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Econ 204B
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Problem Set # 2

Problem 1: Power Utility

Consider an economy with an infinitely lived representative agent. Lifetime utility is given by:

$$\sum_{t=0}^{\infty} \beta^t \Big(\frac{c_t^{1-\gamma} - 1}{1-\gamma} \Big)$$

The economy has Cobb-Dougals production $f(k_t) = k_t^{\alpha}$ and depreciation rate δ.

(a) Write down the representative agent's problem sequentially.

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t \Big(\frac{c_t^{1-\gamma} - 1}{1-\gamma} \Big) \\ \text{s.t.} \quad & c_t + a_{t+1} = w_t + a_t(1+r_t) \\ & c_t, k_t \geq 0 \ \forall \text{ t} \\ & k_0 > 0 \text{ given} \end{aligned}$$

(b) Write down the Bellman equation for the agent's problem.

Plugging in the constraint we have:

$$V(a) = \max_{a'} \Big\{ \Big(\frac{(w + a(1+r) - a')^{1-\gamma} - 1}{1-\gamma} \Big) + \beta V(a') \Big\}$$

(c) Assume full depreciation and derive the Euler equation.

The FOC with respect to a' yields

$$\frac{1}{(w + a(1+r) - a')^{\gamma}} = \beta \frac{dV(a')}{da'} \qquad (\spadesuit)$$

Now we need to apply envelope theorem, namely the Benveniste-Sheinkman theorem:

$$\frac{dV(a)}{da} = \frac{1}{(w + a(1+r) - a')^{\gamma}} (1+r)$$

Pushing forward one period we have

$$\frac{dV(a')}{da'} = \frac{1}{(w' + a'(1+r') - a'')^{\gamma}}(1+r') \tag{✉}$$

Then plugging (✉) into (♠) we have the Euler equation

$$\frac{1}{(w + a(1+r) - a')^{\gamma}} = \beta(1+r')\frac{1}{(w' + a'(1+r') - a'')^{\gamma}} \tag{♣}$$

(d) Find the steady state values of k, c, and production y.

We have $r = F_k(k,1) - \delta = \alpha k^{\alpha-1} - 1$, and in the steady state we have that $c = c'$ which implies $(w + a(1+r) - a') = (w' + a'(1+r') - a'')$. Therefore in the steady state (♣) becomes

$$\begin{aligned}
\frac{1}{\beta} &= (1 + r_{ss}) \\
\frac{1}{\beta} &= (1 + \alpha k_{ss}^{\alpha-1} - 1) && \text{(Plugging in for r)} \\
k_{ss}^{\alpha-1} &= \frac{1}{\alpha\beta} \\
k_{ss} &= (\alpha\beta)^{\frac{1}{1-\alpha}}
\end{aligned}$$

Then we have

$$\begin{aligned}
y_{ss} &= k_{ss}^{\alpha} = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} \\
c_{ss} &= y_{ss} - k_{ss} = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}
\end{aligned}$$

Problem 2: Cake Eating

(a) Solve the cake eating problem analytically and on the computer. Assume the cake will go bad in 30 periods. Let k_t be the amount of cake and let c_t be the amount of cake you eat in time t. Also assume that the initial size of the cake is $k_0 = 1$ and that you have log utility with a discount factor of $\beta = 0.85$. Submit your code, and plot the size of the cake, consumption, and utility *on the same graph.*

The problem we are asked to solve is the following:

$$\begin{aligned}
\max_{\{c_t,\, k_{t+1}\}_{t=0}^{T}} \quad & \sum_{t=0}^{T} \beta^t ln(c_t) \\
\text{s.t.} \quad & k_{t+1} = k_t - c_t \\
& k_0 \text{ given} \\
& k_t \geq 0 \ \forall t
\end{aligned}$$

Now plugging in our budget constraint to eliminate our choice variable c_t we have

$$\max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t ln(k_t - k_{t+1})$$

Let's write out the sum to make things more clear:

$$\max_{\{k_{t+1}\}_{t=0}^{T}} \beta^0 ln(k_0 - k_1) + \beta^1 ln(k_1 - k_2) + \cdots + \beta^T ln(k_T - k_{T+1}) \tag{$\blacklozenge$}$$

We begin by finding the Euler equation by differentiating our objective function ($\blacklozenge$) with respect to the choice variable in some arbitrary period, k_{t+1}. Note that k_{t+1} shows up in two consecutive elements in the sum (this is why actually writing the sum out can sometimes be helpful), so we have

$$\underbrace{k_{t+1} - k_{t+2}}_{c_{t+1}} = \beta(\underbrace{k_t - k_{t+1}}_{c_t})$$

Next we start working back from the terminal period. Since the cake is perishable we have that $k_{T+1} = 0$ in the optimum (i.e. we shouldn't leave any cake to go bad, we should eat it all by the end of period T). This means that in period T we will consume whatever cake is left, therefore we have

$$\begin{aligned}
c_T &= \beta c_{T-1} \\
k_T - k_{T+1} &= \beta(k_{T-1} - k_T) \\
k_T - 0 &= \beta(k_{T-1} - k_T) \\
(1+\beta)k_T &= \beta k_{T-1} \\
k_T &= \frac{\beta}{1+\beta} k_{T-1}
\end{aligned}$$

Now we can go back one more period to $T-2$ and hopefully we'll see a pattern emerging

$$\begin{aligned}
c_{T-1} &= \beta c_{T-2} \\
k_{T-1} - k_T &= \beta(k_{T-2} - k_{T-1}) \\
k_{T-1} - \frac{\beta}{1+\beta} k_{T-1} &= \beta k_{T-2} - \beta k_{T-1} \\
\Big(\frac{1+\beta}{1+\beta} - \frac{\beta}{1+\beta}\Big) k_{T-1} &= \beta k_{T-2} - \beta k_{T-1} \\
\frac{1}{1+\beta} k_{T-1} + \beta k_{T-1} &= \beta k_{T-2} \\
\Big(\frac{1}{1+\beta} + \frac{\beta(1+\beta)}{1+\beta}\Big) k_{T-1} &= \beta k_{T-2} \\
\Big(\frac{1+\beta+\beta^2}{1+\beta}\Big) k_{T-1} &= \beta k_{T-2} \\
k_{T-1} &= \frac{\beta+\beta^2}{1+\beta+\beta^2} k_{T-2}
\end{aligned}$$

We can repeat this calculation all the way back to the initial period. Noting the pattern above we have

$$
\begin{aligned}
c_1 &= \beta c_0 \\
k_1 - k_2 &= \beta(k_0 - k_1) \\
&\vdots \\
k_1 &= \frac{\beta + \cdots + \beta^T}{1 + \beta + \cdots + \beta^T} k_0 \qquad \text{(🦇)}
\end{aligned}
$$

Now let $S = 1 + \beta + \cdots + \beta^T$. Then

$$
\begin{aligned}
S - \beta S &= (1 + \beta + \cdots + \beta^T) - \beta(1 + \beta + \cdots + \beta^T) \\
(1-\beta)S &= (1 + \beta + \cdots + \beta^T) - \beta - \beta^2 - \cdots - \beta^{T+1}) \\
(1-\beta)S &= 1 - \beta^{T+1} \\
S &= \frac{1 - \beta^{T+1}}{1 - \beta} \qquad \text{(⚽)}
\end{aligned}
$$

Then since S is the denominator of (🦇), and since in the numerator of (🦇) we can simply factor out a β to end up with a similar series, we can use (⚽) to rewrite (🦇) as

$$
\begin{aligned}
k_1 &= \frac{\frac{\beta(1-\beta^T)}{1-\beta}}{\frac{1-\beta^{T+1}}{1-\beta}} k_0 \\
k_1 &= \frac{\beta(1 - \beta^T)}{1 - \beta^{T+1}} k_0 \qquad \text{(☣)}
\end{aligned}
$$

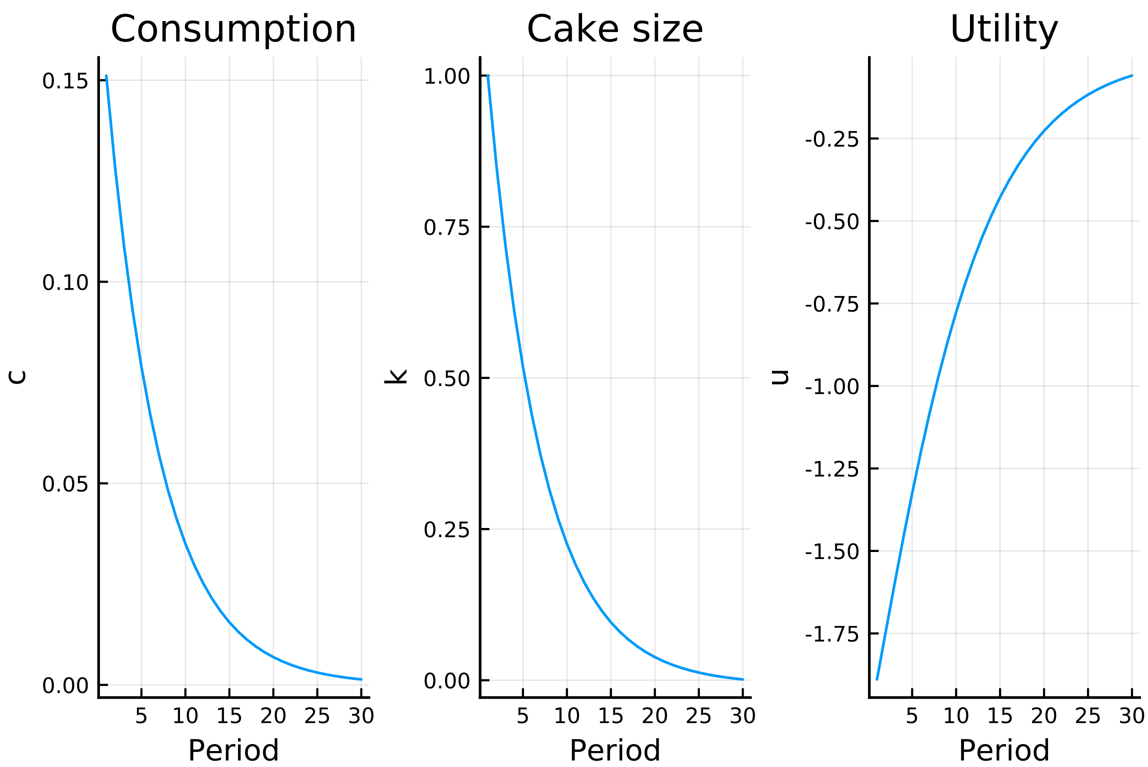
We can generalize (☣) to any period $t+1$

$$
k_{t+1} = \frac{\beta(1 - \beta^{T-t})}{1 - \beta^{T-t+1}} k_t \qquad \text{(Policy Function)}
$$

Now we can solve for consumption in the first period using the budget constraint and the policy function:

$$
\begin{aligned}
c_0 + k_1 &= k_0 \\
c_0 + \frac{\beta(1 - \beta^T)}{1 - \beta^{T+1}} k_0 &= k_0 \qquad \text{(Using ☣)} \\
c_0 = \Big(\frac{1 - \beta^{T+1}}{1 - \beta^{T+1}} - \frac{\beta(1 - \beta^T)}{1 - \beta^{T+1}}\Big) k_0 & \\
c_0 &= \frac{1 - \beta}{1 - \beta^{T+1}} k_0 \qquad (\blacktriangle)
\end{aligned}
$$

Given any initial amount of cake k_0 and number of periods T, we can use the Euler equation and ($\blacktriangle$) to solve for the entire path of c_t and $u(c_t)$, and the policy function to solve for the entire path of k_t.



(b) Suppose when you get your cake you put it in the fridge until the next day when you will begin to eat it. Each day after you eat your portion of cake, you put the remaining cake in the fridge until the next day. However, your sister has a condition known as "sleep eating", and every night she sleep walks down to the fridge and eats a fraction s of your cake (including the night before you ate your first piece). Does this change the amount of cake that you eat? Submit your code, and plot the size of the cake, consumption, and utility for $s = 0.01$, $s = 0.10$, $s = 0.20$, and $s = 0.40$

Here we are solving the cake eating problem with depreciation. Note that here our budget constraint changes, so our problem becomes

$$\begin{aligned} \max_{\{c_t,\, k_{t+1}\}_{t=0}^{T}} \quad & \sum_{t=0}^{T} \beta^t ln(c_t) \\ \text{s.t.} \quad & k_{t+1} = (1-s)k_t - c_t \\ & k_0 \text{ given} \\ & k_t \geq 0 \; \forall t \end{aligned}$$

Solving for the Euler equation as we did in part (a) we have

$$\underbrace{(1-s)k_{t+1} - k_{t+2}}_{c_{t+1}} = \beta(1-s)(\underbrace{(1-s)k_t - k_{t+1}}_{c_t})$$

This time we'll solve for the optimal path of consumption first. Noting again that optimality implies that $k_{T+1} = 0$, using our constraint we have

$$
\begin{aligned}
c_T + k_{T+1} &= (1-s)k_T \\
c_T + 0 &= (1-s)k_T \\
k_T &= \frac{c_T}{(1-s)} \qquad (\heartsuit)
\end{aligned}
$$

Then using our constraint again we have

$$
\begin{aligned}
k_T &= (1-s)k_{T-1} - c_{T-1} \\
\frac{c_T}{(1-s)} &= (1-s)k_{T-1} - c_{T-1} && \text{(Using } \heartsuit\text{)} \\
\frac{\beta(1-s)c_{T-1}}{(1-s)} &= (1-s)k_{T-1} - c_{T-1} && \text{(Using EE)} \\
\beta c_{T-1} + c_{T-1} &= (1-s)k_{T-1} \\
k_{T-1} &= \frac{(1+\beta)}{(1-s)}c_{T-1}
\end{aligned}
$$

Going back one more period we have

$$
\begin{aligned}
k_{T-1} &= (1-s)k_{T-2} - c_{T-2} \\
\frac{(1+\beta)}{(1-s)}c_{T-1} &= (1-s)k_{T-2} - c_{T-2} \\
\frac{(1+\beta)}{(1-s)}\beta(1-s)c_{T-2} &= (1-s)k_{T-2} - c_{T-2} && \text{(Using EE)} \\
(1+\beta)\beta c_{T-2} &= (1-s)k_{T-2} - c_{T-2} \\
(1+\beta)\beta c_{T-2} + c_{T-2} &= (1-s)k_{T-2} \\
(1+\beta+\beta^2)c_{T-2} &= (1-s)k_{T-2} \\
k_{T-2} &= \frac{(1+\beta+\beta^2)}{(1-s)}c_{T-2}
\end{aligned}
$$

Using the emerging pattern we have

$$
c_0 = \frac{(1-s)}{1+\beta+\beta^2+\cdots+\beta^{29}}k_0
$$

Using what we derived in the last problem we know that the denominator can be written as $\frac{1-\beta^{T+1}}{1-\beta}$. Then we have

$$
c_0 = (1-s)\frac{1-\beta}{1-\beta^{T+1}}k_0 \qquad (\blacktriangle)
$$

Then for a given amount of cake k_0, we can find the entire path of c_t using the Euler equation:

$$
c_{t+1} = \beta(1-s)c_t
$$

Then we can find the entire path of k_t by plugging our period consumption values into the budget constraint:

$$
c_t + k_{t+1} = (1-s)k_t
$$

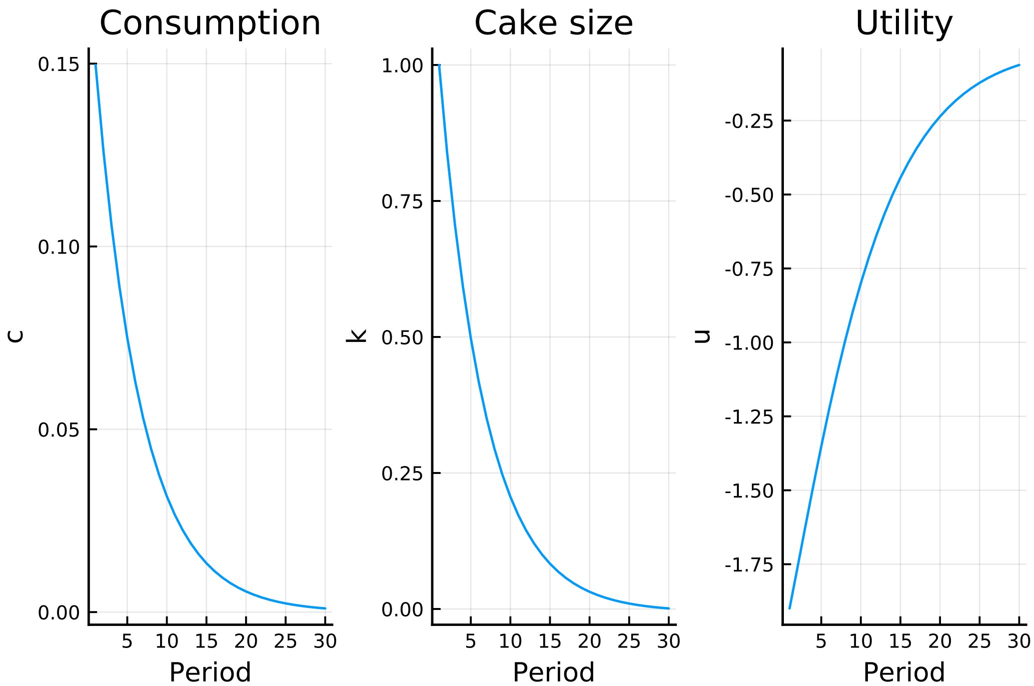


Figure 1: s = 0.01

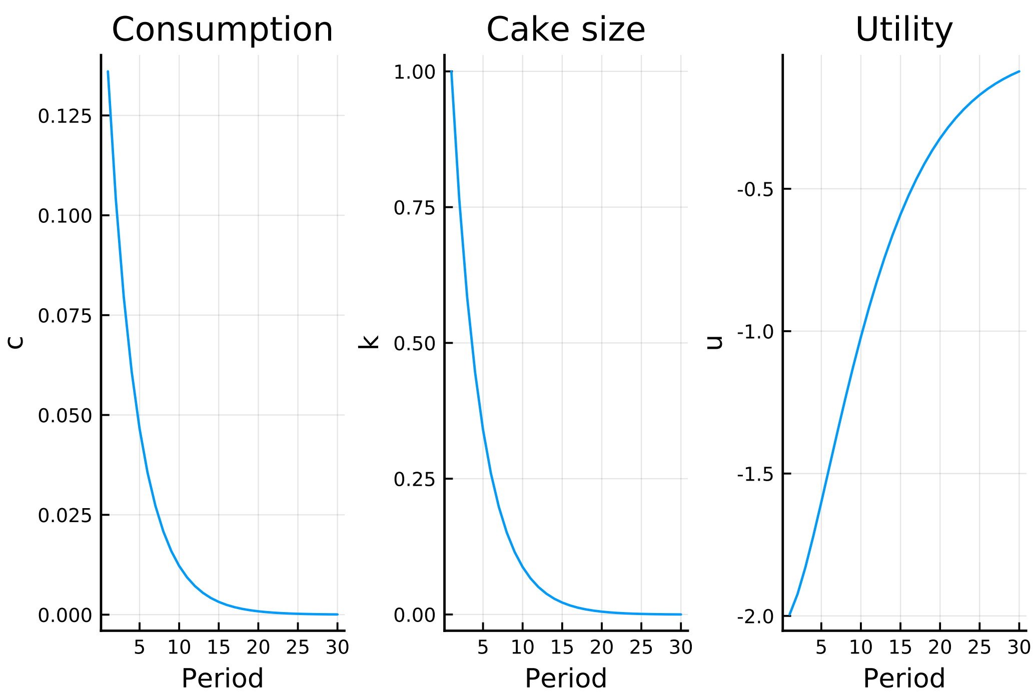


Figure 2: s = 0.10

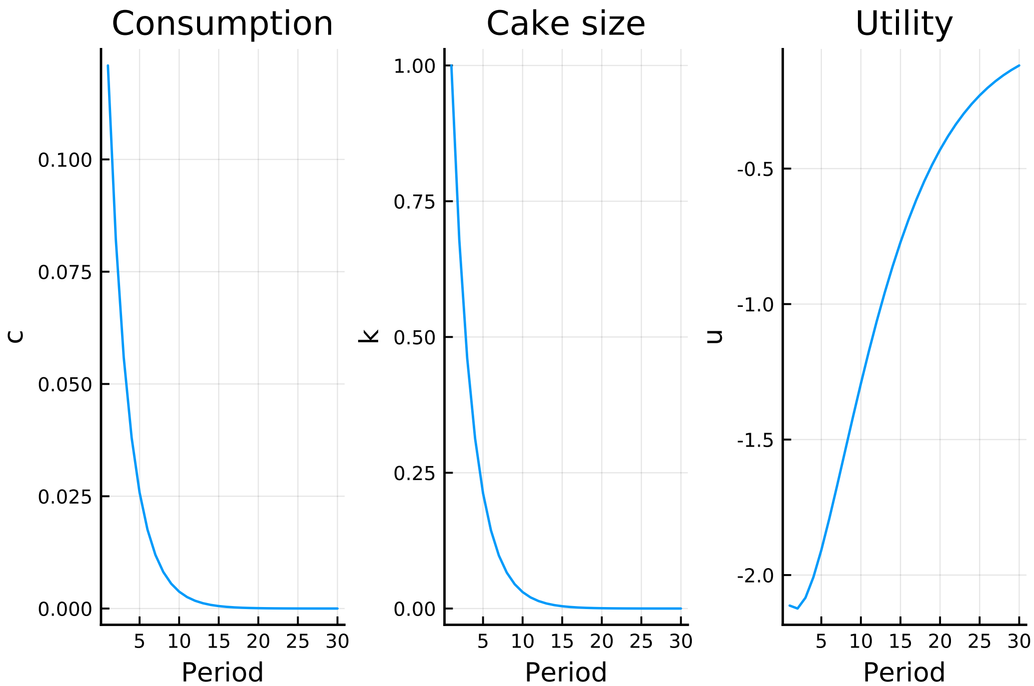


Figure 3: s = 0.20

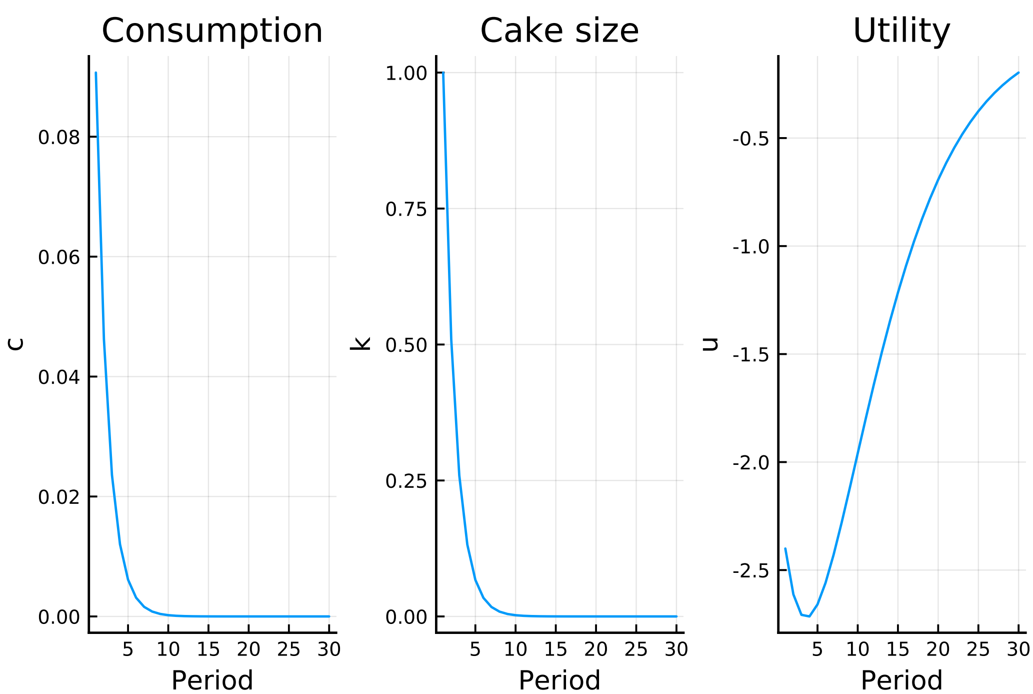


Figure 4: s = 0.40