NEXT: Core Class Material

Have this slide up the moment that class begins. Give students 20-25 minutes to work on the problems. While students work, provide support, and seek out students whose work is worthy of full-group presentation. Ask students to present and let them know their work will be valued.

If available, take photos of student work to be presented, so students don't have to duplicate work in front of the room.

Getting Started

Join a group of 3-4 students and start working on today's problems.

Start of Core Class Material

NEXT: compute

If students are stuck on a and b, ask them what squaring means. If they are stuck on c, point out that squaring undoes the operation of a square root so c and b are related. If needed, point out that for d we want them to make a general rule for the product of two roots. Encourage writing a mathematical statement.

Key ideas: Don't accept $\sqrt{18}\cdot\sqrt{8}=\sqrt{18\cdot 8}$ - develop the property with logic

"Compute $\left(\sqrt{18}\cdot\sqrt{8}\right)^2$."

IF STUDENTS NEED A PUSH: **"What does squaring mean?"**

$$\begin{aligned}\left(\sqrt{18}\cdot\sqrt{8}\right)^2 &= \left(\sqrt{18}\cdot\sqrt{8}\right)\left(\sqrt{18}\cdot\sqrt{8}\right)\\ &= \sqrt{18}\cdot\sqrt{18}\cdot\sqrt{8}\cdot\sqrt{8}\\ &= 18\cdot 8 = \boxed{144}\end{aligned}$$

"Compute $\sqrt{(\sqrt{18}\cdot\sqrt{8})^2}$." From part a we have $\sqrt{(\sqrt{18}\cdot\sqrt{8})^2}=\sqrt{144}=12$

"Compute $\sqrt{18}\cdot\sqrt{8}$." We have $\sqrt{18}\cdot\sqrt{8}=\sqrt{18\cdot 8}=\sqrt{144}=\boxed{12}$

"So, what does this tell us about what $\sqrt{x}\cdot\sqrt{y}$ equals?"

Write these equations on screen:

$$\sqrt{18}\cdot\sqrt{8}=\sqrt{18\cdot 8}=\sqrt{144}.$$

$$\sqrt{x}\cdot\sqrt{y}=\sqrt{xy} \text{ for all nonnegative } x,y$$

Problem 1

Compute

a. $\left(\sqrt{18}\cdot\sqrt{8}\right)^2$.

b. $\sqrt{(\sqrt{18}\cdot\sqrt{8})^2}$.

c. $\sqrt{18}\cdot\sqrt{8}$.

d. What can we say about $\sqrt{x}\cdot\sqrt{y}$?

NEXT: Problem 2:compute order of operations

"Compute $\left(3\sqrt{8}\right)\left(5\sqrt{2}\right)$."

Key ideas: Don't simplify $\sqrt{8}$ unless students suggest it

If a student suggests that $\sqrt{8} = 2\sqrt{2}$, make the student explain why: using the rule we just learned,

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2 \cdot \sqrt{2}.$$

Otherwise, we'll get to simplifying square roots later today.

We rearrange to multiply the square roots, since we see that $8 \cdot 2$ is a perfect square:

$$\begin{aligned}\left(3\sqrt{8}\right)\left(5\sqrt{2}\right) &= 3 \cdot \sqrt{8} \cdot 5 \cdot \sqrt{2} \\ &= (3 \cdot 5) \cdot \left(\sqrt{8} \cdot \sqrt{2}\right) \\ &= 15 \cdot \sqrt{8 \cdot 2} \\ &= 15 \cdot \sqrt{16} \\ &= 15 \cdot 4 = \boxed{60}.\end{aligned}$$

Compute $\left(3\sqrt{8}\right)\left(5\sqrt{2}\right)$.

NEXT: Problem 3: compute division

Remind students of the order of operations and their discovery in problem 1. Some students will take $\sqrt{4\cdot 9} = \sqrt{36}$ others may use the property from problem 1 to do $\sqrt{4\cdot 9} = \sqrt{4}\cdot\sqrt{9}$. Show and discuss both routes. If no one used the second method present it and discuss.

Key ideas: Don't be scared by weird notation. Break it down into pieces you understand.

"This is pretty weird-looking. What piece do we know how to handle?"

We know how to handle the product of the square roots.

"What does that product give us?"

$\sqrt{3}\cdot\sqrt{27} = \sqrt{3\cdot 27} = \sqrt{81} = 9$, so

$$\sqrt{4\left(\sqrt{3}\cdot\sqrt{27}\right)} = \sqrt{4\cdot 9}.$$

"Ah-ha! How do we finish?"

$$\sqrt{4\cdot 9} = \sqrt{36} = \boxed{6}.$$

Summarize: **"Don't be afraid of problems with lots of mathematical notation. Focus on smaller pieces that you understand."**

Problem 2

Compute $\sqrt{4\left(\sqrt{3}\cdot\sqrt{27}\right)}$.

NEXT: compute division

We use a similar process here to derive the quotient property. You can have students present each sub problem individually or have one student show how parts a and b allow us to solve c as $\frac{\sqrt{96}}{\sqrt{6}} = \sqrt{\frac{96}{6}}$.

Key ideas: Don't let them assume the $\sqrt{\frac{96}{6}} = \frac{\sqrt{96}}{\sqrt{6}}$.

a. $\left(\frac{\sqrt{96}}{\sqrt{6}}\right)^2 = \left(\frac{\sqrt{96}}{\sqrt{6}}\right) \cdot \left(\frac{\sqrt{96}}{\sqrt{6}}\right) = \frac{\sqrt{96} \cdot \sqrt{96}}{\sqrt{6} \cdot \sqrt{6}} = \frac{96}{6} = \boxed{16}$.

b. $\left(\sqrt{\frac{96}{6}}\right)^2 = \frac{96}{6} = \boxed{16}$.

c. $\frac{\sqrt{96}}{\sqrt{6}}$.

Since $\left(\frac{\sqrt{96}}{\sqrt{6}}\right)^2 = \frac{96}{6} = \left(\sqrt{\frac{96}{6}}\right)^2$. we must have $\frac{\sqrt{96}}{\sqrt{6}} = \sqrt{\frac{96}{6}}$. So, $\frac{\sqrt{96}}{\sqrt{6}} = \sqrt{\frac{96}{6}} = \sqrt{16} = \boxed{4}$. (we could also square then square root the expression).

"Now we know that we can divide square roots in much the same way we multiply them."

Write on screen: $\frac{\sqrt{m}}{\sqrt{n}} = \sqrt{\frac{m}{n}}$

Problem 3

Compute

a. $\left(\frac{\sqrt{96}}{\sqrt{6}}\right)^2$.

b. $\left(\sqrt{\frac{96}{6}}\right)^2$.

c. $\frac{\sqrt{96}}{\sqrt{6}}$.

NEXT: which is greater problem

"**Compute** $\frac{\sqrt{75}}{\sqrt{12}}$."

Key ideas: Use the result from the end of the last slide.

$$\begin{aligned}\frac{\sqrt{75}}{\sqrt{12}} &= \sqrt{\frac{75}{12}}\\ &= \sqrt{\frac{25}{4}}\\ &= \sqrt{\left(\frac{5}{2}\right)^2}\\ &= \boxed{\frac{5}{2}}.\end{aligned}$$

Alternatively, we could have finished the calculation by "reversing" the rule:

$$\sqrt{\frac{25}{4}} = \frac{\sqrt{25}}{\sqrt{4}} = \boxed{\frac{5}{2}}.$$

Compute $\dfrac{\sqrt{75}}{\sqrt{12}}$.

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NEXT: Problem 4

"Which is greater, $\sqrt{50}+\sqrt{85}$ or $\sqrt{135}$?"

Key ideas: Addition and subtraction with square roots doesn't work like multiplication and division work.

IF STUDENTS NEED A PUSH:

Key ideas: Estimate each square root

"About how large is $\sqrt{135}$? Which integers is it between?"

Since $11^2 = 121$ and $12^2 = 144$, we know that $11 < \sqrt{135} < 12$.

"How large are $\sqrt{50}$ and $\sqrt{85}$?"

Since $7^2 = 49$ and $8^2 = 64$, we have $7 < \sqrt{50} < 8$.

Similarly, $9^2 = 81$ and $10^2 = 100$, so $9 < \sqrt{85} < 10$.

We have $\sqrt{50}+\sqrt{85} > 7+9 > 12 > \sqrt{135}$.

Therefore, $\boxed{\sqrt{50}+\sqrt{85}}$ is greater.

"In particular, we can say that $\sqrt{50}+\sqrt{85}$ is not equal to $\sqrt{135}$."

Write on screen (using words and not $\neq$ for clarity): $\sqrt{50}+\sqrt{85}$ is NOT equal to $\sqrt{135}$.

Which is greater, $\sqrt{50}+\sqrt{85}$ or $\sqrt{135}$?

NEXT: Problem 5a

If students try to make up a rule, suggest they also simplify with order of operations to see if they get the same thing. If appropriate, allow this incorrect solution to be presented and discussed in comparison to the correct solution.

"Compute $\sqrt{8^2+15^2}$**."**

Key ideas: Addition and subtraction with square roots doesn't work like multiplication and division work.

If students say $\sqrt{8^2+15^2}=\sqrt{8^2}+\sqrt{15^2}=8+15=23$*:*

"Can you justify that rule? After the last problem, we should be careful with square roots and addition."

Key ideas: Simplify under the radical

$$\begin{aligned}\sqrt{8^2+15^2} &= \sqrt{64+225}\\ &= \sqrt{289}=\boxed{17}.\end{aligned}$$

"Notice that the answer is not $8+15=23$**."**

Please reinforce that we don't have rules for addition and subtraction that work just like our rules for multiplication and division.

Problem 4

Compute $\sqrt{8^2+15^2}$.

NEXT: Simplify radicals

We want students to solve this one of two ways. If students are stuck or there is an imbalance of solution types, try the following: To prompt solution 1, remind students of the definition "The square root of a is the nonnegative number b such that $b^2 = a$. " To prompt solution 2, ask "How can we write 3 in order to use our rule for multiplying square roots?" Have a student present each solution.

"$3\sqrt{2}$ is equal to the square root of what integer?"

Start by explaining that $3\sqrt{2}$ is the same as $3 \cdot \sqrt{2}$.

Follow the students' lead, but make sure to cover Solution 2 at some point.

$\boxed{18}$

[Solution 1:] Square $3\sqrt{2}$

Squaring $3\sqrt{2}$ gives

$$\left(3\sqrt{2}\right)^2 = 3^2\left(\sqrt{2}\right)^2 = 9 \cdot 2 = 18.$$

Since $\left(3\sqrt{2}\right)^2 = 18$, we know that $3\sqrt{2} = \sqrt{18}$.

[Solution 2:] Use the rule for multiplying square roots

Since $3 = \sqrt{9}$, we have $3\sqrt{2} = \sqrt{9} \cdot \sqrt{2} = \sqrt{9 \cdot 2} = \sqrt{18}$, so our answer is $\boxed{18}$.

Problem 5

a. $3\sqrt{2}$ is equal to the square root of what integer?

NEXT: 5b: simplify radicals

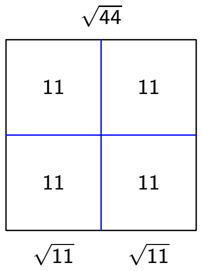
The "greater than 1" condition avoids cheap answers like $1 \cdot \sqrt{44}$.

"We're essentially asked here to reverse the process from the previous problem. How do we do it?"

We note that $44 = 4 \cdot 11$, so we can use our rule for multiplying square roots *backwards*.

$$\sqrt{44} = \sqrt{4 \cdot 11} = \sqrt{4} \cdot \sqrt{11} = \boxed{2\sqrt{11}}.$$

We can also think about this process geometrically. $\sqrt{44}$ is the side length of a square with area 44. If we split the square into 4 smaller squares of equal area, each has area 11. Then, the side length of each of the smaller squares is $\sqrt{11}$. Since two of the small sides equal the side of the big square, we see that $\sqrt{44} = 2\sqrt{11}$.



Key ideas: Define *simplifying a radical* as a follow-up.

[Follow-up:] "We say that we *simplify a radical* when we pull out as many square factors as we can. Let's take a look at another simplification."

How can we express $\sqrt{44}$ as the product of an integer greater than 1 and the square root of an integer?

NEXT: 5c: simplify radical sums

If students are stuck, suggest they factor to look for perfect square factors and then use the product property. Have a student present each solution. If students don't fully simplify, use that as a discussion point on how to know a radical is fully simplified.

"Simplify $\sqrt{1188}$."

There are a couple of good approaches. Cover at least one of them.

[Solution 1:] Pull out one square factor at a time

At each step, ask the students if we're done and how we know.

1188 is divisible by 4, which is a square, so:

$$\sqrt{1188} = \sqrt{4 \cdot 297} = \sqrt{4} \cdot \sqrt{297} = 2\sqrt{297}.$$

297 is divisible by 9, which is a square:

$$2\sqrt{297} = 2\sqrt{9 \cdot 33} = 2\sqrt{9} \cdot \sqrt{33} = 2 \cdot 3 \cdot \sqrt{33} = \boxed{6\sqrt{33}}.$$

The prime factorization of 33 is $3 \cdot 11$, so we see that we can't pull out any more squares.

[Solution 2:] Prime factorization

Key ideas: Start by factoring out the 11

The prime factorization of 1188 is $2^2 \cdot 3^3 \cdot 11$.

Now we pull out squares: $1188 = 2^2 \cdot 3^2 \cdot 3 \cdot 11$.

Then $\sqrt{1188} = \sqrt{2^2} \cdot \sqrt{3^2} \cdot \sqrt{3 \cdot 11} = 2 \cdot 3\sqrt{3 \cdot 11} = \boxed{6\sqrt{33}}$.

Problem 5

Simplify

b. $\sqrt{1188}$.

NEXT: Advance to the next slide to start the Extensions.

If students are stuck, make sure they simplify each radical first then ask "what would happen if the radicals were variables?"

"What would be a good first step?"

We should simplify the square roots.

"What do we get when we simplify the square roots?"

$$\sqrt{75} = \sqrt{25 \cdot 3} = \sqrt{25} \cdot \sqrt{3} = 5\sqrt{3},$$
$$\sqrt{27} = \sqrt{9 \cdot 3} = \sqrt{9} \cdot \sqrt{3} = 3\sqrt{3},$$
$$\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}.$$

"How does that help?"

Now we have

$$\begin{aligned}\sqrt{75} - \sqrt{27} - \sqrt{12} &= 5\sqrt{3} - 3\sqrt{3} - 2\sqrt{3} \\ &= (5-3-2)\sqrt{3} \\ &= 0 \cdot \sqrt{3} = \boxed{0}.\end{aligned}$$

"It sure didn't look like 0! This is one reason we like to simplify square roots!"

Problem 5

Simplify

c. $\sqrt{75} - \sqrt{27} - \sqrt{12}$.

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Extensions

Please pass out the Extensions handout. Ideally, the students will work in small groups, but they may choose to work alone. Meanwhile, the instructor circulates to offer guidance, challenging the high-flyers while providing extra support to students who struggled with earlier content.

Extensions

Compute each of the following:

(a) $\left(\sqrt{3}\cdot\sqrt{5}\cdot\sqrt{7}\right)^2$.

(b) $\sqrt{45}\cdot\sqrt{20}$.

(c) $\dfrac{\sqrt{12}}{\sqrt{147}}$.

(a) $\left(\sqrt{3}\cdot\sqrt{5}\cdot\sqrt{7}\right)^2 = 3\cdot 5\cdot 7 = \boxed{105}$.

(b) $\sqrt{45}\cdot\sqrt{20} = \sqrt{900} = \boxed{30}$.

(c) $\dfrac{\sqrt{12}}{\sqrt{147}} = \sqrt{\dfrac{12}{147}} = \sqrt{\dfrac{4}{49}} = \boxed{\dfrac{2}{7}}$.

(a) $\left(-2\sqrt{12}\right)\left(6\sqrt{75}\right) = (-2\cdot 6)\left(\sqrt{12}\cdot\sqrt{75}\right) = -12\left(2\sqrt{3}\cdot 5\sqrt{3}\right) = (-12)(10)(3) = \boxed{-360}$.

(b) $\dfrac{3\sqrt{200}}{2\sqrt{18}} = \dfrac{3\left(10\sqrt{2}\right)}{2\left(3\sqrt{2}\right)} = \dfrac{30}{6}\cdot\dfrac{\sqrt{2}}{\sqrt{2}} = \boxed{5}$.

Compute each of the following:

(a) $\left(-2\sqrt{12}\right)\left(6\sqrt{75}\right)$.

(b) $\dfrac{3\sqrt{200}}{2\sqrt{18}}$.

(a) $\sqrt{2\frac{7}{9}} - \sqrt{1\frac{7}{9}} - 1 = \sqrt{\frac{25}{9}} - \sqrt{\frac{16}{9}} - 1 = \frac{5}{3} - \frac{4}{3} - 1 = \boxed{-\frac{2}{3}}.$

(b) $\sqrt{0.09} = \sqrt{\frac{9}{100}} = \frac{3}{10} = \boxed{0.3}.$

Compute each of the following:

(a) $\sqrt{2\frac{7}{9}} - \sqrt{1\frac{7}{9}} - 1.$

(b) $\sqrt{0.09}.$

(a) $\sqrt{700} = \sqrt{100 \cdot 7} = \boxed{10\sqrt{7}}$.

(b) $\dfrac{\sqrt{32}-\sqrt{24}}{\sqrt{8}} = \dfrac{\sqrt{32}}{\sqrt{8}} - \dfrac{\sqrt{24}}{\sqrt{8}} = \sqrt{\dfrac{32}{8}} - \sqrt{\dfrac{24}{8}} = \sqrt{4} - \sqrt{3} = \boxed{2-\sqrt{3}}$.

(c) We have

$$\begin{aligned}
\left(2\sqrt{45}-\sqrt{20}\right)\left(4\sqrt{20}\right) &= \left(2\cdot 3\sqrt{5}-2\sqrt{5}\right)\left(4\cdot 2\sqrt{5}\right) \\
&= \left(6\sqrt{5}-2\sqrt{5}\right)\left(8\sqrt{5}\right) \\
&= 6\cdot 8\left(\sqrt{5}\right)^2 - 2\cdot 8\left(\sqrt{5}\right)^2 \\
&= (48-16)(5) = \boxed{160}.
\end{aligned}$$

Simplify each of the following:

(a) $\sqrt{700}$.

(b) $\dfrac{\sqrt{32}-\sqrt{24}}{\sqrt{8}}$.

(c) $\left(2\sqrt{45}-\sqrt{20}\right)\left(4\sqrt{20}\right)$.

$$\frac{12}{\sqrt{3}} = \frac{\sqrt{144}}{\sqrt{3}} = \sqrt{\frac{144}{3}} = \sqrt{\boxed{48}}.$$

$\dfrac{12}{\sqrt{3}}$ equals the square root of what integer?

(a) Jenny's integer could be any integer of the form $2n^2$, where n is a nonnegative integer. Jenny's integer must be of this form.

(b) Since Jenny's integer must be of the form $2n^2$, where n is a nonnegative integer, dividing the square root of her integer by $\sqrt{2}$ gives

$$\frac{\sqrt{2n^2}}{\sqrt{2}} = \sqrt{\frac{2n^2}{2}} = \sqrt{n^2} = n.$$

So, the quotient must be an integer.

Jenny multiplies the square root of her favorite integer by $\sqrt{2}$. The resulting product is an integer. For example, 8 could be Jenny's favorite integer because $\sqrt{2}\cdot\sqrt{8} = \sqrt{16} = 4$, which is an integer.

(a) Name three integers that could be Jenny's favorite integer.

(b) Suppose Jenny instead divides the square root of her favorite integer by $\sqrt{2}$. Must the resulting quotient be an integer?

(a) $\boxed{n}$.

In this problem, we see how we can write $\sqrt{n}$ as a power of n. We will assume that n is nonnegative throughout the problem.

(a) Fill in the box in this equation:

$$\sqrt{n} \cdot \sqrt{n} = \boxed{}.$$

(b) $\boxed{n^x \cdot n^x = n^1}$.

(b) Suppose $\sqrt{n} = n^x$. Use this to rewrite the equation from Part (a) so that n^x appears in place of $\sqrt{n}$ on the left side, and so that the right side of the equation is also written as a power of n.

(c) Applying our laws of exponents, we have $n^x \cdot n^x = n^{x+x} = n^{2x}$. So, $n^x \cdot n^x = n^1$ gives $n^{2x} = n^1$, from which we have $2x = 1$ and $x = \frac{1}{2}$. This suggests we write $\sqrt{n}$ as $\boxed{n^{\frac{1}{2}}}$.

(c) Thinking about the law of exponents mentioned above, what is the appropriate value for the exponent x from part (b)? In other words, to be consistent with the laws of exponents, we should write $\sqrt{n}$ as what power of n?

End of Class Activity

NEXT: Go to next slide to start marking numbers you know are possible.

Read the instructions to the students, and make sure they understand the examples given.

Unlike stricter versions of Four Fours, many more operations are allowed here, including inserting a negative symbol before a 4, or writing a decimal point without a zero preceding it, as in

$$\sqrt{(.\overline{4})^{-\frac{4}{4}}} \times 4 = 3.$$

Rules for the activity to follow:

Divide the class into pairs or triples. Give students **20 minutes (or as much time as you have)** to produce as many numbers between 1 and 100 as possible. Have students write out their expressions on a sheet of paper which cannot be modified at the end of the allotted time.

At the end of the game, ask groups to present all their expressions. Make sure the presented solution obeys order-of-operations precisely! Try not to pick the same student twice.

You can keep score, if you'd like: The group that correctly produces the most numbers between 1 and 100 wins.

If you are required to practice social distancing: Treat this as a 'class challenge', where the class works collaboratively to make as many numbers between 1 and 100 as possible. To keep track of which numbers have been made, use the hundreds chart on the next slide, crossing out numbers that have already been found.

Four Fours: Square Roots Edition

Combine four fours using mathematical operations, parentheses, and concatenation (putting 4 and 4 together to make 44) to get as many numbers between 1 and 100 as possible.

The one math symbol you **must** use in each expression is the square root symbol $\sqrt{}$.

For example, here are some ways we could get 0:

$$\left(\sqrt{.\overline{4} - .\overline{4}}\right)^{44} = 0$$

$$\sqrt{(4-4)!} - \frac{4}{4} = 0$$

$$-\sqrt{44} + \sqrt{44} = 0$$

NEXT: Last slide.

If you are required to practice social distancing: Treat this as a ‘class challenge’, where the class works collaboratively to make as many numbers between 1 and 100 as possible. To keep track of which numbers have been made, use this hundreds chart, crossing out numbers that have already been found.

Optionally, challenge students to make use of certain math symbols, such as the repeating decimal symbol, factorial, nonpositive exponents, etc.

Four Fours: Square Roots Edition

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

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