Week 6

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Overview

- Today we will begin studying search models
- Search models are used in situations where it takes time to find something, and this time is costly
- The things that prevent trade from being executed "smoothly" are referred to as *frictions*, and can explain why some markets might not clear
- With respect to job search, *frictions* (like the time it takes to find a job) can explain why *in equilibrium*
  - some agents would like to work by aren't currently (involuntary unemployment)
  - some agents turn down job offers (reservation wages)
  - otherwise similar people have different wages (wage dispersion)

Search Theory

- For this class we will focus on **labor search**
- The basic models can be cast in discrete time or continuous time
- We will begin with a simple model of **random search** with an **exogenous wage distribution** in discrete time
  - random search (as opposed to directed search) means that unemployed workers are equally likely to locate any job opening
  - when a worker finds a job, a wage is drawn from some exogenous distribution of wages; other models take wage determination more seriously (e.g. bargaining, posting, etc.)
- One key object that we are looking for in this model is the *reservation wage*, the wage at which an agent will accept a job offer

Preliminaries

- First we'll start with some characteristics of **nonnegative random variables**
- Let p be a nonnegative random variable with cdf $F(P) = P(p \leq P)$ such that $F(0) = 0$ (i.e. p is nonnegative), $F(\infty) = 1$, and F nondecreasing and continuous from the right
- Assume upper bound $B < \infty$ such that $F(B) = 1$ so that p is bounded with probability 1
- Then the mean of p is defined as

$$\mathbb{E}[p] = \int_0^B p \; dF(p)$$

Integration by Parts

Recall the integration by parts formula:

$$\int u \; dv = uv - \int v \; du$$

Now let $u = 1 - F(p)$ and $v = p$ and we have

$$\begin{aligned}
\int_0^B p \; dF(p) &= pF(p)\Big|_0^B - \int_0^B F(p)dp \\
&= BF(B) - \cancel{0F(0)} - \int_0^B F(p)dp \\
&= B - \int_0^B F(p)dp \\
&= \int_0^B 1dp - \int_0^B F(p)dp \\
&= \int_0^B [1 - F(p)]dp
\end{aligned}$$

Setup

- Now let's look at an example
- We will consider the case of an unemployed worker who faces a probability distribution of wage offers from which a limited number of offers are drawn each period
- Given her perception of the distribution of offers, the worker must devise a strategy for deciding when to accept an offer
- It turns out that this type of model is an *optimal stopping rule* problem, where the optimal stopping rule is to accept a job if the wage offereing is at least as great as the worker's reservation wage

A Basic Model of Job Search

Consider an individual searching for a job in discrete time, taking market conditions as given. She seeks to maximize

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t x_t$$

where $x_t = w$ if employed and $x_t = b > 0$ if unemployed. We can interpret w as the wage, but more broadly it could be interpreted some notion of desirability of the job . b is often referred to as unemployment benefits (like UI), but it can also include the value of leisure / home production.

Each period, an unemployed agent samples once from an exogenous wage distribution $F(w)$ with a support $(\underline{w}, \overline{w})$. The agent can either accept or reject the offer. If the agent accepts, she is employed for the rest of time at a wage w. If she rejects, she remains unemployed for another period.

We can write the Bellman equation for being employed at some wage w as

$$V(w) = w + \beta V(w) \qquad \Longrightarrow \qquad V(w) = \frac{w}{1-\beta}.$$

An unemployed worker's Bellman is given by

$$U = b + \beta \int_{\underline{w}}^{\overline{w}} max\left\{V(w), U\right\} dF(w).$$

Because $V(w)$ is (strictly) increasing in w, there will exist a *reservation wage*, denoted w_R, such that the agent will accept an offer only if $w \geq w_R$.

Note

Reservation Wage

The reservation wage is defined by the w_R that makes a worker *indifferent* between being employed or staying unemployed. That is, $V(w_R) = U$.

$$\underbrace{\frac{w_R}{1-\beta}}_{V(w_R)} = \underbrace{b + \beta \int_{\underline{w}}^{\overline{w}} max\{V(w), U\} dF(w)}_{U}$$

$$\frac{w_R}{1-\beta} = b + \beta \int_{\underline{w}}^{\overline{w}} \max\left\{ \underbrace{\frac{w}{1-\beta}}_{V(w)}, \underbrace{\frac{w_R}{1-\beta}}_{U} \right\} dF(w)$$

Now all that's left to do is simplify. There are a few different ways of representing the reservation wage, each having it's own purpose.

$$w_R = (1-\beta)b + \beta \int_{\underline{w}}^{\overline{w}} \max\{w, w_R\} dF(w)$$

Now subtract βw_R from both sides and rewrite.

$$(1-\beta)w_R = (1-\beta)b + \beta \int_{\underline{w}}^{\overline{w}} \max\{w - w_R, 0\} dF(w)$$

$$w_R = b + \frac{\beta}{1-\beta} \int_{\underline{w}}^{\overline{w}} \max\{w - w_R, 0\} dF(w)$$

$$= b + \frac{\beta}{1-\beta} \int_{w_R}^{\overline{w}} (w - w_R) dF(w)$$

Integration by Parts

Another important way of writing the reservation wage utilizes *integration by parts* to rewrite the integral. Recall again the integration by parts formula:

$$\int u dv = uv - \int v du.$$

Now, let's apply it to the above integral in the same way we did in the "Preliminaries" section.

$$\int_{w_R}^{\overline{w}} (w - w_R) dF(w) \qquad \Longrightarrow \qquad \begin{array}{ll} u = w - w_R & v = F(w) \\ du = dw & dv = dF(w) \end{array}$$

$$
\begin{aligned}
\int_{w_R}^{\overline{w}} (w - w_R) dF(w) &= (w - w_R)F(w)\Big|_{w_R}^{\overline{w}} - \int_{w_R}^{\overline{w}} F(w)dw \\
&= (\overline{w} - w_R)F(\overline{w}) - \cancel{(w_R - w_R)F(w_R)} - \int_{w_R}^{\overline{w}} F(w)dw \\
&= (\overline{w} - w_R) - \int_{w_R}^{\overline{w}} F(w)dw \\
&= \int_{w_R}^{\overline{w}} 1dw - \int_{w_R}^{\overline{w}} F(w)dw \\
&= \int_{w_R}^{\overline{w}} [1 - F(w)]dw
\end{aligned}
$$

Thus we can rewrite the reservation wage as

$$w_R = b + \frac{\beta}{1-\beta} \int_{w_R}^{\overline{w}} [1 - F(w)] dw.$$

Before continuing, you might notice that the above (in any of its versions) is not an explicit function for w_R. To solve for w_R, one would find the root of

$$G(w_R) \equiv w_R - b - \frac{\beta}{1-\beta} \int_{w_R}^{\overline{w}} [1 - F(w)] dw.$$

This typically cannot be done analytically (it depends on the distributional assumption for F), so one would use some numerical root finding algorithm.

Continuous Time Derivation

Generalize the length of a period to dt and let $\beta = \frac{1}{1+rdt}$. Further, assume that the probability that an agent gets an offer in a period is αdt.[1] The Bellman equations for agents who are employed and unemployed are

$$V(w) = wdt + \frac{1}{1+rdt}V(w) \qquad \Longrightarrow \qquad V(w) = \frac{w(1+rdt)}{r}$$

$$U = bdt + \frac{\alpha dt}{1+rdt}\int_{\underline{w}}^{\overline{w}} \max\left\{V(w), U\right\} dF(w) + \frac{1-\alpha dt}{1+rdt}U.$$

[1] An alternative way is to model job arrivals as a Poisson random variable with a mean of αdt.

We can simplify U a little bit with some algebra.

$$\frac{rdt}{1+rdt}U = bdt + \frac{\alpha dt}{1+rdt}\int_{\underline{w}}^{\overline{w}} \max\left\{V(w), U\right\} dF(w) - \frac{\alpha dt}{1+rdt}U$$

$$\frac{rdt}{1+rdt}U = bdt + \frac{\alpha dt}{1+rdt}\int_{\underline{w}}^{\overline{w}} \max\left\{V(w) - U, 0\right\} dF(w)$$

$$U = \frac{(1+rdt)b}{r} + \frac{\alpha}{r}\int_{\underline{w}}^{\overline{w}} \max\left\{V(w) - U, 0\right\} dF(w)$$

To get the continuous time equations, take the limit: $dt \to 0$.

$$rV(w) = w$$

$$rU = b + \alpha \int_{\underline{w}}^{\overline{w}} \max\{V(w) - U, 0\}\, dF(w)$$

While $V(w)$ and U are the value of employment / unemployment, notice that we typically multiply the value functions by r. Written as such the two equations are interpreted as the *flow* values of employment and unemployment.

The flow value of being unemployed is equal to the instantaneous payoff b plus the expected value of any changes in the value of the worker's state (the probability that she gets an offer times the expected increase in the value associated with that offer).

Reservation Wage

As before, we can find the reservation wage. Recall that this is defined by $rV(w_R) = rU$.

$$
\begin{aligned}
w_R &= b + \alpha \int_{\underline{w}}^{\overline{w}} \max\left\{\frac{w - w_R}{r}, 0\right\} dF(w) \\
&= b + \frac{\alpha}{r} \int_{w_R}^{\overline{w}} (w - w_R) dF(w) \\
&= b + \frac{\alpha}{r} \int_{w_R}^{\overline{w}} [1 - F(w)] dw \qquad \text{(using integration by parts)}
\end{aligned}
$$

Some Simple Results from the Basic Model

In frictionless models, a worker is assumed to costlessly and immediately find a job and work for as many hours as she desires at the market wage. Here, though, we can start to think about things like unemployment duration (which is observed to be > 0 in the real world).

To think about unemployment duration, we'll need to know about the **hazard rate**: the rate at which a worker leaves unemployment.

$$H = \text{rate of job arrival} \times \text{prob. accept offer} = \alpha[1 - F(w_R)]$$

The probability that an agent has not had an offer after a period of time of length t is e^{-Ht}. Thus, the expected duration of an unemployment spell is

$$D = \int_0^\infty tHe^{-Ht}dt = \frac{1}{H}.$$

Note that the result above utilizes integration by parts. Further, noting that H is a function of w_R, which is a function of b, we can determine what the effect an increase in UI has on the average duration of unemployment.

First, let's start with the effect a change in b has on w_R...

Comparative Statics

Recall: $$w_R = b + \frac{\alpha}{r}\left[(\overline{w} - w_R) + \int_{w_R}^{\overline{w}} F(w)dw\right]$$

$$\frac{dw_R}{db} = 1 + \frac{\alpha}{r}\left[-\frac{dw_R}{db} - F(w_R)\frac{dw_R}{db}\right]$$

$$= \left\{1 + \frac{\alpha}{r}[1 - F(w_R)]\right\}^{-1} > 0$$

That is, an increase in unemployment benefits *increases* the reservation wage. We can now look at unemployment duration.

Recall: $$D = \frac{1}{H} = \frac{1}{\alpha[1 - F(w_R)]}$$

$$\frac{dH}{dw_R} = -\alpha \frac{dF(w_R)}{dw_R} \frac{dw_R}{db} < 0$$

$$\frac{dD}{db} = -\frac{1}{H^2} \frac{dH}{dw_R} \frac{dw_R}{db} > 0$$

Unsurprisingly, an increase in unemployment benefits increases the average duration of unemployment.

For more practice, test out your abilities with other comparative statics (try seeing how α and r affect w_R, H, and D)

Conclusion

- Today we looked at a simple model of job search, but this model can be extended to incorporate additional (more realistic) features
  - ex: Quitting, getting fired, on the job search
- In the coming weeks we will look at extending our model to include some of these features
- While we looked specifically at job search, search models can be used in many other contexts
  - ex: Housing markets, crime, dating, monetary theory

Appendix

Note that we can think about the value function of a worker who is currently unemployed and is deciding to take a job in this period as follows:

$$v(w) = \max_{\text{accept, reject}} \left\{ \frac{w}{1-\beta}, b + \beta \int_{\underline{w}}^{\overline{w}} v(w') dF(w') \right\}$$

Bellman Equations