Week 10

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Today

- Finite horizon problems with continuous choice variables
- Finite horizon problems with discrete choice variables
- Practice question

Finite Horizon Problems

- We started this class with a finite horizon problem: Cake eating in T periods
- Let's use this example again to think about how finite time problems differ from infinite horizon problems
- With finite horizon problems, we are still interested in finding policy functions
- In infinite horizon problems, the policy function is the same in every period, but is this the case in finite horizon models?

- No, our optimal policy can change each period when we're working with a finite horizon
- To see this, consider the cake eating problem with 3 periods and log utility, but assume that the cake depreciates each period by a factor of δ and assume $0 < \beta < 1$:

$$\max_{\{c_t \ k_{t+1}\}} \sum_{t=1}^{3} \beta^t ln(c_t)$$

$$\text{s.t. } k_{t+1} = (1-\delta)(k_t - c_t)$$

$$c_t \ , \ k_t \geq 0$$

$$k_0 \text{ given}$$

- Now let's write the Bellman Equation

- Let s denote the time remaining in the problem, so that the last period we can consume is $s = 1$. Then we have

$$V_s(k) = \max_{c,k'}[ln(c) + \beta V_{s-1}(k')]$$

$$\text{s.t. } k' = (1-\delta)(k-c)$$

- We can plug in our constraint and get

$$V_s(k) = \max_{c}[ln(c) + \beta V_{s-1}((1-\delta)(k-c))]$$

- First note the subscript on the value function, why do we need this?
- To see why this is necessary, let's solve the problem. With finite horizon problems we find the solution through **backward recursion**

- We start with time period $s = 0$. What is the value of having an arbitrary amount of cake left in this period, $V_0(k)$?
- Since we can't consume in this period, the value of any amount of cake is zero: $V_0(k) = 0$, therefore the optimal policy must be that in period $s = 0$ we should have $k = 0$
- Now we can go back one period and solve for $V_1(k)$:

$$V_1(k) = \max_c [ln(c) + \beta V_0((1 - \delta)(k - c))]$$

$$V_1(k) = \max_c [ln(c) + 0]$$

- Note that we are at a corner, but we know that in the optimum we should have $k' = 0 \implies 0 = (1 - \delta)(k - c) \implies c = k$ (i.e. you eat whatever cake you have left)

- Thus we have

$$V_1(k) = ln(k)$$

- Now let's go back one more period

$$V_2(k) = \max_c [ln(c) + \beta V_1((1-\delta)(k-c))]$$

$$V_2(k) = \max_c [ln(c) + \beta ln((1-\delta)(k-c))]$$

- Now we can find our FOC with respect to c

$$\frac{1}{c} = \beta \frac{(1-\delta)}{(1-\delta)(k-c)}$$

$$\beta c = k - c$$

$$c = \frac{k}{1+\beta}$$

- Now we can plug this in and obtain $V_2(k)$

$$V_2(k) = ln(\frac{k}{1+\beta}) + \beta \underbrace{ln((1-\delta)(k - \frac{k}{1+\beta}))}_{V_1(k')}$$

$$V_2(k) = ln(\frac{k}{1+\beta}) + \beta ln((1-\delta)(\frac{\beta k}{1+\beta}))$$

- Now note that $V_2 \neq V_1 \neq V_0$. In the infinite horizon case we do not need to index $V(k)$, but here the value of having a cake of size k changes over time
- An important point to note is that our policy function changes each period as well. This makes sense, the decision of how much cake to eat should be different if you can eat more tomorrow vs if you can't

- Ex: If you had a final exam tomorrow and you knew the world was going to end in two days, you probably wouldn't study.
- Note that in infinite horizon problems we face an infinite sum subject to an equation of motion for our state variable:

$$\sum_{t=0}^{\infty} \beta^t u(c_t) \qquad\qquad \text{s.t. } k_{t+1} = g(k_t)$$

- In order for us to solve these types of problems, we need the future terms in the infinite sum to be going to zero (we don't want the sum to diverge)

- There are cases where β can be equal to 1, for example in many finite horizon problems
- For example, consider again the cake eating problem but this time the cake grows at a rate of $1 + r$, there are only two periods, you have linear utility, and $\beta = 1$:

$$\max_{\{c_t \ k_{t+1}\}} \sum_{t=1}^{3} c_t$$

$$\text{s.t. } k_{t+1} = (1 + r)(k_t - c_t)$$

$$c_t \ , \ k_t \geq 0$$

$$k_0 \text{ given}$$

- Let's think about this problem. With linear utility we will be at a corner solution, so we can compare our two options
- We can eat the cake in the first period and receive a utility stream of $k_0 + 0 = k_0$, or we can eat the cake in the second period and receive a utility stream of $0 + (1 + r)k_0 = (1 + r)k_0 > k_0$.
- Thus we can solve this problem when $\beta = 1$
- Think about this problem in the infinite horizon case and try to think about how $\beta = 1$ could cause problems for finding a solution
- Note that $\beta = 1$ implies that you care as much about tomorrow as you do today, the validity of this assumption depends on the particular problem we are trying to solve

Practice Question

Consider a tree whose growth is described by the function h. Suppose that the size of the tree in period t is $k_{t+1} = h(k_t)$ for all $t = 0, 1, \cdots$. Assume that the price of wood is $p = 1$ and the interest rate r are both constant over time. Assume that it is costless to cut down the tree. Let β be the discount factor.

(a) Write down the Bellman equation if the tree cannot be replanted.

(b) Guess that the value function takes the form $V(k) = k$. Find the policy function $k' = h(k)$ that makes the agent indifferent between cutting the tree down and not cutting it down.

(c) Suppose that when the tree is cut down, another can be replanted in its place. When this tree is cut down, another can be replanted and so on. Assume that the cost of replanting $c > 0$ is constant over time. What is the Bellman equation in this case?

Part (a)

(a) Write down the Bellman equation if the tree cannot be replanted.

Solution:

Our choice variable is $d = \{0, 1\}$ where 0 is don't cut and 1 is cut. Our state variable is the size of the tree at the beginning of the period k. Then we have

$$V(k) = \max_{\{\text{cut, don't cut}\}} \{k, \beta V(k')\}$$
$$\text{s.t. } k' = h(k)$$

Part (b)

(b) Guess that the value function takes the form $V(k) = k$. Find the policy function $k' = h(k)$ that makes the agent indifferent between cutting the tree down and not cutting it down.

Solution:

Note that this is similar to a job search question where we're asked to find the reservation wage. Thus using our Bellman equation from part (a) we have

$$
\begin{aligned}
k &= \beta V(h(k)) \\
k &= \beta h(k) && \text{(Plugging in guess)} \\
h(k) &= \frac{k}{\beta}
\end{aligned}
$$

Therefore when the next period's tree size is $k' = \frac{k}{\beta}$, you are indifferent between cutting down the tree and not cutting it down.

Part (c)

(c) Suppose that when the tree is cut down, another can be replanted in its place. When this tree is cut down, another can be replanted and so on. Assume that the cost of replanting $c > 0$ is constant over time. What is the Bellman equation in this case?

Solution:

Our Bellman equation becomes:

$$V(k) = \max_{\{\text{cut, don't cut, cut and replace}\}} \{k, \beta V(h(k)), k - c + \beta V(h(0))\}$$