Week 2

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Equilibria

1. **Arrow-Debreu Equilibria:** agents trade contingent claims (Arrow-Debreu securities) in period zero to fund each other in any state of the world
2. **Sequential Equilibria:** agents make savings and consumption decisions *each period* given the observed state
3. **Recursive Equilibria:** Simplifies the sequential problem down to a decision today, given that we will optimize in the future

Recall that you saw the equivalence theorem relating Arrow-Debreu and Sequential Equilibria in class. Under certain conditions Sequential and Recursive economies have the same solution.

Today

- Recall in the last section I mentioned that we could write our sequential problem recursively
- Today we will show that the solutions to the two problems are the same
- We will also have a deeper discussion about the properties of $V(\cdot)$

Intuition

Before showing a more general result, we will start with some intuition. Consider a two period problem in which you are given an initial endowment of capital k_0 and you must decide how to optimally consume it over the two periods. The problem can be written sequentially as follows:

$$\max_{\{c_t\}_{t=0}^{1}} \sum_{t=0}^{1} \beta^t u(c_t)$$

$$\text{s.t. } c_t + k_{t+1} = k_t$$

We can plug in our budget constraint and our problem becomes

$$V_0(k_0) = \max_{\{0 \leq k_{t+1} \leq k_t\}_{t=0}^{1}} \sum_{t=0}^{1} \beta^t u(k_t - k_{t+1})$$

Intuition Cont.

- Since we have only two periods, we can write out the summation to see our problem a little more clearly:

$$V_0(k_0) = \max_{\{0 \leq k_{t+1} \leq k_t\}_{t=0}^{1}} u(k_0 - k_1) + \beta u(k_1 - k_2)$$

Intuition Cont.

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- Let's think about period 2. In period 2 we have k_1 left over (where $0 \leq k_1 \leq k_0$), and we need to consider how much of k_1 to consume and how much to save.

Intuition Cont.

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- What should k_2 be?

Intuition Cont.

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- What should k_2 be?
  - $k_2 = 0$ (why save for tomorrow when there is no tomorrow?)

Intuition Cont.

- We can rewrite our problem as

$$V_0(k_0) = \max_{0 \leq k_1 \leq k_0} u(k_0 - k_1) + \beta u(k_1)$$

- Note that we now only need to choose k_1 to solve the problem
- This is because we optimized in the last period, therefore we could write our original problem as

$$V_0(k_0) = \max_{0 \leq k_1 \leq k_0} \left[u(k_0 - k_1) + \beta \max_{0 \leq k_2 \leq k_1} u(k_1 - k_2) \right]$$

or equivalently as

$$V_0(k_0) = \max_{0 \leq k_1 \leq k_0} \left[u(k_0 - k_1) + \beta V_1(k_1) \right]$$

General Version

- Consider the following problem written sequentially:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t. } c_t + k_{t+1} = f(k_t) + (1-\delta)k_t$$

- We want to show that we can write our problem recursively as follows

$$V(k) = \max_{c}\{u(c) + \beta V(k')\}$$

$$\text{s.t. } c + k' = f(k) + (1-\delta)k$$

Showing Equivalence

Theorem: Solution to sequential problem also solves the recursive problem

$$V(k_0) = \overbrace{\max_{\{0 \leq k_{t+1} \leq f(k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1})}^{sequential\ \ problem}$$

$$= \max_{\{0 \leq k_{t+1} \leq f(k_t)\}_{t=0}^{\infty}} [U(f(k_0) - k_1) + \beta \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t) - k_{t+1})]$$

$$= \max_{0 \leq k_1 \leq f(k_0)} [U(f(k_0) - k_1) + \beta \max_{\{0 \leq k_{t+1} \leq f(k_t)\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(f(k_t) - k_{t+1})]$$

$$= \underbrace{\max_{0 \leq k_1 \leq f(k_0)} [U(f(k_0) - k_1) + \beta V(k_1)]}_{recursive\ \ problem}$$

Showing Equivalence

Theorem: If $\lim_{T\to\infty} \beta^{T+1} V(k_{T+1}) = 0$ for all $\{k_{T+1}\}$ s.t. $0 \leq k_{T+1} \leq f(k_{T+1})$, then V satisfies SP

$$V(k_0) = \overbrace{\max_{0\leq k_1 \leq f(k_0)} [U(f(k_0) - k_1) + \beta V(k_1)]}^{recursive\ \ problem}$$

$$= \max_{0\leq k_1 \leq f(k_0)} \{U(f(k_0) - k_1) + \beta \max_{0\leq k_2 \leq f(k_1)} [U(f(k_1) - k_2) + \beta V(k_2)]\}$$

$$= \max_{\{0\leq k_{t+1} \leq f(k_t)\}_{t=0}^{1}} \{\sum_{t=0}^{1} \beta^t [U(f(k_t) - k_{t+1}) + \beta^2 V(k_2)\}$$

$$\vdots$$

$$= \max_{\{0\leq k_{t+1} \leq f(k_t)\}_{t=0}^{T}} \{\sum_{t=0}^{T} \underbrace{\beta^t [U(f(k_t) - k_{t+1})}_{sequential\ \ problem} + \beta^{T+1} V(k_{T+1})\}$$

Dynamic Programming Overview

- The key idea underlying dynamic programming is that we have a finite dimensional state which at every point in time fully characterizes all information necessary to make choices
  - k_t in our two period problem
- When working with dynamic programming problems we will be able to solve for a *policy function* that will tell us how to act in each period given the current state
  - This means we don't need to keep track of an infinite history in order to act optimally, we just need to know the current state of the world (in our example problem, how much k I have left)

Bellman Equation

Our recursive formulation utilizes the *Bellman Equation* which is a *functional* equation as it maps the space of functions into functions. It's general form is as follows:

$$V(x) = sup_{y \in \Gamma(x)}\{F(x, y) + \beta V(y)\}$$

- $F(x, y)$ is current period objective function (profit function, utility function, etc.)
- x is a finite dimensional vector we call the **state**
- y is defined as the **choice** and Γ denotes the set of choices available given that the state is x

Policy Correspondence

- We can now define the **policy correspondence** as:

$$G(x) = \{y \in \Gamma(x) : V(x) = F(x, y) + \beta V(y)\}$$

- Given state x, set of all values of y that attain the maximum of $F(x, y) + \beta V(y)$
- We call $G(x)$ the **policy function** if it is single valued (i.e. unique optimum)

Thinking carefully about the problem

- Questions:
  1. Does the Bellman Equation have a solution?
  2. If so, is it unique?
- The key to answering the first question is to set up our Bellman Equation as a fixed-point problem
- But first we need to make two important assumptions:
  - **Assumption 4.3 (SLP):** X is a convex subset of $\mathbb{R}^l$, and the correspondence $\Gamma : X \to X$ is nonempty, compact-valued, and continuous
  - **Assumption 4.4 (SLP):** The function $F : A \to \mathbb{R}$ is bounded and continuous, and $0 < \beta < 1$

A Fixed-Point Problem

- To answer question 1, define the **Bellman Operator**:

$$(Tf)(x) = sup_{y \in \Gamma(x)} \{F(x,y) + \beta f(y)\}$$

- Define metric space (S, ρ):

$$S \equiv \{f : X \to \mathbb{R} \text{ continuous and bounded}\}$$

with the sup norm

$$\|f\| = sup_{x \in X} |f(x)|$$

$$\rho(f,g) = \|f - g\| = sup_{x \in X} |f(x) - g(x)|$$

Theorem of the Maximum

Theorem of the Maximum (SLP): Let $X \subseteq \mathbb{R}^l$ and $Y \subseteq \mathbb{R}^m$, let $f : X \times Y \to \mathbb{R}$ be a continuous function, and let $\Gamma : X \to Y$ be a compact-valued and continued correspondence. Then the function $h : X \to \mathbb{R}$ (i.e. $h(x) = \max_{y \in \Gamma(x)} f(x, y)$) is continuous, and the correspondence $G : X \to Y$ (i.e. $G(x) = \{y \in \Gamma(x) : f(x, y) = h(x)\}$) is nonempty, compact-valued, and upper hemi-continous (see SLP page 56 for discussion of u.h.c and l.h.c)

Properties of the Bellman Operator

- $(Tf)(x)$ is continuous
  - Follows from applying the Theorem of the Maximum with $f(x, y) = F(x, y) + \beta f(y)$ and $h(x) = (Tf)(x)$
- $(Tf)(x)$ is bounded
  - Follows since F is bounded by Assumption 4.4 and f is bounded
- We need to show that T has a unique fixed point
  - First show that T is a contraction using Blackwell's sufficient conditions
  - Second use Contraction Mapping Theorem to show V is the unique fixed point of T in S

Contraction Mappings

Definition (pg. 50 SLP): Let (S, ρ) be a metric space and $T : S \to S$ be a function mapping S into itself. T is a *contraction mapping* (with *modulus* β) if for some $\beta \in (0, 1)$, $\rho(Tx, Ty) \leq \beta\rho(x, y)$, for all $x, y \in S$.

- The idea is that T is a contraction mapping if, when applied to x and y, the distance gets "closer"
- Can be difficult to show that T is a contraction mapping, but luckily we have some handy sufficient conditions we can use

Blackwell's Sufficient Conditions

Blackwell's sufficient conditions for a contraction (SLP): Let $X \subseteq R^l$, and let $B(X)$ be a space of bounded functions $f : X \to \mathbb{R}$, with the sup norm. Let $T : B(X) \to B(X)$ be an operator satisfying

(a) (monotinicity) $f, g \in B(X)$ and $f(x) \leq g(x)$ for all $x \in X$, implies $(Tf)(x) \leq (Tg)(x)$, for all $x \in X$

(b) (discounting) there exists some $\beta \in (0, 1)$ such that $T(f + a)(x) \leq (Tf)(x) + \beta a$ for all $f \in B(X)$, $a \geq 0$, $x \in X$ (where $(f + a)(x) = f(x) + a)$

Then T is a contraction with modulus β.

Bellman Operator is a contraction mapping

Proof: Recall the form of the Bellman Operator

$$(Tf)(x) = sup_{y\in\Gamma(x)}\{F(x,y) + \beta f(y)\}$$

Assume, without loss of generality that $f(x) \leq g(x)$. Then

$$\begin{aligned}(Tf)(x) &= sup_{y\in\Gamma(x)}\{F(x,y) + \beta f(y)\}\\ (Tg)(x) &= sup_{y\in\Gamma(x)}\{F(x,y) + \beta g(y)\}\end{aligned}$$

Part (a):

$$\begin{gathered}\beta f(x) \leq \beta g(x)\\ F(x,y) + \beta f(x) \leq F(x,y) + \beta g(x)\\ sup_{y\in\Gamma(x)}\{F(x,y) + \beta f(y)\} \leq sup_{y\in\Gamma(x)}\{F(x,y) + \beta g(y)\}\\ (Tf)(x) \leq (Tg)(x)\end{gathered}$$

Part (b):

$$\begin{aligned}T(f+a)(x) &= sup_{y\in\Gamma(x)}\{F(x,y) + \beta[f(y) + a]\}\\ &= sup_{y\in\Gamma(x)}\{F(x,y) + \beta f(y)\} + \beta a\\ &= (Tf)(x) + \beta a\end{aligned}$$

Contraction Mapping Theorem

Contraction Mapping Theorem (SLP): If (S, ρ) is a complete metric space and $T : S \to S$ is a contraction mapping with modulus β, then

(a) T has exactly one fixed point V in S

(b) for any $v_0 \in S$, $\rho(T^n V_0, V) \leq \beta^n \rho(V_0, V)$, $n = 0, 1, 2, ...$

Thus, given that assumptions 4.3 and 4.4 hold, and that the Bellman Operator is a contraction mapping (by Blackwell's sufficient conditions), from the contraction mapping theorem we have that T has a unique fixed point in S (i.e. there is a unique continuous bounded function that solves the Bellman Equation)

Some Properties of V

- Some additional assumptions
  - **Assumption 4.5 (SLP):** $F(x, y)$ is strictly increasing in x
  - **Assumption 4.6 (SLP):** Γ is monotone in the sense that $x \leq x'$ implies $\Gamma(x) \subseteq \Gamma(x')$
  - **Assumption 4.7 (SLP):** $F(x, y)$ is strictly concave
  - **Assumption 4.8 (SLP):** Γ is convex
- **Theorem 4.7 (SLP):** If assumptions 4.3-4.6 are satisfied,, then V is strictly increasing
- **Theorem 4.8 (SLP):** If assumptions 4.3-4.4 and 4.7-4.8 are satisfied, then V is strictly concave and G is a continuous and single-valued function

Next time

- Another important property of V is differentiability which we'll talk about next week
- We will also look at solution techniques
- We will also start thinking about how to add uncertainty to our problems