# Applications of Tropical Geometry to the Theory of Algebraic Curves

Matthew Prashker

October 10, 2021

**Abstract**

We aim to give an expository account of a complete proof of the classical Brill-Noether theorem using techniques coming from Tropical Geometry. The strategy is to reduce the algebro-geometric statement into a combinatorial statement about certain graphs, and then prove these results directly. We lastly give an exposition of other directions and problems these ideas and techniques lead to.

## 1 Introduction

Brill-Noether Theory is ultimately concerned with how smooth projective curves of a given genus $g$ may be embedded in various projective spaces. Given a curve $X$ of genus $g$. Define $W_r^d$ as the subscheme of $\mathrm{Pic}_d(X)$, consisting of equivalence classes of those degree $d$ line bundles which admit at least $r+1$ global sections. The non-emptiness of $W_r^d$ implies that some open subscheme of our original curve $X$ can be realized as a degree $d$ subscheme of $\mathbb{P}^r$ (The open subscheme if given by the complement of the base locus of the line bundle). In particular, if $W_r^d$ is empty, then our curve $X$ cannot possibly be realized as a degree $d$ curve in $\mathbb{R}^r$. The following numerical quantity will turn out to govern the behavior of the dimension of $W_r^d$, and will play a key role in what follows

**Definition 1.1** *Given integers $g, r, d \geq 0$, we define the Brill-Noether number as $\rho(g, r, d)$ by*

$$\rho(g, r, d) = g - (r+1)(g - d + r).$$

The version of the Brill Noether Theorem we will aim for, using tropical techniques, is the following

**Theorem 1.1** *Fix non-negative integers $g, r, d$. Then*
*(i): If $\rho(g, r, d) \geq 0$, then every smooth curve $X/\mathbb{C}$ of genus $g$ has a divisor of degree $d$ and rank at least $r$. Moreover, we have that $dim(W_r^d(X)) \leq \rho(g, r, d)$.*
*(ii): If $\rho(g, r, d) < 0$, then on a general smooth curve of genus $g$, there is no degree $d$ divisor with rank at least $r$.*

Note that there is a more precise statement of part (i) of the above Theorem, namely that $dim(W_r^d) = min(g, \rho(g, r, d))$, although we are only concerned with the upper bound in this thesis. We think of $g$ as being fixed here as the genus of our curve, and looking at this expression for varying $r$ and $d$. If $\mathcal{M}_g$ denotes the moduli space of curves of genus $g$, then the function on $\mathcal{M}_g$ mapping $X$ to the dimension of $W_r^d$ is upper semicontinuous. Because this dimension is non-negative, in order to show that a general curve of genus $g$ admits no linear series of degree $d$ with at least $r+1$ global sections when $\rho(g, r, d) < 0$, it suffices to show the existence of a single curve in $\mathcal{M}_g$ with no such line bundle. In a similar vein, in order to show that the upper bound

$$\dim(W_r^d) \leq \rho(g, r, d)$$

holds for a general smooth curve $X$ of genus, it suffices to show that this inequality holds for a single curve $g$.

We note that there are curves which have negative Brill Noether number but still admit such line bundles, so the generality assumption in the second part of the theorem is necessary. As simple examples, we may consider hyperelliptic curves, i.e. those curves which can be realized as a degree 2 branched cover of $\mathbb{P}^1$. Such curves exist of any genus $g$ - indeed, such a curve may be written down explicitly as

$$y^2 = (x - z_1) \cdots (z - z_{2g+1}),$$

where the $z_i$ are distinct complex numbers (we are really interested in the projective completion of such a curve). Because this curve admist a degree 2 map to $\mathbb{P}^1$, it must have a line bundle of degree 2 which admits at least 2 global sections (so the rank is at least 1). But the Brill Noether number in this case is $g - 2(g-1)$, which is negative for large $g$, which shows that the second part of the Brill-Noether Theorem cannot possibly hold for every curve. The point is that hyperelliptic curves and other special classes of curves (give examples) are the exceptions to part (ii) of the Theorem, and the difficulty in proving part (ii) of the Theorem stems from the fact that it is very difficult to explicitly write down and work with a "general" curve in the sense of moduli. Most of the proofs of the theorem historically have relied on "deformations", which involves putting a curve in an algebraic family and looking and examining a special fiber of the family.

To give an overview of the thesis, in Section 2 we will introduce the relationship between curves and graphs, which is motivated by the tropical viewpoint of algebraic geometry, although using tropical terminology is not strictly necessary. We will lay down the terminology we need and prove some basic results. In section 3, we will state and proof Baker's Specialization Lemma, which allows us to compare information between the two sides of this analogy. In section 4, we will use to Lemma to give a proof of the Brill Noether Theorem (part (i) of the theorem and the upper bound on the dimension of $W_r^d(X)$ for a curve $X$). The proof relies heavily on the Specialization Lemma, together with some analysis of the combinatorics of specific types of graphs. Finally, in section 5, we will discuss some further applications in algebraic geometry of these ideas.

# 2 Analogies Between Curves and Graphs

## 2.1 Definitions

Our goal in this section is to develop analogs for classical notions from the theory of algebraic curves for so called *metric graphs*. Metric graphs are essentially combinatorial objects which are interpreted topologically. We first make the basic definition

**Definition 2.1** *A Weight graph $G$ will be a connected multigraph without loops, in which each edge is assigned a positive real number. Associated to such a $G$, we naturally get a compact topological space $\Gamma(G)$, obtained by viewing the edges of $G$ as intervals, which we refer to as the metric graph associated to $G$. Two such metric graphs are considered to be equivalent if they admit a common subdivision of their edges.*

These metric graphs will play the role of algebraic curves for us, so they will be central objects in what follows. In fact, such a metric graph can naturally be viewed as a so called tropical curves as in [ref], although the latter is a more general type of object. The reason we need to pass to metric graphs instead of graphs is essentially the same reason we need to consider curves, originally defined over some discretely valued, instead of arbitrarily ramified extensions of the field $k$. In particular, every metric graph, as we have defined them, may naturally be viewed as a tropical curve, although we will not need this terminology.

Given a metric graph $\Gamma$, we want a notion of a divisor and a rational function on $\Gamma$. A divisor is simply defined to be an element of the free abelian group on the points of $\Gamma$, and an effective divisor is one with all non-negative coefficients. The definition of a rational function is slightly more complicated. As motivation for the definition, we look at rational functions on algebraic curves. Such rational functions can be locally expressed as the ratio of two polynomials. In the tropical setting, multiplication is replaced by addition, and addition is replaced by the *max* operation, so the natural analog for a rational function on a metric graph is a piecewise linear function (with finitely many pieces) with integer slopes, where edges of the metric graph are identified with real intervals. We formally record this definition as

**Definition 2.2** *Given a metric graph $\Gamma$, the space of rational functions $\mathcal{M}(\Gamma)$ on $\Gamma$ is the collections of those real valued functions $f$ on the topological space underlying $\Gamma$ such that there exists some subdivision of $\Gamma$ with the property that $f$ is piecewise linear with integer slope on each interval of the subdivison.*

Given such a piecewise linear function $\phi$ on $\Gamma$, we want to associate to $\phi$ a divisor $\mathrm{div}(\phi)$ on $\Gamma$, in much the same way that we can associate a divisor to any non-zero rational function on a smooth project curve. Divisors of this form on $\Gamma$ will play the role of principal divisors. We denote the space of rational functions on a metric graph $\Gamma$ by $\mathcal{M}(\Gamma)$. Note that we can always replace $G$ by some sub-division to assume without loss of generality that $f$ is affine on the interior of each edge of $G$. In this case, we see that the associated divisor $(f)$ of some $f \in \mathcal{M}(\Gamma)$ is always supported on the vertices of $\Gamma$. Whenever we are considering a divisor on some metric graph, we always assume that we are working with a model of the metric graph in which the divisor is supported on the vertices of the graph. TODO: Given an example of some rational functions on metric graphs and their associated divisors.

It is a classical fact that on a compact Riemann surface, the sum of the zeros and poles, counted with appropriate multiplicities, is zero. On a metric graph, we have the following

**Lemma 2.1** *Let $\Gamma$ be a metric graph and $f \in \mathcal{M}(\Gamma)$. Then $deg((f)) = 0$*

**Proof.** Take an equivalent subdivision of $G$ in which $f$ is affine on the interior of each edge. Then for every edge $e$ which is directed from $v_1$ to $v_2$ with the slope of $f$ on the interior of $e$ given by $n$, we see that $e$ contributes $n$ to $v_2$ and $-n$ to $v_1$ in the sum defining $\deg((f))$.

**Remark 2.1** *Whenever we consider a divisor on some metric graph $\Gamma$, we always assume that we are working with a model of $\Gamma$ with the property that the divisor is supported on the vertices of $\Gamma$.*

Defining the notion of the rank of a divisor on a $\Gamma$ is slightly more complicated. Recall that in the case of an algebraic curve $X$ defined over some field $k$, the rank of a divisor $D$, denoted $r(D)$, is defined to be one less than the $k$ dimension of the vector space $H^0(X, \mathcal{O}_X(D))$, i.e. it is the dimension of the associated projective space $\mathbb{P}H^0(X, \mathcal{O}_X(D))$. Equivalently, it is one less than the dimension of the vector space of rational functions $f$ on $X$ such that $(f) - D$ is effective. If we try to make the analogous definition for a metric graph $\Gamma$, we run into the problem that the set of piecewise linear functions $\phi$ on $\Gamma$ with $(\phi) - D$ effective is not a vector space. (Give an example). Instead, we can use the following alternative definition of the rank of a divisor on $X$ to motivate the definition of the rank of a divisor on a metric graph. Given a divisor $D$, we can consider the associated line bundle $\mathcal{L}(D)$. Then $D$ has rank $r$ iff given any divisor $D'$ of degree $r$, we can find some section $s \in H^0(X, \mathcal{L}(D))$ such that $(s) - D'$ is effective. Concretely, in the case when all $r$ points are distinct, this means we can find some global section $s$ of $\mathcal{L}(D)$ which vanishes at these $r$ points. In other words, a divisor $D$ has rank at least $r$ iff for every divisor $D'$ of degree $r$, the complete linear system $|D - D'|$ is non-empty. This motivates the following definitions of the rank of a divisor on graph and metric graphs

**Definition 2.3** *Let $G$ be a graph. Then a divisor $D$ has rank at least $r$ if for every effective divisor $D'$ of degree $r$, the divisor $D - D'$ is equivalent to some effective divisor. If $\Gamma_G$ is a metric graph, we use the same definition for a divisor, except we allow subdividing the metric graph, replacing it by an equivalent one, so the support of all divisors considered are vertices of the metric graph, following our conventions. Finally, a divisor has rank exactly $r$ if it has degree at least $r$ but not at least $r + 1$. By convention, we define the rank of a non-effective divisor to be $-1$.*

The following combinatorial operation defined on these graphs and metric graphs will also play a key role in what follows - it will turn out to be play the role on the graph theoretic side of linear equivalence of divisors on the algebro-geometric side of our analogy.

**Definition 2.4** *Given a graph or metric graph $G$, a divisor $D$ on $G$, and a vertex $v \in G$, we can define a new divisor $D'$ on $G$ as follows. If $v_1, \cdots v_k$ are the vertices adjacent to $v$ in $G$, then we add 1 to the coefficient of each $v_i$, and we subtract $k$ from the coefficient of $v$. We refer to the divisor so obtained as the result of performing a chip firing operation at the vertex $v$*

It is worth mentioning here that the rank of a divisor has a very nice interpretation in terms of chip firing operations. A divisor $D$ on $G$ has rank at least $r$ if and only if the following condition is satisfied. Suppose $D = \sum_i a_i v_i$, with $a_i \in \mathbb{Z}$ and $v_i$ vertices of $G$. Now remove a total of $r$ from all of the coefficients of the $v_i$. For instance, if $r = 3$, we may subtract 2 form vertex $v_1$ and 1 from vertex $v_2$ (we may also subtract negative quantities here). Then we can always perform a finite number of chip-firing operations on the vertices so that every vertex has a non-negative coefficient.

# 3 Specialization Lemma

Our goal in this section is to define Baker's specialization map and state and prove his Specialization Lemma from [Bak08], which will be our main tool for relating geometric information about a curve to certain combinatorial information. First, by way of setup, throughout this section we let $R$ denote a complete discrete valuation ring, with field of fractions $K$ and algebraically closed residue field $\kappa$. We also give ourselves a smooth, projective, geometrically connected curve $X$ over $K$, which is our main geometric object of interest. We want to fill in a diagram of the form

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X} \arrow[d] & \mathcal{X}_\kappa \arrow[l, hook'] \arrow[d] \\ \mathrm{Spec}(K) \arrow[r, hook] & \mathrm{Spec}(R) & \mathrm{Spec}(\kappa) \arrow[l, hook'] \end{tikzcd}$$

such that $\mathcal{X}$, which we think of as the total space of an infinitesimal family of curves over Spec$(R)$, is regular and flat over Spec$(R)$, and the special fiber $\mathcal{X}_\kappa$ is reduced with only ordinary double points as singularities and has smooth irreducible components $\{C_1, \cdots C_k\}$. Such a $\mathcal{X}$ is called a *strongly regular semistable model* of the curve $X$ (*strong* refers to the smoothness of the irreducible components $C_i$). Both the regularity and the flatness of $\mathcal{X}$ over Spec$(R)$ will be used crucially in order to define the specialization map, as we will explain.

It will be very useful in what follows to observe that Weil divisors on $\mathcal{X}$ (which are naturally in bijection with Cartier divisors as $\mathcal{X}$ is regular) fall into two categories. There are those which are supported on the special fiber $\mathcal{X}_\kappa$, which are commonly referred to as *vertical* divisors, and there are those which are Zariski closures of divisors supported on the generic fiber, referred to as *horizontal* divisors. Every divisor on $\mathcal{X}$ may be written uniquely as a sum of vertical and horizontal divisors.

The combinatorial information referenced in the previous paragraph will be the so-called *dual graph* of the special fiber $\mathcal{X}_\kappa$. This graph, denoted $G(\mathcal{X}_\kappa)$, has one vertex $v_i$ for each component $C_i$, and one edge between vertices $v_i$ and $v_j$ for each point of intersection of the components $C_i$ and $C_j$. Note that because each component $C_i$ is smooth by assumption, the graph $G(\mathcal{X}_\kappa)$ has no loops, although it may have multiple edges, as per our conventions in [2]

We will need the following lemma on extending line bundles from the generic fiber in order to define Baker's Specialization Map:

**Lemma 3.1** *Notation as above, any line bundle $L$ on the generic fiber $X$ extends to a line bundle $\mathcal{L}$ on the total space $\mathcal{X}$. Furthermore, such an extension is unique up to twisting by components of the special fiber, i.e. is of the form $\mathcal{L} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_\mathcal{X}(C_i)$ after fixing one extension $\mathcal{L}$.*

**Proof.** To see the existence of an extension of $L$ to $\mathcal{X}$, represent $L$ by some Weil Divisor. Then the line bundle associated to the Zariski closure in $\mathcal{X}$ of this Weil divisor will give a desired extension. Because every Weil divisor on $\mathcal{X}$ can be written uniquely as a sum of horizontal and vertical divisors, we see that this extended Weil divisor is unique up to adding vertical divisors. But adding a vertical divisor exactly corresponds to twisting the associated line bundle by a component $C_i$ of the special fiber.

We will now show how chip-firing enters the picture. The idea is that associated to each line bundle $L \in \mathrm{Pic}(X)$, we will produce a divisor on the graph $G(\mathcal{X}_\kappa)$. This proceeds by first extending $L$ to $\mathcal{X}$, and then restricting $\mathcal{L}$ to each component $C_i$. Because each $C_i$ is a smooth curve, the restriction of $\mathcal{L}$ to $C_i$ has a well defined degree. This degree will be the coefficient of the vertex $v_i$ corresponding to the component $C_i$ in the divisor we are producing. We formalize this in the following definition.

**Definition 3.1** *Given a line bundle $\mathcal{L} \in Pic(\mathcal{X})$, the divisor $D(\mathcal{L}) \in Div(G(\mathcal{X}_\kappa))$ is defined by*

$$D(\mathcal{L}) := \sum_{i=1}^{k} deg(\mathcal{L}|_{C_i}) v_i,$$

*where $v_i$ is the vertex corresponding to the component $C_i$.*

If we want to extend this definition to map line bundles on $X$ to divisors on $G(\mathcal{X}_\kappa)$, the issue we run into is that the extension $\mathcal{L}$ in not unique. However, by Lemma 3.1, the extension is unique up to twisting by a component of the special fiber. Thus, in order to understand the failure of this non-uniqueness, we just need to understand how $D(\mathcal{L})$ changes when we twist $\mathcal{L}$ by some $C_i$. For this, we have the following lemma, which brings the combinatorics of chip-firing into the picture.

**Lemma 3.2** *Given a line bundle $\mathcal{L} \in Pic(\mathcal{X})$ and a component $C_i$ of $\mathcal{X}_\kappa$, the divisor*

$$Div(\mathcal{L} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_\mathcal{X}(C_i))$$

*is obtained from the divisor $Div(\mathcal{L})$ by performing a chip-firing move the vertex $v_i$.*

**Proof** It will be convenient for the proof to work in terms of Weil divisors on $\mathcal{X}$ instead of line bundles, which is allowed as $\mathcal{X}$ is regular. So suppose $\mathcal{L}$ corresponds to some divisor $D$. Then the divisor corresponding to the line bundle $\mathcal{L} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_\mathcal{X}(C_i)$ is just $D + C_i$. The intersection number of this divisor with some component $C_j$ with $i \neq j$ is just $(D, C_j) + (C_i, C_j)$. But $(C_i, C_j)$ is exactly the number of edges from $v_i$ to $v_j$ in the dual graph of the special fiber, so this number corresponds with the number of chips at vertex $v_j$ after performing a chip firing move at vertex $v_i$. It remains to show that the number of chips at vertex $v_i$ decreases by its degree. For this, we need to use the fact that we have the equality

$$\sum_{i=1}^{n} (C_i, C_j) = 0,$$

as follows from the fact that the divisor $C_1 + \cdots C_n$ is the pullback of the special point $\mathrm{Spec}(k)$ under the proper structure morphism $\pi : \mathcal{X} \to \mathrm{Spec}(R)$, and is therefore algebraically equivalent to zero as a divisor, so its intersection number with any other divisor must be zero. It follows that the number of chips at vertex $v_i$ in the chip configuration associated to the twisted line bundle is given by

$$(D, C_j) + (C_j, C_j) = (D, C_j) - \sum_{i \neq j} (C_i, C_j),$$

which is exactly the number of chips we would expect at vertex $v_i$ after firing vertex $v_i$ with initial chip configuration given by $D(\mathcal{L})$.

Lemma 3.2 suggests that we define a group $\mathrm{Pic}(G)$ associated to any graph $G$ as follows. We first consider the abelian group of all divisors on $G$, which is isomorphic to the free abelian group on a number of generators equal to the number of vertices of $G$, and we quotient out by the relation of two divisors (or equivalently chip configurations) if one can be reached from the other by a sequence of chip-firing operations (i.e. we quotient the group of all divisors on $G$ by those which are chip-fire equivalent to the zero divisor). It is worth mentioning yet another interpretation of the Picard group of a graph. Recall that given a finite graph $G$ on $n$ vertices, we can define its graph Laplacian, $\Delta_G$ as an $n \times n$ matrix, with the ith diagonal entry equal to the negative of the degree of vertex $v_i$, and the $(i, j)$ entry with $i \neq j$ equal to 1 if there is an edge between vertices $v_i$ and $v_j$, and 0 otherwise. We can then observe that if $e_i$ denotes the indicator vector with 1 in the $i$th coordinate and 0s in every other coordinate, then $\Delta_G e_i$ can be identified with the chip configuration obtained by starting at the zero chip configuration and firing vertex $v_i$. More generally, applying the operator $\Delta$ to some column vector $[a_1, \cdots a_{V(G)}]^T$ of integers, thought of as a chip configuration, will result in the column vector corresponding to the chip configuration obtained by starting at the zero chip configuration, firing vertex $v_1$ $a_1$ times, firing vertex $v_2$ $a_2$ times, and so on. Thus, we have that $\mathrm{Pic}(G)$ is the quotient of $\mathrm{Div}(G)$ by the image of the laplacian $\Delta_G$. Combining Lemma 3.1 and Lemma 3.2, we can define a map

$$\sigma : \mathrm{Pic}(X) \to \mathrm{Pic}(G(\mathcal{X}_\kappa))$$

by first extending a line bundle $L \in \text{Pic}(X)$ to $\mathcal{X}$, and then taking the associated divisor. we will refer to this map $\sigma$ as *Baker's Specialization map*, and will play the key role of relating the geometry of $X$ to the combinatorics of chip firing.

Recall from section 2 that we have a well defined notion of rank for elements on both sides of $\sigma$. The following Lemma gives a strong comparison result for how the rank changes under $\sigma$, and will be the key tool for allowing us to deduce geometric results about $X$ from the combinatorics of the dual graph.

**Lemma 3.3** *Let $L \in Pic(X)$ be a line bundle with rank $r$. Then the rank of $\sigma(L)$ is at least $r$ (i.e. rank can only jump up under $\sigma$).*

**Proof:** In the proof we let $G := G(\mathcal{X}_\kappa)$, we let $r(L)$ denote the rank of a line bundle $L$, and we let $r_G(C)$ denote the rank of a chip configuration on $G$. Also, we let $D$ be the divisor associated to the line bundle $L$, so we have $r(D) = r(L)$. Recall that by convention we defined the rank of a non-effective divisor on $G$ to be $-1$. Thus, if $L$ admits no global sections (i.e. has rank $-1$), the result is immediate. Thus, we may assume $r(L) \geq 0$. We claim then that $r(\sigma(L)) \geq 0$. Indeed, the fact that $r(L) \geq 0$ means that $D$ is linearly equivalent to some effective divisor, and because $\sigma$ preserves linear equivalence, we see that $r_G(\sigma(L)) \geq 0$.

Now assume that for some positive integer $k$ we have that $r(L) \geq k$. It suffices to show that $r_G(\sigma(D)) \geq k$. For this, it suffices to show that for all $v \in V(G)$, we have $r_G(\sigma(D) - v) \geq k - 1$. But this follows immediately from the fact that the map from $X(K) = \mathcal{X}(R) \to V(G)$ is surjective, and from the fact that $r(D - P) \geq r(D) - 1$ for all $P \in X(K)$, with equality holding for at least one $P$. (Note that the equality $X(K) = \mathcal{X}(R)$ uses that $X$ is projective.

We will also need the following lemma on comparing the degrees of the restrictions of line bundles on $\mathcal{X}$ to the generic and special fibers:

**Lemma 3.4** *Notation as above, let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Then we have the equality*

$$deg(\mathcal{L}|_X) = deg(\mathcal{L}_{\mathcal{X}|_\kappa})$$

**Proof.** Using Riemann-Roch on the smooth curve $X$, we have

$$\deg(\mathcal{L}|_X) = \mathcal{X}(\mathcal{L}_X) + g - 1$$

Because $\mathcal{X}$ is flat over $\text{Spec}(R)$, using the fact that the Euler characteristic of line bundle is constant for flat families, we have

$$\mathcal{X}(\mathcal{L}|_X) = \mathcal{X}(\mathcal{L}_{\mathcal{X}_\kappa})$$

By considering the pullback of $\mathcal{L}|_{\mathcal{X}_\kappa}$ to the normalization of the special fiber, we conclude that

$$\mathcal{X}(\mathcal{L}_{\mathcal{X}_\kappa}) = \sum_{i=1}^{k} \mathcal{X}(\mathcal{L}_{C_i}) = k - \sum_{i=1}^{k} g(C_i)$$

Because we are ultimately interested in the geometric setting, we need to understand how the specialization map behaves under base change, so we can study curves defined over an algebraically closed field. For this, we will need to leave the world of finite graphs and use the machinery of metric graphs introduced in Section 1.

Consider now an arbitrary nodal curve $C$ over the residue field $\kappa$. In our situation, we have in mind the special fiber of a strongly regular semistable model of a curve over $K$, although the arguments we give here work more generally for all nodal curves. We can similarly associate to $C$ a dual graph (although now this graph may contain loops if a component intersects itself). If $C_1, \cdots, C_n$ denote the components of $C$, $\nu_1, \cdots, \nu_k$ denote the nodal points of $C$, and $\pi : \tilde{C} \to C$ denotes the normalization of $C$, then we have an exact sequence of sheaves on $C$

$$0 \to \mathcal{O}_C \to \mathcal{O}_{\tilde{C}} \to \sum_{i=1}^{k} \mathbb{C}_{\nu_i} \to 0,$$

where $\mathbb{C}_{\nu_i}$ denotes the skyscraper sheaf at the node $\nu_i$. Taking long exact sequences, we get that

$$(i) : g(C) = \sum_{i=1}^{n} g(C_i) + [n - k + 1] = \sum_{i=1}^{n} g(C_i) + g(\Gamma_C),$$

where $\Gamma_C$ denotes the dual graph of $C$. We can now observe that the term $[n-k+1]$ is simply the genus of the dual graph $C$, as the number of vertices in the dual graph is $n$, and the number of edges is $k$ as there is one edge for each node. Thus, we see that if each component $C_i$ is a rational curve, the genus of $C$ coincides with the genus of the dual graph.

Although it will not be necessary for proving Brill Noether, we mention that arguably the most import result in the study of algebraic curves, namely the Riemann-Roch Theorem, has an analog for tropical curves. If $X$ is a smooth curve, the classical Riemann-Roch Theorem states that

$$r(D) - r(K - D) = \deg(D) - g + 1.$$

The exact same result for tropical curves holds under a suitable definition of the canonical divisor. If we define the canonical divisor of a metric graph to be $\sum_v (\deg(v) - 2)$, where the sum is over all vertices of the graph, then the Riemann Roch Theorem holds. This is a purely combinatorial result which is a direct analog of the algebro-geometric Riemann Roch. If our tropical curve (i.e. metric graph) arises as the dual graph of a nodal curve, all of whose components are rational, then we can take the canonical divisor to be $\sum_{i=1}^{k} (\deg(v_i) - 2)$, where the sum is over vertices corresponding to each connected component on the curve. However, if the curve has components which are not rational, we need to adjust the canonical divisor, essentially for the same reason that we do not have the equality $g(C) = g(\Gamma_C)$ when some of the the components of $C$ have positive genus, as demonstrated in $(i)$. The correction we need to make to the canonical divisor, following [AC13] is

$$\sum_{i=1}^{k} (\deg(v_i) - 2) + \sum_{i=1}^{k} 2g(C_i)v_i,$$

where $C_i$ denotes the curve component corresponding to the vertex $v_i$ in the dual graph. Under this definition, the canonical divisor of the nodal curve indeed specializes to the above divisor.

# 4 Non-existence of Special Divisors

In this section, we will show that, for a smooth projective $X$ over $K$, is the Brill Noether number $\rho(g, r, d) < 0$, then there is no linear series of rank $r$ and degree $d$ on $X$.

In order to show the non-existence of certain divisors on curves, we will need to upper bound the rank of certain line bundles. By the specialization lemma, it will suffice to upper bound the rank of the associated divisor on the dual graph. The key to obtaining upper bounds for the ranks of these divisors is coming up with certain distinguished representatives of divisor classes in $\mathrm{Pic}(G)$. Thus, our first goal in this section will be to construct canonical representatives of each divisor class in $\mathrm{Pic}(G)$ for a finite graph $G$, which we think of as a dual graph of some curve as in Section 3. More precisely, for each divisor class $D$ on $G$ and each vertex $v_0 \in G$, we will construct a unique representative of $D$ which is so called $v_0$-*reduced*. The precise definition is as follows

**Definition 4.1** *For a divisor class $D$ on $G$ and a vertex $v_0 \in G$, we say that a divisor $D'$ is a $v_0$-reduced representative of $D$ if $D$ and $D'$ are linearly equivalent, $D$ is effective away from $v_0$, and firing any non-empty collection of vertices $A \subseteq V(G) - \{v_0\}$ results in a divisor which is not effective away from $v_0$.*

The following Lemma is why this notion is useful

**Lemma 4.1** *For every divisor $D$ on $G$ and vertex $v_0 \in G$, there is a $v_0$-reduced representative of $D$.*

**Proof** To begin with, we observe that $D$ is linearly equivalent to some divisor which is effective away from $v_0$. Intuitively, we may just fire $v_0$ a sufficient number of times to make all neighbors of $v_0$ have an arbitrary number of chips, and then fire those vertices, and so on, until every vertex other than $v_0$ has a positive number of chips. A more rigorous proof of this fact uses that the Jacobian of $G$ (degree 0 divisors on $G$ modulo chip-firing equivalence), is finite. In particular, for every vertex $v \in G$, we can find a positive integer $m_v$ so that $m_v[v - v_0]$ is zero in the Jacobian. Thus, starting with any divisor $D$, we may add a sufficiently large multiple of $m_v(v-v_0)$ to $D$ so arrive at a linearly equivalent divisor which is effective away from $v$.

So now replace $D$ with a linearly equivalent divisor which is effective away from $v_0$. We first aim to show that $D$ is equivalent to some $v_0$ reduced divisor. The idea is to set up an ordering on divisors so that while $D$ is not $v_0$ reduced, firing the set $A$ violating the non-reducedness of the divisor results in a strictly larger divisor. To do this, we first choose an ordering of the vertices so that every vertex other than $v_0$ has a neighbor which precedes it in this ordering. This can be achieved, for instance, by performing a breadth first search of the graph starting at $v_0$ (Recall our conventions that $G$ is connected). Now we just take the induced lexicographic ordering on divisors. Suppose there is some non-empty collection of vertices which violates the definition of $D$ being $v_0$ reduced, so that firing all vertices of $A$ still results in a divisor which is effective away from $v_0$. Because every element of $A$ has a neighbor which precedes it, there is a smallest such neighbor over elements of $A$, and firing all vertices of $A$ gives a positive number of chips to this neighbor. Thus, the result of firing every element of $A$ results in a strictly larger divisor. Because chip firing preserves degree, it suffices to show that there are only finitely many divisors which are effective away from $v_0$ and are greater than or equal to a given divisor $D$. Say $\sum_{i=0}^{k} a_i v_i$ is such a divisor. Then each $a_i \geq 0$ for $i \neq 0$, and because $v_0$ is first in the ordering, the condition that the divisor is at least $D$ gives a lower bound on $a_0$. Thus we have lower bounds on all $a_i$ and we know that their sum is fixed, which immediately implies that there are only finitely many options for the $a_i$, thus finishing the proof.

**Remark 4.1** Note that the proof of the above lemma actually gives an algorithmic way to produce $v_0$ reduced divisors on some metric graph $\Gamma$. Indeed, starting with some divisor, we may fire $v_0$ a sufficient number of times to make all vertices away from $v_0$ effective. However, at this point, all vertices away from $v_0$ may have a very large number of chips on them, so it is unlikely that the divisor will satisfy the second part of the definition of a reduced divisor. Intuitively, in order this satisfy this part of the definition, we want to move as many chips as possible as close to $v_0$ is possible - this way, when we fire some subset $A \subseteq V(G) - \{v_0\}$ of vertices, as many chips as possible move to $v_0$, which will make it more likely the divisor to no longer be effective away from $v_0$. To use this observation, we use the ordering on the vertices used in the proof, and start a breadth first search of the graph starting at $v_0$. We mark $v_0$ as red. When we reach a new vertex in the Breath first search, we continue the search at this new vertex iff that the number of chips at this vertex is smaller than the number of edges incident to this vertex which we have explored so far. After this procedure is over, let $A$ be the set of unreached vertices. If $A$ is empty, then the divisor is already $v_0$ reduced. Otherwise, we fire all vertices of $A$ and continue. This procedure terminates and the resulting divisor is $v_0$ reduced. To see that this process terminates, we just use the same observation that the result of firing vertices of $A$ results in a strictly larger divisor. This algorithm is referred to as *Dhar's burning algorithm.* Let us give an example to see the algorithm in action.

**Example 4.1** We consider a graph formed by two vertices, $v_0$ and $v_0'$, joined together by two edges. Consider a divisor $D = av_0 + bv_0'$, and we ask what condition on $a, b$ are there for this divisor to be $v_0$ reduced. (The notation here is to be consistent with later notation in this section). We first can fire $v$ a sufficient number of times to make sure that $b$ is positive, so we can assume without loss of generality that $b$ is positive. We now run Dhar's burning algorithm from $v_0$. Because there are 2 edges incident on $v_0'$, we see that the algorithm will terminate iff $b < 2$, otherwise we fire $b$ and continue the process. Thus, our divisor $D$ is $v_0$ reduced iff $b \in \{0, 1\}$. It can be shown using Riemann Roch that the divisor $v_0 + v_0'$ has rank 1. Because it has degree 2, we see that this graph is "hyperelliptic" in a certain sense. We will later generalize this example to chains of $g$ loops.

We can take this example further and now consider $G$ as a metric graph. That is, our divisor can have support anywhere on the loop, not necessarily the two vertices. If we run Dhar's burning algorithm starting at $v_0$ again, we see that the algorithm will terminate after one step iff either there is zero or one vertices placed anywhere other than $v_0'$. Indeed, every vertex will have two incoming edges, so there are can at most one chip at each vertex. Furthermore, if there are at least two vertices other than $v_0$ with chips placed on them, then the edge between those chips which does not include $v_0$ will cause the Algorithm not to terminate after one step, so the original divisor could not have been $v_0$ reduced.

We will prove that if the Brill Noether Number for a generic chain of $g$-loops is non-negative, then this generic chain of $g$-loops cannot have any divisors of degree $d$ and rank $r$. By Specialization, we will have shown that if a curve this graph as its dual graph, then it itself cannot have any divisors of degree $d$ and rank $r$.

Recall that $W_r^d$ denote the Brill Noether Variety. By semicontinuity, it suffices to show the existence of a single curve $X$ with $\dim(W_r^d) \leq \rho(g,r,d)$.

From now on, we take for out graph a generic chain of $g$ loops, and we want to characterize $v_0$ reduced divisors on this graph. We let $\gamma$ denote a chain of $g$ loops with generic edge lengths, joined by $g-1$ bridges. Suppose the bridges are labeled $\gamma_1, \cdots, \gamma_{g-1}$, and $\gamma_i$ connects $v_i$ to $v_{i+1}'$. We also let $v_0$ and $v_g'$ denote the leftmost and rightmost vertices, respectively. Thus every loop has vertices given by $v_i$ and $v_{i+1}'$ for some $i$. As $\Gamma$ is a metric graph, we have edge lengths associate to each loop, and we assume that the top arc of the path from $v_i$ to $v_{i+1}'$ has length $\ell_i$, and the bottom arc has length $m_i$, with the top arc being oriented from $v_i$ to $v_{i+1}'$ clockwise, and the bottom arc being oriented from $v_{i+1}'$ to $v_i$, also clockwise. Given this very explicit structure on our graph, we can actually characterize completely the $v_i$-reduced divisors for any $i$. The following characterization will be key in proving Brill Noether

**Theorem 4.2** *Let $g \geq 1$ be an integer, and consider the metric graph $\Gamma_g$, and fix a vertex $v_0$. Then the $v_0$ reduced divisors on $\Gamma_g$ are exactly those divisors which have at most one chip on each loop not incident to $v_0$.*

**Proof** The prove is essentially a consquence of running Dhar's burning algorithm from the vertex $v_0$. We essentially just have to ask when the algorithm will not temrinate after one step.

**Definition 4.2** *Let $r > 0$ be an integer. A Lingering Lattice Path in $\mathbb{Z}^r$ is a sequence of points $p_1, p_2, \cdots$ in this lattice such that $P_{i+1} - P_i$ is either the zero vector, a standard basis vector, or the veector $(-1, -1, \cdots, -1)$.*

The idea is that the sequence of points either stays the same or moves on the lattice, but we are also allowed to shift the entire lattice down in all $r$ directions at any step. Given any $v_0$ reduced divisor $D$ on $G$, we may associate to it a lingering lattice path in $\mathbb{Z}^r$ by the following construction. (Note that $r$ here is arbitrary with respect to $G$ - i.e. we do not assume that $r$ is the rank of $D$).

We can now prove the following Theorem, following [CDPR12], whose statement looks extremely similar to Theorem 1.1:

**Theorem 4.3** *Let $\Gamma_g$ be a generic chain of $g$ loops, and let $r, d$ be non-negative integers. Then*
*(i): If $\rho(g,r,d) < 0$, then $\Gamma_g$ admits no divisors of degree $d$ and rank at least $r$.*
*(ii): If $\rho(g,r,d) \geq 0$, then $\Gamma_g$ admits no divisors of degree $d$ and rank at least $r$ satisfying $D \geq (r+\rho+1)v_0$.*

We have now developed enough machinery to give quick proof part (ii) of Theorem 1.1. Indeed, we just need to start with a well chosen smooth curve $X$ of genus $g$ defined over some discretely valued field which is a subfield of $\mathbb{C}$, and then consider its base change to $\mathbb{C}$. For instance, we may consider the subfield

$$\mathbb{Q}((t)) \hookrightarrow \mathbb{C}.$$

Such an embedding must exist because the algebraic closure of $\mathbb{Q}((t))$ is isomorphic to $\mathbb{C}$ as any two algebraically closed fields of the same characteristic are isomorphic (note that this isomorphism is an isomorphism of fields and does not respect the natural topologies on both sides). Now we know choose

$X$ such that the dual graph of its special fiber is $\Gamma_g$. Whether or not $X$ admits a divisor of degree $d$ and rank $r$ does not change whether we view $X$ as being defined over $\mathbb{Q}(t))$ or over $\mathbb{C}$. Suppose for contradiction $X$ had a divisor of degree $d$ and rank $r$. Then its image under Baker's Specialization map must be a divisor of degree $d$, and by the specialization Lemma, would have rank at least $r$. But this is impossible by part (i) of Theorem 4.3, completing the proof.

We will now give a proof of the upper bound $\dim(W_r^d(X)) \leq \rho(g,r,d)$ when $\rho(g,r,d) \geq 0$ and $X$ is general. Again, because the dimension of a variety does not change under base change of the field of definition, it is enough to prove the result when $X$ is defined over a discretely valued field. In the proof that follows, we will be constructing certain special divisors. These divisors may not be defined over $K$, but we can always pass to a finite extension $K'/K$ over which these divisors are defined. Under this base change, the edges in the associated metric graph of $X$ will be split into $e$ equal length segments, where $e$ is the ramification index of the field extension. But such a subdivision is still equivalent to a generic chain of $g$ loops, meaning we will still be able to apply the results of Theorem 4.3, so we may assume from the start that all of the special divisors which we construct in the proof are defined over $K$.

**Proof of Thm 1.1 (i)**: Suppose $D$ is a divisor of degree $d$ and rank $r$ on $X$. This data is equivalent to giving some embedding of $X$ into $\mathbb{P}^r$ and giving a hyperplane in $\mathbb{P}^r$ whose intersection with $X$ is exactly the divisor $D$. To get a handle on the dimension of $W_r^d(X)$, we first make the simple observation that through any $r$ points in $\mathbb{P}^r$, we can find some hyperplane passing through these $r$ points. Thus, because every such hyperplane section is linearly equivalent to $D$, we see that any effective divisor of degree $r$ on $X$ is contained in a divisor linearly equivalent to $D$. To bring $W_r^d(X)$ into the picture, we can now considering varying $D$ in an algebraic family of divisors of degree $d$ and rank at least $r$ (i.e. we consider $D$ as a point in some one dimensional subvariety of $W_r^d$). Now, we can guarantee that given any $r+1$ points on $X$, we can find some divisor in our one dimensional family which contains these $r+1$ points.

Proceeding in a similar manner, we see that any effective divisor of rank $r + \dim(W_r^d)$ is contained in a divisor which is equivalent to a divisor in $W_r^d(X)$. Suppose now for contradiction that $\dim(W_r^d) > \rho(g,r,d)$. Then every effective divisor of rank $r + 1 + \rho(g,r,d)$ would be contained in a divisor equivalent to a divisor of degree $d$ and rank at least $r$. But this contradicts part (ii) of Theorem 4.3 as follows. Given $v_0$ in the dual graph of $X$, we can find a point in $p_0 \in X$ which specializes to this point. Thus the divisor $(r+\rho(g,r,d)+1)p_0$ has image $(r+\rho(g,r,d)+1)v_0$ under Baker's Specialization map. But if $(r+\rho(g,r,d)+1)p_0$ were contained in an effective divisor of degree $d$ and rank $r$, then so would its image $(r+\rho(g,r,d)+1)v_0$, as the rank of divisor can only increase under specialization. (Note that we are implicitly using that the specialization map respects linear equivalence of divisors). This contradicts part (ii) of Theorem 4.3, so we must have that $\dim(W_r^d) \leq \rho(g,r,d)$.

# 5 Further Directions

We

# References


[AC13] O Amini and L. Caporaso. Riemann-roch theory for weighted graphs and tropical curves. *Advances in Mathematics*, 240(2013):1–23, 2013.

[Bak08] Matthew Baker. Specialization of linear systems from curves to graphs. *Algebra Number Theory*, 2(6):613–653, 2008.

[CDPR12] F. Cools, J. Draisma, S. Payne, and E. Robeva. A tropical proof of the brill-noether theorem. *Advances in Mathematics*, 230(2012):759–776, 2012.