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I Introduction

As we all know, the current administration has made getting American boots on the surface of the moon a priority. To help accomplish this, our engineers have developed the Saturn V rocket, the most powerful and capable rocket ever built. Before we launch the first crewed mission on Apollo 1, it is imperative that all parties involved have a working understanding of the gravitational environment of the Earth-moon system and the capabilities of the first stage of the Saturn V rocket.

With new advancements in computer technology such as the computational package known as *numpy* and the visualization package *matplotlib*, we can now quickly produce visualizations which can be directly used in the planning of future missions.

II The Gravitational Potential of the Earth-moon system

One of the most important concepts in orbital mechanics is gravitational potential, described by the following equation:

$$\Phi(\vec{r}) = -\frac{GM}{r} \quad [1]$$

Where Φ is the gravitational potential and r is the distance between the position where you are evaluating the function and the object providing the potential (e.g. the Earth or the moon). Gravitational potential is a particularly useful way of describing a the Earth-moon system as the potential of the system as a whole is simply the sum of the potentials of the Earth and moon

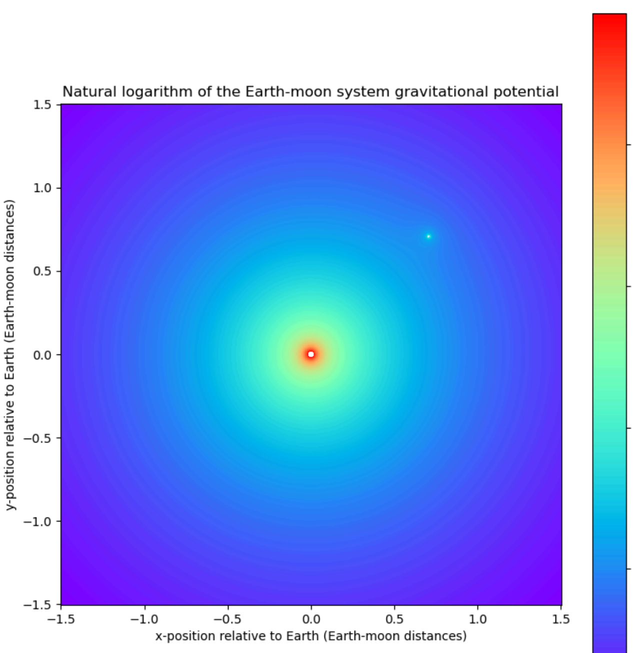

Figure 1: The gravitational potential of points in a cross-section of the Earth-moon system. The cross-section of the Earth is displayed as white.

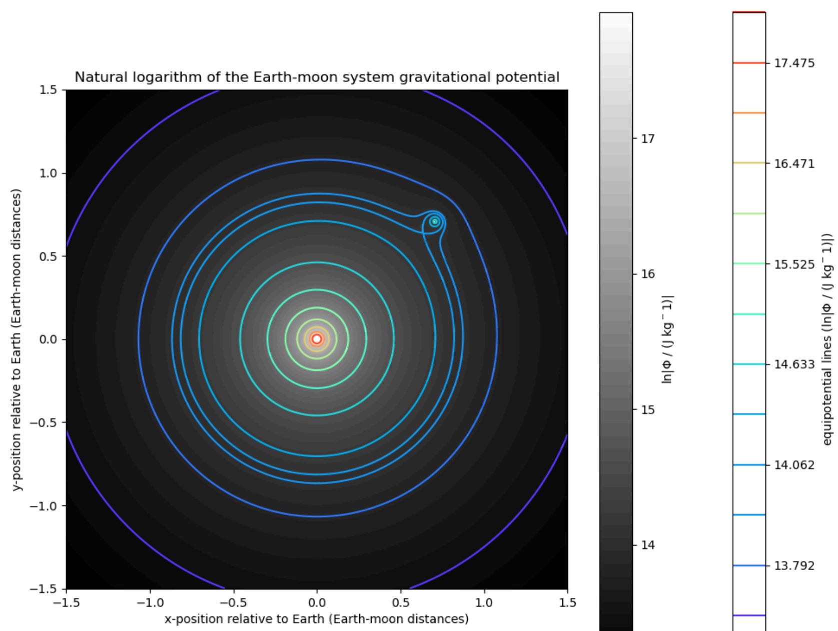

Figure 2: The gravitational potential of points in a cross-section of the Earth-moon system with equipotential lines.

In figures 1 and 2 numpy was used to evaluate the gravitational potential at many points in the Earth-moon system. These values were then mapped to colors and displayed in a figure using the package matplotlib. In figure 2, lines of equipotential were drawn, similar to an elevation map. Generally, the closer the lines are together, the greater the gravitational force in that region, but two additional lines were drawn at potentials of e^{14} joules per kilogram and $e^{14.062}$ Joules per kilogram to showcase how the L1 point of the Earth-moon system and how the Moon's presence distorts the equipotential lines of the Earth's gravitational field. By convention, gravitational potential is negative, meaning that gravitational potential energy (which is gravitational potential times the mass of the object you are looking at) decreases the closer you get to an object, but for purposes of display, the absolute value of the potential was used.

III The Gravitational Force of the Earth-moon system

While gravitational potential can be very helpful for finding how much energy needs to be added to a system, forces can often be more useful for finding how spacecraft will move through a system. The formula for gravitational force is as follows:

$$\vec{F}(\vec{r}) = -\frac{GMm}{r^2}\hat{r} \qquad [2]$$

Where M is the mass of the more massive object, m is the mass of the spacecraft, G is the gravitational constant, r is the distance between the spacecraft and the massive object, and $\hat{r}$ is the direction from the massive object to the spacecraft. What this tells us is that planets and moons attract other objects, with more massive objects being attracted more strongly and more distant objects being attracted more weakly. Force, unlike gravitational potential, is a vector quantity, meaning it has a direction in addition to a magnitude. This means that to visualize it, we have to draw some form of arrows.

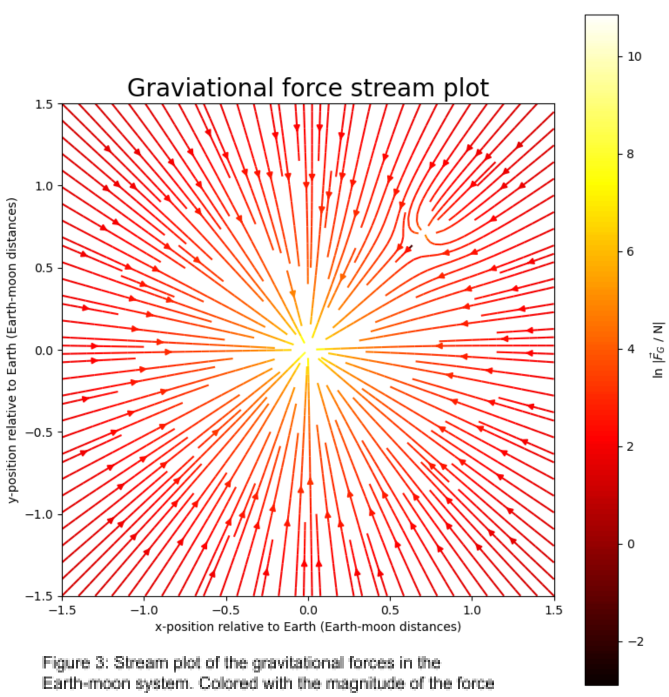

Figure 3: Stream plot of the gravitational forces in the Earth-moon system. Colored with the magnitude of the force vector.

In Figure 3, we see many of the arrows of the gravitational force connected together to form what is known as a stream plot, named since a spacecraft in this gravitational field will tend to flow along the stream lines. The strength of the force is indicated with the color of the streamlines. Notice how the direction of the force lines changes around the moon. There is a special point in the Earth-moon system called L1 where if we get our spacecraft past, it will stop being pulled towards the Earth and start being pulled towards the moon, potentially saving us fuel required to get the lunar lander to the moon.

IV Projected Performance of the Saturn V First Stage

All rockets work using the conservation of momentum; essentially, rockets propel themselves forward by quickly throwing burned exhaust backwards. One of the most important concepts in orbital mechanics is Δv (pronounced: Delta V) which is how much a rocket can change its velocity. When launching from the ground like the first stage of the Saturn V rocket will, the Δv can be described using the Tsiolkovsky rocket equation:

$$\Delta v\ (t)\ = v_e \ln(\frac{m_0}{m(t)})\ -\ gt \qquad [3]$$

Where v_e is the speed at which gas exits the rocket nozzle, m_0 is the mass of the fully fueled rocket including payload, g is the gravitational acceleration at the surface of the Earth, and t is the time you have let the rocket burn for. $m(t)$ is the mass of the rocket at a given time, and is given by the following equation:

$$m(t)\ = m_0 - \dot{m}t \qquad [4]$$

Where $\dot{m}$ is the rate at which the rocket is going through fuel. These equations combined mean that the longer the rocket burns, the lighter it gets, and the faster it gains Δv.

The total amount of time the Saturn V first stage can burn is given by the equation

$$T\ =\ \frac{m_0 - m_f}{\dot{m}} \qquad [5]$$

For the Saturn V first stage, $m_0\ =\ 2.8\ \times\ 10^6$ kg, $m_f\ =\ 7.5\ \times\ 10^5$ kg, and $\dot{m}\ =\ 1.3\ \times\ 10^4$ kg/s which means that we can run the engines for 158 seconds. The height that the rocket reaches after completing its first stage burn can be found by integrating the Δv equation from 0 to 158 seconds using a helpful code from a package known as *scipy* from which we find that the first stage will be approximately 74 kilometers above the ground when its engines cut off.

V Discussion and Future Work

While the calculations for burn time and height above the ground at first stage engine cutoff align very well with the values found from the first test of the Saturn V (burn time of 158 sec vs 160 sec and height of 74 km vs 70 km), there were a few assumptions made to keep the calculations easy. For example, we assumed that the mass flow rate was constant, which isn't necessarily true when the rocket throttles down for max-Q which should mean our calculated value is an under-estimate. Additionally, we did not simulate the impact of aerodynamic pressure on the rocket during ascent which would mean that our calculated height is an overestimate of the realistic value.

In our analysis, we only took the first stage into account and assumed that gravitational acceleration was constant. In reality, there would be multiple stages and gravitational acceleration would decrease slightly with height meaning that we could climb higher, potentially all the way to L1 and the moon. If funding for NASA is continued, we could very well have a practical flight plant to get American boots on lunar soil before the decade is out.