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Final Exam
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Problem 1: Gone Fishing (60 points)

There is a lake near UCSB called Lake Santa Barbara that contains only one type of fish, the Bellman Fish. The Bellman Fish is extremely easy to catch, and because of this the population in the lake was nearly fished to extinction. The California Department of Fish and Wildlife has taken control of the situation, and is attempting to ensure the population returns to a sustainable level. In order to limit the amount of fishing in the lake, the Department decided to sell a permit that gives the permit holder the sole right to fish in Lake Santa Barbara for j years, after which the permit will be awarded to a new recipient (a permit holder cannot reapply for the permit). All Bellman Fish eggs hatch on the first day of each new year, and after four months the fish are fully grown. In the fifth month of the year, the fish lay eggs. Bellman Fish reproduce with perfect replacement (e.g.: 100 young Bellman Fish will produce 100 eggs).

After applying for the permit, the California Department of Fish and Wildlife just informed you that you will be awarded the permit! In order to ensure that the population of Bellman Fish continues to grow, the Department tells you that you are only allowed to fish in June, July, and August (i.e. months 6, 7, and 8) to ensure that the Bellman Fish have a chance to lay eggs before you catch them. Let x_t denote the stock of fish (in pounds) in the lake on the second day of year t and let c_t denote the amount of fish (in pounds) that you catch in year t. Assume that you can catch the exact amount of fish that you want in a fishing season (recall that the Bellman Fish is easy to catch) and that your discount factor β is equal to one. Further assume that **after** you're done fishing, during the last four months of the year a *constant* fraction, $(1-\alpha)$, of the fish die, where $0 < \alpha \leq 1$. Assume you consume all fish that you catch in a given period, and that your utility is linear in fish: $u(c) = c$.

a.) (5 points) What is the law of motion for the stock of fish? In other words, given that there are x_t pounds of fish available at the beginning of year t, what is x_{t+1}?

$$x_{t+1} = x_t + \alpha(x_t - c_t)$$

b.) (10 points) Suppose that when you receive your permit, there are x_1 fish in the lake at the start of your first fishing year. Write down your maximization problem *sequentially*.

$$\begin{aligned}
\max_{\{c_t, x_{t+1}\}_{t=1}^{j}} \quad & \sum_{t=1}^{j} c_t \\
\text{s.t.} \quad & x_{t+1} = x_t + \alpha(x_t - c_t) \quad \forall t \\
& 0 \le c_t \le x_t \quad \forall t \\
& x_1 \text{ given}
\end{aligned}$$

c.) (10 points) Denote s as the number of years before the permit expires so that $s = 1$ denotes the last year that you are allowed to fish, $s = 2$ denotes the second to last year you are allowed to fish, etc (maintain this notation for the rest of the problem). (i) Write down the Bellman Equation for your maximization problem for an arbitrary period s. (ii) Identify the state variable(s).

(i)

$$V_s(x) = \max_{0 \le c \le x} [c + V_{s-1}(x + \alpha(x - c))]$$

(ii) State Variable: x

d.) (5 points) Recalling that when $s = 0$ your permit is expired and you can no longer fish, what is $V_0(x)$?

Since you are not able to consume anything in $s = 0$, the value of any stock of fish in this period is zero:

$$V_0(x) = 0$$

e.) (10 points) Find $V_1(x)$ and $V_2(x)$ (i.e. the Bellman equations when $s = 1$ and $s = 2$).

First we find $V_1(x)$:

$$\begin{aligned}
V_1(x) &= \max_{0 \le c \le x} [c + V_0(x + \alpha(x - c))] \\
V_1(x) &= \max_{0 \le c \le x} [c + 0] \\
V_1(x) &= \max[0, x] = x
\end{aligned}$$

Next we find $V_2(x)$:

$$\begin{aligned}
V_2(x) &= \max_{0 \le c \le x} [c + V_1(x + \alpha(x - c))] \\
V_2(x) &= \max_{0 \le c \le x} [c + (x + \alpha(x - c))]
\end{aligned}$$

Now note that since $c + (x + \alpha(x - c))$ is linear in c, the maximum occurs at either $c = 0$ or $c = x$. Then we have

$$\begin{aligned}
V_2(x) &= \max[0 + (x + \alpha(x - 0)), x + (x + \alpha(x - x))] \\
V_2(x) &= \max[x + \alpha x, x + x] \\
V_2(x) &= \max[(1 + \alpha)x, 2x] \\
V_2(x) &= \max[1 + \alpha, 2]x
\end{aligned}$$

f.) (10 points) Guess that $V_s(x) = \psi_s x$ where ψ is a function of α, s, and ψ_{s-1} (hint: it might be helpful to find $V_3(x)$)

Note that from part c we have

$$V_2(x) = \max[1 + \alpha, 2]x$$

Looking at our guess, we have that $\psi_2 = \max[(1 + \alpha), 2]$. Let's try finding $V_3(x)$:

$$\begin{aligned}
V_3(x) &= \max_{0 \le c \le x} [c + V_2(x + \alpha(x - c))] \\
V_3(x) &= \max_{0 \le c \le x} [c + \max[(1 + \alpha), 2](x + \alpha(x - c))]
\end{aligned}$$

Again noting the linearity we have:

$$\begin{aligned}
V_3(x) &= \max[0 + \max[(1 + \alpha), 2](x + \alpha(x - 0)), x + \max[(1 + \alpha), 2](x + \alpha(x - x))] \\
V_3(x) &= \max[\max[(1 + \alpha), 2](1 + \alpha)x, x + \max[(1 + \alpha), 2]x] \\
V_3(x) &= \max[\max[(1 + \alpha), 2](1 + \alpha), 1 + \max[(1 + \alpha), 2]]x \\
V_3(x) &= \max[\max[(1 + \alpha), 2](1 + \alpha), 1 + \max[(1 + \alpha), 2]]x
\end{aligned}$$

Substituting in $\psi_2 = \max[(1 + \alpha), 2]$ we have:

$$\begin{aligned}
V_3(x) &= \max[\psi_2(1 + \alpha), 1 + \psi_2]x \\
V_3(x) &= \max[\psi_2 + \alpha\psi_2, 1 + \psi_2]x \\
V_3(x) &= (\psi_2 + \max[\alpha\psi_2, 1])x \\
V_3(x) &= \psi_3 x
\end{aligned}$$

Then our guess implies that

$$\psi_s = \psi_{s-1} + \max[\alpha\psi_{s-1}, 1]$$

Now let's verify that our policy function guess is correct:

$$\begin{aligned}
V_s(x) &= \max_{0\leq c\leq x}[c+\psi_{s-1}(x+\alpha(x-c))] \\
V_s(x) &= \max[0+\psi_{s-1}(x+\alpha(x-0)), x+\psi_{s-1}(x+\alpha(x-x))] \\
V_s(x) &= \max[\psi_{s-1}(1+\alpha)x, x+\psi_{s-1}x] \\
V_s(x) &= \max[\psi_{s-1}(1+\alpha), 1+\psi_{s-1}]x \\
V_s(x) &= \max[\psi_{s-1}+\alpha\psi_{s-1}), 1+\psi_{s-1}]x \\
V_s(x) &= (\psi_{s-1}+\max[\alpha\psi_{s-1}), 1])x \\
V_s(x) &= \psi_s x
\end{aligned}$$

This verifies our guess.

g.) (10 points) What is the policy function?

From part e we have

$$\psi_s = \psi_{s-1} + \max[\alpha\psi_{s-1}, 1]$$

Then note that

$$\begin{aligned}
\psi_1 &= \psi_0 + \max[\alpha\psi_0, 1] \\
\psi_1 &= 0 + \max[0, 1] \\
\psi_1 &= 1
\end{aligned}$$

Also note

$$\begin{aligned}
\psi_2 &= \psi_1 + \max[\alpha\psi_1, 1] \\
\psi_2 &= 1 + \max[\alpha, 1] \\
\psi_2 &= \max[1+\alpha, 2] \\
\psi_2 &= 2 && \text{(Since } 0<\alpha\leq 1\text{)}
\end{aligned}$$

Finally note

$$\begin{aligned}
\psi_3 &= \psi_2 + \max[\alpha\psi_2, 1] \\
\psi_3 &= 2 + \max[2\alpha, 1] \\
\psi_3 &= \max[2+2\alpha, 3]
\end{aligned}$$

Note that the last line above cannot be evaluated without knowing α. If $\alpha > 0.5$, then $2+2\alpha > 3$ and if $\alpha \leq 0.5$, then $2+2\alpha \leq 3$. Note that you choose $2+2\alpha$ when $c = 0$, or when you don't fish at all. Therefore we have that when you don't fish $\psi_s = \psi_{s-1} + \alpha\psi_{s-1}$, and when you fish all available fish $\psi_s = s$. Thus you should fish when

$$\psi_{s-1} + \alpha\psi_{s-1} < s$$

and not fish otherwise. This leads to a a cutoff period before which you don't fish and after which you fish all available fish. Let s^* denote the period after which you begin to fish all available fish in every period. Then we have

$$\psi = \begin{cases} s & \text{for } s \le s^* \\ (1+\alpha)^{s-s^*} s^* & \text{for } s > s^* \end{cases}$$

Then s^* is the smallest integer such that

$$1 + s^* \le (1+\alpha) s^*$$

Problem 2: Searching for ideas

After passing prelims, you enter your second year of the Ph.D. program at UCSB and begin searching for paper ideas. You will be a student for 5 years, and at the end of your 5th year you will go on the job market and get a job.

Part I

Assume that you need only one paper to graduate, which is called your job market paper. After you graduate, you get a job with quality equal to that of your job market paper. Thus the utility that you receive from getting your job in year 5 is equal to the quality of your job market paper. You work on ideas in years 2, 3, 4, and 5, and at the end of year 5 you go to the job market and get a job. Let t denote the time remaining until you go to the job market. Then $t = 0$ is your fifth year, $t = 1$ is your fourth year, $t = 3$ is your 3rd year, and $t = 4$ is your 2nd year. Each period you receive instantaneous utility equal to the quality of the paper you choose. Thus if you choose to use your paper idea in period t, your instantaneous utility is p_t. If you do not have a paper idea, your instantaneous utility is zero.

At the beginning of each year $t \in \{4, 3, 2, 1\}$ you get an idea for a new paper of quality p_t. Your ideas come from a distribution $F(p)$ with support 0 to $\bar{p}$, and you develop your paper idea over the course of the year. Assume that when you accept a new paper idea, you must keep that paper idea every year until you graduate and that every year *after* you choose your new idea (including year 5) the quality of the paper grows by θ where $\theta > 1$. Then accepting a paper idea in year $t = k$ gives the following utility stream:

$$\sum_{i=0}^{k-1} \beta^i \theta^i p_k$$

a.) (10 points) Write out your Bellman equation in discrete time in period $t = 0$, assuming that you haven't chosen a paper idea in the previous periods. Denote this Bellman Equation $U_0(p)$. What is the reservation paper quality p_R when $t = 0$?

$$U_0(p) = \max_{\text{accept, reject}} [0, p] \implies p_R = 0$$

If you come in to year 5 with no job market paper, you will take any paper idea that you draw since $0 \leq p \leq \bar{p}$ (in other words, you can't be any worse off than having no paper at all). Therefore the reservation paper quality in period $s = 0$ is $p_R = 0$.

b.) (10 points) Write out your Bellman equation in discrete time in period $t = 1$, assuming that you haven't chosen a paper idea in the previous periods. What is the reservation paper quality p_R when $t = 1$?

$$U_1(p) = \max_{\text{accept, reject}} [\beta \int_0^{\bar{p}} U_0(p')dF(p'),\ p + \beta\theta p]$$
$$U_1(p) = \max_{\text{accept, reject}} [\beta \int_0^{\bar{p}} \max[0, p']dF(p'),\ p + \beta\theta p]$$
$$U_1(p) = \max_{\text{accept, reject}} [\beta \int_0^{\bar{p}} p'dF(p'),\ p + \beta\theta p] \qquad (\spadesuit)$$

We can now find the reservation paper quality:

$$p_R + \beta\theta p_R = \beta \int_0^{\bar{p}} p'dF(p')$$
$$p_R = \frac{\beta}{1 + \beta\theta} \int_0^{\bar{p}} p'dF(p')$$

For a different representation of the reservation wage, we can use integration by parts and find:

$$\begin{aligned} \int_0^{\bar{p}} p'dF(p') &= p'F(p')\Big|_0^{\bar{p}} - \int_0^{\bar{p}} F(p')dp' \\ &= \bar{p}F(\bar{p}) - 0 \cdot F(0) - \int_0^{\bar{p}} F(p')dp' \\ &= \bar{p} - \int_0^{\bar{p}} F(p')dp' \\ &= \int_0^{\bar{p}} 1dp' - \int_0^{\bar{p}} F(p')dp' \\ &= \int_0^{\bar{p}} (1 - F(p'))dp' \end{aligned} \qquad (\clubsuit)$$

Then plugging ($\clubsuit$) into ($\spadesuit$) we have

$$U_1(p) = \max_{\text{accept, reject}} [\beta \int_0^{\bar{p}} (1 - F(p'))dp',\ p + \beta\theta p]$$

Now we can find the reservation wage for period $s = 1$

$$
\begin{aligned}
p_R + \beta\theta p_R &= \beta \int_0^{\bar{p}} (1 - F(p'))dp' \\
p_R(1 + \beta\theta) &= \beta \int_0^{\bar{p}} (1 - F(p'))dp' \\
p_R &= \frac{\beta}{1 + \beta\theta} \int_0^{\bar{p}} (1 - F(p'))dp'
\end{aligned}
$$

c.) (5 points) Using your results from part b, what is the effect of an increase in θ on the reservation paper quality p_R? Briefly explain the intuition.

$$
\begin{aligned}
p_R &= \frac{\beta}{1 + \beta\theta} \int_0^{\bar{p}} p' dF(p') \\
\frac{dp_R}{d\theta} &= -\beta^2(1 + \beta\theta)^{-2} \int_0^{\bar{p}} p' dF(p')
\end{aligned}
$$

An increase in θ decreases the reservation paper quality. The intuition is that if I know my paper quality is going to grow next period by θ and θ is large, then I am willing to accept a lower paper quality now since the reward next period is larger and thus the cost of waiting another period to accept an idea is larger.

Part II

For the following questions assume that if you choose a paper idea, you can continue to look for paper ideas while working on your current paper (think of this as 'on the academic job search'). As before if you have not accepted a paper idea your current period utility is zero, and if you accept a paper idea in this period your current paper quality increases by θ in each subsequent period. You come up with one paper idea at the beginning of each period when you have not yet accepted a paper idea. Now, when you accept a paper idea, you have a probability of ω of coming up with another paper idea next period.

d.) (5 points) Write out your Bellman equation in discrete time in period $t = 0$, assuming that you haven't chosen a paper idea in the previous periods. What is the reservation paper quality p_R when $t = 0$? Let $V_0(p)$ denote the value of entering period $t = 0$ when you have already accepted a paper idea in a previous period and you have a paper of quality p in hand. Write out $V_0(p)$.

Note that the the Bellman equation and reservation paper quality when $s = 0$ and you haven't accepted a paper idea remains unchanged:

$$U_0(p) = \max_{\text{accept, reject}} [0, p] \implies p_R = 0$$

Also note that if you have already been working on a paper idea that you received in period t, then you enter the final period with a paper of quality $\theta^t p_t$ and you have in hand an offer of quality p. Then

$$V_0(p) = \max_{\text{accept, reject}} [p,\ \theta^t p_t]$$

e.) (10 points) Write out your Bellman equation in discrete time in period $t = 1$, assuming that you haven't chosen a paper idea in the previous periods. What is the reservation paper quality p_R when $t = 1$?

Note that the value of not accepting an idea in $s = 1$ is:

$$\beta \int_0^{\bar{p}} U_0(p')dF(p')$$

The value of accepting the current period idea now takes into account that you can switch to a new idea next period. Thus we have

$$U_1(p) = \max_{\text{accept, reject}} [\beta \int_0^{\bar{p}} U_0(p')dF(p'),\ p + \beta[\omega \int_0^{\bar{p}} \max[p', \theta p]dF(p') + (1-\omega)\theta p]$$

Then our reservation wage satisfies

$$\begin{aligned}
\beta \int_0^{\bar{p}} U_0(p')dF(p') &= p_R + \beta[\omega \int_{\theta p_R}^{\bar{p}} [p' - \theta p_R]dF(p') + \theta p_R] \\
p_R + \beta\theta p_R &= \beta \int_0^{\bar{p}} U_0(p')dF(p') - \beta\omega \int_{\theta p_R}^{\bar{p}} [p' - \theta p_R]dF(p') \\
p_R &= \frac{\beta}{1+\beta\theta}\Big(\int_0^{\bar{p}} U_0(p')dF(p') - \omega \int_{\theta p_R}^{\bar{p}} [p' - \theta p_R]dF(p')\Big) \\
p_R &= \frac{\beta}{1+\beta\theta}\Big(\int_0^{\bar{p}} p'dF(p') - \omega \int_{\theta p_R}^{\bar{p}} [p' - \theta p_R]dF(p')\Big) \qquad \text{(Plugging in } U_0(p'))
\end{aligned}$$

Note that using integration by parts on our two integrals leads to the following (equivalent) representation of the reservation paper quality:

$$p_R = \frac{\beta}{1+\beta\theta}\Big(\int_0^{\bar{p}} (1 - F(p'))dp' - \omega \int_{\theta p_R}^{\bar{p}} [1 - F(p')]dp'\Big)$$