Classification of Root Systems

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Chapter 1

Something Something

1.1 Root Systems

Definition 1. Let E be a finite-dimensional Euclidean space with an inner product $\langle \cdot, \cdot \rangle$.

A **root system** in E is a tuple (E, Φ), where Φ is a finite set of non-zero vectors (called roots) satisfying the following properties:

1. Φ spans E.

2. For every root $\alpha \in \Phi$, the set Φ is closed under reflection through the hyperplane orthogonal to α. That is, for any two roots $\alpha, \beta \in \Phi$, the set Φ contains the element

$$\sigma_\alpha(\beta) = \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

Add a figure here to show the reflection through the hyperplane

For convenience and in contexts where the Inner Product Space is clear, the root system is often referred to simply as Φ.

Example 2. The set $R_0 = \{\pm\alpha\}$, where α is any fixed real number, are roots in $\mathbb{R}$.

Definition 3. If a root system the condition that the only multiples of a root, α, that are in the root system are $\pm\alpha$, then the root system is said to be **reduced**.

Definition 4. If a root system satisfies the intgerality condition below, then it is said to be **crystallographic**.

$$[\beta, \alpha] := \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \quad \text{for all } \alpha, \beta \in \Phi.$$

For the following examples, denote e_i as the i-th standard basis vector in $\mathbb{R}^n$. Then, in combinations such as $\pm e_i \pm e_j$, the signs may be chosen independently.

Example 5. The set R_1, shown below, is a root system in $\mathbb{R}^2$ that is neither reduced nor crystallographic.

$$R_1 = \{\pm e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}), (\pm\sqrt{3}, \pm 1)\}$$

R_1 spans $\mathbb{R}^2$ and is closed under reflection through the hyperplane orthogonal to any root, hence it is a root system.

However, is is not a **reduced** root system since a scalar multiple of an element in R_1, namely $2 \cdot (\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$, is contained in R_1 itself. It is also not a **crystallographic** root system because $[e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})] = \frac{\sqrt{3}}{2} \notin \mathbb{Z}$.

Example 6. If we remove the redundant multiple in R_1 above, we obtain a reduced, non-crystallographic root system R_2.

$$R_2 = \{\pm e_1, (\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2})\}$$

One can also construct examples of non-reduced crystallographic root systems. Consider the following example,

Example 7. The set R_3 is a root system in $\mathbb{R}^2$ that is crystallographic but not reduced.

$$R_3 = \{\pm e_1, \pm e_2, \pm 2e_1\}$$

R_3 spans $\mathbb{R}^2$ and is closed under reflection through the hyperplane orthogonal to any root, hence it is a root system. It is a **crystallographic** root system because $[ke_1, e_2] = 0$ and $[ke_1, k'e_1] = kk' \in \mathbb{Z}$, where $k, k \in \{\pm 1, \pm 2\}$. However, is is not a **reduced** root system since $2e_1 \in R_3$.

Example 8. The set R_4 is a root system in $\mathbb{R}^2$ that is reduced and crystallographic.

$$R_4 = \{\pm e_1, \pm e_2\}$$

Therefore, we see that a root system may be reduced, crystallographic, both, or neither.

This paper concerns itself primarily with reduced, crystallographic root systems, simply referred to as root systems henceforth, unless otherwise specified.

Definition 9. The rank of a root system Φ is the dimension of the Euclidean space E.

Ambigious definition for irreducible root systems.

Definition 10. Two root systems can be combined to form a new root system by regarding the Euclidean spaces they span as mutually orthogonal subspaces of a common Euclidean space. A root system which does not arise from such a combination is said to be irreducible. Otherwise, for systems that do arise from such a combination, such as R_4 from R_0 they are said to be reducible.

Example 11. The set BC_n is the only irreducible non-reduced root system (upto isomorhism) in $\mathbb{R}^n$ [Source: textcolorredRevealed in a dream].

$$BC_n = \{\pm e_i, \pm e_i \pm e_j, \pm 2e_i\}$$

More examples ...

Since we aim to classify all root systems, upto isomorhism, it is important to understand when two root systems are isomorphic.

Definition 12. Two root systems (E, Φ) and (F, Ψ) are said to be isomorphic if there exists a linear isomorphism $\varphi : E \to F$ such that $\varphi(\Phi) = \Psi$ and preserves the number $\langle x, y \rangle$ for each pair of roots.

Examples here ...