)}{\partial t} + u(x,y)\frac{\partial F}{\partial x} + v(x,y)\frac{\partial F}{\partial y} = 0 \tag{1}$$

This report details the modeling of Burger's equation in 2D using finite difference techniques while comparing the robustness of those techniques with respect to stability and flux conservation. We compare Forward Time Centered Space (FTCS), Lax-Wendroff and Mimetic Discretization methods to obtain a comprehensive and consistent model of Burger's equation in 2D up to shock formation.

2 An Overview of Numerical Schemes

2.1 FTCS in 2D

The FTCS equation in 2D is first order in time and second order in space and has the form,

$$U_{i,j}^{n+1} = U_{i,j}^{n} - \frac{u\Delta t}{\Delta x^2}(U_{i-1,j}^{n} - U_{i+1,j}^{n}) - \frac{v\Delta t}{\Delta y^2}(U_{i-1,j}^{n} - U_{i+1,j}^{n}) \tag{2}$$

where U represents the discretization of the equation of motion, u is the x component of the velocity and v is the y component. Although u and v are dependent on x and y, the treatment of the velocity will be detailed in the Numerical Treatment section of the report.

2.2 Lax-Wendroff in 2D

The Lax-Wendroff numerical method takes the Taylor expansion of the equation of motion about x_i and $t_i + \Delta t$. We then discretize the function and its derivatives as centered differences to derive the full scheme to second order in space and time [3].

$$F(x, y, t + \Delta t) = F + \Delta t F_t + \frac{1}{2}\Delta t^2 F_{tt} + ... \tag{3}$$

Since F is a function of the velocity, we can separate F_t into the velocity and space components by equation (1),

$$F_t = -(uF_x + vF_y) \tag{4}$$

$$\begin{aligned} F_{tt} = {} & u(u_x F_x + uF_{xx}) + u(v_x F_y + uF_{yx}) \\ & + v(u_y F_x + uF_{xy}) + v(v_y F_y + uF_y y)) \end{aligned} \tag{5}$$

which can then be discretized by centered difference techniques.

$$\begin{aligned} F_x &\approx \frac{1}{2\Delta x}(U_{i+1,j}^{n} - U_{i-1,j}^{n}) \\ F_y &\approx \frac{1}{2\Delta y}(U_{i,j+1}^{n} - U_{i,j-1}^{n}) \\ F_{xx} &\approx \frac{1}{\Delta x^2}(U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n}) \\ F_{yy} &\approx \frac{1}{\Delta y^2}(U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}) \\ F_{xy} = F_{yx} &\approx \frac{1}{4\Delta x\Delta y}[(U_{i+1,j+1}^{n} - U_{i-1,j+1}^{n}) - (U_{i+1,j-1}^{n} - U_{i-1,j-1}^{n})] \end{aligned} \tag{6}$$

With (3) - (6) we get the full Lax-Wendroff scheme as follows,

$$\begin{aligned}
U_{i,j}^{n+1} = U_{i,j}^{n} &- \frac{\Delta t}{2\Delta x}u(U_{i+1,j}^{n} - U_{i-1,j}^{n}) - \frac{\Delta t}{2\Delta y}v(U_{i,j+1}^{n} - U_{i,j-1}^{n}) \\
&+ \frac{\Delta t^2}{2}[\frac{u}{\Delta x}(\frac{u_x}{2}(U_{i+1,j}^{n} - U_{i-1,j}^{n}) + \frac{u}{\Delta x}(U_{i+1,j}^{n} - 2U_{i,j}^{n} + U_{i-1,j}^{n})) \\
&+ \frac{u}{2\Delta y}(v_x(U_{i,j+1}^{n} - U_{i,j-1}^{n}) + \frac{v}{2\Delta x}(U_{i+1,j+1}^{n} - U_{i-1,j+1}^{n} - U_{i+1,j-1}^{n} + U_{i-1,j-1}^{n})) \\
&+ \frac{v}{2\Delta x}(u_y(U_{i+1,j}^{n} - U_{i-1,j}^{n}) + \frac{u}{2\Delta y}(U_{i+1,j+1}^{n} - U_{i-1,j+1}^{n} - U_{i+1,j-1}^{n} + U_{i-1,j-1}^{n})) \\
&+ \frac{v}{2\Delta y}(\frac{v_y}{2}(U_{i,j+1}^{n} - U_{i,j-1}^{n}) + \frac{v}{\Delta y}(U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}))]
\end{aligned} \tag{7}$$

with U, u and v as described in the FTCS section. Again, the discretization of u and v will be detailed in the Numerical Treatment section of this report. Additional thanks to Jakob Beran for the Lax-Wendroff 2D code [1].

2.3 Mimetic Discretization

The Mimetic Method utilizes mimetic operators which discretize and mimic operators from vector calculus on an interpolated grid [4]. In the case of Burger's equation, we would like to maintain conservation of flux throughout the simulation. To do so, we use the mimetic divergence operator,

$$\nabla \cdot F = D(U) \tag{8}$$

which operates solely on spatial terms of the PDE [2, 4]. The divergence acts on the centers of the grid taking information from the edges as in Figure 1. To address the time dependence, we use forward time which gives the scheme first order in time. In space, the order of accuracy can be selected in the code.

To apply the mimetic operators, we used the MOLE library which contains 1D - 3D operators as well as interpolators for the selected dimensionality [4]. We detail its implementation in the Numerical Treatment and Analysis section below.

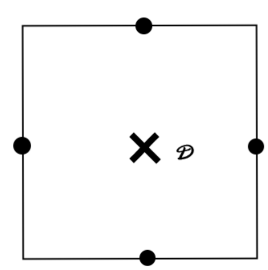

Figure 1: One node of a two dimensional grid. The circles represent the values used to calculate the divergence while the cross is where the divergence is actually calculated.

3 Numerical Treatment and Analysis

3.1 Initial and Boundary Conditions

We chose to use open boundaries as in Figure 2 such that the wave was allowed to travel outside the grid space and an initial Gaussian waveform centered at (x,y) = (0,0) in some direction θ. For dt we chose 0.2 over 250 grid points such that there would be about 50 time steps which was right at or before the discontinuity. The mimetic method required an extra step as per the interpolation which had no effect on the model.

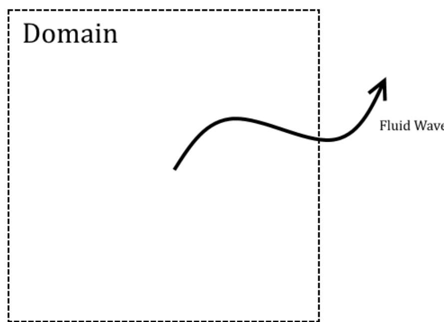

Figure 2: Example of an open boundary as used for this model.

We utilized a symmetric system such that $dx = dy$ and our x-grid and y-grid are equivalent. This simplified our approach greatly especially in the case of the mimetic method in which we extended the 1D approach to 2D with proper application of the 1D mimetic divergence operator and interpolator. The 2D operators were considered but difficulties arose when attempting to properly implement the scheme in 2D as conservation was not maintained.

The components of the velocity were treated as proportional to the equation of motion such that the magnitude of the velocity vector is equivalent to F. In doing so, we may discretize both components in the same way that we discretized F previously while applying the direction by way of trigonometry.

$$u(x_i, y_j, t_n) = U_{i,j}^n cos\theta \tag{9}$$

$$v(x_i, y_j, t_n) = U_{i,j}^n sin\theta \tag{10}$$

In choosing u and v in this manner, we can define the discretization of the derivative of each, as appears in (7), as

$$\begin{aligned} u_x &\approx F_x cos\theta \\ u_y &\approx F_y cos\theta \\ v_x &\approx F_x sin\theta \\ v_y &\approx F_y sin\theta \end{aligned} \tag{11}$$

where F_x and F_y can be found in (6).

3.2 Conservation

As described in the Numerical Scheme section, the mimetic divergence operator maintains conservation which can be used as a metric to determine proper implementation. In the conservative form of Burgers equation, the Lax-Wendroff and FTCS schemes should be likewise conservative [3].

To calculate the conservation of the flux, we calculated the volume underneath the wave at a time step by taking the norm of the solution matrix. We then compared each time step and took the standard deviation between the norms to generate a final conservation number. If that number was 0 then we could say that the scheme maintained conservation as should be expected [2].

4 Results

We implemented the numerical schemes in MATLAB and generated movies which display the propagation of the wave over the grid space up to and slightly beyond the shock wave. It should be noted that although the boundary condition was taken as open, the wave never approached the boundary before the shock developed so the choice of boundary condition had no effect on any of the implementations of the numerical schemes.

N	dx	C-FTCS	C-LW	C-M
40	0.5	3.4×10^{-4}	5.08×10^{-4}	0
100	0.2	0.0017	0.0023	0
500	0.04	0.019	0.027	0

Table 1: Three different runs at N = 40, 100 and 500 and the corresponding conservation number results. In all three runs, dt = 0.2 and the number of temporal grid points is 250.

As can be seen in Figures 3 - 5 and Table 1, modifying the grid points N and hence $dx = dy$, had little to no effect on the stability of the schemes. All were stable within this range with these specific initial and boundary conditions. What did change however was the conservation number for the FTCS and Lax-Wendroff schemes.

As N increased, dx increased as well having been defined by the number of steps across fixed x and y minimums and maximums. This means that as the grid was refined, the FTCS and Lax-Wendroff schemes lost their ability to further conserve the flux or area underneath the solution curves. This could be indicative of conservation errors propagating at each space step although more work will have to be done to understand the conservation loss.

The mimetic method performed as expected up to and a little past the shock which occurred about 5 time units sooner than in the other two methods (around 50 for dt = 0.02) as can be seen in the videos linked in Figure 4. Based on the conservation assessment, it is possible that the time difference arises from the conservation errors as the waves propagate although again, more work will have to be done to understand why this is.

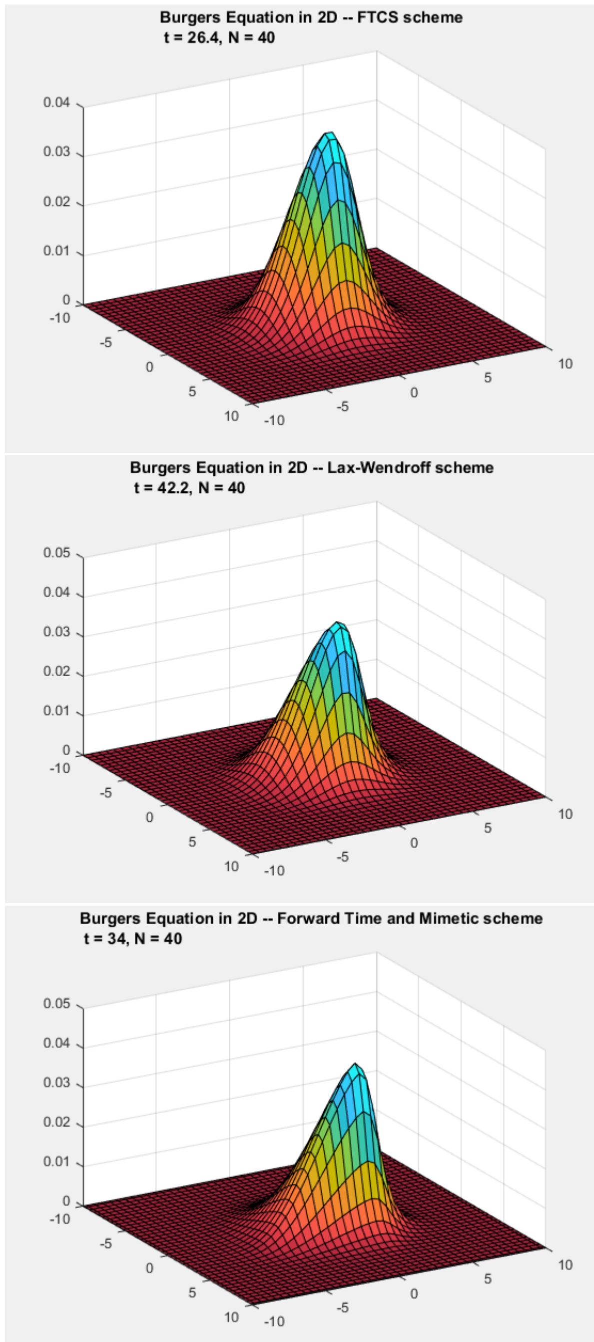

Figure 3: All three schemes at various time steps for N = 40. All snapshots were taken at various times before shock development to show stability.

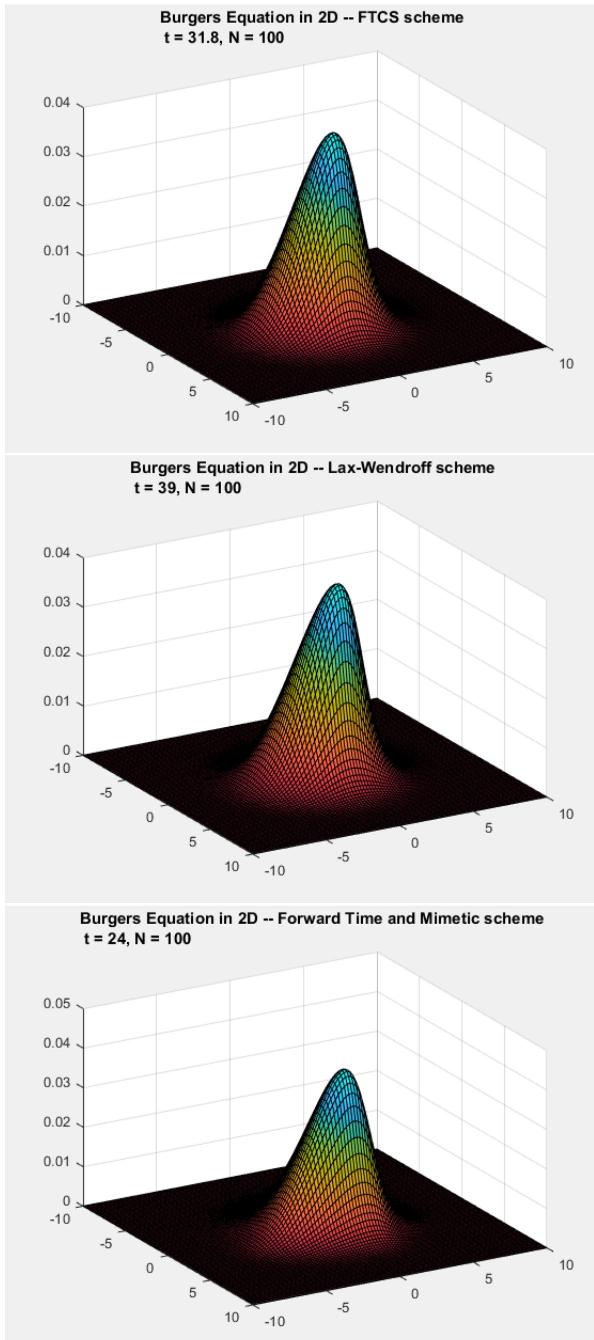

Figure 4: All three schemes at different time steps for N = 100. Full videos of each model for N = 100 can be found via the hyperlinks: FTCS, Lax-Wendroff, Mimetic. Other videos can be made by request to the author at mportman@sdsu.edu.

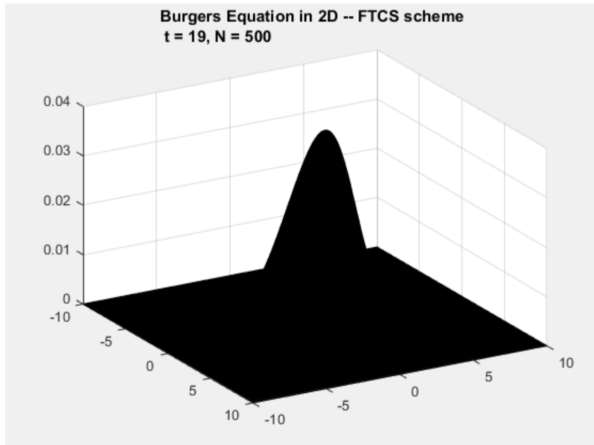

Figure 5: FTCS at a time step for N = 500. The other two were not included because the grid is too dense to see well.

5 Conclusions and Final Remarks

5.1 Conclusion

To solve Burgers' equation in 2D, we implemented two classic finite difference schemes, Forward Time Centered Space and the Lax-Wendroff scheme, to compare against mimetic methods. Based on the performance of the different schemes for the conditions used, we assert that the mimetic method is the most robust of the three. It maintained flux conservation across a range of spatial step sizes and even retained conservation through a few time steps past the shock discontinuity. A natural extension would be to attempt to model the shock and check for entropy conservation up to and through the shock using the mimetic method by utilizing an adaptive grid [3]. Another would be to model the viscid Burgers' equation in 2D.

Difficulties were encountered in coding the mimetic scheme using full-blown 2D mimetic operators. Although the solution propagated similarly, the conservation was not maintained. Future work could implement the 2D mimetic operators properly such that the code could be extended to use different x and y grids as well as perhaps further extending the full 2D code to also include an adaptive grid which can be made using the mimetic library. This would be the first step in modeling the shock although much more work would be necessary to create a full-blown simulation.

5.2 Other Remarks

As mentioned in the Results section, both the Lax-Wendroff and FTCS schemes did not maintain conservation for as-of-yet unknown reasons. What is almost more interesting is that the Lax-Wendroff scheme was slightly worse at maintaining the conservation despite the second order term of the approximation. One possible reason for this is improper numerical implementation. Another is perhaps a dispersive term that arises from extra terms in the Taylor expansion or truncation error [3]. Further work is necessary to understand these issues.

Finally, it would be interesting to be able to validate the implementation by manufacturing a solution or otherwise testing against other methods of which we are unfamiliar. Although it wasn't done for this project, it would be easy to see how changing the order of accuracy in the mimetic method could effect the solution as the library was built such that the order can be chosen [4]. For higher orders of accuracy, we could then test the mimetic method against other higher order numerical schemes.

References

[1] Jakob Beran. Online. June 2015.

[2] Angel Boada. Private Conversation. Dec. 2017.

[3] R. J. LeVeque. *Finite Volume Methods for Hyperbolic Problems*. Cambridge University Press, 2002.

[4] J. E. Castillo; G. F. Miranda. *Mimetic Discretization Methods*. Chapman and Hall/CRC, 2013.