Classification of Root Systems

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Chapter 1

Root Systems

1.1 Root Systems

Definition (Root System). Let E be a finite-dimensional Euclidean space with an inner product $\langle \cdot, \cdot \rangle$.

A **root system** in E is a tuple (E, Φ), where Φ is a finite, non-empty set of non-zero vectors (called **roots**) satisfying the following properties:

(R1) Φ spans E.

(R2) For every root $\alpha \in \Phi$, the set Φ is closed under reflection through the hyperplane orthogonal to α. That is, for any two roots $\alpha, \beta \in \Phi$, the set Φ contains the element

$$\sigma_\alpha(\beta) = \beta - 2 \operatorname{proj}_\alpha(\beta).$$

where $\operatorname{proj}_\alpha(\beta)$ is the projection of β on α as shown below.

$$\operatorname{proj}_\alpha(\beta) := \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha$$

The **rank** of the root system is the dimension of the Euclidean space E.

For convenience and in contexts where the inner product space is clear, the root system is often referred to simply as Φ.

Example 1. *$R_0 = \{\pm\alpha\}$, where α is any fixed non-zero real number, constitutes a root system in $\mathbb{R}$.*

Definition (Reduced Root System). If a root system satisfies the condition that the only multiples of a root, α, that are in the root system are $\pm\alpha$, then the root system is said to be **reduced**.

Definition (Crystallographic Root System). If a root system satisfies the integrality condition below, then it is said to be **crystallographic**.

$$\langle\langle \beta, \alpha \rangle\rangle = 2 \frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Phi$$

Throughout this text, denote e_i as the i-th standard basis vector in $\mathbb{R}^n$. Then, in combinations such as $\pm e_i \pm e_j$, the signs may be chosen independently.

Example 2. *The set R_1, shown below, is a root system in $\mathbb{R}^2$ that is neither reduced nor crystallographic.*

$$R_1 = \{\pm e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}), (\pm\sqrt{3}, \pm 1)\}$$

R_1 spans $\mathbb{R}^2$ and is closed under reflection through the hyperplane orthogonal to any root, hence it is a root system.

However, is is not a ***reduced*** *root system since a scalar multiple of an element in R_1, namely $2 \cdot (\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$, is contained in R_1 itself. It is also not a* ***crystallographic*** *root system because $\langle\langle e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})\rangle\rangle = \frac{\sqrt{3}}{2} \notin \mathbb{Z}$.*

Example 3. *If we remove the redundant multiple in R_1 above, we obtain a reduced, non-crystallographic root system R_2.*

$$R_2 = \{\pm e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})\}$$

One can also construct examples of non-reduced crystallographic root systems. Consider the following example,

Example 4. *The set R_3 is a root system in $\mathbb{R}^2$ that is crystallographic but not reduced.*

$$R_3 = \{\pm e_1, \pm e_2, \pm 2e_1\}$$

R_3 spans $\mathbb{R}^2$ and is closed under reflection through the hyperplane orthogonal to any root, hence it is a root system. It is a ***crystallographic*** *root system because $\langle\langle ke_1, e_2\rangle\rangle = 0$ and $\langle\langle ke_1, k'e_1\rangle\rangle = kk' \in \mathbb{Z}$, where $k, k \in \{\pm 1, \pm 2\}$. However, is is not a* ***reduced*** *root system since $2e_1 \in R_3$.*

Example 5. *The set R_4 is a root system in $\mathbb{R}^2$ that is reduced and crystallographic.*

$$R_4 = \{\pm e_1, \pm e_2\}$$

Therefore, we see that a root system may be reduced, crystallographic, both, or neither.

It should be noted that for crystallographic root systems, the integrality condition implies the second condtion (R2) in root systems since $\sigma_\alpha(\alpha) = -\alpha$. The integrality condition can be interpretted geometerically as follows–the projection of β on α is an integer or half-integer multiple of α since,

$$\text{proj}_\alpha(\beta) = \frac{1}{2}\langle\langle\beta, \alpha\rangle\rangle\alpha$$

In fact, this is the most restrictive condition since,

$$\langle\langle\beta, \alpha\rangle\rangle = 2\frac{\langle\alpha, \beta\rangle}{\langle\alpha, \alpha\rangle} = 2\frac{\|\beta\|\|\alpha\|\cos\theta}{\|\alpha\|^2} = 2\frac{\|\beta\|}{\|\alpha\|} \in \mathbb{Z},$$

where θ is the angle between α and β.

Furthermore, since $\langle\langle\beta, \alpha\rangle\rangle$ and $\langle\langle\alpha, \beta\rangle\rangle$ must be integers,

$$\langle\langle\beta, \alpha\rangle\rangle \cdot \langle\langle\alpha, \beta\rangle\rangle = 4\cos^2\theta \in \mathbb{Z}$$

We restate this result as the following lemma.

Lemma 6. *For any two roots $\alpha, \beta \in \Phi$, the product $\langle\langle\beta,\alpha\rangle\rangle \cdot \langle\langle\alpha,\beta\rangle\rangle$ is an integer. More precisely, $4\cos^2\theta \in \{0,1,2,3,4\}$. If $4\cos^2\theta = 4$, then $\theta = 0$ or π and $\beta = \pm\alpha$ which is exactly condition $(R2)$ for a root system.*

The possible values for $4\cos^2\theta$ and the corresponding angles between the roots are displayed in the table below.

$4\cos^2\theta$	$\langle\langle\alpha,\beta\rangle\rangle$	$\langle\langle\beta,\alpha\rangle\rangle$	$\lVert\alpha\rVert/\lVert\beta\rVert$	θ
0	0	0	N/A	$\pi/2$
1	1	1	1	$\pi/3$
	-1	-1	1	$2\pi/3$
2	1	2	$\sqrt{2}$	$\pi/4$
	-1	-2	$\sqrt{2}$	$3\pi/4$
3	1	3	$\sqrt{3}$	$\pi/6$
	-1	-3	$\sqrt{3}$	$5\pi/6$

Table 1.1: The possible angles between roots where, without loss of generality, the root α is no longer than the root β under the induced norm in E. A computation also reveals that $|\langle\beta,\alpha\rangle| \geq |\langle\alpha,\beta\rangle|$.

Since we aim to classify all root systems, upto isomorhism, it is important to understand when two root systems are isomorphic.

Definition (Root System Isomorphism). Two root systems (E,Φ) and (F,Ψ) are said to be isomorphic if there exists a linear isomorphism $\varphi : E \to F$ such that $\varphi(\Phi) = \Psi$ and preserves the number $\langle\langle x,y\rangle\rangle$ for each pair of roots.

Example 7. *The root systems $R = \{\pm e_1, \pm e_2\}$ and $R' = \{\pm e_1 \pm e_2\}$ are isomorphic.*

Example 8. *The root systems $S = \{\pm e_1, \pm e_2, \pm e_1 \pm e_2\}$ and $S' = \{\pm e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})\}$ are not isomorphic.*

Finally, we wish to exclude root systems that can be constructed from direct sums of smaller root systems. This motivates the following definition.

Definition (Decomposable Root System). A Root System Φ is said to be **decomposable** if there is a proper decomposition $\Phi = \Phi_1 \cup \Phi_2$ such that Φ_1 and Φ_2 are root systems in E and $\Phi_1 \perp \Phi_2$ (i.e. $\forall \alpha_1 \in \Phi_1, \forall \alpha_2 \in \Phi_2, \langle\alpha_1,\alpha_2\rangle = 0$).

Otherwise, Φ is said to be **indecomposable**. Often, in literature, such root systems are also called **irreducible**. We adopt the term indecomposable to avoid confusion with the notion of reduced root systems.

Example 9. *The root system $R_4 = \{\pm e_1, \pm e_2\}$ is an indecomposable, reduced, crystallographic root system.*

Example 10. *The root system $R_5 = \{\pm e_1, \pm e_2\} \cup \{\pm 2e_1\}$ is a decomposable root system.*

This report concerns itself primarily with reduced, crystallographic root systems, simply referred to as root systems henceforth, unless otherwise specified. [1]

1.2 Classifying Root Systems of Small Rank

Theorem (Classification of Rank 1 Root Systems). The only indecomposable, reduced, crystallographic root systems of rank 1 (upto isomorhism) is $A_1 = \{\pm\alpha\}$ [See Figure 1.1], where α is any fixed non-zero real number.

Proof. Let Φ be a reduced, crystallographic root system of rank 1. Let $\alpha \in \Phi$ be a non-zero root. Since Φ is reduced, the only multiples of α in Φ are $\pm\alpha$. Therefore, $\Phi = \{\pm\alpha\}$. Clearly, if the root system depends on α where $\Phi_\alpha = \{\pm\alpha\}$, then for any non-zero α_1, α_2 in the Euclidean space, Φ_{α_1} is isomorphic to Φ_{α_2}. $\square$

Figure 1.1: Root System A_1

Theorem (Classification of Rank 2 Root Systems). There exist only 4 reduced, crystallographic root systems (upto isomorphism). Namely,

$$\begin{aligned}
&A_2 := \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2)\} \\
&B_2 := \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2)\} \\
&D_2 := \{\pm\alpha_1, \pm\alpha_2\} \\
&G_2 := \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2)\}
\end{aligned}$$

where α_1 and α_2 are roots satisfying the condition that $\|\alpha_1\| \leq \|\alpha\|$ for all α in the root system under the induced norm (i.e. α_1 is the shortest root) and the angle between α_1 and α_2, $\theta_{\alpha_1,\alpha_2}$ is no smaller than $\pi/2$.

Proof. Notice that the roots must span the space $\mathbb{R}^2$. Hence, there must be atleast two linearly independent roots. Pick α_1 to be the shortest root (under the inner product), i.e. $\|\alpha_1\| \leq \|\alpha\|$ for all $\alpha \in \mathbb{R}^2$.

Choose α_2 such that the angle between the two chosen roots is no smaller than $\pi/2$. This is always possible, since if the angle were to be smaller than $\pi/2$ then $-\alpha_2$, which must also be a root by (R2), would have an angle no smaller than $\pi/2$ with α_1.

Then Table 1.1 enumerates all possible angles between these roots and given the constraints we deduce that there are only four possible cases. We begin with the linearly independent vectors

[1]The provided definition differs from the more general definition stated in Lean 4's MathLib. In Lean 4, root systems are defined over modules instead of Euclidean spaces among other minor differences. As such, when neccessary the proofs in Lean 4 restrict the more general definition to our definition. Of note, root systems in Lean 4 are assumed to be reduced, crystallographic root systems.

α_1 & α_2 and construct from them a root system with the four possible cases of angles between them.

Case 1. $\theta_{\alpha_1,\alpha_2} = \pi/2$. In this scenario, the set D_2 is already a root system and any further additions would make it a non-reduced root-system.

Case 2. $\theta_{\alpha_1,\alpha_2} = 2\pi/3$. In this case, we begin with the set $\{\pm\alpha_1, \pm\alpha_2\}$ and observe that it is not closed under reflection through the hyperplane othogonal to its roots. It is evident that upon the addition of the element $\pm(\alpha_1 + \alpha_2)$ into the above set, we do get a closed root system.

Case 3. $\theta_{\alpha_1,\alpha_2} = 3\pi/4$. Yet again, begin with the set $\{\pm\alpha_1, \pm\alpha_2\}$. Computing $\sigma_{\alpha_2}(\alpha_1)$ returns $\alpha_1 + \alpha_2$, and $\sigma_{\alpha_2}(\alpha_1)$ gives $2\alpha_1 + \alpha_2$. Thus adding those into our set and performing a final closure check leads to the root system B_2.

Case 4. $\theta_{\alpha_1,\alpha_2} = 5\pi/6$. We repeat the process again, and arrive at $\sigma_{\alpha_2}(\alpha_1) = 3\alpha_1 + \alpha_2$, $\sigma_{\alpha_1}(\alpha_1 + \alpha_2) = 2\alpha_1 + \alpha_2$, $\sigma_{\alpha_2}(3\alpha_1 + \alpha_2) = 3\alpha_1 + 2\alpha_2$. Thus we obtain our final root system G_2. $\square$