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A model equation for steady surface waves over a bump

S.P. SHEN,[1] M.C. SHEN[2,*] and S.M. SUN[2]
[1]Department of Mathematics, Texas A&M University, College Station, TX 77843, USA
[2]Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA (*author for correspondence)

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Abstract. The objective of this paper is to study the solutions of a model equation for steady surface waves on an ideal fluid over a semicircular or semielliptical bump. For upstream Froude number $F > 1$, we show that the numerical solution of the equation has two branches and there is a cut-off value of F below which no solution exists. For $F < 1$, the problem is reformulated to overcome the so-called infinite-mass dilemma. A branch of solutions and a cut-off value of F, above which no solution exists, are found. Furthermore, we also obtain a branch of hydraulic-fall solutions which decrease monotonically from upstream to downstream.

1. Introduction

The problem considered here concerns steady surface waves on a two-dimensional, incompressible, inviscid fluid flow over a small bump on a flat bottom. Assume that the depth $H^*_{-\infty}$ and the speed c of the fluid flow far upstream are constant and an upstream Froude number F is defined by $F = c/(gH_0)^{1/2}$ where H_0 is the constant depth of the fluid flow as the size of the bump becomes zero. We call a solution to this problem supercritical (subcritical) if $F > 1$ ($F < 1$). Numerical computations of steady solutions to exact equations for a semicircular bump (Forbes and Schwartz [1], Vanden-Broeck [2], and Forbes [3]) indicate the following results. For $1 < F_+ < F$ there are two branches of supercritical solutions and no solution exists below F_+. Each supercritical solution behaves like a solitary wave. As the size of the bump tends to zero, one branch approaches the uniform state far upstream and the other branch approaches a solitary wave. As F increases, the branch of larger solutions approaches a limiting configuration with a 120° angle at the crest. For $F < F_- < 1$, only one branch of subcritical solutions is found and no solution exists above F_-. They exhibit a quiescent region upstream and a Stokes wave train downstream. In $F_- < F < F_+$ even if no steady solution exists unsteady waves can appear. Recently a solution which behaves like a hydraulic fall with $F < 1$ and the downstream Froude number $F_d = C_\infty/(gH_0)^{1/2} > 1$ has been found [3], where C_∞ is the constant speed at $x = \infty$. The solution remains almost constant up to the obstacle, then decreases monotonically to a constant value far downstream.

If $F = 1 + \varepsilon F_1$ is close to unity where ε is a small positive parameter, an inhomogeneous nonlinear ordinary differential equation can be derived as a model equation for the study of the steady surface waves over the bump. The purpose of this paper is to study systematically the solutions of the model equation and compare them with the numerical solutions to the exact equations ([1] to [3]). In the following discussion, we shall restrict ourselves to the case of a semicircular or semielliptical bump. However, the general method may be applied to an arbitrary obstacle with a compact support. For $F_1 > 0$ it has been indicated in [4] that multiple supercritical solutions for a given bump may exist. Here we show numerically that there appear two supercritical solutions for each F_1 greater than some cut-off value. Both behave like a solitary wave. One approaches the uniform state and the other, a solitary wave, as the bump size tends to zero. For each $F_1 < 0$ below some cut-off value, there exists a solution equal to zero upstream and expressed in terms of the cn function downstream. However, the mean depth of the cn function is not zero; there is infinite mass increase or decrease and the perturbation scheme fails. This is the so-called infinite-mass dilemma. To overcome this difficulty, we assume that the perturbed free surface approaches a nonzero constant H_1 far upstream and impose the condition that the sum of H_1 and the mean depth of the downstream periodic solution vanishes. Under the condition the model equation still has one branch of subcritical solutions below a cut-off value of $F_1 < 0$, and both H_1 and the period of the downstream solution can now be determined. Assume again $F_1 < 0$ is given and relax the condition to resolve the infinite-mass dilemma. We may construct a solution with constant H_1 up to the obstacle and expressed in terms of the cn function downstream of the obstacle. As F_1 tends to a cut-off value from below, the downstream cnoidal solution approaches to a monotone profile with $F_d > 1$. It is found that there is only one value of H_1 such that the solution tends to $-H_1$ far downstream. These results agree well with the exact solutions obtained in [1], [2] and [3] except that the supercritical solution with a 120° angle at the crest is beyond the reach of the model equation. The model equation with $H_1 = 0$ was also derived in [5] and [6] among others. However, the question concerning the infinite-mass dilemma has not been discussed in the literature.

We formulate the problem and derive the model equation for steady surface waves in Section 2. Although the derivation is rather straightforward, we intend to make the needed modification of the usual formulation to overcome the infinite-mass difficulty. The numerical solutions are discussed and presented in Section 3.

2. Formulation

The configuration of the fluid flow over a bump is shown in Fig. 1. The governing equations are the following:

$$u^*_{x^*} + v^*_{y^*} = 0 . \tag{1}$$

$$\rho(u^* u^*_{x^*} + v^* u^*_{y^*}) = -p^*_{x^*} , \tag{2}$$

$$\rho(v^* v^*_{x^*} + v^* v^*_{y^*}) = -p^*_{y^*} - \rho g , \tag{3}$$

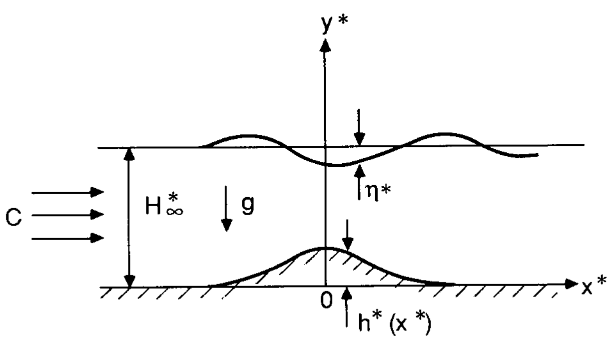

Fig. 1. Configuration of the fluid domain.

$$\text{at } y^* = H + \eta^*: \quad u^*\eta^*_{x^*} - v^* = 0\,, \qquad p^* = 0\,, \tag{4}$$

$$\text{at } y^* = h^*(x^*): \quad u^*h^*_{x^*} - v^* = 0 \tag{5}$$

where (u^*, v^*) is the velocity, p^* is the pressure, ρ is the constant density, g is the constant gravitational acceleration, and $y^* = H_0 + \eta^*$, $y^* = h^*$ are respectively the equations of the free surface and the bottom. We introduce the nondimensional variables

$$(u, v) = (u^*, \varepsilon^{-1/2}v^*)/(gH_0)^{1/2}\,, \qquad (x, y) = (\varepsilon^{1/2}x^*, y^*)/H_0\,,$$

$$p = p^*/(\rho g H_0)\,, \qquad \eta = \eta^*/H_0\,, \qquad h = \varepsilon^{-2}h^*/H_0\,,$$

$$H_{-\infty} = H^*_{-\infty}/H_0\,, \qquad \varepsilon^2 = H_0/L$$

where ε is a small positive parameter and L is the horizontal length scale. In terms of the nondimensional variables, (1) to (5) take the form

$$u_x + v_y = 0\,, \tag{6}$$

$$uu_x + vu_y = -p_x\,, \tag{7}$$

$$\varepsilon(uv_x + vv_y) = -p_y - 1\,, \tag{8}$$

$$\text{at } y = 1 + \eta: \quad u\eta_x - v = 0\,, \qquad p = 0\,, \tag{9}$$

$$\text{at } y = \varepsilon^2 h: \quad \varepsilon^2 uh_x - v = 0\,. \tag{10}$$

Assume that u, v, p and η possess asymptotic expansions of the form

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \cdots\,. \tag{11}$$

We also expand $H_{-\infty}$ in an asymptotic series as

$$H_{-\infty} = 1 + \varepsilon H_1 + \varepsilon^2 H_2 + \cdots\,, \tag{12}$$

and note that H_1 here plays a crucial role to resolve the infinite-mass dilemma. Without loss of generality, we let

$$F = F_0 + \varepsilon F_1 \tag{13}$$

where F_0 (called the critical Froude number) has to be determined. Substitution of (11) in (6) to (10) will yield a sequence of equations and boundary conditions for the successive approximations ϕ_n. The zeroth approximations are assumed to be given,

$$(u_0^{(0)}, v_0^{(0)}) = (F_0; 0), \qquad p_0^{(0)} = -y + 1\,, \qquad \eta_0^{(0)} = 0\,. \tag{14}$$

The equations for the first approximations are

$$u_{1x} = v_{1y} = 0\,, \tag{15}$$

$$F_0 u_{1x} + p_{1x} = 0\,, \tag{16}$$

$$p_{1y} = 0\,, \tag{17}$$

$$F_0 \eta_{1x} - v_1 = 0\,, \qquad p_1 = \eta_1 \quad \text{at } y = 1\,, \tag{18}$$

$$v_1 = 0 \quad \text{at } y = 0\,. \tag{19}$$

We see from (17) that p_1 is a function of x only and $p_1 = \eta_1$. It follows from (16) by integration with respect to x and the boundary conditions $u_0 \to F_0$, $\eta_0 \to H_1$ as $x \to -\infty$ that

$$u_1 = -F_0^{-1}(\eta_1 - H_1) + F_1\,. \tag{20}$$

Then, from (15), (19) and (20), we have

$$v_1 = F_0 \eta_{1x} y\,. \tag{21}$$

(18) and (21) then yield

$$F_0^2 = 1\,. \tag{22}$$

In the following we shall always choose $F_0 = 1$. The equations for the second approximations are

$$u_{2x} + v_{2y} = 0\,, \tag{23}$$

$$u_{2x} + u_1 u_{2x} + v_1 u_{1y} + p_{2x} = 0\,, \tag{24}$$

$$v_{1x} + p_{2y} = 0\,, \tag{25}$$

$$\left.\begin{aligned} &\eta_{2x} + u_1 \eta_{1x} - v_2 - v_{1y}\eta_1 = 0 \quad (26)\\ &p_2 = \eta_2 \quad (27) \end{aligned}\right\}\text{at } y = 1\,,$$

$$h_x - v_2 = 0 \quad \text{at } y = 0\,. \tag{28}$$

It follows from (21), (25) and (27) that

$$p_2 = -\eta_{1xx}[(y-1)^2/2 + (y-1)] + \eta_2\,. \tag{29}$$

Then from (20), (21), (24) and (29) we obtain an expression for u_{2x} in terms of η_1 and η_2, and substitute it in (23). Integration of the resulting equation with respect to y from $y = 0$ to $y = 1$ and use of (26), (28) and $F_0 = 1$ yield the equation for η_1,

$$(F_1 + H_1)\eta_{1x} - (3/2)\eta_1 \eta_x - (1/6)\eta_{1xxx} = h_x/2\,. \tag{30}$$

The above equation can be further integrated to yield

$$(F_1 + H_1)\eta_1 - (3/4)\eta_1^2 - (1/6)\eta_{1xx} = h/2 + (F_1 + H_1)H_1 - (3/4)H_1^2 \,. \tag{31}$$

where we have used the boundary condition $\eta_1 \to H_1$, $\eta_{1xx} \to 0$ as $x \to -\infty$. (30) is the equation to be studied in the next section and H_1 is an unknown constant to be determined as part of the solution.

3. Supercritical and subcritical solutions

Case 1. Supercritical solutions

In this case we look for solutions of (31) symmetric with respect to x, which together with its first derivative approach zero as $x \to \pm\infty$. Therefore we set $H_1 = 0$ and (30) becomes

$$F_1\eta_1 - (3/4)\eta_1^2 - (1/6)\eta_{1xx} = h/2 \,. \tag{32}$$

For $|x| \geq 1$, $h = 0$, and (32) can be further integrated to yield

$$F_1(\eta_1)^2 - (1/2)\eta_1^3 - (1/3)\eta_{1x} = 0 \,, \tag{33}$$

which possesses the well-known solitary-wave solution

$$\eta_1 = 2F_1 \operatorname{sech}^2[(6F_1)^{1/2}(x - x_0)/2] \,, \tag{34}$$

where x_0 is the phase shift. To find a solution in $|x| \leqslant 1$, we need only consider (32) in $-1 \leqslant x \leqslant 0$ subject to (33) at $x = -1$ and $\eta_{1x} = 0$ at $x = 0$. The above problem can be solved numerically by a shooting method and the phase shift x_0 is then determined by (34) for $x = -1$. In Fig. 2 we show the relationship between $A = \max_{-\infty<x<\infty} |\eta_1|$ and F_1 for $h = \alpha(1 - x^2)^{1/2}$, $|x| \leqslant 1$ and $h = 0$ otherwise, where $\alpha = 0.1$, 0.5 and 1. Two typical solutions corresponding to $F_1 = 1.4$, $\alpha = 1$ are shown in Fig. 3.

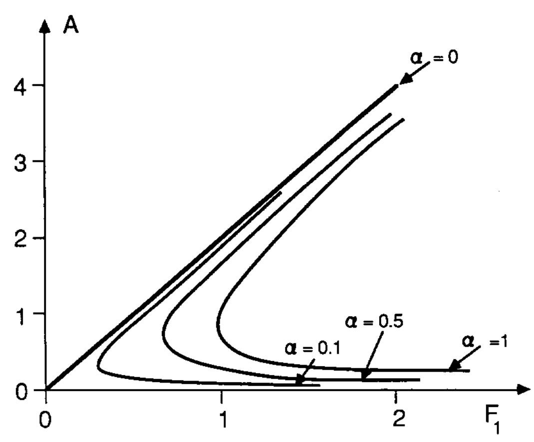

Fig. 2. Relationship between A and F_1 for supercritical solutions.

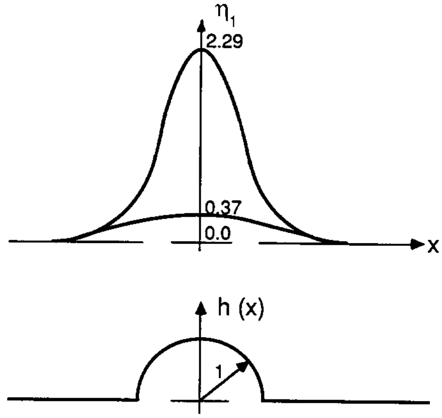

Fig. 3. Two typical supercritical solutions, $F_1 = 1.4$.

Case 2. Subcritical solutions

In this case we construct a solution which is constant for $x \leqslant -1$ and periodic for $x \geqslant 1$. However, for $H_1 = 0$, that is $\eta_1 \equiv 0$ in $x \leqslant -1$, there is no periodic solution of (32) for $x \geqslant 1$, whose mean value over one period is zero. Therefore, we impose the asymptotic condition of conservation of volume

$$X^{-1} \int_1^{X+1} \eta_1 \, \mathrm{d}x + H_1 = 0 , \tag{35}$$

where X is the period of η_1 in $x \geqslant 1$. (35) may be expressed in terms of the values of η_1, η_{1x} at $x = 1$. We multiply (31) by η_{1x} and integrate the resulting equation from 1 to x to obtain

$$\eta_{1x}^2 = -3(\eta_1 - H_1)^2(\eta + 2\lambda_1^2) + d = f(\eta_1) , \tag{36}$$

where $\lambda_1^2 = -F_1$ and

$$d = \eta_{1x}(1) + 3[\eta_1(1) - H_1][\eta_1(1) + 2\lambda_1^2] .$$

We require

$$4(2\lambda_1^2 + H_1)^3/9 > d > 0 , \tag{37}$$

so that $f(\eta_1) = 0$ has three distinct roots

$$\begin{aligned} \xi_0 &= (2/3)(2\lambda_1^2 + H_1) \cos \theta_1 - (2/3)(2/3)(\lambda_1^2 - H_1) , \\ \xi_1 &= -(2/3)(2\lambda_1^2 + H_1) \cos(\theta_1 + \pi/3) - (2/3)(\lambda_1^2 - H_1) , \\ \xi_2 &= -(2/3)(2\lambda_1^2 + H_1) \cos(\theta_1 - \pi/3) - (2/3)(\lambda_1^2 - H_1) , \\ \theta &= (1/3)\{\pi - \arccos[1 - (9/2)d(2\lambda_1^2 + H_1)^3]\} > 0 . \end{aligned} \tag{38}$$

The solution η_1 of (36) for $x \geq 1$ can be expressed as

$$\eta_1 = \xi_0 \cos^2 \tau + \xi_1 \sin^2 \tau \,, \tag{39}$$

where

$$(\alpha)^{1/2} x = \int_0^{\tau} (1 - \beta^2 \sin^2 \zeta)^{-1/2} \, d\zeta \,, \tag{40}$$

$$\alpha = (3/4)(\xi_0 - \xi_2) \,, \qquad \beta^2 = (\xi_0 - \xi_1)/(\xi_0 - \xi_2) \,. \tag{41}$$

The mean value of η_1 in $x \geq 1$ over a period X is

$$\left[2\xi_2 \int_0^{\pi/2} (1 - \beta^2 \sin^{-2} \zeta)^{-1/2} \, d\zeta + 2(\xi_0 - \xi_2) + \int_0^{\pi/2} (1 - \beta^2 \sin^2 \zeta)^{1/2} \, d\zeta \right] (\alpha^{1/2} X) \,, \tag{42}$$

where

$$X = 2(\alpha)^{-1/2} \int_0^{\pi/2} (1 - \beta^2 \sin^2 \tau)^{-1/2} \, d\tau \,. \tag{43}$$

Combining (35) and (42), we obtain the boundary condition at $x = 1$:

$$(H_1 + \xi_2) \int_0^{\pi/2} (1 - \beta^2 \sin^2 \tau)^{-1/2} \, d\tau + (\xi_0 - \xi_2) \int_0^{\pi/2} (1 - \beta^2 \sin^2 \tau)^{1/2} \, d\tau = 0 \,. \tag{44}$$

(36) subject to $\eta_{1x} = 0$ at $x = -1$ and (44) at $x = 1$ can also be solved by a shooting method. For a negative F_1 less than some cut-off value, we find one subcritical solution, which is equal to H_1 for $x \leq -1$ and periodic for $x \geq 1$. Here we consider the same h as in the supercritical case. In Fig. 4, we show the relationship between $A = \max_{-\infty<x<\infty} |\eta_1|$ and F_1 for two cases $\alpha = 0.5$, 1, and note that for each branch of solutions there is a cut-off value of F_1 above which no subcritical solution exists. As α tends to zero, a subcritical solution also approaches zero. A typical solution corresponding to $F_1 = -1.3$, $\alpha = 1$ is presented in Fig. 5.

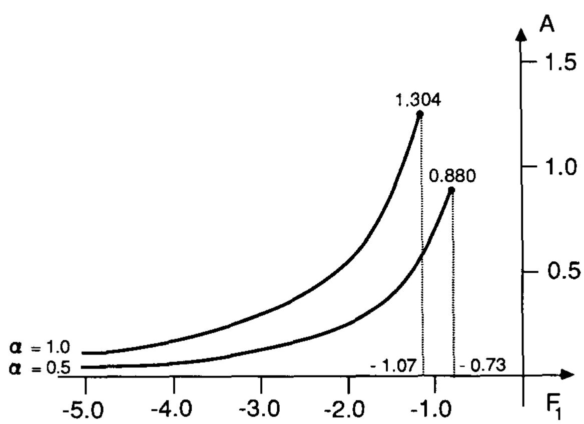

Fig. 4. Relationship between A and F_1 for subcritical solutions.

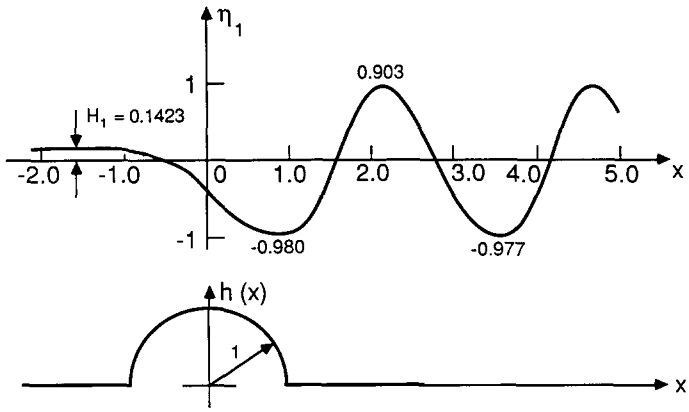

Fig. 5. A typical subcritical solution, $F_1 = -1.3$.

We also note that at a cut-off value of F_1 the solution generally remains periodic downstream, but for $F > F_1$ the amplitude of the downstream solution becomes unbounded.

We now relax the condition (35) and assume $\alpha = 1$. For a given H_1 and F_1 below some cut-off value $F_c < 0$, there exists a solution η_1, equal to H_1 for $x \leqslant -1$ and periodic for $x \geqslant 1$ as given by (38) to (41). As $F \to F_c$ from below, by (37) to (41), $\beta \to 1$, $X \to \infty$, $d \to (4/9)(2\lambda_1^2 + H_1)^3$, and the limiting solution of η_1 for $x \geqslant 1$ is

$$\eta_1 = -(2/3)(2\lambda_1^2 - H_1/2) + (2\lambda_1^2 + H_1)\,\mathrm{sech}^2[(\lambda_1^2 + H_1/2)^{1/2}x]\,.$$

As $x \to \infty$,

$$\eta_1 \to -(2/3)(2\lambda_1 - H_1/2) = \eta_1(\infty)\,.$$

In Fig. 6, we plot both H_1 and $-\eta_1(\infty)$ against F_c. The two lines intersect at $F_c = -0.8610$ and $F_c + H_1 \leqslant 0$ if $F_c \leqslant -0.8610$ and >0 if $F_c > -0.8610$. By conservation of mass flux, we have

$$F_c + H_1 = F_d + \eta_1(\infty)\,.$$

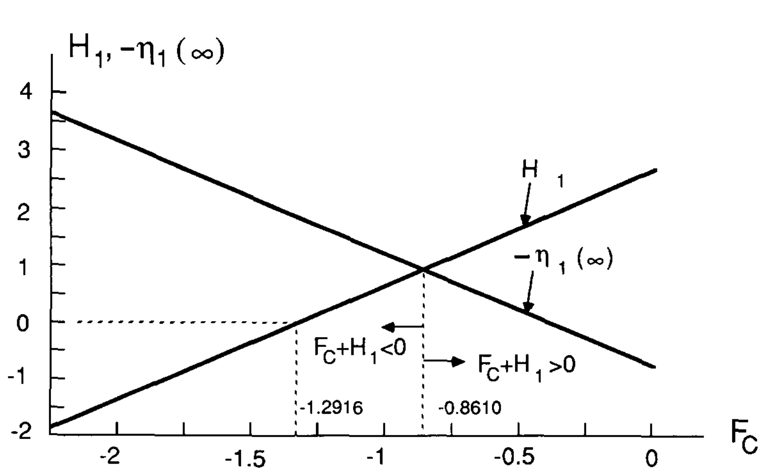

Fig. 6. H_1 and $-\eta_1(\infty)$ vs. F_c, $\alpha = 1$.

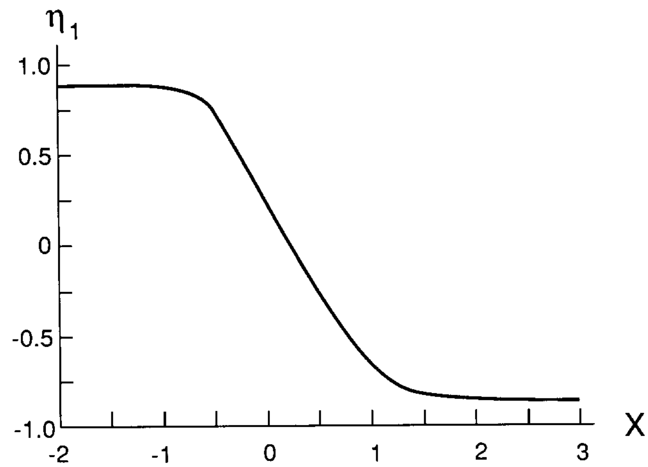

Fig. 7. Free-surface profile for $F_c = -0.8610$, $\alpha = 1$.

At $F_c = -0.8610$, $H_1 = -\eta_1(\infty) = F_d = 0.8610$. The flow represents a hydraulic-fall solution, which is subcritical upstream and supercritical downstream and satisfies asymptotically the condition of conservation of volume. The free-surface profile is shown in Fig. 7.

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