# The Application of Linear Algebra in Netflix Algorithms

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**Abstract**

In this paper we will investigate the topic of content recommendation for Netflix. Specifically, we will be investigating the algorithms Netflix uses to find movies and shows based on what the user had previously watched. The algorithms that Netflix - along with many other websites, such as Facebook, Reddit, YouTube, Instagram, and more - are based in linear algebra. Coding the software that creates the recommendations produced by these websites would not be possible without the knowledge of matrices and vectors. At the end of this paper, we will display a basic example of a linear algebra based algorithm in Python.

Based on the observational nature of this paper, all members of the group were required to sufficiently research and understand the topic. As a result all three members contributed in analyzing the results and preparing the report. Matthew Teta also implemented the algorithm in python in this paper.

# Contents

# 1 Introduction

The application of linear algebra in user recommendations is utilized as a way for companies to engage users in their product. In other words, these algorithms entice us to continuously consume product. In addition to keeping viewers, or website users, from logging off and doing something else, companies will utilize their findings by collecting data. With these algorithms, Netflix and other websites with the same features have the ability to make connections between the type of content consumed and a person's location, ethnicity, income, age, among other data points that they may have access to (think: when you post on Instagram and tag your location, when you give your address to Netflix for billing, when you put your birthday into Facebook, etc.).

In order to fully understand how linear algebra is used for these algorithms, the reader of this paper should be knowledgeable of matrices, including what they look like and their associated properties, vectors and how they can interact with matrices, and the concept of data analysis. In addition to basic knowledge of linear algebra and data analysis, prior research on the ideas of back propagation and gradient descent will improve the reader's grasp of how the algorithm works.

In the Examples and Numerical Results section of this paper we demonstrate an example of the functionality of this algorithm in code. We have downloaded some movie data from the official Netflix data bank in order to prove the concepts described in the paper.

# 2 Mathematical Formulation

Netflix and other websites that model consumer algorithms need places to store the data that they collect. The collected data is placed into vectors, then into matrices, and then stored and tied to your unique account ID.

Consider all TV shows that are available to watch in Netflix. TV shows with similar genres are represented by similar numerical values, for example, Love is Blind and The Ultimatum may be represented by 20 and 21, respectively.

Let the following matrix, Matrix A, represent all of the dating shows a user may have watched in a certain period of time, for example, 6 months. These data points for all dating shows are roughly around the number 20. Matrix A may look something like:

$$A = \begin{bmatrix} 21 & 22 & 23 \\ 20 & 25 & 22 \end{bmatrix}$$

Let Matrix B represent horror shows, most of which are represented by the number 10:

$$B = \begin{bmatrix} 11 & 10 & 14 \\ 12 & 12 & 13 \end{bmatrix}$$

Finally, let Matrix C represent cooking shows, most of which are represented by the number 30.

$$C = \begin{bmatrix} 30 & 32 & 33 \\ 35 & 31 & 34 \end{bmatrix}$$

The matrices A, B, and C are the principal/typical components of the set of movies offered. This means that each matrix represents a specific aspect underlying the data (in this case, the aspects are genres). Utilizing matrix addition and singular value decomposition (SVD), Netflix can determine what shows to suggest to any given user. If the user clicks on dating shows more often than horror shows, but clicks on cooking shows more than horror shows, Netflix's formula for its algorithm may look something like:

$$AM = A + 0.5B + .001C \tag{1}$$

With AM representing the final "Algorithm Matrix". The result of this matrix addition and multiplication is as follows:

$$AM = \begin{bmatrix} 21 & 22 & 23 \\ 20 & 25 & 22 \end{bmatrix} + .5 * \begin{bmatrix} 11 & 10 & 14 \\ 12 & 12 & 13 \end{bmatrix} + .001 \begin{bmatrix} 30 & 32 & 33 \\ 35 & 31 & 34 \end{bmatrix} \tag{2}$$

$$AM = \begin{bmatrix} 26.53 & 27.032 & 30.033 \\ 26.035 & 31.031 & 28.534 \end{bmatrix} \tag{3}$$

As shown in the final algorithm matrix, Netflix now knows what types of shows to suggest. Four of those values are in the 20s, suggesting to Netflix that this user likes dating shows. There are two values in the 30s, suggesting that this user may also like horror shows. The more that the user clicks on cooking shows as opposed to horror and dating, the values in this matrix will change. Starting from a completed AM matrix, these calculations can be done in reverse order using principal component analysis (PCA) to find the typical components, A, B, and C.

This strategy works well for forming AM when the principal components are known and the user has submitted a rating for every show. The question now, is how to create AM when the user has not watched all the shows. Because the matrix AM is now sparse, the SVD of AM is now undefined.

Assume a $u \times t$ rating matrix $R$ of a set of movies is dense (every user has submitted a rating for every title on Netflix). The PCA of R and the transpose of R will give typical users and movies as vectors that can then be formed into a set of matrices U and M. The SVD of R can be represented by,

$$R = M\Sigma U^T$$

$$R = MU^T$$

With $U$ being the $u \times f$ matrix of typical users and $M$ the $t \times f$ matrix of typical movies. Importantly, the common dimension $f$ is the number of latent factors that describes the users and titles. This process reduces the size of the data significantly, and it allows us to represent our original matrix $R$ with two much smaller matrices. $\Sigma$ can be neglected as it is just a diagonal matrix to scale the rows and columns of the U and M matrices. Considering the rating of user $u$ for movie $i$ gives the form,

$$\begin{bmatrix} r_{ui} \end{bmatrix} = \begin{bmatrix} -p_u- \end{bmatrix} \begin{bmatrix} | \\ q_i \\ | \end{bmatrix}$$

$$r_{ui} = p_u \cdot q_i$$

Now, in the Netflix data case, R is no longer dense. It is not possible to use traditional SVD methods, because it is not even possible to pull out the eigenvalues from the original matrix. Furthermore, the (sparse) matrix is about 99% empty. However impossible it may seem to generate so many remaining values from the limited amount of data known, there is a method which has proven to be very successful. Since the minimization is only depends on the known values of the $R$ matrix, the resulting factorization can be used to extrapolate estimates for the unknown values later on. Simon Funk developed the minimization function used in the algorithm.

$$\min_{p_u, q_i} \sum_{r_{ui} \in R} (r_{ui} - p_u \cdot q_i)^2 \tag{4}$$

Finding exact values of $p_u$ and $q_i$ that minimize the sum is a very hard process since it cannot be solved directly, and computational estimates are inefficient on large data. Instead we will approximate the values using the stochastic gradient descent (SGD) technique. This technique repeatedly computes the partial derivatives of the the error function in order to adjust the values in the solution matrices and trend toward (a) minimum. The algorithm uses back propagation to find gradients needed for SGD. Because the minimization can be written as a function of $p_u$ and $q_i$, that is,

$$f(p_u, q_i) = \sum_{r_{ui} \in R} (r_{ui} - p_u \cdot q_i)^2 = f_{ui}(p_u, q_i) \tag{5}$$

The SGD procedure requires the partial derivative of $f$ with respect to both $p_u$ and $q_i$ be taken for all known values of $r_{ui}$. Starting out by randomly initializing $p_u$ and $q_i$, we take the partial of a specific known $r_{ui}$ value and then update $p_u$ and $q_i$ according to the following rule,

$$p_u \leftarrow p_u + \alpha \cdot q_i(r_{ui} - p_u \cdot q_i)$$

$$q_i \leftarrow q_i + \alpha \cdot p_u(r_{ui} - p_u \cdot q_i)$$

The method is applied for all known values, $r_{ui}$, and repeated for a number of epochs. The learning rate, $\alpha$, is a constant that determines how large the iterations steps are. The smaller the learning rate, the more epochs are necessary for reaching a minimum. A smaller learning rate also provides the benefit of following the surface of the minimization function more closely, so high frequency details won't be skipped over. The goal is to reach the absolute minimum, not a local minimum. This can be tough with this technique, since it is highly dependent on the starting points for $p$ and $q$, and fluctuations in the terrain will lead to many false / local minimums. The model can be trained multiple times until the best solution is found. Once all the $p_u$ and $q_i$ vectors have been computed to satisfaction, a completed rating matrix R with approximate rating values can be created from the simple formula,

$$\hat{r}_{ui} \approx p_u \cdot q_i$$

$$R \approx PQ$$

We denote $\hat{r}_{ui}$ with a hat because the values are just estimates and not exact. With this algorithm, the idea of approximating values based on known data can be used more generally outside of just Netflix's recommendations.

## 3 Examples and Numerical Results

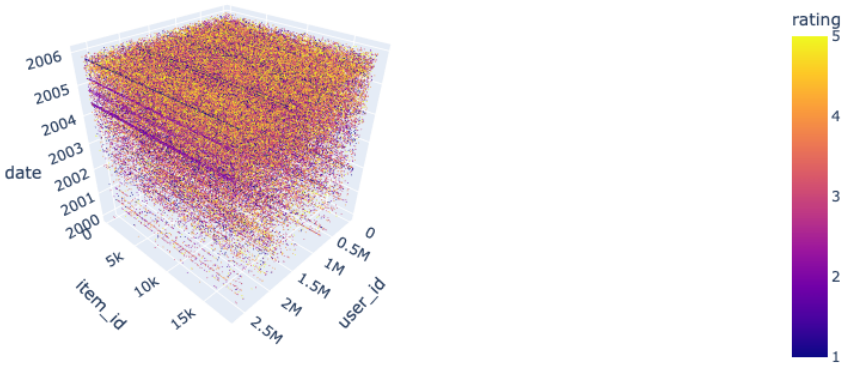
One of the most challenging aspects of this project was the data import aspect. The Netflix ratings dataset contaings 100,480,507 unique user ratings. The data is provided in four 500Mb files which each contain sections for movie id's and rows containing a user id, rating, and date of rating. For example,

```
189:
1943087,5,2001-10-20
306466,4,2001-09-21
1966165,5,2004-02-10
```

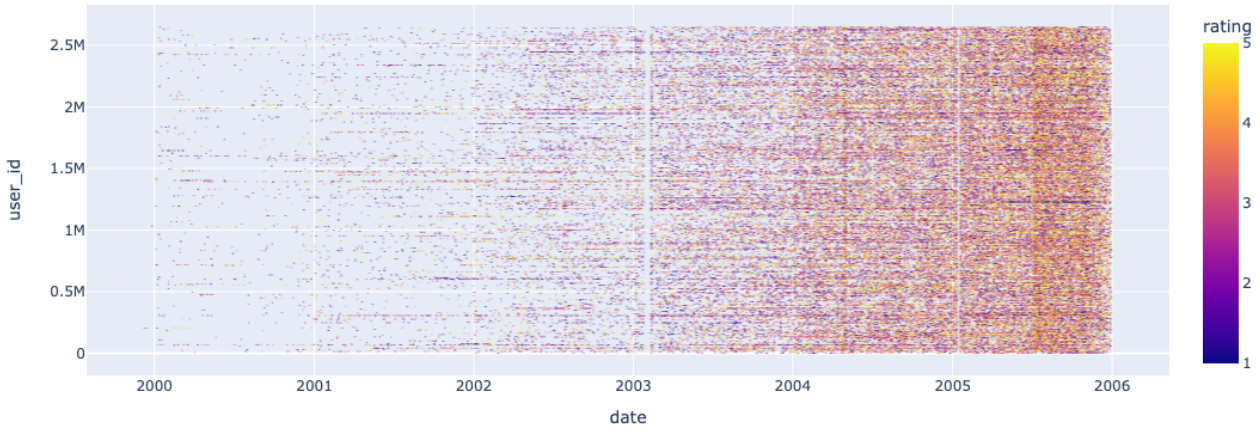
I had to write a custom parser that reads in the data files line by line and pulls out the columns. The data is then formatted into a data frame for ease of use with the Plotly graphing library. In addition, since the user id's are randomly assigned, I had to map them to array indecies to reduce the size of the p and q matrices later.

```python
import numpy as np
import pandas as pd
import plotly.express as px
from typing import List
from datetime import datetime

def load_data(filepaths: List[str], import_date=False):
    '''Generate a list of all data entries from the given list of file paths
    '''
    data = dict(
        item_id=[],
        user_id=[],
        item_index=[],
        user_index=[],
        rating=[]
    )
    if import_date:
        data['date'] = []
    user_id_map = {}
    item_id_map = {}
    item_id = 0
    for fname in filepaths:
        with open(fname, 'r') as input_file:
            for line in input_file:
                if line[-2] == ':':
                    # Set item_id when we encouter a new section of the data
                    item_id = int(line[:-2])
                else:
                    # Line has a user rating in it, capture the values and store in array
                    [user_id, rating, date] = line.split(',')
                    user_id = int(user_id)
                    if import_date:
                        date = datetime.strptime(date.rstrip(), '%Y-%m-%d')

                    if (not item_id in item_id_map):
                        item_id_map[item_id] = len(item_id_map)
                    if (not user_id in user_id_map):
                        user_id_map[user_id] = len(user_id_map)

                    rating = float(rating)

                    data['item_id'].append(item_id)
                    data['user_id'].append(user_id)
                    data['item_index'].append(item_id_map[item_id])
                    data['user_index'].append(user_id_map[user_id])
                    data['rating'].append(rating)
                    if import_date:
                        data['date'].append(date)
    data = pd.DataFrame(data)
    return (dict(
        n_items=len(item_id_map),
        n_users=len(user_id_map),
        data=data
    ),
    user_id_map,
    item_id_map)
```

I decided to pare down the data into a manageable size so that I could work with it quickly for testing and development purposes. This will affect the results that I get later on, but I had to do this because I haven't had the time to run the code for multiple days of training and/or optimize the code for hardware acceleration. I did work on a C implementation for a while, but ultimately decided that fine tuning the algorithm in Python would yield more interesting results than debugging my code for hours. I decided to only include the first 100 ratings for each movie id. The following plot gives you an idea of the size of the data set. There is an interactive version of the plot that can be seen by cloning the Jupyter Notebook from my GitHub.




This plot is useful to show that the user id field is normally distributed, and the number of Netflix users increased dramatically around 2005.



Once the data is imported into python, we can feed it into the gradient descent algorithm,

```
def gradient_descent(dataset, n_factors=10, learning_rate=0.01, n_epochs=10):
    '''Learn the reduced dimension matricies p and q from the data points in the data
    array
    '''
    # initialize p and q randomly
    # p is user matrix and q is the movie matrix
    p = np.random.normal(0, .1, (dataset['n_users'], n_factors))
    q = np.random.normal(0, .1, (dataset['n_items'], n_factors))

    data = dataset['data']

    # optimize
    for _ in range(n_epochs):
        for ind in data.index:
            p_ind = data['user_index'][ind]
            q_ind = data['item_index'][ind]
            rating = data['rating'][ind]
            # Pull values onto stack for efficiency
            # Calculate gradient descent error
            err = rating - np.dot(p[p_ind], q[q_ind])
            adj = err * learning_rate
            # Apply descent correction
            p[p_ind] += adj * q[q_ind]
            q[q_ind] += adj * p[p_ind]

    # return the matricies
    return (p, q)
```

This training process takes the longest time out of all of the computations. When I attempted to run it on the full dataset, it took over 40 hours and eventually I killed the process. Interestingly, it would be possible to save the $p$ and $q$ matrices to use them as the starting point for training when new data becomes available. It would also be possible to modify the algorithm to only use new ratings to update the existing prediction model. Optimizing in this way causes the overall speed of the algorithm to improve significantly. In fact, it would be very fast to update the model instantly any time a rating is made on the Netflix platform. With the trained $p$ and $q$ models in memory, it becomes time to make some rating predictions! We can use the following functions to generate reccomendations.

```
def estimate(user_factors, item_factors, user_id, item_id, user_id_map, item_id_map):
    return np.dot(user_factors[user_id_map[user_id]], item_factors[item_id_map[item_id]])

def estimate_with_index(user_factors, item_factors, user_index, item_index):
    return np.dot(user_factors[user_index], item_factors[item_index])
```

There are two versions of the function because I remapped the indecies of the user id's and title id's. Taking a dot product is a very fast process and ratings can be generated in milliseconds even in Python. I estimated the predictions from the model for all of the known ratings and put them into a dataframe consistent with the first one,

```
# Generate data frame identical to the actual values but with the predictions
pred_ratings = np.zeros(len(dataset['data']['rating']))
for ind in dataset['data'].index:
    user_index = dataset['data']['user_index'][ind]
    item_index = dataset['data']['item_index'][ind]
    pred_ratings[ind-1] = estimate_with_index(user_factors, movie_factors, user_index,
    item_index)
predictionset = pd.DataFrame(dict(
    rating=pred_ratings,
    user_index=dataset['data']['user_index'],
    item_index=dataset['data']['item_index'],
    user_id=dataset['data']['user_id'],
    item_id=dataset['data']['item_id']
))

err = dataset['data']['rating'] - predictionset['rating']
```

```
df2 = pd.DataFrame(dict(err=err, user_index=predictionset['user_index'], item_index=
    predictionset['item_index']))
```

In order to analyze the success of the matrix factorization and gradient decent minimization, I generated a number of error measurements and summary statistics using this error dataframe and the `df2` dataframe. Using Plotly, I generated this histogram which shows the distribution of the error in the model.

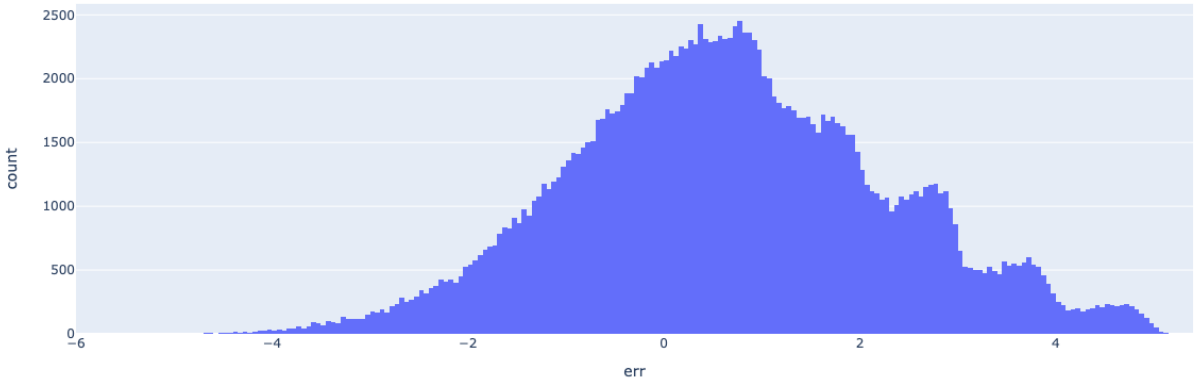



Figure 1: $\mu = 0.667$, $\sigma = 1.62$

## 4 Discussion and Conclusion

Something interesting to consider is how the rating algorithm can be modified to inlude other factors such as cursor hover time, time spent reading a caption, time spent watching, number of episodes watched, common cast between titles, title budget, director, search terms, etc. There is a lot more data collected by these social media platforms that can be utilized to improve the reccomendations.

Also, there should be more criteria used to filter out the data that is fed into the rating training algoithm. Bad data can ruin the utility of a model, and there are internet trolls. It would be best to preliminaryily filter out data from users who haven't rated more than a few titles or who have rated titles that they haven't actually watched. Also, since the algorithm influences the titles that people watch and then rate, there is a positive feedback that can occur where the reccomendations are reinforced by the effect of reccomending titles to users. Some strategies that I imagine for minimizing this effect is to either change the weight of ratings that were a direct result of common reccomendations or to throw in random titles that cause a user to be shown more options.

## References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