**NEXT: Next slide**

***Have this slide up the moment class begins. Spend no more than five minutes on it. Have students work independently, then briefly discuss.***

Students should recall how we built up the reflection expression using other isometries: We translate the line to the origin, rotate it to be the real axis, reflect the point, rotate it back, and translate it back to its original spot.

The only thing that needs to change is that after the reflection of the point, we must translate it $k$ units. The net effect is adding a term of $ke^{i\theta}$ to our reflection expression, giving us a three-term quadratic in $e^{i\theta}$.

**STEPS**

1. Translate line: $z-w$
2. Rotate line to real axis: $(z-w)e^{-i\theta}$
3. Reflect point: $\overline{(z-w)e^{-i\theta}}$
4. Translate $k$ units along real line: $\overline{(z-w)e^{-i\theta}}+k$
5. Rotate line back: $\left(\overline{(z-w)e^{-i\theta}}+k\right)e^{i\theta}$
6. Translate line back: $\left(\overline{(z-w)e^{-i\theta}}+k\right)e^{i\theta}+w$

Simplifying a bit, we get $\boxed{\overline{z-w}\cdot e^{2i\theta}+ke^{i\theta}+w}$.

# Warm up!

A fourth isometry of the complex plane is a *glide reflection*, which is a reflection across a line followed by a translation parallel to the line.

Recall that a reflection of $z$ through a line that intercepts the real axis with angle $\theta$ at a point $w$ is given by $\overline{z-w}\cdot e^{2i\theta}+w$.

How can we express a glide reflection through the same line, with translation of $k$ units?

# Start of Core Class Material

NEXT: Next slide

**“In two dimensions, a vector is an ordered pair of real numbers. We draw them as arrows.”**

*We will cover 3-dimensional vectors after the midterm.*

**“Here we have two vectors, $\mathbf{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.”**

*As a convention, we will write all vectors in boldface font to distinguish them from real number variables.*

**Key ideas:** Define *magnitude* and *direction*

**“All vectors carry two pieces of information:**

- **their *magnitude*, or size, and**
- **their *direction*.**

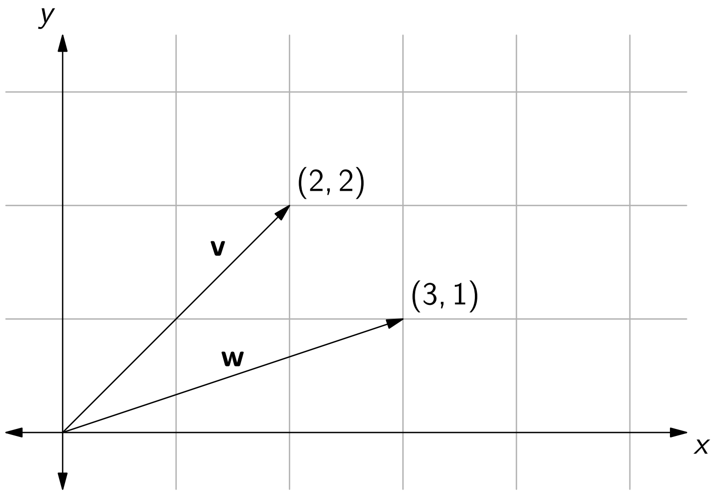
**What about the arrow tells us the magnitude of the vector?”**

Its length.

**“And how do we indicate its direction?”**

It is the direction the arrow points.

## Vectors in 2D



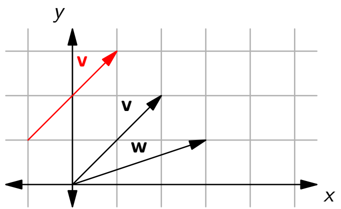
NEXT: Next slide

**"We can think of vectors as representing movements. For example, v represents the movement from the origin to $(2,2)$."**

**"If we started at $(-1,1)$ and followed v, where would we end up?"**

$(1,3)$

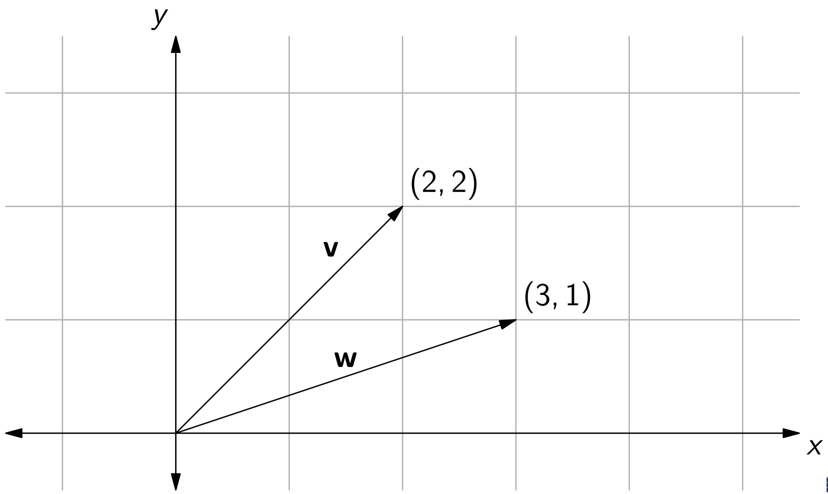
Since vectors have no position (just magnitude and direction), we can start them wherever we want.




**Key ideas:** Vectors have no position; **v** and **v** are the same vector

**"In general, we can translate any vector to get an equivalent one."**

Let $\mathbf{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ be vectors.



**NEXT: Next slide**

**"What is $\mathbf{v}+\mathbf{w}$?"**

*IF STUDENTS NEED INSPIRATION:*

**"Vectors represent movements. What movement results from combining v and w?"**

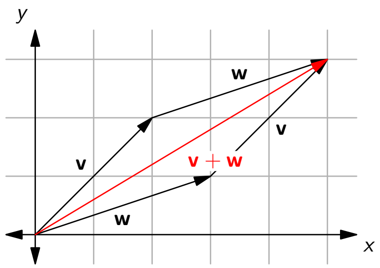
$\boxed{\begin{pmatrix}5\\3\end{pmatrix}}$

Applying the movement should get us from the origin to $(2,2)$ to $(5,3)$.

**"What is $\mathbf{w}+\mathbf{v}$?"**

Also $\begin{pmatrix}5\\3\end{pmatrix}$.

**Key ideas:** Vector addition is commutative




**Key ideas:** Vector addition is *component-wise*

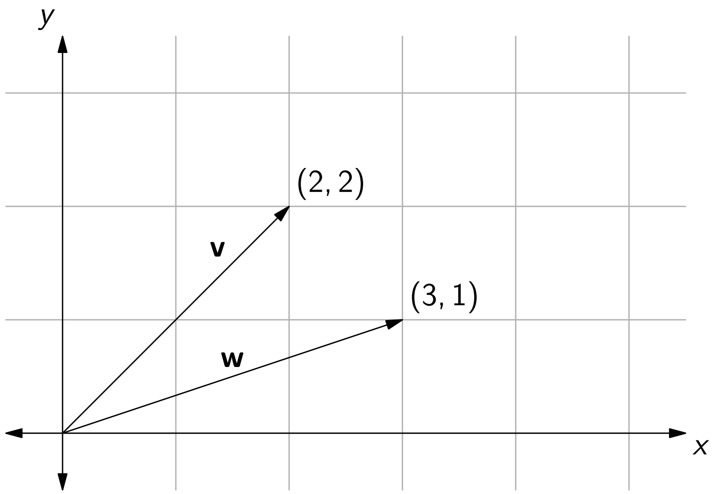
**"Algebraically, we add vectors *component-wise*, meaning that we add the components separately:**

$$\mathbf{v}+\mathbf{w}=\begin{pmatrix}2\\2\end{pmatrix}+\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}2+3\\2+1\end{pmatrix}=\begin{pmatrix}5\\3\end{pmatrix}.$$

**Geometrically, we add vectors head-to-tail. Since we can translate vectors, we can translate v so that its tail matches the head of w. The resulting vector is then $\mathbf{v}+\mathbf{w}$. "**

*We can also translate the vector* **w**, *as shown above.*

Let $\mathbf{v}=\begin{pmatrix}2\\2\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}3\\1\end{pmatrix}$ be vectors. What is $\mathbf{v}+\mathbf{w}$?

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**NEXT: Next slide**

**"What is $2\mathbf{w}$?"**

We have $2\mathbf{w} = \mathbf{w} + \mathbf{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 6 \\ 2 \end{pmatrix}}$.

**"What is $3\mathbf{w}$?"**

We can say that $3\mathbf{w} = 2\mathbf{w} + \mathbf{w} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 9 \\ 3 \end{pmatrix}}$.

**"What is $-\mathbf{w}$?"**

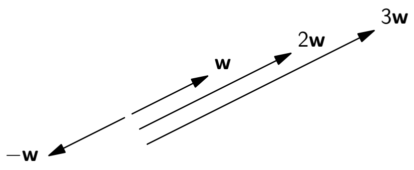
We need $\mathbf{w} + (-\mathbf{w}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $-\mathbf{w} = \boxed{\begin{pmatrix} -3 \\ -1 \end{pmatrix}}$.

**"In general, for a real number $c$, what is $c\mathbf{v}$?"**

If $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, then

$$c\mathbf{v} = \begin{pmatrix} ca \\ cb \end{pmatrix}.$$

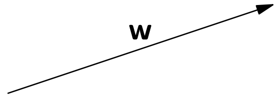
In other words, we simply multiply the components in $\mathbf{v}$ by $c$.




**Key ideas:** Define *scalar multiplication*

**"This operation is called *scalar multiplication*, because the factor $c$ scales the vector $\mathbf{v}$."**

For the vector $\mathbf{w} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, what is $2\mathbf{w}$? What is $3\mathbf{w}$? What is $-\mathbf{w}$?



**NEXT: Next slide**

**"What is $\mathbf{v}-\mathbf{w}$?"**

*LET THE STUDENTS DISCUSS. Cover both approaches.*

*IF STUDENTS NEED A PUSH:* **"With real numbers, $7-2$ is the number you have to add to $2$ to get $7$."**

**Key ideas: $\mathbf{w}+(\mathbf{v}-\mathbf{w})=\mathbf{v}$**

**[Approach 1:] Algebra**

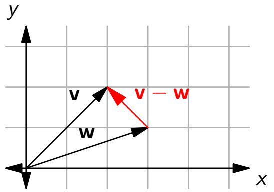
The vector $\mathbf{v}-\mathbf{w}$ must satisfy

$$\begin{pmatrix}3\\1\end{pmatrix}+(\mathbf{v}-\mathbf{w})=\begin{pmatrix}2\\2\end{pmatrix},$$

so $\mathbf{v}-\mathbf{w}=\begin{pmatrix}2\\2\end{pmatrix}-\begin{pmatrix}3\\1\end{pmatrix}=\boxed{\begin{pmatrix}-1\\1\end{pmatrix}}$.

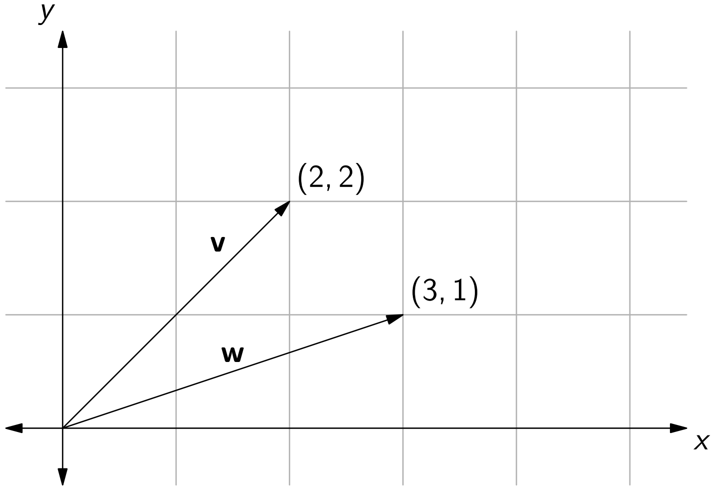
**[Approach 2:] Geometry**

If we draw the tail of $\mathbf{v}-\mathbf{w}$ at the tip of $\mathbf{w}$, then the tip of $\mathbf{v}-\mathbf{w}$ aligns with the tip of $\mathbf{v}$.




Visually, we see that $\mathbf{v}-\mathbf{w}=\boxed{\begin{pmatrix}-1\\1\end{pmatrix}}$.

Let $\mathbf{v}=\begin{pmatrix}2\\2\end{pmatrix}$ and $\mathbf{w}=\begin{pmatrix}3\\1\end{pmatrix}$ be vectors. What is $\mathbf{v}-\mathbf{w}$?

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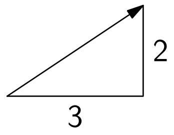
**NEXT: Next slide**

**"How can we find the length of the vector?"**

We can a form a right triangle, where the length of the vector is the hypotenuse, and the legs are 3 and 2.




Then by Pythagorean Theorem, the length of the vector is $\sqrt{3^2+2^2} = \boxed{\sqrt{13}}$.

**Key ideas:** Define *norm* of a vector, introduce notation for it

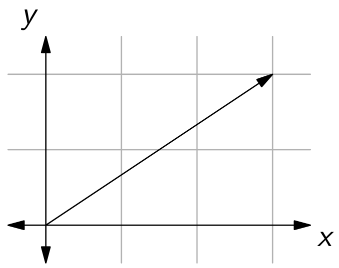
**"We call the length of a vector v its *norm*, and we denote it by $\|\mathbf{v}\|$."**

**"What is the norm of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$?"**

Using the Pythagorean Theorem again, the norm of the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is

$$\left\|\begin{pmatrix} a \\ b \end{pmatrix}\right\| = \sqrt{a^2+b^2}.$$

What is the length of the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$?



**NEXT: Next slide**

**"Suppose that $\mathbf{v} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then what is $\|c\mathbf{v}\|$?"**

$$\|c\mathbf{v}\| = \left\| c \begin{pmatrix} a \\ b \end{pmatrix} \right\| = \left\| \begin{pmatrix} ca \\ cb \end{pmatrix} \right\| = \sqrt{c^2a^2 + c^2b^2}.$$

**"What can we do with this expression?"**

We can factor $c^2$ out of the radical to get

$$\sqrt{c^2a^2 + c^2b^2} = \sqrt{c^2}\sqrt{a^2+b^2} = \boxed{|c| \cdot \|\mathbf{v}\|}.$$

*NOTE: $\sqrt{c^2}$ does not simplify to $c$, because $c$ could be negative.*

Express $\|c\mathbf{v}\|$ in terms of $c$ and $\mathbf{v}$.

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**NEXT: Next slide**

**"How can we find $c$?"**

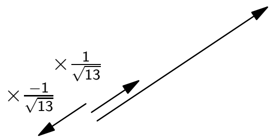
**Key ideas:** Apply the results of the last two slides

We know that

$$\left\| c \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\| = |c| \cdot \left\| \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\| = |c|\sqrt{13}.$$

We want this expression to evaluate to 1, so

$$|c|\sqrt{13} = 1$$
$$|c| = \frac{1}{\sqrt{13}}.$$




Therefore, $c = \boxed{\pm \frac{1}{\sqrt{13}}}$.

**Key ideas:** Define *unit vector*

**"We call a vector whose norm is 1 a *unit vector*. Here, we call $\frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ the *unit vector in the direction of* $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ because it points the same way as $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$."**

**"In general, for a vector $\mathbf{v}$, what is a scalar $c$ so that $c\mathbf{v}$ is a unit vector?"**

$\pm \frac{1}{\|\mathbf{v}\|}$

Find a scalar $c$ so that the norm of $c \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is 1.

**NEXT: Next slide**

**"How does $\|\mathbf{v}+\mathbf{w}\|$ relate to $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$?"**

**Key ideas:** Use an inequality

$\|\mathbf{v}+\mathbf{w}\| \leq \|\mathbf{v}\|+\|\mathbf{w}\|$

The three norms are the side lengths of a triangle. We apply the fact that

$$(\text{one side length of a triangle}) \leq (\text{sum of other two side lengths}).$$

**Key ideas:** State the *Triangle Inequality*

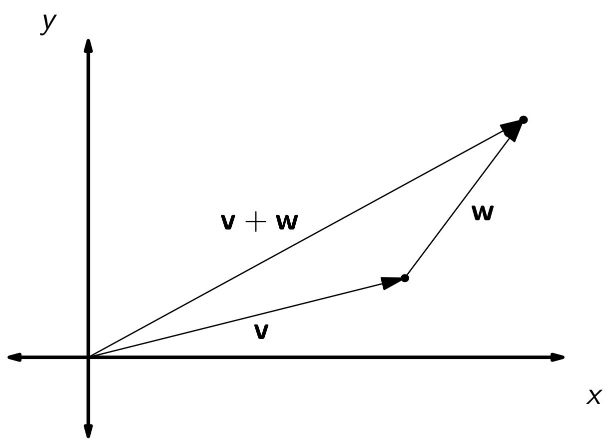
**"We call this inequality the *Triangle Inequality*."**

**"When do we have equality? In other words, when is it true that $\|\mathbf{v}+\mathbf{w}\| = \|\mathbf{v}\|+\|\mathbf{w}\|$?"**

when **v** and **w** have the same direction

In that case, all three sides of the "triangle" are collinear.

# How does $\|\mathbf{v}+\mathbf{w}\|$ relate to $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$?



**Multi-page: Continued on next slide**

**NEXT: Next slide**

**"How might we find the angle between v and w?"**

*LET THE STUDENTS DISCUSS.*

**Key ideas:** Complete the triangle

We can complete a triangle and then solve for the angle using the Law of Cosines.

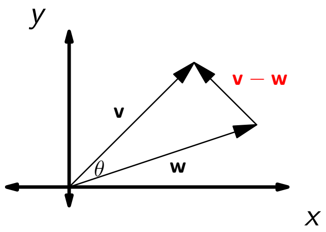
**"What vector completes the triangle?"**

$\mathbf{v} - \mathbf{w}$

**"What does the Law of Cosines say about this triangle?"**

**Key ideas:** Use norms to represent side lengths

$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\,\|\mathbf{v}\|\,\|\mathbf{w}\|\cos(\theta)$



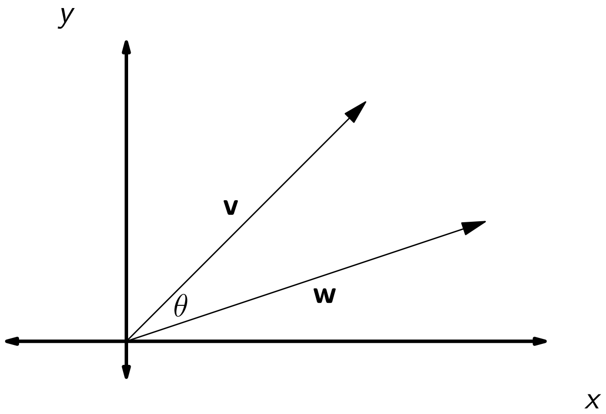

**"What do we get when we solve that equation for $\cos(\theta)$?"**

$$\cos(\theta) = \frac{\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - \|\mathbf{v} - \mathbf{w}\|^2}{2\,\|\mathbf{v}\|\,\|\mathbf{w}\|}.$$

**"Let's see if we can simplify this equation any further."**

*Advance to the next slide.*

# What is the angle between two vectors, **v** and **w**?

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**Multi-page ends this slide.**

**NEXT: Next slide**

**"Suppose that $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$. What are $\|\mathbf{v}\|^2$, $\|\mathbf{w}\|^2$, and $\|\mathbf{v}-\mathbf{w}\|^2$?"**

$\|\mathbf{v}\|^2 = v_1^2 + v_2^2$

$\|\mathbf{w}\|^2 = w_1^2 + w_2^2$

$\|\mathbf{v}-\mathbf{w}\|^2 = (v_1 - w_1)^2 + (v_2 - w_2)^2 = v_1^2 + w_1^2 - 2v_1 w_1 + v_2^2 + w_2^2 - 2v_2 w_2$

**"How does our numerator simplify?"**

$2v_1 w_1 + 2v_2 w_2$

All squared terms in $\|\mathbf{v}\|^2$ and $\|\mathbf{w}\|^2$ cancel those in $\|\mathbf{v}-\mathbf{w}\|^2$.

**"So what does our fraction for $\cos(\theta)$ become?"**

$\cos(\theta) = \dfrac{v_1 w_1 + v_2 w_2}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$

**Key ideas:** Define *dot product*

**"That numerator, $v_1 w_1 + v_2 w_2$, is an important quantity; we call it the *dot product* of v and w."**

**"We write the dot product of v and w as $\mathbf{v} \cdot \mathbf{w}$."**

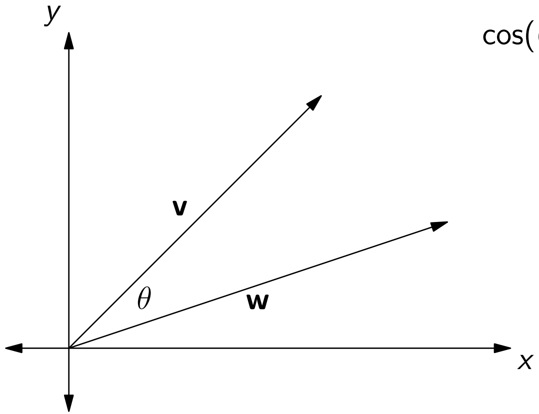
**"Is $\mathbf{v} \cdot \mathbf{w}$ a vector?"**

**Key ideas:** Dot products are scalars

No! $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$ is a sum of product of real numbers, which is also a real number.

**"Using dot products, we can write $\cos(\theta) = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\,\|\mathbf{w}\|}$."**

# What is the angle between two vectors, **v** and **w**?




$$\cos(\theta) = \frac{\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - \|\mathbf{v}-\mathbf{w}\|^2}{2\,\|\mathbf{v}\|\,\|\mathbf{w}\|}$$

**NEXT: Next slide**

*This slide is primarily a summary.*

For vectors $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, their dot product is

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2$$

and the angle $\theta$ between them satisfies

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|}.$$

**NEXT: Next slide**

**"What is the angle between v and itself?"**

Intuitively, $0^\circ$

**Key ideas:** Apply the dot product formula

From the last slide, we know that $\cos(\theta) = \dfrac{\mathbf{v}\cdot\mathbf{v}}{\|\mathbf{v}\|\,\|\mathbf{v}\|}$.

**"Can we simplify $\mathbf{v}\cdot\mathbf{v}$?"**

**Key ideas:** $\mathbf{v}\cdot\mathbf{v}$ is the square of $\|\mathbf{v}\|$

If $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then $\mathbf{v}\cdot\mathbf{v} = v_1^2 + v_2^2 = \left(\sqrt{v_1^2+v_2^2}\right)^2 = \|\mathbf{v}\|^2$.

**"So what is $\cos(\theta)$?"**

We have $\cos(\theta) = \dfrac{\mathbf{v}\cdot\mathbf{v}}{\|\mathbf{v}\|\,\|\mathbf{v}\|} = \dfrac{\|\mathbf{v}\|^2}{\|\mathbf{v}\|^2} = 1$.

**"What is $\theta$?"**

Since $\cos(\theta) = 1$, we have $\theta = 360^\circ \cdot k$ for some integer $k$.

**Key ideas:** The angle between vectors is between $0^\circ$ and $180^\circ$

*From here on, we won't worry about the "$+360^\circ k$" in such problems.*

Geometrically, the angle between the vectors is $\boxed{0^\circ}$

Let **v** be a nonzero vector. What is the angle between **v** and itself?

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**NEXT: Next slide**

**"How can we find the angle between two vectors?"**

We use the dot product formula!

**"What do we get?"**

**Key ideas:** No need to compute the norms

Let $\mathbf{v} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$, and let $\theta$ be the angle between $\mathbf{v}$ and $\mathbf{w}$.

We have

$$\begin{aligned} \cos(\theta) &= \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|} \\ &= \frac{3 \cdot 8 + (-2) \cdot 12}{\|\mathbf{v}\| \, \|\mathbf{w}\|} \\ &= \frac{0}{\|\mathbf{v}\| \, \|\mathbf{w}\|} = 0. \end{aligned}$$

**"So what is $\theta$?"**

The angle $\theta$ is $\boxed{90^\circ}$.

**Key ideas:** Define *orthogonal*

**"When the angle between two non-zero vectors is $90^\circ$, we say that they are *orthogonal*."**

**Key ideas:** Check whether $\mathbf{v} \cdot \mathbf{w} = 0$ for orthogonality

**"Thus, two non-zero vectors are orthogonal if and only if their dot product is equal to 0."**

Find the angle between the vectors $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 8 \\ 12 \end{pmatrix}$.

**NEXT: Advance to Slide 17 to start the Extensions.**

**"How should we start?"**

Let $\theta$ be the angle between the vectors. Then

$$\begin{aligned}\cos(\theta) &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\\ &= \frac{2\cdot 4+3\cdot -3}{\sqrt{2^2+3^2}\cdot\sqrt{4^2+(-3)^2}}\\ &= \frac{-1}{\sqrt{13}\cdot 5}.\end{aligned}$$

**"So what is $\theta$?"**

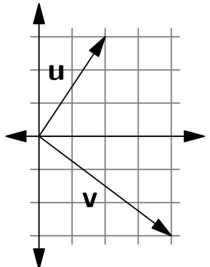
**Key ideas:** Use inverse trig

$$\theta = \boxed{\arccos\left(-\frac{1}{5\sqrt{13}}\right)}$$

**"Without computing it, what can we say about that angle?"**

- It is obtuse, since its cosine is negative.
- It is close to a right angle since $\frac{1}{5\sqrt{13}} < \frac{1}{15}$ is fairly small.

*For the curious,* $\theta \approx 93.18°$.




Find the angle between the vectors $\mathbf{u} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

# Extensions

Please pass out the Extensions handout. Ideally, the students will work in small groups, but they may choose to work alone. Meanwhile, the instructor circulates to offer guidance, challenging the high-flyers while providing extra support to students who struggled with earlier content.

# Extensions

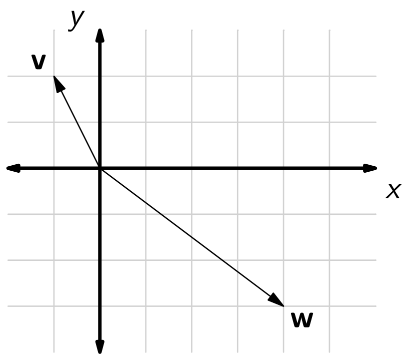
(a) $\mathbf{v}+\mathbf{w}=\begin{pmatrix}-1\\2\end{pmatrix}+\begin{pmatrix}4\\-3\end{pmatrix}=\boxed{\begin{pmatrix}3\\-1\end{pmatrix}}$.

(b) $3\mathbf{v}-2\mathbf{w}=3\begin{pmatrix}-1\\2\end{pmatrix}-2\begin{pmatrix}4\\-3\end{pmatrix}=\boxed{\begin{pmatrix}-11\\12\end{pmatrix}}$.

(c) $\|\mathbf{v}\|=\sqrt{(-1)^2+2^2}=\boxed{\sqrt{5}}$.

(d) The unit vector in the direction of $\mathbf{v}$ is

$$\frac{\mathbf{v}}{\|\mathbf{v}\|}=\frac{\begin{pmatrix}-1\\2\end{pmatrix}}{\sqrt{5}}=\boxed{\begin{pmatrix}-\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{pmatrix}}.$$




(a) Find $\mathbf{v}+\mathbf{w}$.

(b) Find $3\mathbf{v}-2\mathbf{w}$.

(c) Find $\|\mathbf{v}\|$.

(d) Find the unit vector in the direction of $\mathbf{v}$.

Let $\mathbf{v} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 2-\sqrt{3} \\ 1+2\sqrt{3} \end{pmatrix}$, and $\theta$ be the angle between them. Then

$$\|\mathbf{v}\| = \sqrt{4^2+2^2} = \sqrt{20} = 2\sqrt{5}$$

and

$$\|\mathbf{w}\| = \sqrt{(2-\sqrt{3})^2+(1+2\sqrt{3})^2} = \sqrt{(7-4\sqrt{3})+(13+4\sqrt{3})} = \sqrt{20} = 2\sqrt{5},$$

so

$$\cos(\theta) = \frac{\mathbf{v}\cdot\mathbf{w}}{2\sqrt{5}\cdot 2\sqrt{5}} = \frac{8-4\sqrt{3}+2+4\sqrt{3}}{20} = \frac{10}{20} = \frac{1}{2}.$$

The angle $\theta$ is $\boxed{60^\circ}$.

Find the angle between the vectors $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2-\sqrt{3} \\ 1+2\sqrt{3} \end{pmatrix}$.

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$.

(a) Writing each vector in terms of its components and applying commutativity of real number multiplication, we get

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_1 v_1 + u_2 v_2 \\ &= v_1 u_1 + v_2 u_2 \\ &= \mathbf{v} \cdot \mathbf{u}. \end{aligned}$$

(b) For the first half, we expand each vector in terms of its components:

$$\begin{aligned} (\mathbf{u}+\mathbf{v}) \cdot \mathbf{w} &= \left[ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right] + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ &= (u_1 + v_1) w_1 + (u_2 + v_2) w_2 \\ &= (u_1 w_1 + u_2 w_2) + (v_1 w_1 + v_2 w_2) \\ &= \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}. \end{aligned}$$

For the second half, we apply the first half of Part (b) with the result of Part (a).

(c) All three expressions are equal to $a u_1 v_1 + a u_2 v_2$.

Prove that the dot product operation:

(a) is commutative, so that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.

(b) is distributive, so that $(\mathbf{u}+\mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$ and $\mathbf{w} \cdot (\mathbf{u}+\mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v}$.

(c) satisfies $a(\mathbf{u} \cdot \mathbf{v}) = (a\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (a\mathbf{v})$ for all scalars $a$.

The vector $\overrightarrow{AB}$ represents the vector pointing from $A$ to $B$, and the vector $\overrightarrow{CD}$ represents the vector pointing from $C$ to $D$. If these two vectors are equal, then $\overline{AB}$ and $\overline{CD}$ are equal in length and parallel, so quadrilateral $ABDC$ is a parallelogram.

If $A$, $B$, $C$, and $D$ are points in the Cartesian plane such that $\vec{AB} = \vec{CD}$ and the four points are not all on the same line, then must $ABDC$ be a parallelogram?

(Problem 9.1.4 in the textbook. *Hint: See Problem 9.5.*)

We have $\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|} = -1$, so $\theta = 180^\circ$. Therefore, the two vectors point in the $\boxed{\text{opposite direction}}$.

Suppose $\mathbf{x} \cdot \mathbf{y} = -\|\mathbf{x}\| \, \|\mathbf{y}\|$. What can you say about the directions of vectors $\mathbf{x}$ and $\mathbf{y}$?

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors such that $\|\mathbf{v}\| = 2$, $\|\mathbf{w}\| = 5$, and $\|2\mathbf{v} + 3\mathbf{w}\| = 11$. Find $\mathbf{v} \cdot \mathbf{w}$.

The dot product of any vector with itself is the square of the vector's norm, so $\mathbf{v} \cdot \mathbf{v} = 4$, $\mathbf{w} \cdot \mathbf{w} = 25$, and $(2\mathbf{v} + 3\mathbf{w}) \cdot (2\mathbf{v} + 3\mathbf{w}) = 121$. We distribute in the last of these equations to get

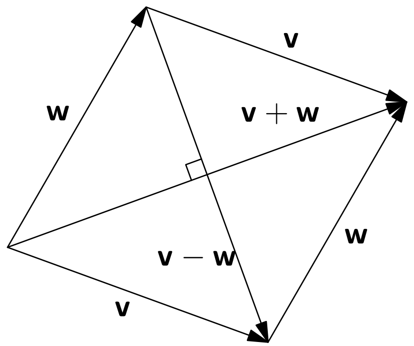
$$\begin{aligned} 121 &= (2\mathbf{v} + 3\mathbf{w}) \cdot (2\mathbf{v} + 3\mathbf{w}) \\ &= 4\mathbf{v} \cdot \mathbf{v} + 9\mathbf{w} \cdot \mathbf{w} + 6\mathbf{v} \cdot \mathbf{w} + 6\mathbf{w} \cdot \mathbf{v} \\ &= 4 \cdot 4 + 9 \cdot 25 + 12\mathbf{v} \cdot \mathbf{w}, \end{aligned}$$

so $\mathbf{v} \cdot \mathbf{w} = \dfrac{121 - 16 - 225}{12} = \dfrac{-120}{12} = \boxed{-10}$.

We show that $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are orthogonal by showing that their dot product is equal to 0:

$$\begin{aligned}(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) &= \mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{w}-\mathbf{w}\cdot\mathbf{v}-\mathbf{w}\cdot\mathbf{w}\\ &= \mathbf{v}\cdot\mathbf{v}-\mathbf{w}\cdot\mathbf{w}\\ &= \|\mathbf{v}\|^2-\|\mathbf{w}\|^2=0.\end{aligned}$$

For any two vectors, we can build a parallelogram whose opposite sides are copies of those vectors. In the case that the vectors have the same norm, that parallelogram becomes a rhombus. The diagonals of the rhombus are $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$. Our result is equivalent to the fact that the diagonals of a rhombus are perpendicular to each other.




Let $\mathbf{v}$ and $\mathbf{w}$ be vectors such that $\|\mathbf{v}\|=\|\mathbf{w}\|$. Prove that the vectors $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are orthogonal. Find a geometric interpretation of this result.

(a) $\boxed{\text{No.}}$ For example, consider the case $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Then $\mathbf{a} \neq \mathbf{b}$, but $\mathbf{v} \cdot \mathbf{a} = 2 = \mathbf{v} \cdot \mathbf{b}$.

(b) $\boxed{\text{Yes.}}$ Define $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Suppose that $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. If $\mathbf{i} \cdot \mathbf{a} = \mathbf{i} \cdot \mathbf{b}$, then $a_1 = b_1$. Similarly, if $\mathbf{j} \cdot \mathbf{a} = \mathbf{j} \cdot \mathbf{b}$, then $a_2 = b_2$. So, if $\mathbf{v} \cdot \mathbf{a} = \mathbf{v} \cdot \mathbf{b}$ for all vectors $\mathbf{v}$, then $a_1 = b_1$ and $a_2 = b_2$, which forces $\mathbf{a} = \mathbf{b}$.

(a) If **a** and **b** are vectors such that $\mathbf{v} \cdot \mathbf{a} = \mathbf{v} \cdot \mathbf{b}$ for some non-zero vector **v**, is it necessarily true that $\mathbf{a} = \mathbf{b}$?

(b) If **a** and **b** are vectors such that $\mathbf{v} \cdot \mathbf{a} = \mathbf{v} \cdot \mathbf{b}$ for **all** vectors **v**, is it necessarily true that $\mathbf{a} = \mathbf{b}$?

# End of Class Activity

**NEXT: Next slide**

**"We could solve this problem by letting $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and solving**

$$\sqrt{(v_1-3)^2+(v_2+2)^2} = \sqrt{(v_1+2)^2+(v_2-4)^2},$$

**but let's look for a better way."**

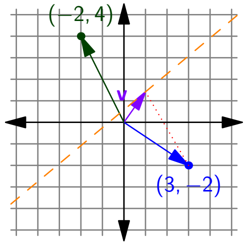
**"Geometrically, what does $\left\| \mathbf{v} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right\|$ represent?"**

The distance between the tip of **v** and the tip of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

**"And $\left\| \mathbf{v} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right\|$?"**

The distance between the tip of **v** and the tip of $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$

**"So what does the equation say?"**
The tip of **v** is equidistant from $(3,-2)$ and $(-2,4)$.




**"Which v satisfy that equation?"**

| |
|---|
| Any vector whose tip is on the perpendicular bisector of $(3,-2)$ and $(-2,4)$. |

*The perpendicular bisector is the orange dashed line above.*

## Geometry of Vector Equations

Find all vectors **v** such that $\left\| \mathbf{v} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \right\| = \left\| \mathbf{v} - \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right\|$.

**NEXT: Next slide**

**"Once again, we could write v in terms of components and rewrite the equation, but let's try something more elegant."**

**"Geometrically, how can we interpret this equation?"**

$\mathbf{v} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$ are vectors with a dot product of 0, so they are orthogonal.

**"What is orthogonal to $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$?"**

The line perpendicular to it at the origin.

*This line is orange and dashed in the diagram.*

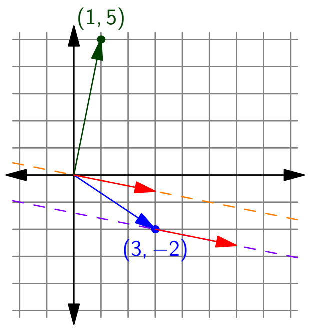
**"So $\mathbf{v} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is a vector on that line."**

*One possibility for $\mathbf{v} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ is the red vector graphed.*

**"Where would v be then?"**

We get $\mathbf{v}$ by adding $\mathbf{v} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ to $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

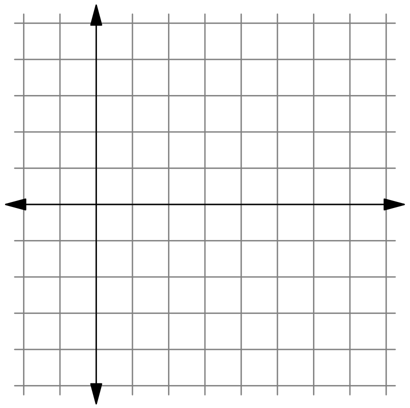
We end up on the purple line graphed.




$\mathbf{v}$ can be $\boxed{\text{any vector from the origin to the purple line.}}$

Find all vectors **v** such that

$$\left[\mathbf{v} - \begin{pmatrix} 3 \\ -2 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 \\ 5 \end{pmatrix} = 0.$$

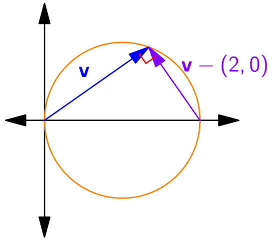


**"How can we approach this problem?"**

**[Approach 1:] Vector Geometry**

We can move both terms to the left side and factor to get

$$\mathbf{v}\cdot\left[\mathbf{v}-\begin{pmatrix}2\\0\end{pmatrix}\right]=0.$$




The vectors $\mathbf{v}$ and $\mathbf{v}-\begin{pmatrix}2\\0\end{pmatrix}$ are orthogonal.

By Thales's Theorem, the tip of possible $\mathbf{v}$ forms a circle with diameter from the origin to $(2,0)$.

Thus, $\mathbf{v}$ can be $\boxed{\text{any vector with its tip on the circle with center } (1,0) \text{ and radius } 1}$.

**[Approach 2:] Component Arithmetic**

Let $\mathbf{v}=\begin{pmatrix}x\\y\end{pmatrix}$. Then the equation becomes

$$\begin{aligned}x^2+y^2&=2x\\x^2-2x+y^2&=0\\x^2-2x+1+y^2&=1\\(x-1)^2+y^2&=1,\end{aligned}$$

so $\mathbf{v}$ is $\boxed{\text{any vector with its tip on the circle centered at } (1,0) \text{ with radius } 1}$.

Find all vectors **v** that satisfy the equation

$$\mathbf{v}\cdot\mathbf{v}=\mathbf{v}\cdot\begin{pmatrix}2\\0\end{pmatrix}.$$