Week 3

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Today

- Last time we worked to show that the Bellman equation has a unique solution
- Today we will talk about solving problems
- First we will go over the 2018 Prelim question you worked on yesterday in class
  - Will also finish our discussion on properties of the Bellman Equation by looking at differentiability in the context of this problem
- Next we will set up an example problem sequentially and recursively and begin talking about solution techniques

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Question #2

Imagine an infinitely-lived representative agent economy where agents derive utility from the wealth of society. This can be interpreted as a form of nationalism. Agents have instantaneous utility of the form

$$U(c,k) = u(c) + \lambda v(k), \quad \lambda > 0$$

where c is consumption and k is aggregate capital. $u(c)$ and $v(k)$ are strictly increasing, strictly concave, twice continuously differentiable functions. Utility is additively separable over time with discount factor $\beta \in (0,1)$. Consumers do not value leisure and supply one unit of labor inelastically. They take wages and interest rate as given. Let a stand for individual asset holdings. Production takes place according to a constant returns to scale Cobb-Douglas technology and no depreciation.

a) Write down the problem of the consumer recursively and derive the Euler equation.

b) Define equilibrium.

c) Assume log-utility for both u and v. What is steady state wealth?

d) If you were the social planner, are there any grounds for government intervention to increase social welfare? If so, what policy would you recommend? Show how you arrived at this recommendation.

Prelim Question: Part A

a. Write down the problem of the consumer recursively and derive the Euler equation.

Solution:

$$
\begin{aligned}
V(a,k) &= \max_{c,a'} u(c) + \lambda v(k) + \beta V(a',k') \\
\text{s.t.} \quad & c + a' = w + a(1+r) \\
& k' = \phi(k)
\end{aligned}
$$

We can plug in the budget constraint to get:

$$V(a,k) = \max_{c,a'} u(w + a(1+r) - a') + \lambda v(k) + \beta V(a',k')$$

Now let's take FOC with respect to a'

$$u'(w + a(1+r) - a') = \beta V_{a'}(a',k') \tag{1}$$

What is $V_{a'}(a',k')$?

Benveniste-Schienkman

Theorem 4.10 (SLP): Let $X \subseteq \mathbb{R}^l$ be a convex set, let $V : X \to \mathbb{R}$ be concave, let $x_0 \in \text{int } X$, and let D be a neighborhood of x_0. If there is a concave, differentiable function $W : D \to \mathbb{R}$, with $W(x_0) = V(x_0)$ and with $W(x) \leq V(x)$ for all $x \in D$, then V is differentiable at x_0, and

$$V_i(x_0) = W_i(x_0), \quad i = 1, 2, .., , l$$

Theorem 4.11 (SLP): Define

$$V(x) = \max_{y \in \Gamma(x)} [F(x, y) + \beta V(y)]$$

$$G(x) = \{y \in \Gamma(x) : V(x) = F(x, y) + \beta V(y)\}$$

Then if assumptions 4.3-4.4 and 4.7-4.9 from (SLP) are met, and $x_0 \in \textit{int } X$ and $g(x_0) \in \textit{int } \Gamma(x_0)$, then V is continuously differentiable at x_0 with derivatives given by

$$V_i(x_0) = F_i[x_0, g(x_0)], \quad i = 1, 2, ..., l.$$

Benveniste-Schienkman

How does this apply to the Bellman Equations we see in our problems? Let's look at two important elements from the proof of Theorem 4.11 in SLP. Let

$$W(x) = F[x, g(x_0)] + \beta V[g(x_0)]$$

Then we have

$$W(x) \leq \max_{y \in \Gamma(x)} [F(x, y) + \beta V(y) = V(x)$$

with equality at x_0. Therefore W and V satisfy the hypotheses from Benveniste-Schienkman.

Why is this important? Taking a derivative of $V(a, k)$ is difficult because we have the *max* operator. Since we have shown that $W(x)$ as defined above satisfies the requirements for Benveniste-Schienkman, we can take the derivative of $W(x)$ and evaluate it at x_0 and we know that this will be the same at $V'(x_0)$. Hence Theorem 4.11 tells us that $V_i(x_0) = F_i[x_0, g(x_0)]$

Prelim Question: Part A

Returning to our problem, we can use Benveniste-Schienkman this envelope condition to find $V_{a'}(a', k')$.

$$V_a(a, k) = u'(w + a(1 + r) - a')(1 + r)$$

Pushing this forward one period gives us

$$V_{a'}(a', k') = u'(w' + a'(1 + r') - a'')(1 + r')$$

Then combining this with (1) we have

$$u'(c) = \beta(1 + r')u'(c') \tag{EE}$$

Recursive Competitive Equilibrium

Definition. Let K denote the state of the aggregate economy and a denote the personal state of an agent. Then a *Recursive Competitive Equilibrium (RCE)* is a set of functions that describe

Quantities: $K' = G(K)$, $a' = g(a, K)$ (agg. and personal policy functions)
Lifetime Utility: $V(a, K)$ (the value function)
Prices: $r(K) = f_K(K, N) + 1 - \delta$, $w(K) = f_N(K, N)$ (competitive prices)

such that:

1. Prices are complete and given
2. $V(a, K)$ and $g(a, K)$ solve the consumer's maximization problem
3. Consistency: $G(K) = g(K, K)$

That is, households must know prices so they can make utility maximizing decisions and if we gave on person all of the capital, their choice would coincide with the aggregation of everyone else's choices.

Prelim Question: Part B

b. Define Equilibrium.

Solution: We can use the definition on the previous page changing:

1. Aggregate capital K to k
2. Law of motion for aggregate capital $G(K)$ to $\phi(k)$

Prelim Question: Part C

c. Assume log-utility for both u and v. What is the steady state wealth?

Solution: In steady state $c = c'$. Also since no depreciation, $r = F_k(k, 1) = \alpha k^{\alpha-1}$. Then with log utility our Euler Equation is:

$$
\begin{aligned}
u'(c) &= \beta(1 + r')u'(c') \\
\frac{1}{c} &= \beta(1 + r)\frac{1}{c} \\
\frac{1}{\beta} &= (1 + r) \\
\frac{1}{\beta} &= 1 + \alpha k^{\alpha-1} \\
k_{ss} &= \left(\frac{\alpha\beta}{1-\beta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}
$$

Prelim Question: Part D

d. If you were the social planner, are there any grounds for government intervention to increase social welfare? If so, what policy would you recommend?

Solution: Set up the social planner's problem:

$$V(k) = \max_{c,k'} u(c) + \lambda v(k) + \beta V(k')$$
$$\text{s.t.} \quad c + k' = k^\alpha + k$$

Taking FOC with respect to k' we have

$$u'(c') = \beta V'(k')$$

Again we utilize Benveniste-Schienkman and find

$$V'(k) = u'(c)(\alpha k^{\alpha-1} + 1) + \lambda v'(k)$$

Pushing this forward by one period we have

$$V'(k') = u'(c')(\alpha k'^{\alpha-1} + 1) + \lambda v'(k') \tag{2}$$

Prelim Question: Part D

Then note that $\alpha k'^{\alpha-1} + 1 = r' + 1$, so we can rewrite (2) as

$$V'(k') = u'(c')(1 + r') + \lambda v'(k')$$

Then plugging this back in we have

$$u(c') = \beta V'(k')$$
$$u(c') = \beta u'(c')(1 + r') + \beta\lambda v'(k')$$

We want to find a government subsidy τ that makes the individual Euler equation the same as the social planner's. Thus the optimal government intervention is to subsidize savings at a rate

$$\tau = \lambda \frac{v'(k')}{u'(c')}$$

Then note that the individual Euler equation becomes:

$$u'(c) = \beta(1 + r' + \tau)u'(c')$$
$$u'(c) = \beta(1 + r')u'(c') + \beta\lambda v'(k')$$

Sequential Formulation

We will use the Neoclassical Growth Model as our running example. Consider an infinitely lived representative agent that maximizes the present discounted utility over consumption in discrete time. The sequential formulation of the representative agent's optimization problem is rewritten:

$$
\begin{aligned}
\max_{\{c_t, k_{t+1}, a_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t. } & c_t + a_t \leq f(k_t) \ \forall t \\
& k_{t+1} = (1-\delta)k_t + a_t \ \forall t \\
& c_t, k_t \geq 0 \ \forall t \\
& k_0 \text{ given}
\end{aligned}
$$

Note that we can substitute the second constraint into the first constraint to get the following constraint:

$$
c_t + k_{t+1} \leq f(k_t) - (1-\delta)k_t
$$

Then we can write:

$$
\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t [f(k_t) + (1-\delta)k_t - c_t - k_{t+1}] \right\}
$$

FOC's:

$$\lambda_t = \beta^t u'(c_t) \ \ \forall t \tag{3}$$

$$\frac{\lambda_t}{\lambda_{t+1}} = f'(k_{t+1}) + (1-\delta) \ \ \forall t \tag{4}$$

Pushing "pushing" (3) forward by one period yields: $\lambda_{t+1} = \beta^{t+1} u'(c_{t+1})$. Plugging this in to (4) gives us:

$$\begin{aligned}
\frac{\beta^t u'(c_t)}{\beta^{t+1} u'(c_{+1})} &= f'(k_{t+1}) + (1-\delta) \ \ \forall t \\
\frac{u'(c_t)}{u'(c_{+1})} &= \beta[\underbrace{f'(k_{t+1}) + (1-\delta)}_{1+r_{t+1}}] \ \ \forall t \\
u'(c_t) &= \beta(1+r_{t+1})u'(c_{t+1}) \ \ \forall t \qquad \text{(EE)}
\end{aligned}$$

Assume utility is logarithmic, production is Cobb-Douglas, and there is full depreciation ($\delta = 1$). Then our Euler Equation is:

$$\frac{1}{c_t} = \alpha\beta k_{t+1}^{\alpha-1} \frac{1}{c_{t+1}} \tag{5}$$

Sequential Markets Equilibrium

Definition: A *SME* is prices $\{\hat{r}_t\}_{t=0}^{\infty}$ and allocations $\{\hat{c}_t{}^i, \hat{a}_{t+1}^i\}_{t=0}^{\infty}$ (where i indexed agents) such that:

1. Given $\{\hat{r}_t\}_{t=0}^{\infty}$, $\forall i$ we have that $\{\hat{c}_t{}^i, \hat{a}_{t+1}^i\}_{t=0}^{\infty}$ solves the agent's maximization problem
2. The allocation is feasible (i.e. $\sum_i \hat{c}_t = \sum_i e_t^i \ \forall t$) and the amount of debt equals the amount of assets (i.e. $\sum_i \hat{a}_{t+1}^i = 0 \ \forall t$)

Note here that e_t^i denotes "income" (and thus is more general than an "endowment"). Also note that in the market clearing statement ($\sum_i \hat{a}_{t+1}^i = 0 \ \forall t$) assets and debt net out in the aggregate. This requires a no-Ponzi condition to ensure agents don't borrow an infinite amount and attain infinite utility (ex: $a_{t+1}^i \geq -\bar{A} \ \forall t$ where A is large) though sometimes this is implicit or simply unstated.

There is also sometimes a *natural debt limit* pinned down by rational agents' understanding of the ability of others to pay back debt given a stream of (expected) income. For example, in a stochastic income problem with some lower limit y_{min} and a constant interest rate r, the debt limit is:

$$a_{t+1}^i \geq -\frac{y_{min}}{r} \quad \forall t$$

Policy functions

Recall that a **policy function** maps the set of state variables into an action by agents

Let's find the policy function of our running example. Plugging in $c_t = k_t^\alpha - k_{t+1}$ to the EE:

$$\frac{1}{k_t^\alpha - k_{t+1}} = \alpha\beta k_{t+1}^{\alpha-1} \frac{1}{k_{t+1}^\alpha - k_{t+2}} \quad \forall t$$

To back out the policy function, assume a finite terminal period T (where we will impose terminal conditions and iterate backward) and then take the solution as $T \to \infty$

Consider the final period T. We know there will be no savings (since there's no tomorrow) so $k_{T+1} = 0$. Then looking at the EE for period $T-1$:

$$\frac{1}{k_{T-1}^\alpha - k_T} = \alpha\beta k_T^{\alpha-1} \frac{1}{k_T^\alpha - 0}$$

Now solve for k_T:

$$\frac{1}{k_{T-1}^{\alpha} - k_T} = \alpha\beta \frac{1}{k_T}$$
$$k_T = \frac{\alpha\beta}{1+\alpha\beta} k_{T-1}^{\alpha}$$

Now we can continue to iterate using the EE. Starting with period $T-2$:

$$\frac{1}{k_{T-2}^{\alpha} - k_{T-1}} = \alpha\beta k_{T-1}^{\alpha-1} \frac{1}{k_{T-1}^{\alpha} - k_T}$$
$$= \alpha\beta k_{T-1}^{\alpha-1} \frac{1}{k_{T-1}^{\alpha} - \left(\frac{\alpha\beta}{1+\alpha\beta} k_{T-1}^{\alpha}\right)}\Bigg)$$

Simplifying we get:

$$k_{T-1} = \frac{\alpha\beta + (\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2} k_{T-2}^{\alpha}$$

Notice the developing pattern

Consider some arbitrary period $t = T - \tau$:

$$k_{T-\tau} = \frac{\alpha\beta + (\alpha\beta)^2 + ... + (\alpha\beta)^{\tau+1}}{1 + \alpha\beta + (\alpha\beta)^2 + ... + (\alpha\beta)^{\tau+1}} k^{\alpha}_{T-\tau-1}$$

1. Consider the series $S = 1 + \alpha\beta + (\alpha\beta)^2 + ... + (\alpha\beta)^{\tau}$
2. Multiply S by $\alpha\beta$: $\alpha\beta S = \alpha\beta + (\alpha\beta)^2 + ... + (\alpha\beta)^{\tau+1}$
3. Then $S - \alpha\beta S = 1 - (\alpha\beta)^{\tau+1} \implies S = \frac{1-(\alpha\beta)^{\tau+1}}{1-\alpha\beta}$

Plug this into the expression above and we have:

$$k_{T-\tau} = \frac{\alpha\beta[1 - (\alpha\beta)^{\tau+1}]}{1 - (\alpha\beta)^{\tau+2}} k_{T-\tau-1}$$

For an arbitrary period $t = T - \tau$, if we take $T \to \infty$, we'll have $\tau \to \infty$ (i.e. any given period gets infinitely far away from the terminal period if the terminal period tends to infinity)

Then since $\alpha, \beta < 1$ we have that $\alpha\beta < 1$, so $(\alpha\beta)^\tau \to 0$ as $\tau \to \infty$. Now consider an arbitrary period $t + 1$. Using the results above we have

$$k_{t+1} = \alpha\beta k_t^\alpha$$

And so we've found the policy function for the problem. This function says given the state in period t, the agent sets $k_{t=1}$ according to some rule (function) $k_{t+1} = \kappa(k_t)$

Then since this relationship is always the same between any two periods in an infinite setting, all we need to know is k_0 to be able to recover the full path of k over time

Guess and Verify

- Guess and Verify (also known as the *method of undetermined coefficients*) requires that solution to the Bellman Equation is unique, and the guess is correct (hence it is generally not available)
- Steps:
  1. Guess the functional form of the value or policy function with stand in coefficients
  2. Verify that the guess is consistent with optimization
  3. Solve for the coefficients of the guess
- General Idea: if our guess is correct, then when "operated on" it should recover that same form and we can back out the coefficients previously left undetermined

Guessing the Policy Function

Consider the following problem solved by the household:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t ln(c_t) \quad s.t. \quad c_t + k_{t+1} = k_t^{\alpha}$$

The problem can be written recursively as follows:

$$V(k) = \max_{k'} \Big\{ ln(k^{\alpha} - k') + \beta V(k') \Big\}$$

Now let's guess that the policy function takes the form $k' = \eta k^{\alpha}$ where η is the undetermined coefficient. Now we solve for η and find the policy function for capital and consumption.

Take FOC:

$$\frac{1}{k^\alpha - k'} = \beta V'(k')$$

Now plug in guess for k':

$$\frac{1}{k^\alpha - (\eta k^\alpha)} = \beta V'(k')$$
$$\frac{1}{(1-\eta)k^\alpha} = \beta V'(k')$$

Note:

$$V'(k) = \frac{1}{k^\alpha - k'}(\alpha k^{\alpha-1})$$

Again pluggin in our guess for k' we have

$$V'(k) = \frac{1}{(1-\eta)k^\alpha}(\alpha k^{\alpha-1})$$
$$= \frac{\alpha}{(1-\eta)k}$$

Pushing $V'(k)$ forward one period we have:

$$\begin{aligned} V'(k') &= \frac{\alpha}{(1-\eta)k'} \\ &= \frac{\alpha}{(1-\eta)\eta k^{\alpha}} \quad \text{(plugging in guess)} \end{aligned}$$

We can plug this in to our FOC and solve for the undetermined coefficient η:

$$\begin{aligned} \frac{1}{(1-\eta)k^{\alpha}} &= \beta V'(k') \\ \frac{1}{(1-\eta)k^{\alpha}} &= \frac{\alpha}{(1-\eta)\eta k^{\alpha}} \end{aligned}$$

Solving this yields:

$$\eta = \alpha\beta$$

which is indeed a constant. Finally, the policy function for k' and c are

$$\begin{aligned} k' &= \alpha\beta k^{\alpha} \\ c &= (1-\alpha\beta)k^{\alpha} \end{aligned}$$

Guessing the Value Function

Now we'll try guessing the value function. Consider the same problem:

$$V(k) = \max_{k'} \Big\{ ln(k^\alpha - k') + \beta V(k') \Big\}$$

Now guess that $V(k) = A + Bln(k)$ where A and B are undetermined coefficients. We can take the FOC of the maximization problem with respect to k' and plug in our guess:

$$\frac{d}{dk} \Big\{ ln(k^\alpha - k') + \beta[A + Bln(k')] \Big\} = 0$$
$$\implies k'^* = \frac{\beta B k^\alpha}{1 + \beta B}$$

Now evaluate the right hand side of the Bellman Equation at the optimum:

$$RHS(k'^\alpha = ln\Big(k^\alpha - \frac{\beta B k^\alpha}{1 + \beta B}\Big) + \beta\Big[A + Bln\Big(\frac{\beta B k^\alpha}{1 + \beta B}\Big)\Big]$$
$$= ln\Big(\frac{k^\alpha}{1 + \beta B}\Big) + \beta A + \beta Bln\Big(\frac{\beta B k^\alpha}{1 + \beta B}\Big)$$

Now we group the constants together and separate out the k terms:

$$RHS(k'^{\alpha} = \underbrace{-ln(1+\beta B) + \beta A + \beta ln\Big(\frac{\beta B}{1+\beta B}\Big)}_{constant} + \underbrace{\alpha(1+\beta B)}_{ln(k)\text{-term coeff.}} \; ln(k)$$

Now let's think about the LHS of our Bellman Equation, V. We know that if our guess is correct, the LHS V will have the same form as our guess, therefore the $RHS(k'^{\alpha}$ we found above will be *equal to* $A + Bln(k)$.

$$A = \text{constant} = -ln(1+\beta B) + \beta A + \beta ln\Big(\frac{\beta B}{1+\beta B}\Big)$$

$$B = ln(k)\text{-term coeff.} = \alpha(1+\beta B)$$

Now we can solve for B using the second equation, and then A from the first:

$$B = ln(k)\text{-term coeff.} = \frac{\alpha}{1-\alpha\beta B}$$

$$A = \frac{1}{1-\beta}\Big[ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}ln(\alpha\beta)\Big]$$

From here we might want to get the policy function. We solved for this earlier and found

$$k' = \frac{\beta B k^{\alpha}}{1 + \beta B}$$

So now we can plug in B and we get:

$$k' = \alpha\beta k^{\alpha}$$

Next time

- Go over additional solution methods
- Discuss how to solve these problems numerically