Matthew Prem
PHYS 265
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Mine crafting report

I. Introduction

One of our company's most prized assets is our ultra-deep gold mine, a 5 meter diameter borehole providing access to the rich ore veins between 3.5 and 4.5 kilometers underground. However, to this day, no accurate measurements of our mine depth have been made. To determine this information, which is necessary to inform how deep our company's next generation mineshafts need to be, chief engineer Roberts has suggested deploying our company's corporate sky-divers and timing how long their fall to the bottom of the shaft is. In this report, I analyzed how including the impacts of changing gravitational acceleration, drag, and the Coriolis effect would change the fall time and therefore the determined depth. Additionally, I investigated whether or not it would be physically possible for a human to fall down the mineshaft to the bottom without hitting the wall of the mineshaft. Further, our company's Grand Plan™ involves the construction of mine shafts through the poles of the Earth and Moon to harvest the concentrated metals in their cores, so I calculated how long it would take to fall through these proposed tunnels with varying density distributions. Finally, I discuss the shortcomings of the analyzed approaches and the future work needed to refine my numerical values.

II. Calculation of Fall Time

The simplest method of calculating how long it takes to fall a vertical distance comes from kinematics and assumes that the only force is a constant vertical gravitational force. This naïve equation and its associated equations are as follows:

$$t = \sqrt{\frac{2h}{g}} \qquad\qquad \frac{d^2y}{dt^2} = -g_0 \tag{1}$$

Both this naïve theoretical equation and the numerical integration of the associated differential equation say that it should take 28.6 seconds to fall a 4 kilometer vertical mineshaft. The value of g_0 used was calculated with $g_0 = \frac{GM_\oplus}{R_\oplus^2}$ so the more complicated density distributions we analyze later simplify to this in the constant density case. Since the Earth is not a point mass, we then assume the Earth has a constant density which yields the following differential equation of motion:

$$\frac{d^2y}{dt^2} = -g_0 \frac{y}{R_\oplus} \tag{2}$$

Numerically integrating this also yields a fall time of 28.6 seconds for a 4 kilometer mineshaft which makes sense, since 4 kilometers is less than 0.1% of the Earth's radius. Finally, when taking drag into account, the associated differential equation is:

$$\frac{d^2y}{dt^2} = -g_0 \frac{r}{R_\oplus} + \alpha \left| \frac{dy}{dt} \right|^\gamma = -g_0 \frac{r}{R_\oplus} - \hat{v}_y |\vec{v}|^\gamma \tag{3}$$

Where γ is equal to 2, stating that drag is proportional to speed squared which is the case for most objects in air and α=0.0039 so that the terminal speed of the skydivers is 50 meters per second, similar to their terminal speed in open air. The fall time taking these effects into account increases to 83.4 seconds.

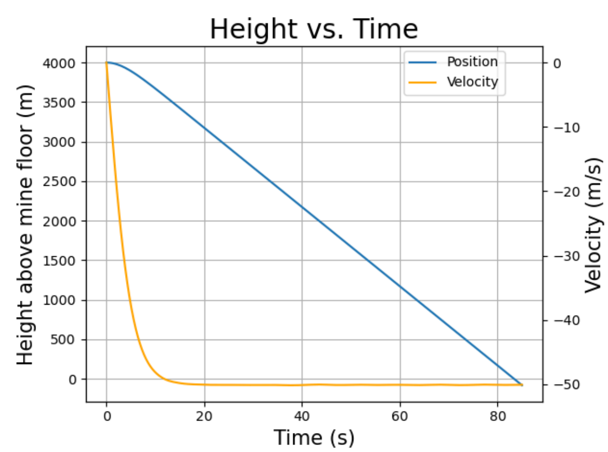

Figure 1: A plot showing the depth and velocity of a skydiver falling down a 4 km mineshaft taking drag and linear gravity

into account. Note how the velocity remains constant after the 20 second mark.

III. Feasibility of Depth Measurement

So far, in my calculations of the fall times, I assumed that the the effects of the rotating Earth, namely the Coriolis effect were negligible, but since our mineshaft is quite narrow relative to its depth (5 meters in diameter compared to roughly 4 kilometers deep), I investigated whether it would cause our skydivers to meet an untimely demise on the walls of the shaft before reaching the bottom. Adding the Coriolis effect to our differential equation are get:

$$\frac{d^2\vec{x}}{dt^2} = -g_0 \frac{r}{R_\oplus}\hat{r} - \alpha\hat{v}|\vec{v}|^\gamma - 2(\vec{\Omega} \times \vec{v}) \quad [4]$$

At first, I ignored the drag force on the skydivers to simplify calculations and found that they would once again reach the bottom of the mine in 28.6 seconds, however, I found that the skydivers, when released from the center of the mineshaft, would collide with the side of the mineshaft due to the coriolis force 21.9 seconds into the fall and 2352.9 meters underground. Since we deeply value human life here at Generic Mine Corp®, and the collisions would introduce uncertainty in when they reach the bottom, this is not a feasible way of determining the depth of the mineshaft.

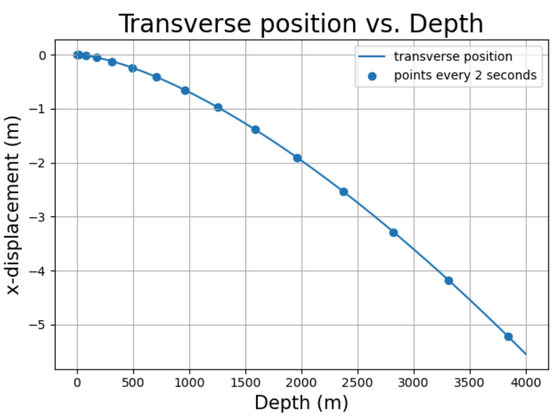

Figure 2: Plot of our skydiver's horizontal displacement as a function of their depth in the mineshaft assuming no drag. Since our mineshaft has a diameter of 5 meters and our skydivers jump from the center of the shaft, they only have to have an x-displacement of -2.5 meters before they crash into the wall.

When taking the effect of drag into account, our skydivers would still hit the mineshaft walls, but it would take them 29.6 seconds to do so during which time they would have fallen 1304.0 meters and they would only reach the bottom of the mineshaft after 83.4 seconds. It is possible that our skydivers could be trained to keep themselves in the center of the mineshaft as they are falling, but that may cause some form of lift that would have to be taken into account. More research is needed in this area, but until then, we can discard this as a feasible way of finding the depth of our mine.

IV. Crossing Times

A major part of our company's Grand Plan™ is a tunnel through the entire Earth to ensure quick access to the metals in the Earth's core. To find exactly how much more efficient this scenario is at extracting minerals, it is important to find the time it takes an object to fall from one side of the Earth or Moon to the other side. When assuming that these bodies have a constant density, we have our mineshaft in a vacuum to avoid drag, and that we build them through the poles of the body to avoid the Coriolis effect, we can use the differential equation described in equation [2] to both symbolically and numerically solve for how our equipment will fall through the planet. This linear differential equation has the solution:

$$y(t) = Y_0\cos(\omega t) + V_0\sin(\omega t) \qquad \omega = \sqrt{\frac{g_0}{R}} \quad [5]$$

Assuming we drop our equipment from rest, we find that the time needed to reach the center of the planet is given by the following equation:

$$t_{center} = \sqrt{\frac{3\pi}{16G\rho}} \quad [6]$$

Due to symmetry, the time it takes to fall from one side of the planet to the other is twice the time it takes to fall from the surface. This shows that under these assumptions, the crossing time for a planet is inversely proportional to the square root of its density. For the Earth, the theoretical crossing time is 2534.7 seconds and the theoretical crossing time of the Moon is 3250.2 seconds. When our equipment crosses the center of the planets, they are moving at high speeds, 7906.0 m/s for the Earth and 1680.0 m/s for the Moon. The other part of the efficiency calculation is finding how long the alternative method of equipment transportation is, which is putting our equipment in orbit and sending them to the other side of the Earth. The orbital speed v_o and orbital period P are given by the following equations:

$$v_o = \sqrt{\frac{2GM}{R}} \qquad P = \frac{2\pi R}{v_o} \qquad [7]$$

For the Earth, the orbital period is 5069.4 seconds, twice the time it takes to fall from one side of the Earth to the other. However, since one full orbit period corresponds to getting back to where you started, it takes the same amount of time to orbit from one side of the planet to the other as it does to fall through the planet meaning we don't experience any increase in efficiency aside from the fact that we actually transfer our equipment through the mine they are designed to be in.

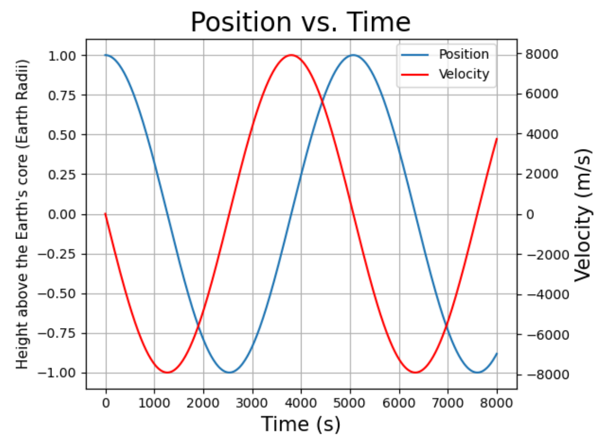

Figure 3: Plot showing the height above the Earth's core assuming constant density, no drag, and no Coriolis effect. Note the sinusoidal nature of position and velocity and how the velocity has a 90 degree phase shift relative to position.

So far in this report, we have assumed that the Earth has constant density, but geology tells us that the density of a planet increases towards its center and can be modeled using the following function family: $\rho(r) = \rho_n(1 - \frac{r^2}{R^2})^n$ which can be numerically integrated to find the mass interior to a certain radius. ρ_n is a normalization constant which is set to ensure that the total mass of the planet is the mass we expect. A constant density has $n = 0$ and when applied to the Earth results in a numerically computed crossing time of 2534.5 seconds while a density distribution of $n = 9$ results in a numerically computed crossing time of 1887.7 seconds. This is smaller due to the greater acceleration throughout more of the time spent in the planet.

V. Discussion and Future Work

In this investigation, I made several simplifying assumptions which are not completely reflected in reality. For example, we assumed that the Earth has a radially symmetric density and is a perfect sphere. In reality, the Earth is an oblate spheroid with regions that are measurable denser than others. Additionally, any very deep hole in the Earth would have some level of outgassing or air leakage resulting in a non-negligible drag with likely different normalization and scaling constants than we assumed by comparison to regular surface skydiving. One partially important aspect of the Earth we should take into account for future work is the different densities of the different layers of the Earth. Due to the density separation that occurred when the Earth first formed, the inner and outer core of the Earth have large and relatively sudden jumps in density which aren't taken into account by our simple density scaling model.