Classification of Root Systems

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Chapter 1

Classifying Root Systems

1.1 Root Systems

Definition (Root System). Let E be a finite-dimensional Euclidean space with an inner product $\langle \cdot, \cdot \rangle$.

A **root system** in E is a tuple (E, Φ), where Φ is a finite, non-empty set of non-zero vectors (called **roots**) satisfying the following properties:

(R1) Φ spans E.

(R2) For every root $\alpha \in \Phi$, the set Φ is closed under reflection through the hyperplane orthogonal to α. That is, for any two roots $\alpha, \beta \in \Phi$, the set Φ contains the element

$$\sigma_\alpha(\beta) = \beta - 2 \operatorname{proj}_\alpha(\beta).$$

where $\operatorname{proj}_\alpha(\beta)$ is the projection of β on α as shown below.

$$\operatorname{proj}_\alpha(\beta) := \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

The **rank** of the root system is the dimension of the Euclidean space E.

For convenience and in contexts where the inner product space is clear, the root system is often referred to simply as Φ.

Example 1. *$R_0 = \{\pm\alpha\}$, where α is any fixed non-zero real number, constitutes a root system in $\mathbb{R}$.*

Definition (Reduced Root System). If a root system satisfies the condition that the only multiples of a root, α, that are in the root system are $\pm\alpha$, then the root system is said to be **reduced**.

Definition (Crystallographic Root System). If a root system satisfies the integrality condition below, then it is said to be **crystallographic**.

$$\langle\langle \beta, \alpha \rangle\rangle = 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Phi$$

It should be noted that for crystallographic root systems, the integrality condition implies the second condtion (R2) in root systems since $\sigma_\alpha(\alpha) = -\alpha$. The integrality condition can be interpretted geometerically as follows–the projection of β on α is an integer or half-integer multiple of α since,

$$\text{proj}_\alpha(\beta) = \frac{1}{2}\langle\langle\beta, \alpha\rangle\rangle\alpha$$

In fact, this is the most restrictive condition since,

$$\langle\langle\beta, \alpha\rangle\rangle = 2\frac{\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} = 2\frac{\|\beta\|\|\alpha\|\cos\theta}{\|\alpha\|^2} = 2\frac{\|\beta\|}{\|\alpha\|} \in \mathbb{Z},$$

where θ is the principal angle between α and β.

Furthermore, since $\langle\langle\beta, \alpha\rangle\rangle$ and $\langle\langle\alpha, \beta\rangle\rangle$ must be integers,

$$\langle\langle\beta, \alpha\rangle\rangle \cdot \langle\langle\alpha, \beta\rangle\rangle = 4\cos^2\theta \in \mathbb{Z}$$

We restate this result as the following lemma.

Lemma 2. *For any two roots $\alpha, \beta \in \Phi$, the product $\langle\langle\beta, \alpha\rangle\rangle \cdot \langle\langle\alpha, \beta\rangle\rangle$ is an integer. More precisely, $4\cos^2\theta \in \{0, 1, 2, 3, 4\}$. If $4\cos^2\theta = 4$, then $\theta = 0$ and $\beta = \pm\alpha$ which is exactly condition R2 for a root system.*

Throughout this text, denote e_i as the i-th standard basis vector in $\mathbb{R}^n$. Then, in combinations such as $\pm e_i \pm e_j$, the signs may be chosen independently.

Example 3. *The set R_1, shown below, is a root system in $\mathbb{R}^2$ that is neither reduced nor crystallographic.*

$$R_1 = \{\pm e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}), (\pm\sqrt{3}, \pm 1)\}$$

R_1 spans $\mathbb{R}^2$ and is closed under reflection through the hyperplane orthogonal to any root, hence it is a root system.

*However, is is not a **reduced** root system since a scalar multiple of an element in R_1, namely $2 \cdot (\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$, is contained in R_1 itself. It is also not a **crystallographic** root system because $\langle\langle e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})\rangle\rangle = \frac{\sqrt{3}}{2} \notin \mathbb{Z}$.*

Example 4. *If we remove the redundant multiple in R_1 above, we obtain a reduced, non-crystallographic root system R_2.*

$$R_2 = \{\pm e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})\}$$

One can also construct examples of non-reduced crystallographic root systems. Consider the following example,

Example 5. *The set R_3 is a root system in $\mathbb{R}^2$ that is crystallographic but not reduced.*

$$R_3 = \{\pm e_1, \pm e_2, \pm 2e_1\}$$

*R_3 spans $\mathbb{R}^2$ and is closed under reflection through the hyperplane orthogonal to any root, hence it is a root system. It is a **crystallographic** root system because $\langle\langle ke_1, e_2\rangle\rangle = 0$ and $\langle\langle ke_1, k'e_1\rangle\rangle = kk' \in \mathbb{Z}$, where $k, k \in \{\pm 1, \pm 2\}$. However, is is not a **reduced** root system since $2e_1 \in R_3$.*

Example 6. *The set R_4 is a root system in $\mathbb{R}^2$ that is reduced and crystallographic.*

$$R_4 = \{\pm e_1, \pm e_2\}$$

Therefore, we see that a root system may be reduced, crystallographic, both, or neither.

Since we aim to classify all root systems, upto isomorhism, it is important to understand when two root systems are isomorphic.

Definition (Root System Isomorphism). Two root systems (E, Φ) and (F, Ψ) are said to be isomorphic if there exists a linear isomorphism $\varphi : E \to F$ such that $\varphi(\Phi) = \Psi$ and preserves the number $\langle\langle x, y\rangle\rangle$ for each pair of roots.

Example 7. *The root systems $R = \{\pm e_1, \pm e_2\}$ and $R' = \{\pm e_1 \pm e_2\}$ are isomorphic.*

Example 8. *The root systems $S = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$ and $S' = \{\pm e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})\}$ are not isomorphic.*

Finally, we wish to exclude root systems that can be constructed from direct sums of smaller root systems. This motivates the following definition.

Definition (Decomposable Root System). A Root System Φ is said to be **decomposable** if there is a proper decomposition $\Phi = \Phi_1 \cup \Phi_2$ such that Φ_1 and Φ_2 are root systems in E and $\Phi_1 \perp \Phi_2$ (i.e. $\forall \alpha_1 \in \Phi_1, \forall \alpha_2 \in \Phi_2, \langle \alpha_1, \alpha_2\rangle = 0$).

Otherwise, Φ is said to be **indecomposable**. Often, in literature, such root systems are also called **irreducible**. We adopt the term indecomposable to avoid confusion with the notion of reduced root systems.

Example 9. *The root system $R_4 = \{\pm e_1, \pm e_2\}$ is an indecomposable reduced, crystallographic root system.*

Example 10. *The root system $R_5 = \{\pm e_1, \pm e_2\} \cup \{\pm 2e_1\}$ is a decomposable root system.*

This report concerns itself primarily with indecomposable, reduced, crystallographic root systems, simply referred to as root systems henceforth, unless otherwise specified. [1]

1.2 Classifying Root Systems of Small Rank

Theorem (Classification of Rank 1 Root Systems). The only indecomposable, reduced, crystallographic root systems of rank 1 are $R_0 = \{\pm\alpha\}$, where α is any fixed non-zero real number.

Proof. Let Φ be a reduced, crystallographic root system of rank 1. Let $\alpha \in \Phi$ be a non-zero root. Since Φ is reduced, the only multiples of α in Φ are $\pm\alpha$. Therefore, $\Phi = \{\pm\alpha\}$. □

[1] The provided definition differs from the more general definition stated in Lean 4's MathLib. In Lean 4, root systems are defined over modules instead of Euclidean spaces among other minor differences. As such, when neccessary the proofs in Lean 4 restrict the more general definition to our definition. Of note, root systems in Lean 4 are assumed to be reduced, crystallographic root systems.