Classification of Root Systems

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Chapter 1

Root Systems

1.1 Defining Root Systems

Definition (Root System). Let E be a finite-dimensional Euclidean space with an inner product $\langle \cdot, \cdot \rangle$.

A **root system** Φ in E is a finite, non-empty set of non-zero vectors (called **roots**) satisfying the following properties:

(R1) Φ spans E.

(R2) For every root $\alpha \in \Phi$, the set Φ is closed under reflection through the hyperplane orthogonal to α. That is, for any two roots $\alpha, \beta \in \Phi$, the set Φ contains the element

$$\sigma_\alpha(\beta) = \beta - 2 \operatorname{proj}_\alpha(\beta).$$

where $\operatorname{proj}_\alpha(\beta)$ is the projection of β on α as shown below.

$$\operatorname{proj}_\alpha(\beta) := \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

The **rank** of the root system is the dimension of the Euclidean space E.

Example 1. *$R_0 = \{\pm\alpha\}$, where α is any fixed non-zero real number, constitutes a root system in $\mathbb{R}$.*

Example 2. *For any root system Φ in E, the set $\alpha\Phi$ (i.e. scaling every root by α) for any non-zero scalar α is also a root system in E*

Definition (Reduced Root System). If a root system satisfies the condition that the only multiples of a root, α, that are in the root system are $\pm\alpha$, then the root system is said to be **reduced**.

Definition (Crystallographic Root System). If a root system satisfies the integrality condition below, then it is said to be **crystallographic**.

$$\langle\langle \beta, \alpha \rangle\rangle := 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Phi$$

Throughout this text, denote e_i as the i-th standard basis vector in $\mathbb{R}^n$. Then, in combinations such as $\pm e_i \pm e_j$, the signs may be chosen independently.

Example 3. *The set R_1, shown below, is a root system in $\mathbb{R}^2$ that is neither reduced nor crystallographic.*

$$R_1 = \{\pm e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}), (\pm\sqrt{3}, \pm 1)\}$$

R_1 spans $\mathbb{R}^2$ and is closed under reflection through the hyperplane orthogonal to any root, hence it is a root system.

However, is is not a reduced root system since a scalar multiple of an element in R_1, namely $2 \cdot (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})$, is contained in R_1 itself. It is also not a crystallographic root system because $\langle\langle e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})\rangle\rangle = \frac{\sqrt{3}}{2} \notin \mathbb{Z}$.

Example 4. *If we remove the redundant multiple in R_1 above, we obtain a reduced, non-crystallographic root system R_2.*

$$R_2 = \{\pm e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})\}$$

One can also construct examples of non-reduced crystallographic root systems. Consider the following example,

Example 5. *The set R_3 is a root system in $\mathbb{R}^2$ that is crystallographic but not reduced.*

$$R_3 = \{\pm e_1, \pm e_2, \pm 2e_1\}$$

R_3 spans $\mathbb{R}^2$ and is closed under reflection through the hyperplane orthogonal to any root, hence it is a root system. It is a crystallographic root system because $\langle\langle ke_1, e_2\rangle\rangle = 0$ and $\langle\langle ke_1, k'e_1\rangle\rangle = kk' \in \mathbb{Z}$, where $k, k \in \{\pm 1, \pm 2\}$. However, is is not a reduced root system since $2e_1 \in R_3$.

Example 6. *The set R_4 is a root system in $\mathbb{R}^2$ that is reduced and crystallographic.*

$$R_4 = \{\pm e_1, \pm e_2\}$$

Therefore, we see that a root system may be reduced, crystallographic, both, or neither.

The integrality condition can be interpretted geometerically as follows–the projection of β on α is an integer or half-integer multiple of α since,

$$\text{proj}_\alpha(\beta) = \frac{1}{2}\langle\langle \beta, \alpha\rangle\rangle\alpha$$

In fact, this is the most restrictive condition since,

$$\langle\langle \beta, \alpha\rangle\rangle = 2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} = 2\frac{\|\beta\|\|\alpha\|\cos\theta}{\|\alpha\|^2} = 2\frac{\|\beta\|}{\|\alpha\|} \in \mathbb{Z},$$

where θ is the angle between α and β.

Furthermore, since $\langle\langle \beta, \alpha\rangle\rangle$ and $\langle\langle \alpha, \beta\rangle\rangle$ must be integers,

$$\langle\langle \beta, \alpha\rangle\rangle \cdot \langle\langle \alpha, \beta\rangle\rangle = 4\cos^2\theta \in \mathbb{Z}$$

We restate this result as the following lemma.

Lemma 7. *For any two roots $\alpha, \beta \in \Phi$, the product $\langle\langle \beta, \alpha \rangle\rangle \cdot \langle\langle \alpha, \beta \rangle\rangle$ is an integer. More precisely, $4\cos^2\theta \in \{0, 1, 2, 3, 4\}$. If $4\cos^2\theta = 4$, then $\theta = 0$ or π and $\beta = \pm\alpha$ which is exactly condition $(R2)$ for a root system.*

The possible values for $4\cos^2\theta$ and the corresponding angles between the roots are displayed in the table below.

$4\cos^2\theta$	$\langle\langle \alpha, \beta \rangle\rangle$	$\langle\langle \beta, \alpha \rangle\rangle$	$\|\alpha\|/\|\beta\|$	θ
0	0	0	N/A	$\pi/2$
1	1	1	1	$\pi/3$
	-1	-1	1	$2\pi/3$
2	1	2	$\sqrt{2}$	$\pi/4$
	-1	-2	$\sqrt{2}$	$3\pi/4$
3	1	3	$\sqrt{3}$	$\pi/6$
	-1	-3	$\sqrt{3}$	$5\pi/6$

Table 1.1: The possible angles between roots where, without loss of generality, the root α is no longer than the root β under the induced norm in E. A computation also reveals that $|\langle \beta, \alpha \rangle| \geq |\langle \alpha, \beta \rangle|$.

Since we aim to classify all root systems, up to isomorphism, it is important to understand when two root systems are isomorphic.

Definition (Root System Isomorphism). Two root systems (E, Φ) and (F, Ψ) are said to be isomorphic if there exists a linear isomorphism $\varphi : E \to F$ such that $\varphi(\Phi) = \Psi$ and preserves the number $\langle\langle x, y \rangle\rangle$ for each pair of roots i.e for any roots $\alpha, \beta \in \Phi$, $\langle\langle \phi(\alpha), \phi(\beta) \rangle\rangle = \langle\langle \alpha, \beta \rangle\rangle$.

Example 8. *The root systems $R = \{\pm e_1, \pm e_2\}$ and $R' = \{\pm e_1 \pm e_2\}$ are isomorphic. Consider the map $\phi(x, y) = (x + y, x - y)$. This maps e_1 to $e_1 + e_2$ and e_2 to $e_1 - e_2$. It is easily verifiable that it satisfies the conditions for a root system isomorphism.*

Example 9. *The root systems $S = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$ and $S' = \{\pm e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})\}$ are not isomorphic.*

Finally, we wish to exclude root systems that can be constructed from direct sums of smaller root systems. This motivates the following definition.

Definition (Decomposable Root System). A Root System Φ is said to be **decomposable** if there is a proper decomposition $\Phi = \Phi_1 \cup \Phi_2$ such that Φ_1 and Φ_2 are root systems in E and $\Phi_1 \perp \Phi_2$ (i.e. $\forall \alpha_1 \in \Phi_1, \forall \alpha_2 \in \Phi_2, \langle \alpha_1, \alpha_2 \rangle = 0$).

Otherwise, Φ is said to be **indecomposable**. Often, in literature, such root systems are also called **irreducible**. We adopt the term indecomposable to avoid confusion with the notion of reduced root systems.

Example 10. *The root system $R_4 = \{\pm e_1, \pm e_2\}$ is an indecomposable, reduced, crystallographic root system.*

Example 11. *The root system $R_5 = \{\pm e_1, \pm e_2\} \cup \{\pm 2e_1\}$ is a decomposable root system.*

This report concerns itself primarily with reduced, crystallographic root systems, simply referred to as root systems henceforth, unless otherwise specified. [1].

1.2 Classifying Root Systems of Small Rank

Theorem (Classification of Rank 1 Root Systems). The only indecomposable, reduced, crystallographic root systems of rank 1 (up to isomorhpism) is $A_1 = \{\pm\alpha\}$ [See Figure 1.1], where α is any fixed non-zero real number.

Proof. Let Φ be a reduced, crystallographic root system of rank 1. Let $\alpha \in \Phi$ be a non-zero root. Since Φ is reduced, the only multiples of α in Φ are $\pm\alpha$. Therefore, $\Phi = \{\pm\alpha\}$. Clearly, if the root system depends on α where $\Phi_\alpha = \{\pm\alpha\}$, then for any non-zero α_1, α_2 in the Euclidean space, Φ_{α_1} is isomorphic to Φ_{α_2}. $\square$

Figure 1.1: Root System A_1

Theorem (Classification of Rank 2 Root Systems). There exist only 4 reduced, crystallographic root systems (up to isomorphism). Namely,

$$\begin{aligned}
A_1 \times A_1 &:= \{\pm\alpha_1, \pm\alpha_2\} \\
A_2 &:= \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\} \\
B_2 &:= \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2)\} \\
G_2 &:= \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2)\}
\end{aligned}$$

where α_1 and α_2 are roots satisfying the condition that $\|\alpha_1\| \leq \|\alpha\|$ for all α in the root system under the induced norm (i.e. α_1 is the shortest root) and the angle between α_1 and α_2, $\theta_{\alpha_1,\alpha_2}$ is no smaller than $\pi/2$.

Proof. Notice that the roots must span the space $\mathbb{R}^2$. Hence, there must be atleast two linearly independent roots. Pick α_1 to be the shortest root (under the inner product), i.e. $\|\alpha_1\| \leq \|\alpha\|$ for all $\alpha \in \mathbb{R}^2$.

Choose α_2 such that the angle between the two chosen roots is no smaller than $\pi/2$. This is always possible, since if the angle were to be smaller than $\pi/2$ then $-\alpha_2$, which must also be a root by (R2), would have an angle no smaller than $\pi/2$ with α_1.

Then Table 1.1 enumerates all possible angles between these roots and given the constraints we deduce that there are only four possible cases. We begin with the linearly independent vectors

[1]The provided definition differs from the more general definition stated in Lean 4's MathLib. In Lean 4, root systems are defined over modules instead of Euclidean spaces among other minor differences. As such, when neccessary the proofs in Lean 4 restrict the more general definition to our definition. Of note, root systems in Lean 4 are assumed to be reduced, crystallographic root systems. The precise axioms can be found in Lean 4's MathLib

α_1 & α_2 and construct from them a root system with the four possible cases of angles between them.

Case 1. $\theta_{\alpha_1,\alpha_2} = \pi/2$. In this scenario, the set $A_1 \times A_1$ is already a root system and any further additions would make it a non-reduced root-system.

Case 2. $\theta_{\alpha_1,\alpha_2} = 2\pi/3$. In this case, we begin with the set $\{\pm\alpha_1, \pm\alpha_2\}$ and observe that it is not closed under reflection through the hyperplane othogonal to its roots. It is evident that upon the addition of the element $\pm(\alpha_1 + \alpha_2)$ into the above set, we do get a closed root system.

Case 3. $\theta_{\alpha_1,\alpha_2} = 3\pi/4$. Yet again, begin with the set $\{\pm\alpha_1, \pm\alpha_2\}$. Computing $\sigma_{\alpha_2}(\alpha_1)$ returns $\alpha_1 + \alpha_2$, and $\sigma_{\alpha_2}(\alpha_1)$ gives $2\alpha_1 + \alpha_2$. Thus adding those into our set and performing a final closure check leads to the root system B_2.

Case 4. $\theta_{\alpha_1,\alpha_2} = 5\pi/6$. We repeat the process again, and arrive at $\sigma_{\alpha_2}(\alpha_1) = 3\alpha_1 + \alpha_2$, $\sigma_{\alpha_1}(\alpha_1 + \alpha_2) = 2\alpha_1 + \alpha_2$, $\sigma_{\alpha_2}(3\alpha_1 + \alpha_2) = 3\alpha_1 + 2\alpha_2$. Thus we obtain our final root system G_2.

Since all these root systems are not ismoprhic, we can conclude that the given root systems categorize all root systems of rank 2. $\square$

Chapter 2

Classifying Root Systems

The classification presented ahead closely follows Humphrey's proof in [2]. Other proofs for the classification of root systems can be found in several textbooks, a few common references include [1], [2] and [5].

2.1 Simple Roots

For a particular root system Φ one may choose a subset (not necesarily unique) of roots called **simple roots** that are, in essence, building blocks of all the other roots in the root system.

For each root, $\alpha \in \Phi$, there exists a unique hyperplane passing through the origin that is orthogonal to this root. Since the root system is finite, the union of all such hyperplanes can never be the whole ambient space E.

As such, it is always possible to find a vector $d \in E$ such that for any root α, $\langle \alpha, d \rangle \neq 0$. One partitions the root system into two sets, $R^+(d)$ and $R^-(d)$, where $R^+(d) = \{\alpha \in \Phi | \langle \alpha, d \rangle > 0\}$ and $R^-(d) = -R^+(d)$.

Definition (Simple Root). Let d be a vector in a finite dimensional inner product space E, such that for any α in the root system Φ, $\langle \alpha, d \rangle \neq 0$. If α belongs to $R^+(d)$ then it is a positive root; otherwise, it is defined to be a negative root. A positive root is called **simple** if it is not the sum of any two other positive roots. The set of all such simple roots of a root system Φ is called a **basis** for Φ.

Fortunately, the choice of the vector d above is not important. Since the following remark is not directly related to the classification of root systems, it is only discussed briefly, but one may refer to the literature read more [4].

For every root α in a root system Φ, there exists a hyperplane orthogonal to the root and the union of all such hyperplanes cut the ambient space E into open, connected regions called **Weyl Chambers**.

Interestingly, one may construct a one-to-one correspondence between all bases for a root system and Weyl Chambers [3].

2.2 Properties of Simple Roots

We establish several useful lemmas that we use later.

Lemma 12. *If the roots α, β in a root system Φ are not proportional and $\langle \alpha, \beta \rangle > 0$, then $\alpha - \beta \in \Phi$.*

Proof. It follows that $\langle\langle \alpha, \beta \rangle\rangle > 0$. Using Table 1.1, we see that either $\langle\langle \alpha, \beta \rangle\rangle$ or $\langle\langle \beta, \alpha \rangle\rangle$ is 1. If $\langle\langle \alpha, \beta \rangle\rangle$ is 1, then $\sigma_\beta(\alpha) = \alpha - \beta \in \Phi$. Otherwise, $\sigma_\alpha(-\beta) = \alpha - \beta \in \Phi$. □

Lemma 13. *Let α and β be distinct simple roots in the root system Φ, then $\langle \alpha, \beta \rangle \leq 0$.*

Proof. Assume to the contrary that $\langle \alpha, \beta \rangle > 0$. Then from the previous lemma (lemma 12), we know that $\gamma := \alpha - \beta \in \Phi$. If γ is a positive root, then $\alpha = \gamma + \beta$ is also positive, contradicting our hypothesis. Otherwise, $-\gamma$ must be a positive root, in which case $\beta = \alpha + (-\gamma)$ is a positive root, yet again contradicting our hypothesis. □

The proofs for the following lemmas can be found in [3].

The following lemma justifies the use of the term basis for the set of all simple roots.

Lemma 14. *The simple roots are linearly independent and span their ambient space.*

And the proceeding lemma links the decomposability of a root system to its basis.

Lemma 15. *A root system is decomposable if and only if its basis is decomposable.*

2.3 Classification of Root Systems

From lemma 13, we may deduce that for any two simple roots, the roots are either orthogonal or the angle between them is obtuse. Much like the proof for the classification of Rank 2 Root Systems, from Table 1.1 we see that the angle between the aforementioned roots must be either $\pi/2$, $2\pi/3$, $3\pi/4$ or $5\pi/6$. This information can be encoded into a graph.

Definition (Coxeter Graph). The **Coxeter Graph** of a root system Φ, is a (multi)graph, with each vertex corresponds to a simple root of Φ and every pair of vertices $\hat{\alpha}, \hat{\beta}$ are connected by $\langle\langle \alpha, \beta \rangle\rangle \langle\langle \beta, \alpha \rangle\rangle = 4cos^2\theta$ edges, where θ is the angle between the simple roots $\alpha, \beta \in \Phi$ corresponding to the vertices in the graph.

Coxeter graphs do not however capture all essential information about the simple roots of a root system. In particular, the relative lengths of the roots (like when the angle between the roots is $3\pi/4$ or $5\pi/6$) are not represented by the Coxeter Graph.

Definition (Dynkin Diagram). A **Dynkin Diagram** of a root system is a Coxeter graph of the root system with arrows on the double or triple edges to indicate the shorter root

In order to classify indecomposable root systems it suffices to classify all possible Coxeter Graphs, since the Dynkin Diagrams can be obtained from the Coxeter graphs by adding the arrows.

Definition (Admissable Configuration). A linearly independent set of n unit vectors $\{v_1, \dots, v_n\}$ that spans the ambient space E is called an admissible configuration if for all $i \neq j$, $\langle v_i, v_j \rangle \leq 0$ and $4\langle v_i, v_j \rangle^2 \in \{0, 1, 2, 3\}$.

Observe that the set of normalized simple roots of any root system is an admissible configuration by lemma 14.

Definition (Admissable Diagram). A Coxeter graph of an admissible configuration is called an admissible diagram.

From lemma 15, we know that the basis of an indecomposable root system cannot be decomposed into mutually orthogonal subsets. Therefore, the corresponding Coxter graph for an indecomposable root system will also be connected. As such, to classify all indecompsable root systems it is sufficient to consider only connected admissible diagrams. This leads directly to the main classification theorem.

Theorem (Classification of Indecompsable Root Systems). The Dynkin Diagram of an indecomposable root system belongs to one of the nine families shown in Fig. 2.3.

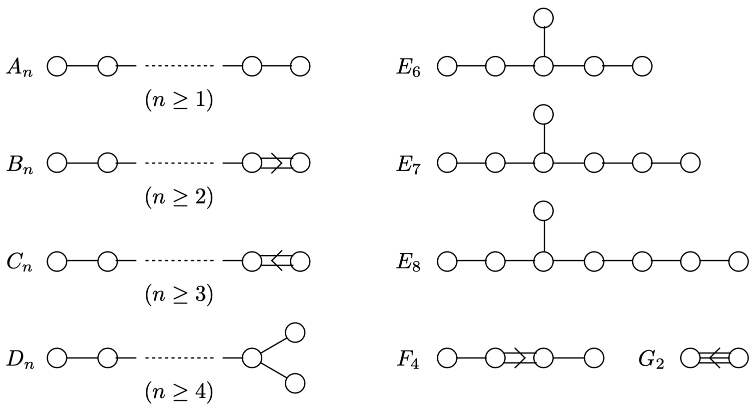

Figure 2.1: All possibilities for The Dynkin Diagram of Indecomposable Root Systems. Four infinite families and five exceptional root systems.

Proof. We begin by classifying connected admissible diagrams and then proceeding to the Dynkin Diagrams.

Since the proof is large, it is split into the following steps.

1. Any subdiagram of an admissible diagram is also admissible. If the set $\{v_1, \ldots, v_n\}$ is an admissible configuration, then clearly any subset of it is also an admissible configuration (in the space it spans). The same holds for admissible diagrams.

2. A connected admissible diagram is a tree. Let $v = \sum_{i=1}^{n} v_i$. Evidently, $v \neq 0$, since the admissible configuration is linearly independent. Then,

$$0 < \langle v, v\rangle = \sum_{i=1}^{n} \langle v_i\rangle + \sum_{i<j} 2\langle v_i, v_j\rangle = n + 2\sum_{i<j} \langle v_i, v_j\rangle$$

If the vertices v_i and v_j are connected, then $2\langle v_i, v_j\rangle$ must take on a value in the set $\{-1, -\sqrt{2}, -\sqrt{3}\}$. In particular, $2\langle v_i, v_j\rangle \leq -1$, hence the number of terms in the sum above cannot exceed $n-1$, so the number of distinct pairs of connected vertices must also be at most $n-1$. Since the diagram is connected, there must be at least $n-1$ such pairs. This leads to the conclusion that the number of distinct connected pairs of vertices is exactly $n-1$ and thus the diagram is a tree.

3. No more than three edges (counting multiplicities) can originate from the same vertex. Let c be any vertex (we refer to its corresponding root as c as well when there is no ambiguity) and $\{v_1, \dots v_k\}$ be all vertices that are connected to c. Since the graph has no cycles, there are no edges between any v_i and v_j. Therefore, $\langle v_i, v_j \rangle = 0$ when $i \neq j$ and $\{v_1, \dots v_k\}$ is an orthonormal set. Morevoer, since the simple roots are linearly independent, c can not be expressed as a linear combination of v_i's. Hence c has a non-zero projection to the orthogonal complement of span$\{v_1, \dots v_k\}$. Normalize this projection and denote it by v_0. Then $\{v_0, v_1, \dots v_k\}$ is an orthonormal set and we can express c as follows:

$$c = \sum_{i=0}^{k} \langle c, v_i \rangle v_i$$

Since c is a unit vector, $\langle c, c \rangle = \sum_{i=0}^{k} \langle c, v_i \rangle^2 = 1$. However $\langle c, v_i \rangle^2 \neq 0$, thus,

$$\sum_{i=1}^{k} 4\langle c, v_i \rangle^2 < 4 \tag{2.1}$$

Recall that the value $4\langle c, v_i \rangle^2$ denotes the number of edges between the vertices c and v_i, hence it follows from equation 2.1 that the number of edges originating from c are no more than 3.

4. The only connected admissible diagram containing a triple edge is G_2 that is shown in Fig. 2.3. This follows from the previous step. From now on we will consider only diagrams with single and double edges.

   Before proceeding any further, we recall the definition for a simple chain.

   Definition 16. *A* ***simple chain*** *in a graph is a non-repeating sequence of vertices such that every two consecutive vertices are connected with a single edge.*

5. Any simple chain $(v_1, \dots, v_l)$ in a connected admissible diagram can be replaced by the single vector $v = \sum_{i=1}^{l} v_i$. [See Fig 2.2]

   It suffices to show that v is a unit vector and the new diagram, obained after the replacement, is connected and admissible. Note that,

$$\langle v, v \rangle = l + \sum_{i<j} 2\langle v_i, v_j \rangle$$

   Since there are no cycles in the diagram, $\langle v_i, v_j \rangle = 0$ for all pairs where $i < j$ except for when $j = i+1$ (i.e consecutive vertices). Then for consecutive vertices, we have $\langle v_i, v_{i+1} \rangle = -1$, which is why

$$\sum_{i<j} 2\langle v_i, v_j \rangle = \sum_{i=1}^{k-1} 2\langle v_i, v_{i+1} \rangle = -(k-1)$$

   And so $\langle v, v \rangle = 1$.

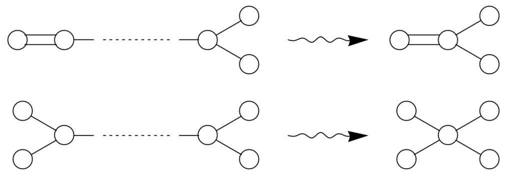

Figure 2.2: A Visualization of collapsing simple chains in step 5.

Furthermore, since the diagram is acyclic, an arbitrary vertex u not in the chain can be connected to at most one vertex (call it v_m) in the chain. Then,

$$\langle u, v\rangle = \sum_{i=1}^{k}\langle u, v_i\rangle = \langle u, v_m\rangle$$

Consequently, when the whole chain is replaced by the single cevtor v, any vertex u that was not in the chain reamains connected to v in the same way it was to v_m. Hence, we can conclude that this new diagram is also connected and admissible.

6. A connected admissible diagram has none of subdiagrams shown in Fig. 2.2. In each case the subdiagram contains a simple chain. According to Step 5 it can be collapsed to a single vertex. To the contrary, Step 3 shows that the obtained subdiagram is not valid, since it has a vertex of degree four. This contradiction Step 1.

7. As such, a connected admissible diagram can contain at most one double edge and at most one branching, but not both of them simultaneously. Excluding, diagram G2 with a triple edge, we can make the following conclusion. There are only three possible types of connected admissible diagrams (see Fig. 2.3):

   T1: a simple chain

   T2: a diagram with a double edge

   T3: a diagram with branching

8. The admissible diagram of type T1 corresponds to the Dynkin diagram A_n in Fig. 2.3, where $n \geq 1$.

9. The only admissible diagrams of type T2 are $B_n \cong C_n$, and F_4. Define $u = \sum_{i=1}^{p} i \cdot u_i$, where p is number of vertices before the double edge (See Figure 2.3). A brief computation reveals,

$$\langle u, u\rangle = \sum_{i=1}^{p} i^2\langle u_i, u_i\rangle + \sum_{i<j} ij \cdot 2\langle u_i, u_j\rangle = p^2 - \frac{p(p-1)}{2} = \frac{p(p+1)}{2} \tag{2.2}$$

Likewise, define $v = \sum_{j=1}^{q} j \cdot v_j$ to get $\langle v, v\rangle = q(q+1)/2$. Then, $\langle u, v\rangle = pq\langle u_p, v_q\rangle$, since the double edge is the only edge between the components of u and v. Then, $\langle u, v\rangle^2 = p^2q^2/2$ since the number of edges $4\langle u_p, v_q\rangle$ is 2.

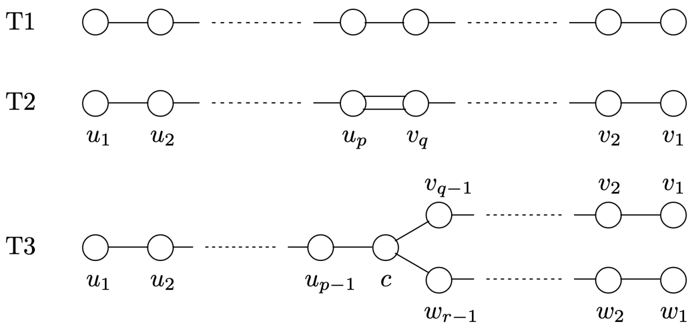

Figure 2.3: Possible types of connected admissible diagrams, excluding G_2.

Then, since u is not a multiple of v Cauchy-Schwarz holds strictly giving us,

$$\frac{p^2q^2}{2} < \frac{p(p+1)}{2} \cdot \frac{q(q+1)}{2}$$

It is known that p and q are positive integers, so the only solutions are $p = 1$ and q is arbitrary (or vice versa) or $p = q = 2$.

The first solution corresponds to either of B_n or C_n (depending upon the choice of the short root), while the second solution corresponds to the Dynkin Diagram F_4 in Fig. 2.3.

If $n = 1$, both solutions correspond to A_1, however if $n = 2$ then we have the case for $B_2 = C_2$. Otherwise, we establish the convention that B_n has $n \geq 2$ while C_n has $n \geq 3$.

10. Finally, we claim that the only admissible diagrams of type T3 are D_n, E_6, E_7 and E_8. As earlier, define $u = \sum_{i=1}^{p} i \cdot u_i$ and $v = \sum_{j=1}^{q} j \cdot v_j$. This time, also define $w = \sum_{k=1}^{r-1} k \cdot w_k$. Observe that there are no direct edges between the components u_i, v_j or w_k so they exist in mutually orthogonal subspaces. This holds also for u, v and w.

    Using a similar argument to step 3, we can conclude that c is not a linear combination of u, v and w, hence:
    $$1 = \langle c, c\rangle > \langle c, \hat{u}\rangle^2 + \langle c, \hat{v}\rangle^2 + \langle c, \hat{w}\rangle^2 \tag{2.3}$$
    where $\hat{u}, \hat{v}$ and $\hat{w}$ are the corresponding unit vectors for u, v and w. So,

    $$\langle c, \hat{u}\rangle^2 = \frac{\langle c, u\rangle^2}{\langle u, u\rangle} \tag{2.4}$$

    Note that no u_i is connected to c except for u_{p-1}, thus $\langle c, u_i\rangle = 0$ for all $i \neq p-1$. Moreover, since they are connected by a single edge, $4\langle c, u_{p-1}\rangle = 1$. Therefore, the numerator in the relation above (Eq. 2.4) is,

    $$\langle c, u\rangle^2 = \sum_{i-1}^{p-1} i^2 \langle c, u_i\rangle^2 = (p-1)^2 \cdot \langle c, u_{p-1}\rangle = \frac{(p-1)^2}{4}$$

p	q	r	Dynkin Diagram
any	2	2	D_n
3	3	2	E_6
4	3	2	E_7
5	3	2	E_8

Table 2.1: All possible integer solutions to inequality 2.5 and their corresponding Dynkin Diagrams of type T3.

Whereas, the denominator (in Eq. 2.4) can be obtained to be $p(p-1)/2$ using Eq. 2.2. Then, Eq. 2.4 transforms into the following after some computation (note that $p \geq 2$).

$$\langle c, \hat{u} \rangle^2 = \frac{1}{2}\left(1 - \frac{1}{p}\right)$$

Repeating this for v and w helps transform Eq. 2.3 into,

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1, \qquad p, q, r \geq 2 \tag{2.5}$$

Without loss of generaltiy, assume $p \geq q \geq r \geq 2$. Observe that $r < 3$ since the sum cannot exceed 1. This forces $r = 2$. If we take $q = 2$ as well, then any feasible p works. If $q = 3$, however, then $p < 6$. No other solutions exist for $q \geq 4$. The correspondence between type T3 Dynkin Diagrams and the solution here are summarized in Table 2.1.

This completes the classification of all indecomposable root systems.

□

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