**NEXT: Core Class Material**

***Have this problem up the minute the class is supposed to start. Spend no more than 5 minutes on it. Have students work, then discuss the solution quickly.***

Expect many different solutions.

The simplest four solutions are $(-2+\sqrt{65},1),(-2-\sqrt{65},1),(-2,1+\sqrt{65}),(-2,1-\sqrt{65})$, all of which are $\sqrt{65}$ units in a cardinal direction from $(-2,1)$.

Students may recognize that $8^2+1^2=65$ and this leads to eight additional points, including $(6,0)$, $(6,2)$, $(-1,9)$, $(-3,9)$, and more.

Eight additional solutions can be found using $7^2+4^2=65$, counting 7 and 4 in any direction, giving $(5,5)$ and $(-9,-3)$ and more.

An infinite number of additional solutions are available, picking $x$ then determining values of $y$ so that the distance is exactly $\sqrt{65}$. Students are less likely to find these, but one such solution is $(0,1+\sqrt{61})$.

Students may also use the distance formula here: $\sqrt{(x+2)^2+(y-1)^2}=\sqrt{65}$. If this happens, you can use it as a lead-in to much of the work introducing the equations of circles.

# Warm Up!

Find as many points as you can that are exactly $\sqrt{65}$ units away from $(-2,1)$.

## Section: Math Contest Information

**As of AY 25-26, every campus takes the contest during the same window.**

Campus-specific contest dates can be found in this sheet:

https://docs.google.com/spreadsheets/d/1U1eUsTBSnDAiJrSR0a0qb6vuenNRIA7Wbqp5vqR1tNs/edit?usp=sharing

**Notes:**

- Students will only be able to access the test during the listed date/time window.
- They will have 55 minutes to complete the test once they have started it.
- Students have 40 minutes for the main contest (15 problems), and
- 10 minutes for a tiebreaker round (2 problems).
- The tiebreaker problems are TIMED: students are encouraged to submit their answers as fast as possible (within the 10 minute window).

You can see details here: https://aopsacademy.org/my-academy/problemsolver to help answer any other student questions.

## Coming Soon: Problem Solvers Showdown!

Your second math competition for the year is coming up! There will be 15 problems for you to solve in 40 minutes, plus a 2-question tiebreaker. You'll have to work quickly! Problems will start out pretty easy and get very hard by the end.

This contest is a **Challenge**. Don't expect to get all problems correct - just do your best and have fun!

- 1st Place - $60
- 2nd Place - $40
- 3rd Place - $20
- 4th Place - $10
- **All students who score above a threshold score of 10 will earn a collectible AoPS pin as well.**

# Start of Core Class Material

# Section: The Distance Formula

**NEXT: Generalizing distance**

*Lean on students' familiarity with the Pythagorean Theorem.*

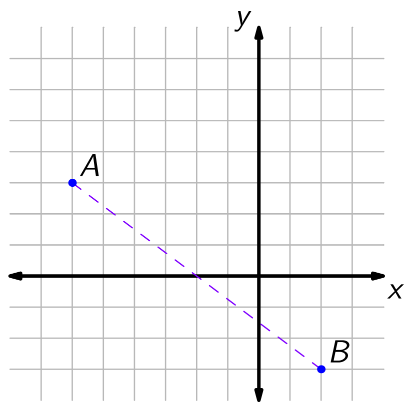
**"How can we calculate distance if it's not horizontal or vertical?"** Build the horizontal and vertical segments, then use the Pythagorean Theorem.

**Have a student come up to demonstrate and visualize the vertical and horizontal segments.**

The distance between $A$ and $B$ is the length of the hypotenuse of this right triangle. The leg lengths are 8 and 6, so the length of the hypotenuse is $\sqrt{8^2+6^2}=\sqrt{100}=\boxed{10}$.

## Distance

What is the distance between $A(-6,3)$ and $B(2,-3)$?




AoPS Academy

**NEXT: Distance formula**

**Key ideas:** The rule for general distances can be built from repeated calculation.

*Students are likely to need graph paper for the second example.*

**"If you're unsure about the last one, follow the same steps you did with numbers."**

**"What changed? What stayed the same?"** The points changed, but the calculation stayed the same.

1. The distance is
$$\sqrt{(6-2)^2+(0-(-3))^2}=\sqrt{16+9}=\sqrt{25}=\boxed{5}.$$

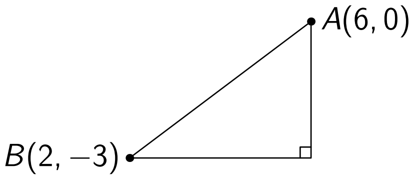
2. The distance is
$$\sqrt{(10-2)^2+(1-(-3))^2}=\sqrt{64+16}=\boxed{\sqrt{80}}=\boxed{4\sqrt{5}}.$$

3. The distance is
$$\sqrt{(x-2)^2+(y-(-3))^2}=\boxed{\sqrt{(x-2)^2+(y+3)^2}}.$$

*Emphasize the use of $y+3$ in this distance as a representation of $y-(-3)$. This will be part of the work in the general equation for circles.*

**"Hopefully you feel confident about calculating the distance between any two points."**

1. What is the distance between $A(6,0)$ and $B(2,-3)$?




2. What is the distance between $(10,1)$ and $(2,-3)$?

3. What is the distance between $(x,y)$ and $(2,-3)$?

**NEXT: Distance formula leads to equations of circles**

*This is not students' first exposure to subscripts, so they should feel pretty sensible here. The textbook uses $x_1$ and $x_2$, and either is fine.*

**"What do you think $x_A$ means?"** It means the $x$-coordinate of point $A$. We use the subscripts because saying "$x$" isn't enough when there are multiple different $x$'s.

**"Do you think it matters if you did point $A$ or point $B$ first?"** No, the distance between $A$ and $B$ is the same as the distance between $B$ and $A$.

**"This formula shouldn't be surprising at all. It just describes what we did with numbers."**

# The Distance Formula

The **distance** between points $A(x_A, y_A)$ and $B(x_B, y_B)$ is

$$d = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}.$$

# Section: Circles

**NEXT: Graphing circles**

**Key ideas:** The equation for a circle can be built directly from the distance formula.

1. No, the distance is $\sqrt{(6-2)^2+(1-(-3))^2} = \sqrt{4^2+4^2} = \sqrt{32}$, which is not equal to 5.

2. No, the distance is $\sqrt{(4-2)^2+(2-(-3))^2} = \sqrt{2^2+5^2} = \sqrt{29}$, which is not equal to 5.

3. The distance from $(x,y)$ to $(2,-3)$ is $\sqrt{(x-2)^2+(y+3)^2}$, so one possible equation is

$$\sqrt{(x-2)^2+(y+3)^2} = 5.$$

**"Should we clean this up a bit?"** Sure, let's square both sides.

$$(x-2)^2+(y+3)^2 = 25.$$

**"What do you think the graph of this equation looks like?"** It's a circle, because the points are all 5 units away from a given point.

1. Is the distance between $(6,1)$ and $(2,-3)$ equal to 5?

2. Is the distance between $(4,2)$ and $(2,-3)$ equal to 5?

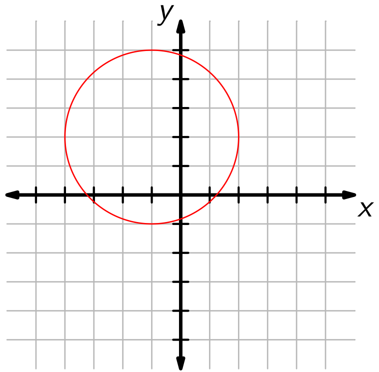
3. Write an equation that must be true for any point $(x,y)$ that is 5 units away from $(2,-3)$.

**NEXT: General circle equation given center and radius**

**Key ideas:** The center-radius form of a circle comes directly from the distance formula.

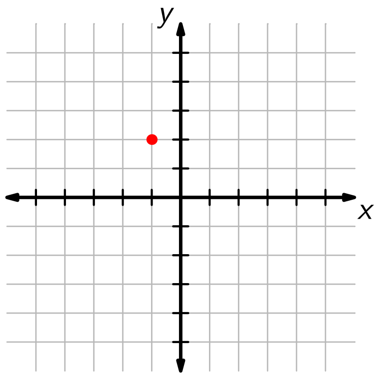
*If students are stuck, start by identifying some points that are 3 units away, notably* $(2,2)$ *and* $(-4,2)$. *Try to get them to think of the distance formula here. If things still are not working, ask them for the distance between* $(x,y)$ *and* $(-1,2)$ *as an expression.*

$$\sqrt{(x+1)^2+(y-2)^2}=3$$
$$(x+1)^2+(y-2)^2=9$$




Consider all points $(x,y)$ such that the distance from $(x,y)$ to $(-1,2)$ is equal to 3.

1. Sketch the graph of these points.
2. What equation describes these points?



**NEXT: Standard form for the equation of a circle**

*Introduce the third equation as a generalization of the first and second.*

**"We just saw that the graph of $(x+1)^2+(y-2)^2=9$ is a circle. What is its center?"** The center is $(-1,2)$.
**"What is its radius?"** The radius is 3, because 9 is the square of the radius.

**"Look at the second equation. What can you figure out?"** The center of this circle is $(7,-2)$, because those are the values of $x$ and $y$ that make each piece zero. The radius is 8, because $\sqrt{64}=8$.

**"Look at the third equation. What can you figure out?"** The center is $(h,k)$ and the radius is $r$.

**"What does $h$ represent?"** It is the $x$-coordinate of the center of the circle.

**"What does $k$ represent?"** It is the $y$-coordinate of the center of the circle.

**"For the equation $(x+1)^2+(y-2)^2=9$, what is $h$, what is $k$, and what is $r$?"** Since the equation uses $x+1$, $h=-1$, $k=2$, and $r=3$.

The graph of each equation is a circle. What is its **center** and **radius**?

$$(x+1)^2+(y-2)^2=9$$

$$(x-7)^2+(y+2)^2=64$$

$$(x-h)^2+(y-k)^2=r^2$$

**NEXT: A hidden circle**

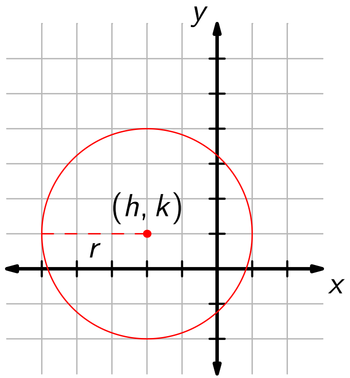
This slide lists the standard equation of a circle.

*IF NEEDED, to further probe students' understanding*: **"What's the equation of the pictured circle?"** $(x+2)^2+(y-1)^2=9$.

# The Equation of a Circle

The standard form for the equation of a circle with center $(h, k)$ and radius $r$ is given by

$$(x-h)^2+(y-k)^2=r^2$$



**NEXT: Completing the square**

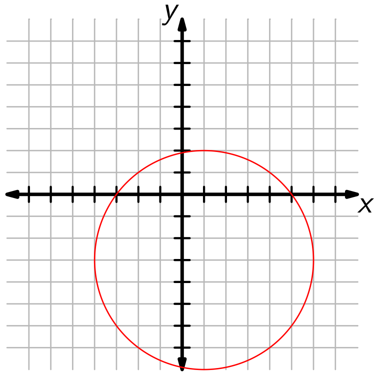
**Key ideas:** This doesn't look like the circle equation, but it does. Perfect square trinomials makes this possible.

**"OK, now we know how to graph circles. Let's try graphing this quadratic."**

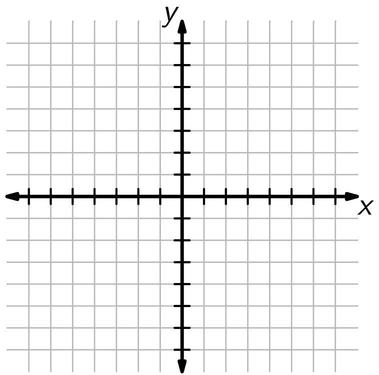
*Give students some time to figure out how to rewrite the equation.*

**"Wait, how do you know this is also a circle?"** The groups of terms form perfect squares, and we can rewrite them: $(x-1)^2+(y+3)^2=5^2$.

We could then take the square root of each side: $\sqrt{(x-1)^2+(y+3)^2}=5$. This proves the equation is a circle, because the distance formula tells us it is all the points that are 5 units away from $(1,-3)$.




Sketch the graph of $x^2-2x+1+y^2+6y+9=25$.



**NEXT: Degenerate case**

**Key ideas:** The center-radius equation can be built by completing the square, twice.

**"What can we do with this equation?"** This looks like it could be a circle, if we complete the square. We will also need to regroup the terms.

$$x^2+y^2+4x-10y=-13$$

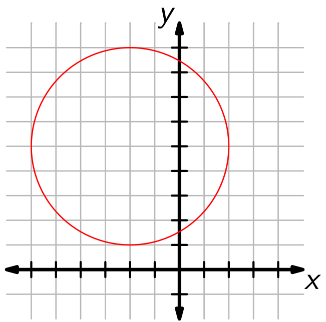
$$x^2+4x+\boxed{\phantom{00}}+y^2-10y+\bigcirc=-13+\boxed{\phantom{00}}+\bigcirc$$

$$x^2+4x+4+y^2-10y+25=-13+4+25$$

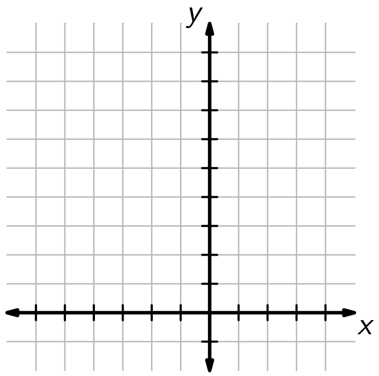
$$x^2+4x+4+y^2-10y+25=16$$

$$(x+2)^2+(y-5)^2=16.$$

It is a circle with center $(-2,5)$ and radius 4.




Sketch the graph of $x^2+y^2+4x-10y=-13$.



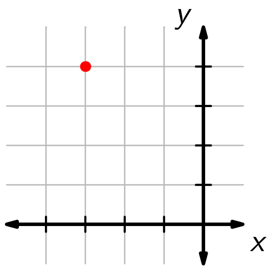
**NEXT: Circles and Inequalities**

*Let students come to the conclusion that they need to complete the square.*

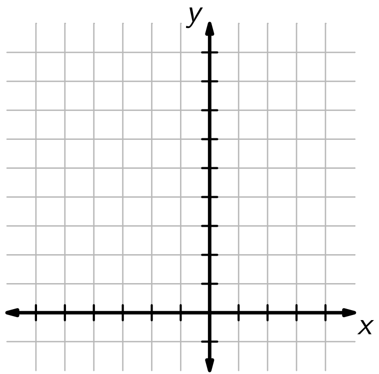
$$x^2+y^2+6x-8y=-25$$
$$x^2+6x+\boxed{\phantom{9}}+y^2-8y+\bigcirc=-25+\boxed{\phantom{9}}+\bigcirc$$
$$x^2+6x+9+y^2-8y+16=-25+9+16$$
$$(x+3)^2+(y-4)^2=0.$$

**"What do you think this graph is?"** It is a circle of radius 0, just the point $(-3,4)$.

**"Could we have a similar equation where there is no graph at all?"** Yes, make the value of $r^2$ negative. Since the coordinate plane only uses real numbers, no imaginary numbers can appear.




Sketch the graph of $x^2+y^2+6x-8y=-25$.



# Section: Quadratic Inequalities

**NEXT: On the outside**

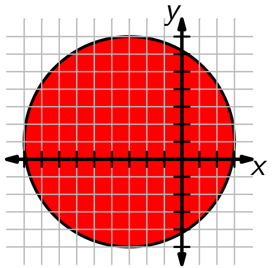
***Expect students' memories of 2D inequalities to be relatively poor, notably about the solid and dotted lines involved.***

**"What makes this different from the previous problems?"** Here we have an **inequality**.

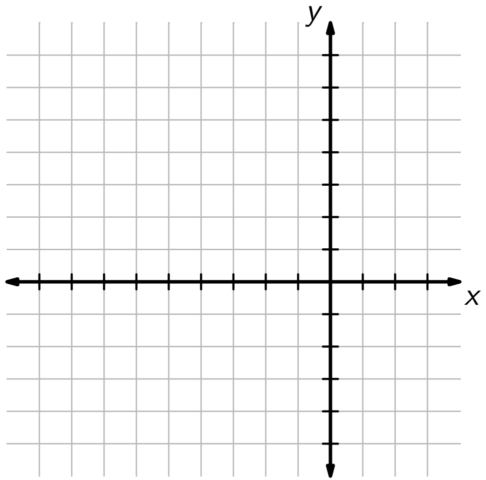
**"How should we start?"** Let's start by sketching the graph of $(x+3)^2+(y-1)^2=36$, which defines a boundary. The points on the boundary make the inequality true, so the circular boundary is solid, not dashed.

**"In terms of the distance formula, what does it mean if $(x+3)^2+(y-1)^2 \le 36$?"** It means that the distance from $(x,y)$ to $(-3,1)$ is less than or equal to 6.

**"How do we draw this on our graph?"** We shade inside the circle. We could also have tested points both inside and outside the circle to decide when the inequality was true. *Have students test some points to see if they make the inequality true.*




Sketch the graph of $(x+3)^2+(y-1)^2 \le 36$.

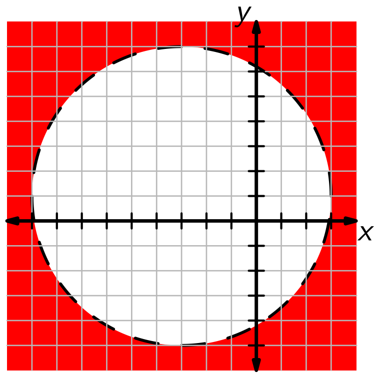


NEXT: Inside a parabola

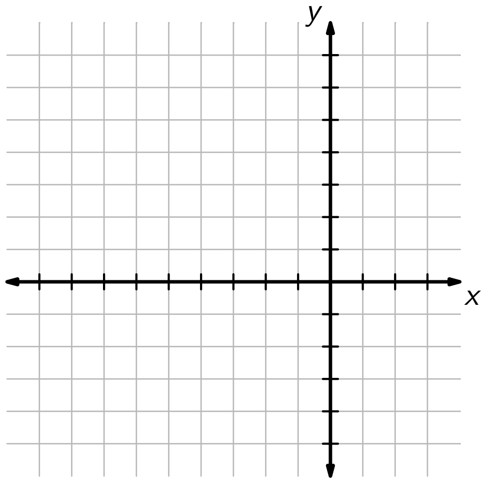
**"How does this question relate to the last one?"** We are going to graph everything that wasn't part of the last graph. We graph the points outside the circle.

The circular boundary is dashed, not solid. This indicate that points on the circle do not make the inequality true.

**"What does this say, in relation to the distance formula?"** The distance from $(x, y)$ to $(-3, 1)$ is greater than 6.




Sketch the graph of $(x+3)^2+(y-1)^2 > 36$.



**NEXT: Extensions**

**Optional: Medium**

**"What's similar, and what's different?"** It's still an inequality, but this time it involves a parabola.

**"How do we start drawing this graph?"** Graph the boundary. The boundary is the graph of $y=(x-3)^2-4$, and since the boundary does not make the inequality true, it is dashed.

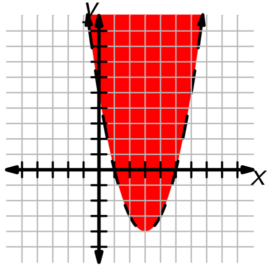
The vertex of the parabola is $(3,-4)$, and it points up with intercepts $(0,5)$, $(1,0)$, and $(5,0)$.

**"Where do we shade?"** A handy way to do this is to pick a point on each side of the boundary, and test:

$$\text{test } (0,0): 0>(0-3)^2-4, \text{ false!}$$

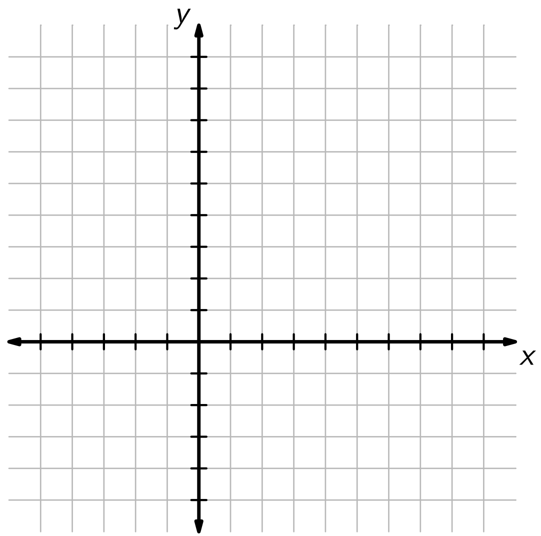
$$\text{test } (3,0): 0>(3-3)^2-4, \text{ true!}$$

So the region containing $(3,0)$ should be shaded, and the region containing $(0,0)$ should not be shaded.




Alternately, you could notice that the equation is written as $y>\cdots$, which means we're looking for the points where $y$ is greater than the corresponding point on the parabola. That means we should shade above it.

Sketch the graph of $y>(x-3)^2-4$.



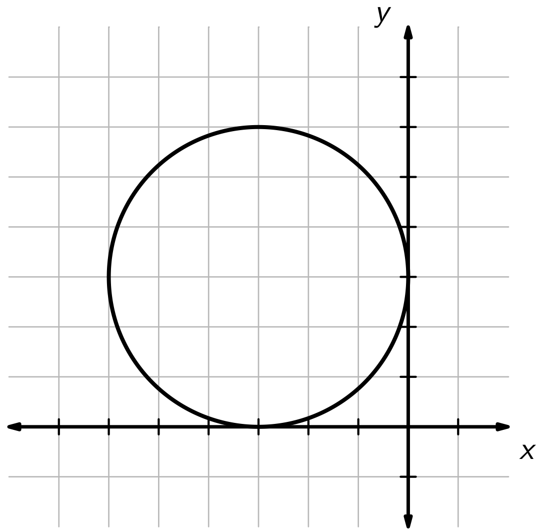
# Extensions

Please pass out the Extensions handout. Ideally, the students will work in small groups, but they may choose to work alone. Meanwhile, the instructor circulates to offer guidance, challenging the high-flyers while providing extra support to students who struggled with earlier content.

# Extensions

This is a circle of radius 3, centered at $(-3,3)$. Its equation is $\boxed{(x+3)^2+(y-3)^2=9}$.

What equation does the following graph represent?



We start by dividing everything by 4 to make the coefficients of $x^2$ and $y^2$ equal to 1. This gives

$$x^2+y^2+x-4y=\frac{7}{4}.$$

Completing the square then gives us

$$\begin{aligned} x^2+x+\frac{1}{4}+y^2-4y+4 &= \frac{7}{4}+\frac{1}{4}+4 \\ \left(x+\frac{1}{2}\right)^2+(y-2)^2 &= 6. \end{aligned}$$

The graph of this equation is a circle with center $\boxed{\left(-\frac{1}{2},2\right)}$ and radius $\boxed{\sqrt{6}}$.

Find the center and radius of the circle that is the graph of the equation $4x^2+4y^2+4x-16y=7$.

(Problem 14.20 in the textbook)

If you are stuck, read Section 14.2 in the textbook.

The information lets us determine the circle's center and radius.

The center is the midpoint of the diameter, which is

$$\left(\frac{(-2)+10}{2}, \frac{3+(-7)}{2}\right) = (4,-2)$$

The radius is the distance between the center and any point on the circle, which is

$$\sqrt{(10-4)^2+(-2-(-7))^2} = \sqrt{36+25} = \sqrt{61}$$

Knowing the center and radius, we can use the form

$$(x-h)^2+(y-k)^2 = r^2$$

In this case the equation is $\boxed{(x-4)^2+(y+2)^2 = 61}$.

Let's verify that the original points work in this equation by letting $(x,y) = (-2,3)$ and $(x,y) = (10,-7)$.

$$(-2-4)^2+(3+2)^2 = (-6)^2+5^2 = 61$$

$$(10-4)^2+(-7+2)^2 = 6^2+(-5)^2 = 61$$

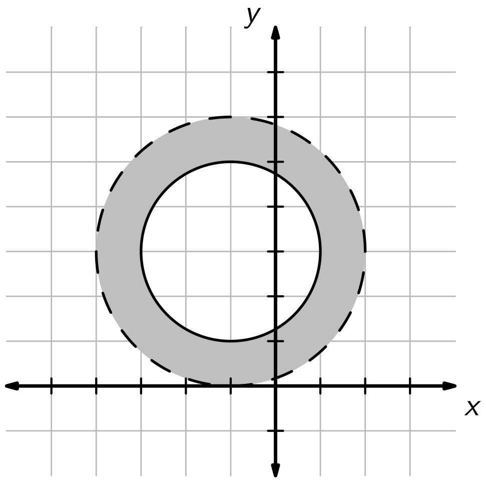
The fact that opposite numbers are seen shows that these points are on a diameter: their midpoint is the center of the circle.

A circle has a diameter with endpoints $(-2,3)$ and $(10,-7)$.

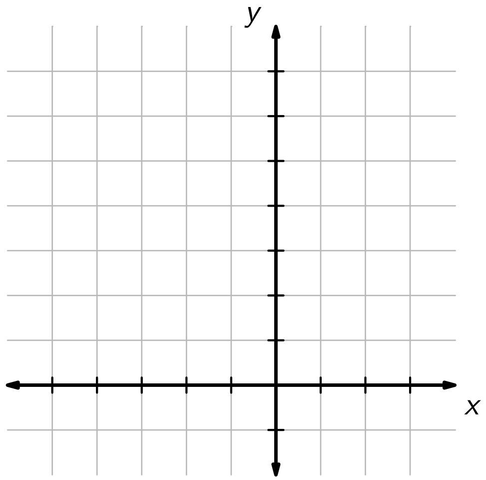
Write an equation for this circle.

These inequalities represent circles centered at $(-1,3)$. The larger circle has radius 3, and the smaller circle has radius 2.

The points we're looking for are outside the smaller circle, but inside the larger circle. The boundary of the smaller circle is included, but the boundary of the larger circle is not.

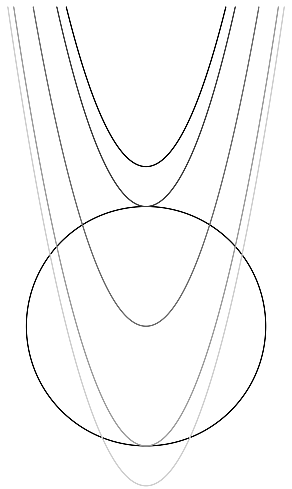



Graph the points for which $4 \le (x+1)^2 + (y-3)^2 < 9$.



There can be 0, 1, 2, 3, or 4 intersections. It can be proven through a later course's algebra work that there can't be more than 4 intersections.

There are multiple ways to do this, but the five parabolas shown illustrate all the possibilities with the one given circle.



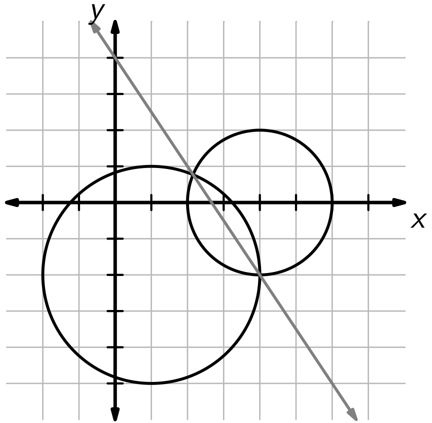
A parabola and a circle are drawn.

What are the possibilities for the number of intersections between their graphs? Draw an example of each.

We multiply the second equation by $-1$ and add it to the first equation, which will remove the $x^2$ and $y^2$ terms. Adding

$$x^2 - 2x + y^2 + 4y = 4$$
$$-x^2 + 8x - y^2 = 12$$

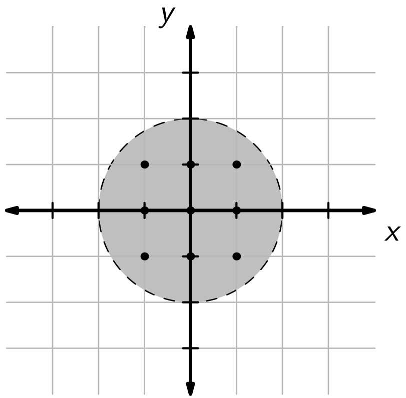
we get $6x + 4y = 16$, which in standard form is $\boxed{3x + 2y = 8}$. This equation must be satisfied by every point that is on the graphs of **both** circles, so this must be the equation of the radical axis.




The circles with equations $x^2 - 2x + y^2 + 4y = 4$ and $x^2 - 8x + y^2 = -12$ intersect at two points. The line through these points is called the **radical axis**. Find its equation.

How many lattice points are on the graph of $x^2 + y^2 < 4$? A **lattice point** is a point where both coordinates are integers.

The graph is the interior of a disc of radius 2. For a point to be on this graph, it must be *inside* the circle, not on it. This is true for any $-1 \le x \le 1, -1 \le y \le 1$, and not otherwise. There are $\boxed{9}$ such points.



***If you skipped the last slide, consider asking students to skip this problem; it will be much more difficult.***

The parabola has its vertex at $(4,-3)$, so its equation is of the form $y=a(x-4)^2-3$. It contains the point $(6,1)$, so $1=a(6-4)^2-3$ and $4a=4$, so $a=1$. Thus, the equation of the parabola is $y=(x-4)^2-3$.

The center of the circle is at $(4,2)$, and its radius is 5, so its equation is $(x-4)^2+(y-2)^2=25$.

Since all of our lines are solid, both of our inequalities are non-strict.

We're looking for the region inside the circle and above the parabola.

Thus, the region is described by

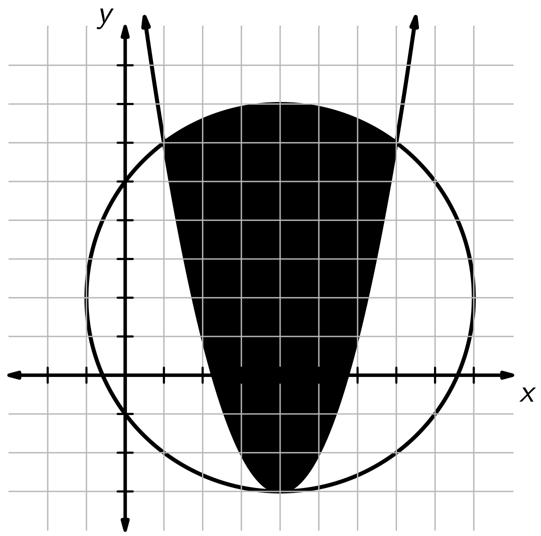
$$\boxed{y \geq (x-4)^2-3, \quad (x-4)^2+(y-2)^2 \leq 25}.$$

The parabola and circle appear to intersect in Quadrant I at $(1,6)$ and $(7,6)$, and we can verify this by substitution:

$$6 \geq (1-4)^2-3, \quad (1-4)^2+(6-2)^2 \leq 25$$

For both, there is equality. And $(7,6)$ will behave the same way, since we replace $(1-4)^2$ with $(7-4)^2$ in each.

# Write a system of inequalities that describes the following region.



## End of Class Activity

**Have students work in pairs.**

**If you are required to practice social distancing:** have students work individually instead of in pairs.

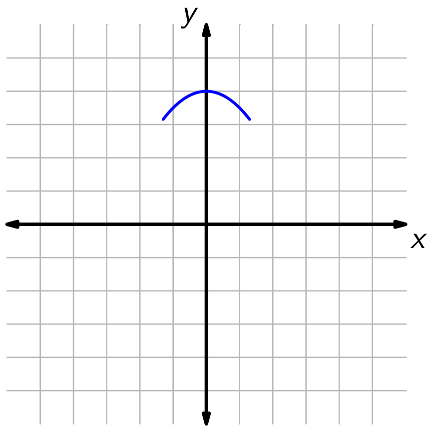
*The structure of this activity has everyone work on the same problem, with the slide showing the graph in blue, then reveal the solution. Since students will have a handout, it is possible for them to work ahead to later problems. You can also use the document camera to display students' possible solutions, as written on the handout.*

*Remind students that they may want to spend a lot of time deciding what type of graph they are looking at, since that will determine the graph's parameters and possibilities.*

# Target Practice

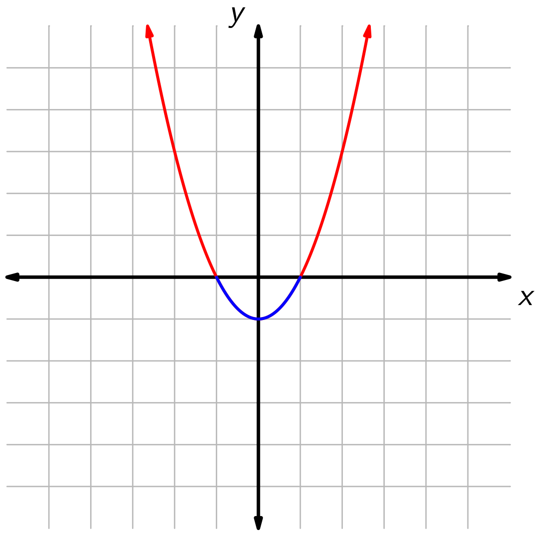
You'll be given a small piece of a graph, and asked to find an equation and sketch the complete graph.

There are no limits to the types of graphs being used, so be mindful that many graphs can look the same when you only see a little of it.

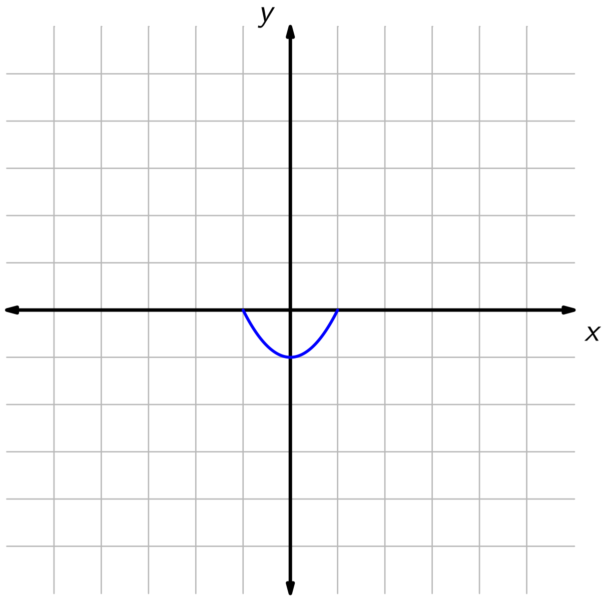


*Each problem has two pages, one with the piece, and one with the complete graph. On this page, a preview of the solution.*

1. This is the graph of $y = x^2 - 1$.




1. Sketch the full graph, and write an equation.



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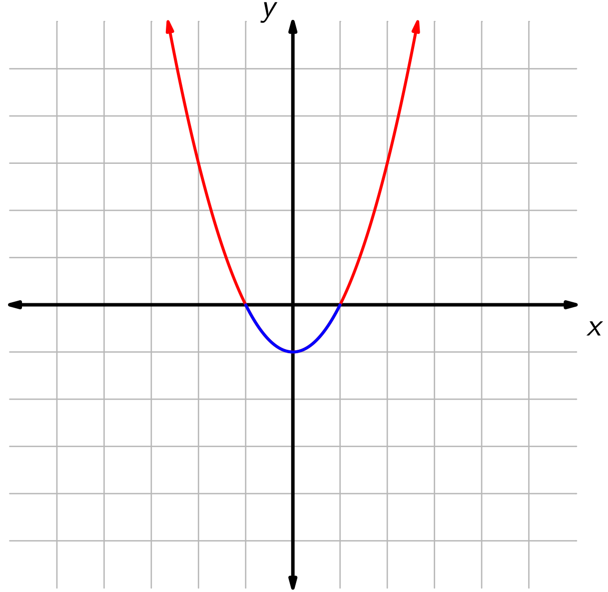
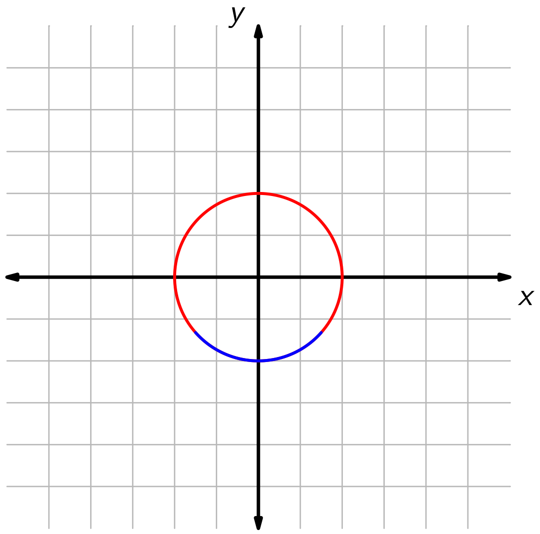


*Each problem has two pages, one with the piece, and one with the complete graph. On this page, a preview of the solution.*

2. This is the graph of $x^2 + y^2 = 4$.

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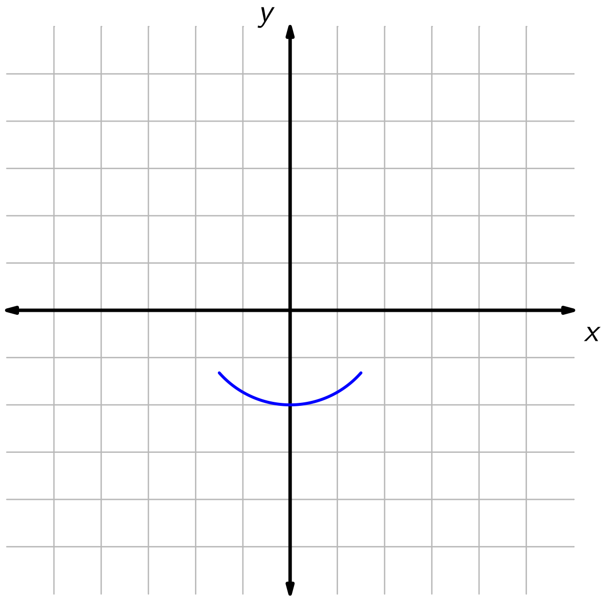

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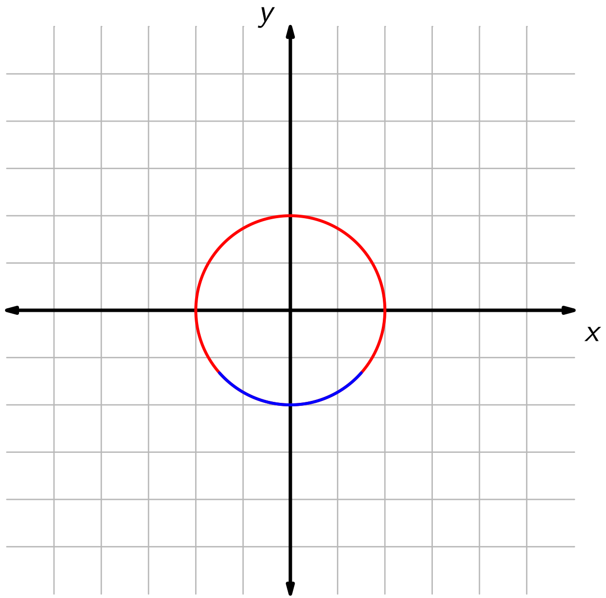

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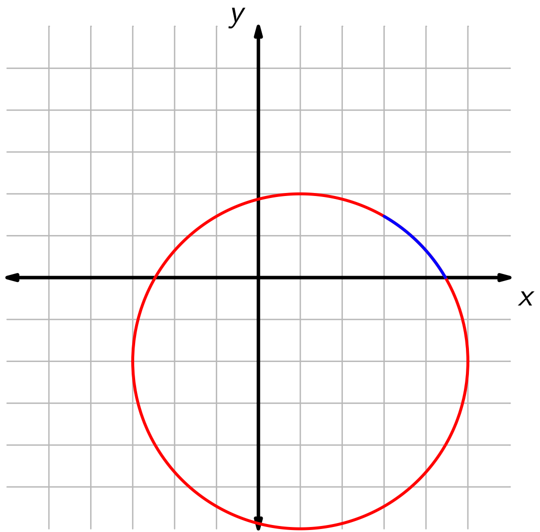
3. This is the graph of $(x-1)^2+(y+2)^2=16$.

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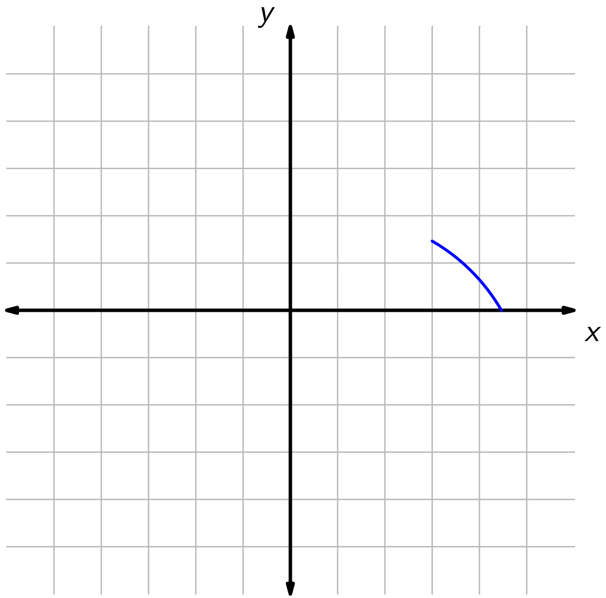

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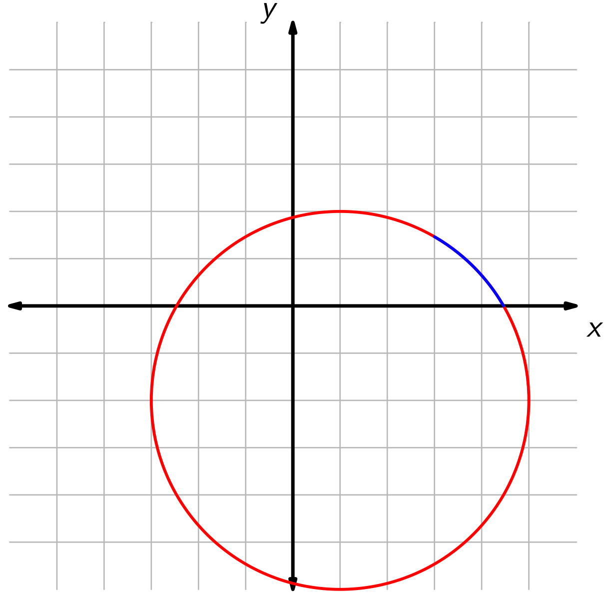
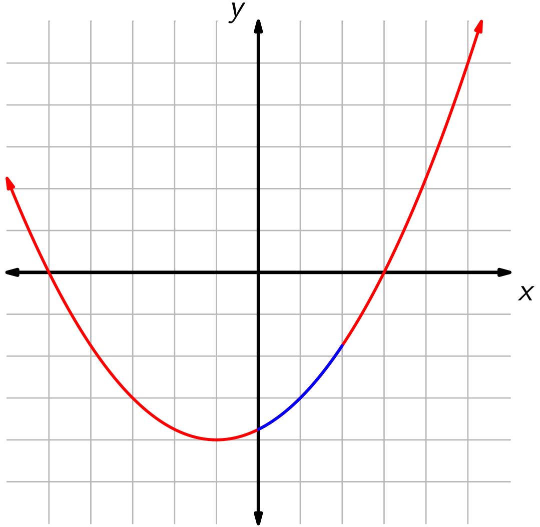


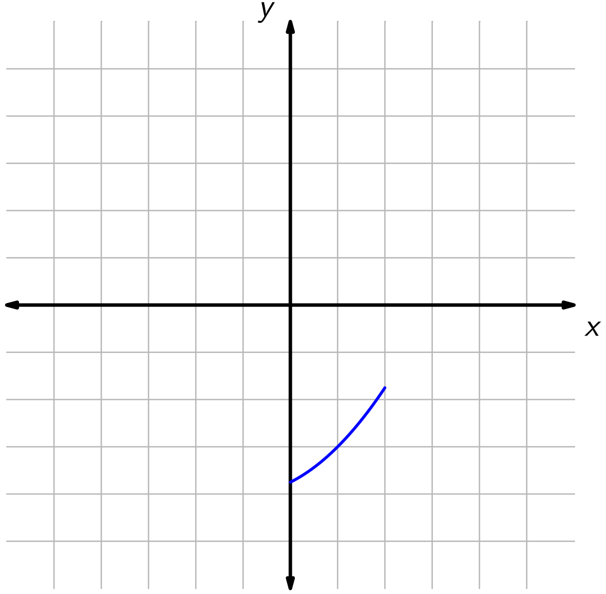
**Graph 4 of 5**

*Each problem has two pages, one with the piece, and one with the complete graph. On this page, a preview of the solution.*

4. This is the graph of $y = \frac{1}{4}(x+1)^2 - 4$.




4. Sketch the full graph, and write an equation.



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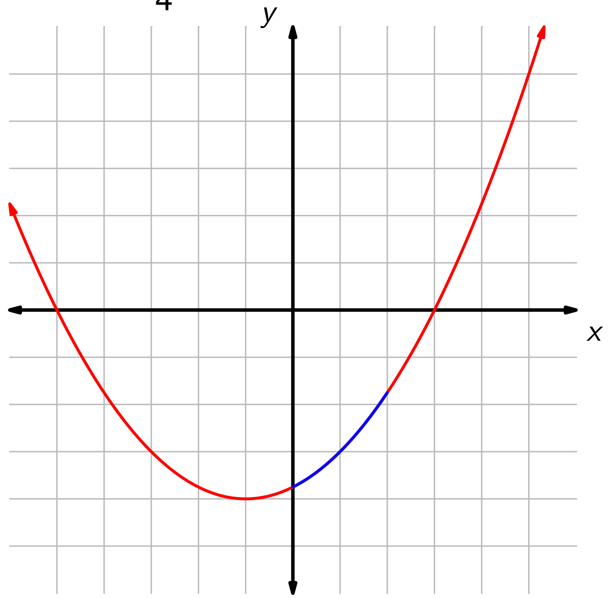
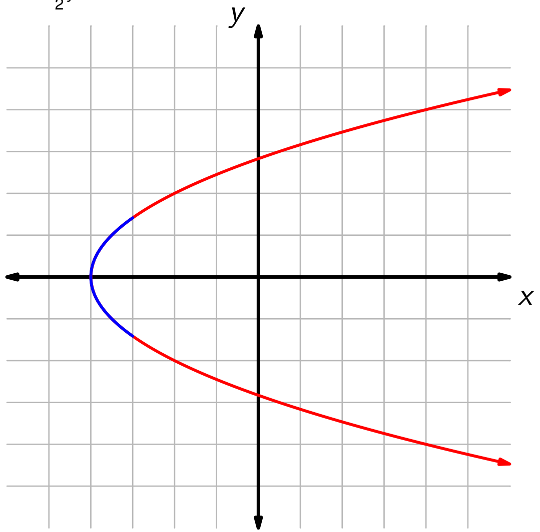


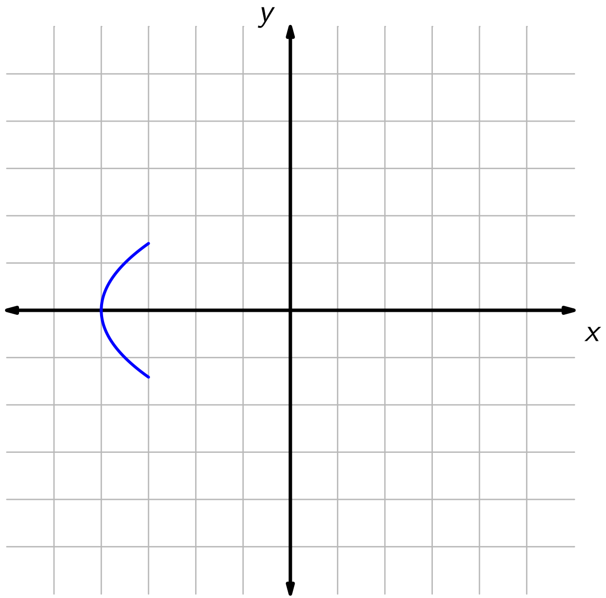
Graph 5 of 5

*Each problem has two pages, one with the piece, and one with the complete graph. On this page, a preview of the solution.*

5. This is the graph of $x = \frac{1}{2}y^2 - 4$.




5. Sketch the full graph, and write an equation.

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