Applications of Tropical Geometry to the Theory of Algebraic Curves

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Abstract

We aim to give an expository account of a complete proof of the classical Brill-Noether theorem using techniques coming from Tropical Geometry.

1 Introduction

2 Analogies between Curves and Graphs

2.1 Definitions

Our goal in this section is to develop analogs for classical notions from the theory of algebraic curves for so called *metric graphs*. Metric graphs are essentially combinatorial objects which are interpreted topologically. First, by a weighted graph, we mean a connected multigraph without loops, in which each edge is assigned a positive real number. Associated to such a metric graph G, we get a compact connected topological space, denoted by $\Gamma(G)$, obtained by viewing edges of G as line segments of distance given by the weight of that edge. We always view such weighted graphs up to equivalence, where two such graphs are equivalent if they admit a common length preserving sub-division. These metric graphs will play the role of algebraic curves for us, so they will be central objects in what follows. In fact, such a metric graph can naturally be viewed as a so called tropical curves as in [ref], although the latter is a more general type of object.

Given a metric graph Γ, we want a notion of a divisor and a rational function on Γ. A divisor is simply defined to be an element of the free abelian group on the points of Γ, and an effective divisor is one with all non-negative coefficients. The definition of a rational function is slightly more complicated. As motivation for the definition, we look at rational functions on algebraic curves. Such rational functions can be locally expressed as the ratio of two polynomials. In the tropical setting, multiplication is replaced by addition, and addition is replaced by the *max* operation, so the natural analog for a rational function on a metric graph is a piecewise linear function (with finitely many pieces) with integer slopes. We denote the space of rational functions on a metric graph Γ by $\mathcal{M}(\Gamma)$. Note that we can always replace G by some sub-division to assume without loss of generality that f is affine on the interior of each edge of G. In this case, we see that the associated divisor (f) of some $f \in \mathcal{M}(\Gamma)$ is always supported on the vertices of Γ. Whenever we are considering a divisor on some metric graph, we always assume that we are working with a model of the metric graph in which the divisor is supported on the vertices of the graph.

It is a classical fact that on a Riemann surface, the sum of the zeros and poles, counted with appropriate multiplicities, is zero. On a metric graph, we have the following

Lemma 2.1 *Let Γ be a metric graph and $f \in \mathcal{M}(\Gamma)$. Then $deg((f)) = 0$*

Proof. Take an equivalent subdivision of G in which f is affine on the interior of each edge. Then for every edge e which is directed from v_1 to v_2 with the slope of f on the interior of e given by n, we see that e contributes n to v_2 and $-n$ to v_1 in the sum defining $\deg((f))$.

Remark 2.1 *Whenever we consider a divisor on some metric graph Γ, we always assume that we are working with a model of Γ with the property that the divisor is supported on the vertices of Γ.*

Defining the notion of the rank of a divisor on a Γ is slightly more complicated. Recall that in the case of an algebraic curve X defined over some field k, the rank of a divisor D, denoted $r(D)$, is defined to be one less than the k dimension of the vector space $H^0(X, \mathcal{O}_X(D))$, i.e. it is the dimension of the associated projective space $PH^0(X, \mathcal{O}_X(D))$. Equivalently, it is the dimension of the vector space of rational functions f on X such that $(f) - D$ is effective. If we try to make the analogous definition for a metric graph Γ, we run into the problem that the set of piecewise linear functions ϕ on Γ with $(\phi) - D$ effective is not a vector space. (Give an example). Instead, we can use the following alternative definition of the rank of a divisor on X to motivate the definition of the rank of a divisor on a metric graph. Given a divisor D, we can consider the associated line bundle $\mathcal{L}(D)$. Then D has rank r iff given any divisor D' of degree r, we can find some section $s \in H^0(X, \mathcal{L}(D))$ such that $(s) - D'$ is effective. Concretely, in the case when all r points are distinct, this means we can find some global section s of $\mathcal{L}(D)$ which vanishes at these r points. In other words, a divisor D has rank at least r iff for every divisor D' of degree r, the complete linear system $|D - D'|$ is non-empty.

In particular, every metric graph, as we have defined them, may naturally be viewed as a tropical curve.

Now fix a metric graph Γ. We want to develop a notion of divisors and rational functions on this graph, in parallel to the theory of divisors and rational functions on a smooth projective curve. The analog of meromorphic functions on a smooth projective curve over the complex numbers (or more generally algebraic rational functions on a smooth projective curve over an arbitrary algebraically closed field) will be played by *piecewise-linear* functions on the metric graph Γ, which are defined as follows. Given such a piecewise linear function ϕ on Γ, we want to associate to ϕ a divisor $\mathrm{div}(\phi)$ on Γ, in much the same way that we can associate a divisor to any non-zero rational function on a smooth project curve. Divisors of this form on Γ will play the role of principal divisors.

3 Specialization Lemma

Our goal in this section is to define Baker's specialization map and state and prove his Specialization Lemma from [Bak08], which will be our main tool for relating geometric information about a curve to certain combinatorial information. First, by way of setup, throughout this section we let R denote a complete discrete valuation ring, with field of fractions K and algebraically closed residue field κ. We also give ourselves a smooth, projective, geometrically connected curve X over K, which is our main geometric object of interest. We want to fill in a diagram of the form

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X}_\kappa \arrow[l, hook'] \arrow[d] \\ \mathrm{Spec}(K) \arrow[r, hook] & \mathrm{Spec}(R) & \mathrm{Spec}(\kappa) \arrow[l, hook'] \end{tikzcd}$$

such that $\mathcal{X}$, which we think of as the total space of an infinitesimal family of curves over $\mathrm{Spec}(R)$, is regular and flat over $\mathrm{Spec}(R)$, and the special fiber $\mathcal{X}_\kappa$ is reduced with only ordinary double points as singularities and has smooth irreducible components $\{C_1, \cdots C_k\}$. Such a $\mathcal{X}$ is called a *strongly regular semistable model* of the curve X (*strong* refers to the smoothness of the irreducible components C_i). Both the regularity and the flatness of $\mathcal{X}$ over $\mathrm{Spec}(R)$ will be used crucially in order to define the specialization map, as we will explain.

It will be very useful in what follows to observe that Weil divisors on $\mathcal{X}$ (which are naturally in bijection with Cartier divisors as $\mathcal{X}$ is regular) fall into two categories. There are those which are supported on the special fiber $\mathcal{X}_\kappa$, which are commonly referred to as *vertical* divisors, and there are those which are Zariski closures of divisors supported on the generic fiber, referred to as *horizontal* divisors. Every divisor on $\mathcal{X}$ may be written uniquely as a sum of vertical and horizontal divisors.

The combinatorial information referenced in the previous paragraph will be the so-called *dual graph* of the special fiber $\mathcal{X}_\kappa$. This graph, denoted $G(\mathcal{X}_\kappa)$, has one vertex v_i for each component C_i, and one edge between vertices v_i and v_j for each point of intersection of the components C_i and C_j. Note that because each component C_i is smooth by assumption, the graph $G(\mathcal{X}_\kappa)$ has no loops, although it may have multiple edges, as per our conventions in [2]

We will need the following lemma on extending line bundles from the generic fiber in order to define Baker's Specialization Map:

Lemma 3.1 *Notation as above, any line bundle L on the generic fiber X extends to a line bundle $\mathcal{L}$ on the total space $\mathcal{X}$. Furthermore, such an extension is unique up to twisting by components of the special fiber, i.e. is of the form $\mathcal{L} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_\mathcal{X}(C_i)$ after fixing one extension $\mathcal{L}$.*

Proof. To see the existence of an extension of L to $\mathcal{X}$, represent L by some Weil Divisor. Then the line bundle associated to the Zariski closure in $\mathcal{X}$ of this Weil divisor will give a desired extension. Because every Weil divisor on $\mathcal{X}$ can be written uniquely as a sum of horizontal and vertical divisors, we see that this extended Weil divisor is unique up to adding vertical divisors. But adding a vertical divisor exactly corresponds to twisting the associated line bundle by a component C_i of the special fiber.

We will now show how chip-firing enters the picture. The idea is that associated to each line bundle $L \in \mathrm{Pic}(X)$, we will produce a divisor on the graph $G(\mathcal{X}_\kappa)$. This proceeds by first extending L to $\mathcal{X}$, and then restricting $\mathcal{L}$ to each component C_i. Because each C_i is a smooth curve, the restriction of $\mathcal{L}$ to C_i has a well defined degree. This degree will be the coefficient of the vertex v_i corresponding to the component C_i in the divisor we are producing. We formalize this in the following definition.

Definition 3.1 *Given a line bundle $\mathcal{L} \in Pic(\mathcal{X})$, the divisor $D(\mathcal{L}) \in Div(G(\mathcal{X}_\kappa))$ is defined by*

$$D(\mathcal{L}) := \sum_{i=1}^{k} deg(\mathcal{L}|_{C_i})v_i,$$

where v_i is the vertex corresponding to the component C_i.

If we want to extend this definition to map line bundles on X to divisors on $G(\mathcal{X}_\kappa)$, the issue we run into is that the extension $\mathcal{L}$ in not unique. However, by Lemma 3.1, the extension is unique up to twisting by a component of the special fiber. Thus, in order to understand the failure of this non-uniqueness, we just need to understand how $D(\mathcal{L})$ changes when we twist $\mathcal{L}$ by some C_i. For this, we have the following lemma, which brings the combinatorics of chip-firing into the picture.

Lemma 3.2 *Given a line bundle $\mathcal{L} \in Pic(\mathcal{X})$ and a component C_i of $\mathcal{X}_\kappa$, the divisor*

$$Div(\mathcal{L} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_\mathcal{X}(C_i))$$

is obtained from the divisor $Div(\mathcal{L})$ by performing a chip-firing move the vertex v_i.

Proof It will be convenient for the proof to work in terms of Weil divisors on $\mathcal{X}$ instead of line bundles, which is allowed as $\mathcal{X}$ is regular. So suppose $\mathcal{L}$ corresponds to some divisor $D = D_h + D_v$, where D_h and D_v are the horizontal and vertical components of D, respectively. Observe that $D(\mathcal{L})$ only depends on D_v, as D_h has intersection number zero with every component of the special fiber. Thus, we may assume without loss of generality that $D = D_v$. Then the divisor corresponding to the line bundle $\mathcal{L} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_\mathcal{X}(C_i)$ is just $D_v + C_i$. The intersection number of this divisor with some component C_j with $i \neq j$ is just $(D_v, C_j) + (C_i, C_j)$. But (C_i, C_j) is exactly the number of edges from v_i to v_j in the dual graph of the special fiber, so this number corresponds with the number of chips at vertex v_j after performing a chip firing move at vertex v_i. It remains to show that the number of chips at vertex v_i decreases by its degree. For this, we need to use the fact that we have the equality

$$\sum_{i=1}^{n}(C_i, C_j) = 0,$$

as follows from the fact that the divisor $C_1 + \cdots C_n$ is the pullback of the special point $\mathrm{Spec}(k)$ under the proper structure morphism $\pi : \mathcal{X} \to \mathrm{Spec}(R)$, and is therefore algebraically equivalent to zero as a divisor, so its intersection number with any other divisor must be zero. It follows that the number of chips at vertex v_i in the chip configuration associated to the twisted line bundle is given by

$$(D_v, C_j) + (C_j, C_j) = (D_v, C_j) - \sum_{i \neq j} (C_i, C_j),$$

which is exactly the number of chips we would expect at vertex v_i after firing vertex v_i with initial chip configuration given by $D(\mathcal{L})$.

Lemma 3.2 suggests that we define a group $\mathrm{Pic}(G)$ associated to any graph G as follows. We first consider the abelian group of all divisors on G, which is isomorphic to the free abelian group on a number of generators equal to the number of vertices of G, and we quotient out by the relation of two divisors (or equivalently chip configurations) if one can be reached from the other by a sequence of chip-firing operations (i.e. we quotient the group of all divisors on G by those which are chip-fire equivalent to the zero divisor). It is worth mentioning yet another interpretation of the Picard group of a graph. Recall that given a finite graph G on n vertices, we can define its graph Laplacian, Δ_G as an $n \times n$ matrix, with the ith diagonal entry equal to the degree of vertex v_i, and the (i, j) entry with $i \neq j$ equal to -1 if there is an edge between vertices v_i and v_j, and 0 otherwise. We can then observe that if e_i denotes the indicator vector with 1 in the ith coordinate and 0s in every other coordinate, then $\Delta_G e_i$ can be identified with the chip configuration obtained by starting at the zero chip configuration and firing vertex v_i. Thus, we have that $\mathrm{Pic}(G)$ is the quotient of $\mathrm{Div}(G)$ by the image of the laplacian Δ_G. Combining Lemma 3.1 and Lemma 3.2, we can define a map

$$\sigma : \mathrm{Pic}(X) \to \mathrm{Pic}(G(\mathcal{X}_\kappa))$$

by first extending a line bundle $L \in \mathrm{Pic}(X)$ to $\mathcal{X}$, and then taking the associated divisor. we will refer to this map σ as *Baker's Specialization map*, and will play the key role of relating the geometry of X to the combinatorics of chip firing.

Recall from section 2 that we have a well defined notion of rank for elements on both sides of σ. The following Lemma gives a strong comparison result for how the rank changes under σ, and will be the key tool for allowing us to deduce geometric results about X from the combinatorics of the dual graph.

Lemma 3.3 *Let $L \in Pic(X)$ be a line bundle with rank r. Then the rank of $\sigma(L)$ is at least r (i.e. rank can only jump under σ).*

Proof: In the proof we let $G := G(\mathcal{X}_\kappa)$, we let $r(L)$ denote the rank of a line bundle L, and we let $r_G(C)$ denote the rank of a chip configuration on G. Also, we let D be the divisor associated to the line bundle L, so we have $r(D) = r(L)$. Recall that by convention we defined the rank of a non-effective divisor on G to be -1. Thus, if L admits no global sections (i.e. has rank -1), the result is immediate. Thus, we may assume $r(L) \geq 0$. We claim then that $r(\sigma(L)) \geq 0$. Indeed, the fact that $r(L) \geq 0$ means that D is linearly equivalent to some effective divisor, and because σ preserves linear equivalence, we see that $r_G(\sigma(L)) \geq 0$.

Now assume that for some positive integer k we have that $r(L) \geq k$. It suffices to show that $r_G(\sigma(D)) \geq k$. For this, it suffices to show that for all $v \in V(G)$, we have $r_G(\sigma(D) - v) \geq k - 1$. But this follows immediately from the fact that the map from $X(K) = \mathcal{X}(R) \to V(G)$ is surjective, and from the fact that $r(D - P) \geq r(D) - 1$ for all $P \in X(K)$, with equality holding for at least one P. (Note that the equality $X(K) = \mathcal{X}(R)$ uses that X is projective.

We will also need the following lemma on comparing the degrees of the restrictions of line bundles on $\mathcal{X}$ to the generic and special fibers:

Lemma 3.4 *Notation as above, let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Then we have the equality*

$$deg(\mathcal{L}|_X) = deg(\mathcal{L}_{\mathcal{X}|_\kappa})$$

Proof. Using Riemann-Roch on the smooth curve X, we have

$$\deg(\mathcal{L}|_X) = \mathcal{X}(\mathcal{L}_X) + g - 1$$

Because $\mathcal{X}$ is flat over Spec(R), using the fact that the Euler characteristic of line bundle is constant for flat families, we have

$$\mathcal{X}(\mathcal{L}|_X) = \mathcal{X}(\mathcal{L}_{\mathcal{X}_\kappa})$$

By considering the pullback of $\mathcal{L}|_{\mathcal{X}_\kappa}$ to the normalization of the special fiber, we conclude that

$$\mathcal{X}(\mathcal{L}_{\mathcal{X}_\kappa}) = \sum_{i=1}^{k} \mathcal{X}(\mathcal{L}_{C_i}) = k - \sum_{i=1}^{k} g(C_i)$$

Because we are ultimately interested in the geometric setting, we need to understand how the specialization map behaves under base change, so we can study curves defined over an algebraically closed field. For this, we will need to leave the world of finite graphs and use the machinery of metric graphs introduced in Section 1.

4 Non-existence of Special Divisors

Our first goal in this section will be to construct canonical representatives of each divisor in Pic(G) for a finite graph G, which we think of as a dual graph of some curve as in Section 3. More precisely, for each divisor D on G and each vertex $v_0 \in G$, we will introduce a unique representative of D which is V_0-*reduced*. The precise definition is as follows

References

[Bak08] Matthew Baker. Specialization of linear systems from curves to graphs. *Algebra Number Theory*, 2(6):613–653, 2008.