Matthew Fitzgerald
Econ 204B
January 31, 2020

Problem Set # 3

Problem 1: Be careful what you study

You just graduated from UCSB and got a job as a two year postdoc in environmental economics at the University of Antarctica (that's what you get for doing your job market paper on emperor penguins). The University pays your room and board, and in addition you receive your entire two year salary of x_0 on your first day [1]. The university is located in a remote town called Bellingshausentown. There is only one bank, Bank of the South Pole, which offers an interest rate of r each month on the savings you deposit.

One day you're having lunch with the other postdocs and they start discussing their compensation packages. It turns out they all have the same payment scheme as you, except each postdoc received a different salary and has been hired by the university for a different number of months. Since you are the only economist in the group, they ask you to tell them the optimal amount to save each month given their compensation packages. Assume each individual discounts the future at a rate of $\beta < 1$ and that none of the postdocs plan on having any savings when they move on to their next job (hopefully somewhere warmer).

(a) Write down the postdoc's saving problem recursively (hint: Let the contract for a given postdoc last T months, and note that you do not receive any interest on your savings in the first period).

$$\begin{aligned} V_t(x) &= \max_{c,x'} \{u(c) + \beta V_{t+1}(x')\} \\ \text{s.t.} \quad & x' = (1+r)(x-c) \\ & x',\ c \geq 0 \end{aligned}$$

(b) Assume each postdoc's period utility function is $u(c) = c$, and define period $T+1$ as the month after the postdoc's contract is over. Solve for V_{T+1} (hint: at what points can the maximum occur for a linear function?).

Using linear utility and plugging our constraint in we have

$$V_t(x) = \max_{0 \leq c \leq x} \{c + \beta V_{t+1}((1+r)(x-c))\}$$

Now let's solve for V_{T+1}. Note that if we are in $T+1$, then there are no periods left. This means that the present value of utility next period will be zero no matter how much capital we choose to save. Mathematically we have $V_{T+1}(x) = 0\ \forall x$.

[1] Not to add insult to injury, but had you listened to Chris Costello in Bren and worked on fisheries management for your job market paper, the University of the Bahamas also had an open postdoc position this year focusing on the study of optimal management of the Caribbean Spiny Lobster fishery which offered a larger compensation plan. Perhaps now you regret telling Chris during your meeting that you were "more of a bird person".

(c) Using your answer from the previous part, write out the postdoc's problem in period T. Solve for V_T.

$$\begin{aligned}
V_T(x) &= \max_{0\le c\le x}\{c+\beta V_{T+1}((1+r)(x-c))\}\\
V_T(x) &= \max_{0\le c\le x}\{c+\beta(0)\}\\
V_T(x) &= \max_{0\le c\le x}\{c\}\\
V_T(x) &= \max\{0,x\}\\
V_T(x) &= x
\end{aligned}$$

(d) Does it ever make sense to consume part of the current period income? Explain the intuition (note that since the University is covering your room and board, zero consumption is fine here if it is optimal).

With linear utility it does not make sense to consume part of the current period income. A constrained optimization problem with a linear objective function will be maximized at one of its end points (i.e. it will be a corner solution), and therefore in our problem in each period the solution will be to either consume everything or consume nothing.

(e) Under what conditions are you indifferent between saving and consuming? Carefully explain your answer.

Let's go back one more period and solve for V_{T-1}.

$$\begin{aligned}
V_{T-1}(x) &= \max_{0\le c\le x}\{c+\beta V_T((1+r)(x-c))\}\\
V_{T-1}(x) &= \max_{0\le c\le x}\{c+\beta(1+r)(x-c)\} && (V_T \text{ is linear})\\
V_{T-1}(x) &= \max\{0+\beta(1+r)(x-0),\ x+\beta(1+r)(x-x)\}\\
V_{T-1}(x) &= \max\{\beta(1+r)x,\ x\}\\
V_{T-1}(x) &= \max\{\beta(1+r),\ 1\}x
\end{aligned}$$

From the above we can see that we are indifferent between saving and consuming when $\beta(1+r)=1$. Note the two competing effects: β is how much I care about tomorrow and $(1+r)$ is the return I get from postponing my consumption at least one more period. Imagine the extreme case when $\beta=0$. Then I don't care about the future at all and therefore I would have no reason to save for tomorrow. If β is larger than zero, but is close to zero, it means that I need to be offered a large return tomorrow in order to save given that I much prefer consumption today to consumption tomorrow (i.e. since waiting to consume is costly for me in the sense that I have a low β, I need to be offered a large interest rate to choose to hold off my consumption until tomorrow).

Now imagine the other extreme when $\beta=1$ (note that the problem already specifies that $\beta<1$, but we can think about this case to help build intuition). Then my decision to save depends on whether $(1+r)>1$, which is the case when I earn some positive interest on my savings. If $\beta=1$, I care about all periods equally. Then since I am indifferent between consuming today and consuming tomorrow, and if I consume tomorrow I get to consume more, I will always choose to save until there is no tomorrow (i.e. the last period). This may be hard to internalize since we typically think of people caring at least a little more about today than tomorrow. To give a more concrete example, imagine that I offer you \$10 right now, or you can wait 15 seconds and I'll give you \$11. Since you probably are fairly indifferent between getting paid now or in 15 seconds, there would be no reason for you to take the \$10 over the \$11 and thus you would wait.

Now back to the problem at hand. If β is not one, but is very close to one, then I care a great deal about consumption tomorrow and therefore I will choose to save even at relatively small interest rates so long as $\beta(1 + r) > 1$. Therefore in order for me to save, as β decreases the rate of return for savings needs to increase to make it worth it for me to wait to consume until a future period.

(f) Assume $\beta = 1$. How does this change your consumption behavior? Explain the intuition.

As was discussed in the answer to part (e), if $\beta = 1$ then I care equally about all periods. Then if I earn positive interest on savings, I will hold off consumption until the last period when I will consume everything.

Problem 2: The Fountain of Youth

Tired of the cold, you decide to go to the Bahamas for spring break. After flying in to the South Bimini airport, you order a Lyft to take you to your hotel. During the ride you start up a conversation with your driver Ponce, a Spanish expat who's lived on the island for many years. Ponce tells you about a beautiful remote area on the island and after dropping your things at the hotel, you decide to check it out. On your way you run out of water and notice a small spring just off the path. You take a sip of water from the spring and instantly feel rejuvenated. It turns out the spring is actually the Fountain of Youth and now you will live forever[2]! That night after dinner you buy a cake to celebrate, but the economist in you instantly realizes your mistake. You now have to decide how to optimally eat the cake over an infinite lifespan.

(a) Write down your maximization problem recursively, taking into account your recently acquired immortality.

$$
\begin{aligned}
V(k) &= \max_{c,k'}\{u(c) + \beta V(k')\} \\
\text{s.t.} \quad & c + k' = k \\
& k',\ c \geq 0
\end{aligned}
$$

We can also plug in the constraint and write

$$V(k) = \max_{k' \geq 0}\{u(k - k') + \beta V(k')\}$$

[2]The interested reader can find a history of the legend of the Fountain of Youth here: https://www.history.com/news/the-myth-of-ponce-de-leon-and-the-fountain-of-youth

(b) Derive the Euler equation.

Taking the FOC we have

$$
\begin{aligned}
-u'(k-k') + \beta V'(k') &= 0 \\
u'(k-k') &= \beta V'(k') \qquad (\spadesuit)
\end{aligned}
$$

Using the Benveniste-Sheinkman Theorem we have

$$
\begin{aligned}
\frac{dV(k)}{k} &= u'(k-k') \\
\frac{dV(k')}{k'} &= u'(k'-k'') \qquad \text{(Pushing forward)}
\end{aligned}
$$

Then plugging the above expression into ($\spadesuit$) we have

$$
u'(c) = \beta u'(c') \qquad \text{(Euler Equation)}
$$

(c) Assume you have log utility. Then guess that the value function takes the following form:

$$
V(k) = A + Bln(K)
$$

Verify that this guess is consistent with optimization, and solve for A and B.

Note that this problem is a special case of the Guess and Verify example from the notes with $\alpha = 1$. With log utility our Bellman Equation can be written as

$$
V(k) = \max_{k'}\{ln(k-k') + \beta V(k')\}
$$

Now we guess that $V(k) = A + Bln(k)$:

$$
A + Bln(k) = \max_{k'}\{ln(k-k') + \beta A + Bln(k')\}
$$

Then taking the FOC yields

$$
\begin{aligned}
\frac{d}{dk}\Big\{ln(k-k') + \beta[A + Bln(k')]\Big\} &= 0 \\
\frac{1}{k-k'} &= \frac{\beta B}{k'} \\
k' &= \beta Bk - \beta Bk' \\
(1+\beta B)k' &= \beta Bk \\
k' &= \frac{\beta B}{(1+\beta B)}k \qquad \text{(Policy Function)}
\end{aligned}
$$

Note that we have found the optimal k' above. Therefore, plugging our expression for k' into the max function allows us to drop the max operator and write

$$\begin{aligned}
A + Bln(k) &= ln\Big(k - \frac{\beta B}{1+\beta B}\Big) + \beta\Big[A + Bln\Big(\frac{\beta B k}{1+\beta B}\Big)\Big] \\
&= ln\Big(\frac{k}{1+\beta B}\Big) + \beta A + \beta B ln\Big(\frac{\beta B k}{1+\beta B}\Big)
\end{aligned}$$

Now we can separate the k terms from the constant terms:

$$A + Bln(k) = \underbrace{-ln(1+\beta B) + \beta A + \beta B ln\Big(\frac{\beta B}{1+\beta B}\Big)}_{constant} + \underbrace{(1+\beta B)}_{ln(k)\text{-term coeff.}} \; ln(k)$$

Then we can line up our constants and solve. Let's start with B:

$$\begin{aligned}
B &= ln(k)\text{-term coeff.} = (1+\beta B) \\
B &= 1 + \beta B \\
B &= \frac{1}{1-\beta}
\end{aligned}$$

Now let's solve for A:

$$\begin{aligned}
A &= \text{constant} = -ln(1+\beta B) + \beta A + \beta B ln\Big(\frac{\beta B}{1+\beta B}\Big) \\
A &= -ln(1+\beta B) + \beta A + \beta B ln\Big(\frac{\beta B}{1+\beta B}\Big) \\
A &= -ln(1+\beta\frac{1}{1-\beta}) + \beta A + \beta\frac{1}{1-\beta} ln\Big(\frac{\beta\frac{1}{1-\beta}}{1+\beta\frac{1}{1-\beta}}\Big) \\
A - \beta A &= -ln(\frac{1}{1-\beta}) + \frac{\beta}{1-\beta} ln\Big(\frac{\frac{\beta}{1-\beta}}{\frac{1}{1-\beta}}\Big) \\
(1-\beta)A &= ln(1-\beta) + \frac{\beta}{1-\beta} ln(\beta) \\
A &= \frac{1}{1-\beta}\Big[ln(1-\beta) + \frac{\beta}{1-\beta} ln(\beta)\Big]
\end{aligned}$$

Plugging the expression for B into the policy function yields

$$\begin{aligned}
k' &= \frac{\beta\frac{1}{1-\beta}k}{1+\beta\frac{1}{1-\beta}} \\
k' &= \frac{\frac{\beta k}{1-\beta}}{\frac{1}{1-\beta}} \\
k' &= \beta k
\end{aligned}$$

(d) Now guess that the policy function takes the following form:

$$k' = \theta k$$

Verify that this guess is consistent with optimization, and solve for θ.

Recall our Euler Equation

$$u'(c) = \beta u'(c')$$
$$k' - k'' = \beta(k - k')$$

Then note that our guess gives us $k'' = \theta k'$, and $k' = \theta k$. Plugging this in we have

$$k' - k'' = \beta(k - k')$$
$$\theta k - \theta^2 k = \beta(k - \theta k)$$
$$(1 - \theta)\theta k = \beta(1 - \theta)k$$
$$\theta = \beta$$

Thus the policy function is

$$k' = \beta k$$

Then plugging this into the budget constraint we obtain

$$c + k' = k$$
$$c + \beta k = k$$
$$c = (1 - \beta)k$$

Problem 3: Fill in the blank value function iteration

Open the file "ValueFunctionIterationIncomplete.jl".

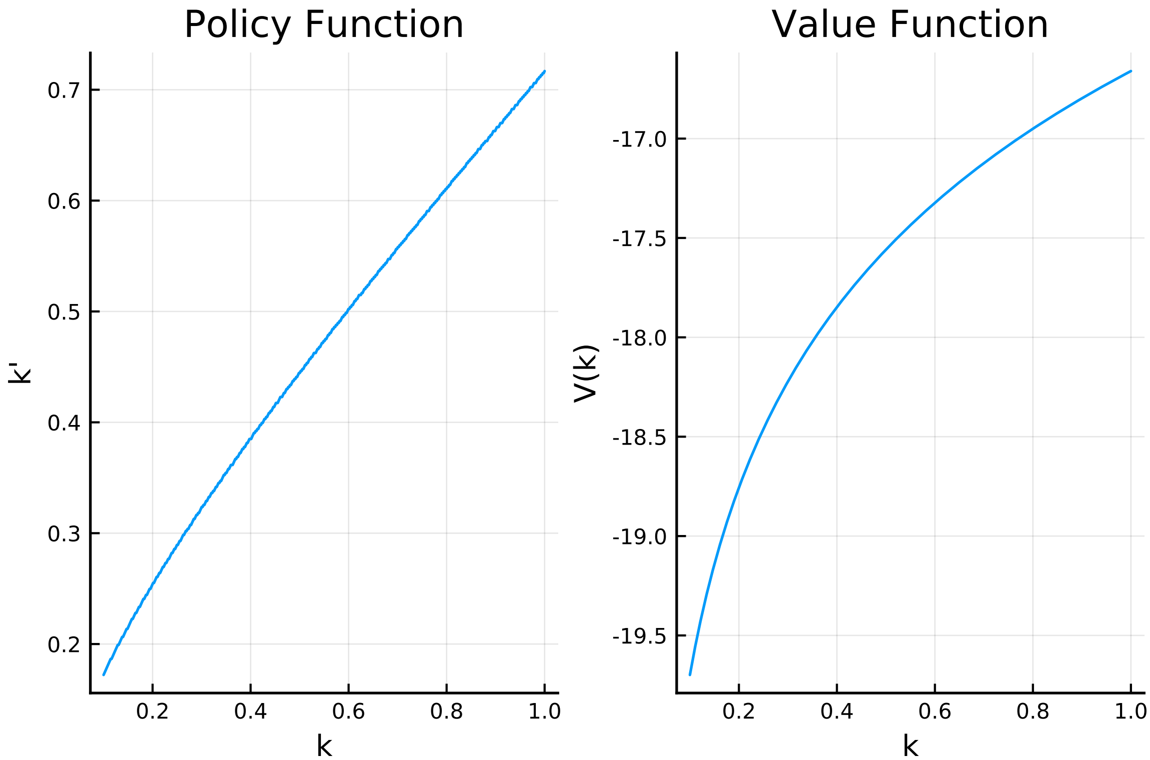
(a) This file has a simple version of value function iteration, but it is incomplete. Fill in the code where necessary and make sure it runs properly.

See script "ValueFunctionIteration.jl"

(b) Try the function "val_fun_iter" using the production function you created in the script, 500 grid points, an α of 0.3, and a δ of 0.5.

See script "ValueFunctionIteration.jl"

(c) Write additional code in the script that plots the value function and the policy function against k in separate graphs (Turn this script in, and rename it "ValueFunctionIterationYOURNAME.jl").



(d) Now try increasing the number of grid points to 2000. Why is *each iteration* slower when the number of grid points increases?

Note that when the number of points in the grid increases, the for loop goes through more iterations which will make the function take longer to complete. However, *each iteration* also takes longer because each iteration is now dealing with larger vectors. In line 76 the length of "k_grid" is now 2000 instead of 500, thus in line 80 the instantaneous utility function is being evaluated on 1500 more points, etc. Therefore increasing the number of grid points increases the amount of time the function takes to run both by creating more iterations in the for loop, and by making each iteration take longer due to the larger length of "k_grid".