PHYS2160 Introductory Computational Physics
2021/22 Project
(Due date: 20 Apr 2022, 5:00 pm)

Each student should work on **any one** of the following problems. You should submit a project report which includes the following parts: (1) theory, (2) algorithm, (3) listing of your program, (4) results, and (5) discussion. Your program will be marked based on the use of Python's modules and packages, programming style, and capability of explaining physical phenomena.

1. According to the principle of superposition, the electric potential at the point $\vec{r}$ due to a continuous charge distribution is given by the formula

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{|\vec{r}-\vec{r}'|}$$

where dq is the charge element located at $\vec{r}'$ and ϵ_0 is the permittivity of free space. Using this formula, one can show that the electric potential at a distance z above the center of a uniformly charged wire of total charge Q and length L is

$$V(z) = \frac{Q}{4\pi\epsilon_0 L}\ln\left[\frac{\sqrt{z^2+(L/2)^2}+L/2}{\sqrt{z^2+(L/2)^2}-L/2}\right]$$

and the electric potential at a distance z above the center of a uniformly charged ring of total charge Q and radius R is

$$V(z) = \frac{Q}{4\pi\epsilon_0\sqrt{z^2+R^2}}$$

Similarly, one can show that the electric potential at a distance z above the center of a uniformly charged disc of total charge Q and radius R is

$$V(z) = \frac{Q}{2\pi\epsilon_0 R^2}(\sqrt{z^2+R^2}-z)$$

and the electric potential at a distance z above the center of a uniformly charged solid sphere of total charge Q and radius R is

$$V(z) = \begin{cases} \dfrac{Q}{8\pi\epsilon_0 R}\left(3-\dfrac{z^2}{R^2}\right) & \text{for } z < R \\ \dfrac{Q}{4\pi\epsilon_0 z} & \text{for } z > R \end{cases}$$

To investigate the properties of the electric potential of these continuous charge distributions, write Python programs to perform the following tasks:

(a) Use the SciPy function `integrate` to numerically evaluate the integral to find the electric potential $V(z)$ at a distance z above the center of the distribution for each of these charge distributions. Compare each numerical solution with the corresponding analytical solution by using the `matplotlib` module to plot them as a function of the distance z on the same graph.

(b) Use the SciPy function `integrate` to numerically evaluate the integral to find the electric potential at arbitrary points on the xz plane and then use the `matplotlib` module to make a plot of the equipotential lines on the xz-plane for each of these charge distributions. Assume the center of all these charge distributions are located at the origin with the ring and disc lying on the xy-plane and the wire lying on the x-axis.

2. A simple but nonlinear pendulum is made up of a particle of mass m suspended by a light rod of length l fixed to a smooth pivot at its upper end. After released from some angle with the vertical, the pendulum swings back and forth in a vertical plane under the influence of gravity. The angular displacement $\theta(t)$ of the pendulum from the vertical is modeled by the nonlinear differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

where g is the the acceleration due to gravity. This equation is not easy to solve due to the nonlinear term $\sin\theta$. It can be shown that the period of oscillation of the nonlinear pendulum for the case $\theta'(0) = 0$ is

$$T = 4\sqrt{\frac{l}{2g}}\int_0^{\theta_m} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_m}}$$

where $\theta_m = \theta(0)$ is the amplitude of oscillation in such case.

However, for small angle θ, we can use the small-angle approximation $\sin\theta \approx \theta$. Then the above differential equation becomes the linear equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

which models a linear pendulum. This equation has the analytical solution

$$\theta_a(t) = \theta_m \cos\left(\sqrt{\frac{l}{g}}t + \delta\right)$$

where θ_m and δ are some constants that depend on the initial conditions $\theta(0)$ and $\theta'(0)$. From this equation, one can also show that the period of small-angle oscillation of the nonlinear pendulum for the case $\theta'(0) = 0$ is approximately given by

$$T_s = 2\pi\sqrt{\frac{l}{g}}$$

To investigate the properties of the nonlinear pendulum, write Python programs to perform the following tasks:

(a) Use SciPy function `odeint` to numerically solve the nonlinear pendulum equation to find the angular displacement $\theta(t)$ of the nonlinear pendulum. Compare it with the analytical solution $\theta_a(t)$ for the linear pendulum by using the `matplotlib` module to plot them as a function of time t on the same graph.

(b) Use the `matplotlib` module and the data obtained in part (a) to plot the phase diagram, i. e. the plot of the angular displacement $\theta(t)$ against the angular velocity $\theta'(t)$, for the linear and nonlinear pendulum on the same graph.

(c) Use the SciPy function `integrate` to determine the period T of the nonlinear pendulum for the case $\theta'(0) = 0$ with the amplitude of oscillation $\theta_m = 1^\circ, 2^\circ, 3^\circ, \ldots, 90^\circ$. Use `matplotlib` module to plot the ratio of the pendulum's true period T to the small-angle period T_s against the amplitude θ_m.

You should repeat the plotting in parts (a) and (b) for different initial conditions $\theta(0)$ and $\theta'(0)$ in order to clearly explain the underlying physical theories.