Lecture 6

Electrostatic field in material space

Boundary conditions

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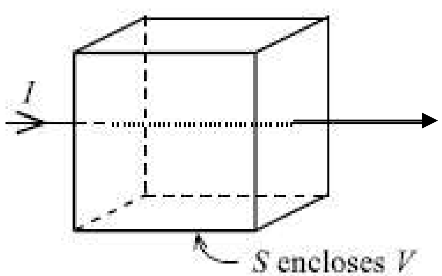
Current Continuity Equation

Current continuity equation defines the basic *conservation of charge* relationship between current and charge. That is, a net current in or out of a given volume must equal the net increase or decrease in the total charge in the volume.

If we define a surface S enclosing a volume V, the **net current** out of the volume (I_{out}) is defined by



$$I_{out} = -\frac{dQ}{dt}$$

$$\oint\!\!\oint_S \boldsymbol{J} \cdot d\boldsymbol{s} = I_{out}$$

$$\boxed{\oint\!\!\oint_S \boldsymbol{J} \cdot d\boldsymbol{s} = -\frac{dQ}{dt}}$$

where $d\boldsymbol{s} = ds\, \boldsymbol{a}_n$ and $\boldsymbol{a}_n$ is the outward normal unit vector.

Current Continuity Equation (differential form)

The previous equation is the integral form of the continuity equation. The differential form of the continuity equation can be found by applying the **divergence theorem** to the surface integral and expressing the total charge in terms of the charge density.

$$\oint\!\!\oint_S \boldsymbol{J}\cdot d\boldsymbol{s} = \iiint_V (\nabla\cdot\boldsymbol{J})\,dv = -\frac{dQ}{dt}$$

$$= -\frac{d}{dt}\iiint_V \rho_v\,dv = -\iiint_V \frac{\partial\rho_v}{\partial t}\,dv$$

Thus,

$$\iiint_V (\nabla\cdot\boldsymbol{J})\,dv = -\iiint_V \frac{\partial\rho_v}{\partial t}\,dv \qquad (\text{valid for any } V)$$

Since the previous equation is valid for any volume V, we equate the integrands

$$\boxed{\nabla\cdot\boldsymbol{J} = -\frac{\partial\rho_v}{\partial t} \qquad (\text{continuity equation})}$$

For *DC* currents, the charge density does not change with time so that the divergence of the current density is always zero.

$$\boxed{\nabla\cdot\boldsymbol{J} = 0 \qquad \text{(steady current)}}$$

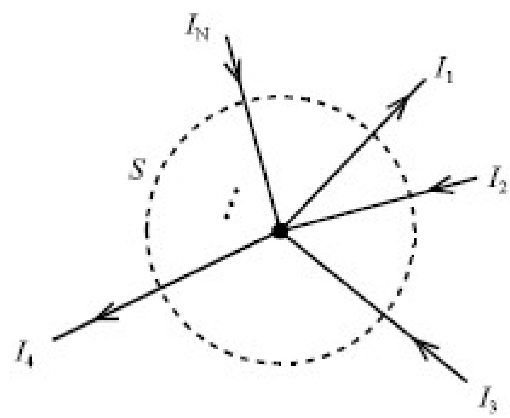
Current continuity equation is the basis for *Kirchhoff's current law.*

Given a circuit node connecting a system of *N* wires (assuming DC currents) enclosed by a spherical surface *S*, the integral form of the continuity equation gives

$$\oint\!\!\!\oint_S \boldsymbol{J}\cdot d\boldsymbol{s} = I_{out} = -\frac{dQ}{dt} = 0$$

$$= I_1 - I_2 - I_3 + I_4 + \dots - I_N$$

$$\text{or} \quad \sum_{n=1}^{N} I_n = 0 \qquad \textit{Kirchoff's current law}$$



The integral form of the continuity equation (and thus Kirchhoff's current law) also holds true for time-varying (AC) currents if we let the volume enclosed by the surface *S* shrink to zero i.e. the size of a point (the node).

Relaxation Time

* If a finite number of individual charges (all positive or all negative) is placed inside a volume of **conducting metal**, an electrostatic field will be generated and the Coulomb forces on the individual charges would cause them to migrate away from each other.

* The result is a charge accumulation on the outer surface of the conductor while the inside of the conductor remains charge-neutral.

* The time required for the conductor to reach this inside-charge-neutral state is known as the ***relaxation time***.

From the continuity equation $\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho_v}{\partial t}$ and

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

We get

$$\sigma(\nabla \cdot \boldsymbol{E}) = -\frac{\partial \rho_v}{\partial t}$$

What is the divergence of ***E*** ?

$$\sigma(\nabla\cdot\boldsymbol{E}) = -\frac{\partial\rho_v}{\partial t}$$

The divergence of the electric field is related to the charge density by

$$\nabla\cdot\boldsymbol{D} = \epsilon(\nabla\cdot\boldsymbol{E}) = \rho_v \qquad \Rightarrow \qquad \nabla\cdot\boldsymbol{E} = \frac{\rho_v}{\epsilon}$$

Inserting this result into previous equation yields

$$\sigma\frac{\rho_v}{\epsilon} = -\frac{\partial\rho_v}{\partial t} \qquad \text{or} \qquad \boxed{\frac{\partial\rho_v}{\partial t} + \frac{\sigma}{\epsilon}\rho_v = 0}$$

The solution to this homogeneous, first order PDE is

$$\rho_v = \rho_{vo}e^{-t/T_r}$$

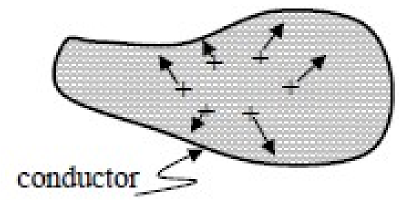
The relaxation time constant describes the rate of decay of the charge inside the conductor.

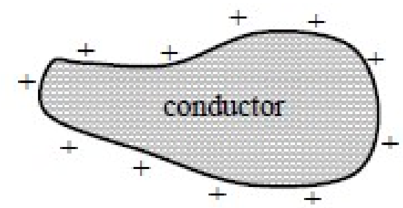
$$\boxed{T_r = \frac{\epsilon}{\sigma}}$$

How many relaxation time constants does it take the conductor to become chare neutral inside?

Conductors in Static Electric Field

- Assume some charges are introduced inside a conductor. These charges will produce electrostatic field which will apply a force on each of them, therefore the charges will be accelerated away from each other.
- This movement will continue until all charges reach the conductor surface (after 5 T_r) .
- In this case the charge and electrostatic field inside the conductor vanish.



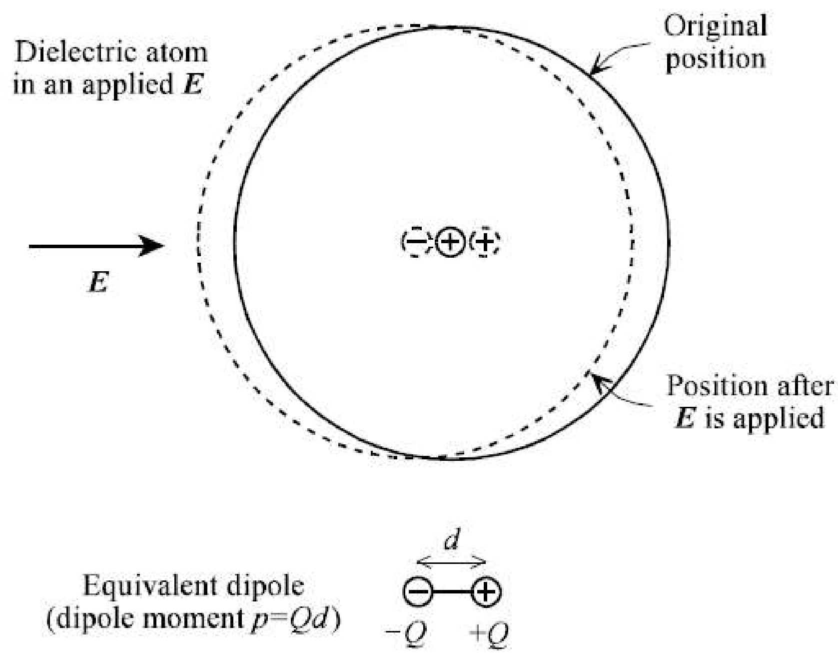


inside the conductor

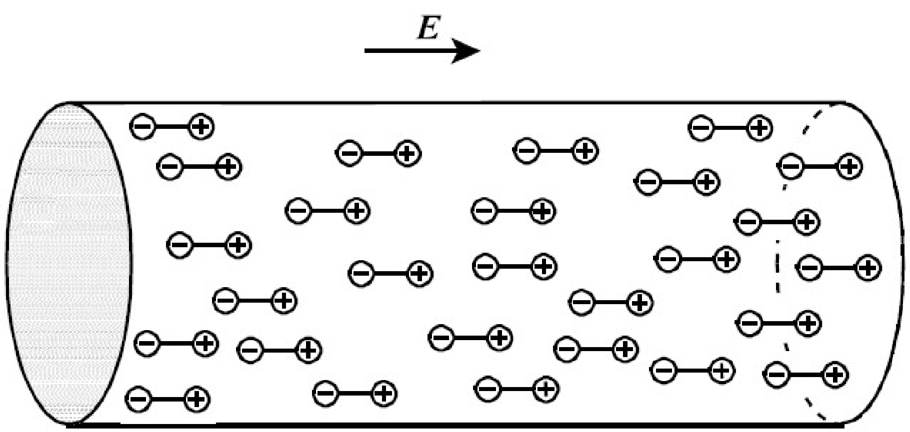
$$\rho_v = 0$$
$$\overline{E} = 0$$

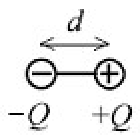
Polarization in Dielectrics

- With no electric field applied to dielectric materials, the atom is electrically neutral and the centroid of the (negative) electron charge is coincident with the centroid of the (positive) nucleus charge.
- When an electric field is applied to the atom, the positively charged nucleus is displaced in the direction of the electric field while the centroid of the negative electron charge is displaced in the direction opposite to the electric field.
- The dielectric atom is thus polarized and can be modeled as an equivalent electric dipole.



The polarization within the dielectric produces an additional electric flux density component which is included in the electric flux density equation as the vector *polarization* $\boldsymbol{P}$.





$$\boldsymbol{D} = \epsilon_o \boldsymbol{E} + \boldsymbol{P}$$

The polarization $\boldsymbol{P}$ is defined as the dipole moment per unit volume such that

$$\boldsymbol{P} \equiv \frac{\text{dipole moment}}{\text{unit volume}} = \frac{nqd}{v} \qquad (\text{C/m}^2)$$

where n is the number of dipoles in the volume v. Assuming that the polarization vector $\boldsymbol{P}$ is proportional to the electric field $\boldsymbol{E}$, we may write

$$\boldsymbol{P} = \chi_e \epsilon_o \boldsymbol{E}$$

where χ_e is defined as the *electric susceptibility* (unitless). Inserting this definition of $\boldsymbol{P}$ into the electric flux equation gives

$$\begin{aligned} \boldsymbol{D} &= \epsilon_o \boldsymbol{E} + \boldsymbol{P} = \epsilon_o (1 + \chi_e) \boldsymbol{E} \\ &= \epsilon_o \epsilon_r \boldsymbol{E} = \epsilon \boldsymbol{E} \end{aligned}$$

where

$$\epsilon_r = 1 + \chi_e \qquad \text{(relative permittivity)}$$

$$\epsilon = \epsilon_o \epsilon_r = \epsilon_o (1 + \chi_e) \qquad \text{(total permittivity)}$$

- Note that the electric susceptibility χ_e and the relative permittivity ϵ_r are both measures of the polarization within a given dielectric.
- The larger the value of χ_e or ϵ_r for the material, the more polarization within the material.

$$\boxed{\boldsymbol{P} = \chi_e \epsilon_o \boldsymbol{E}}$$

- For free space (vacuum), there is no polarization such that $\boldsymbol{P} = 0$, $\chi_e = 0$ or $\epsilon_r = 1$
- The amount of polarization found in air is extremely small, so that we typically model our atmosphere with the free space permittivity.

The total charge density (ρ_T) in an insulating material consists of the *free* conduction charge density (ρ_v) plus the *bound* polarization charge density (ρ_{vp}).

$$\rho_T = \rho_v + \rho_{vp} \qquad \textbf{(total charge density)}$$

From our previous definition of the differential form of Gauss's law, we see that the divergence of the electric flux density yields the free charge density.

$$\nabla \cdot \boldsymbol{D} = \rho_v$$

If we insert the expression for the electric flux density in terms of the polarization and the free charge density in terms of the total charge density, we find

$$\nabla \cdot (\epsilon_o \boldsymbol{E} + \boldsymbol{P}) = \rho_T - \rho_{vp}$$

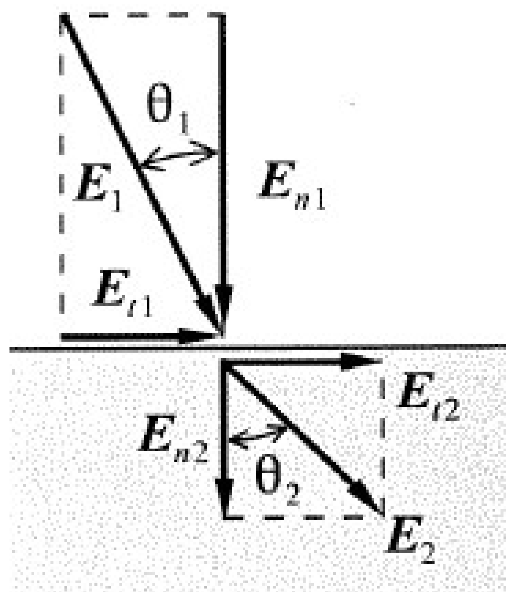
Equating terms yields

$$\nabla \cdot (\epsilon_o \boldsymbol{E}) = \rho_T \qquad \boxed{\nabla \cdot \boldsymbol{P} = -\rho_{vp}}$$

The divergence of the polarization vector gives the negative of the bound polarization charge density.

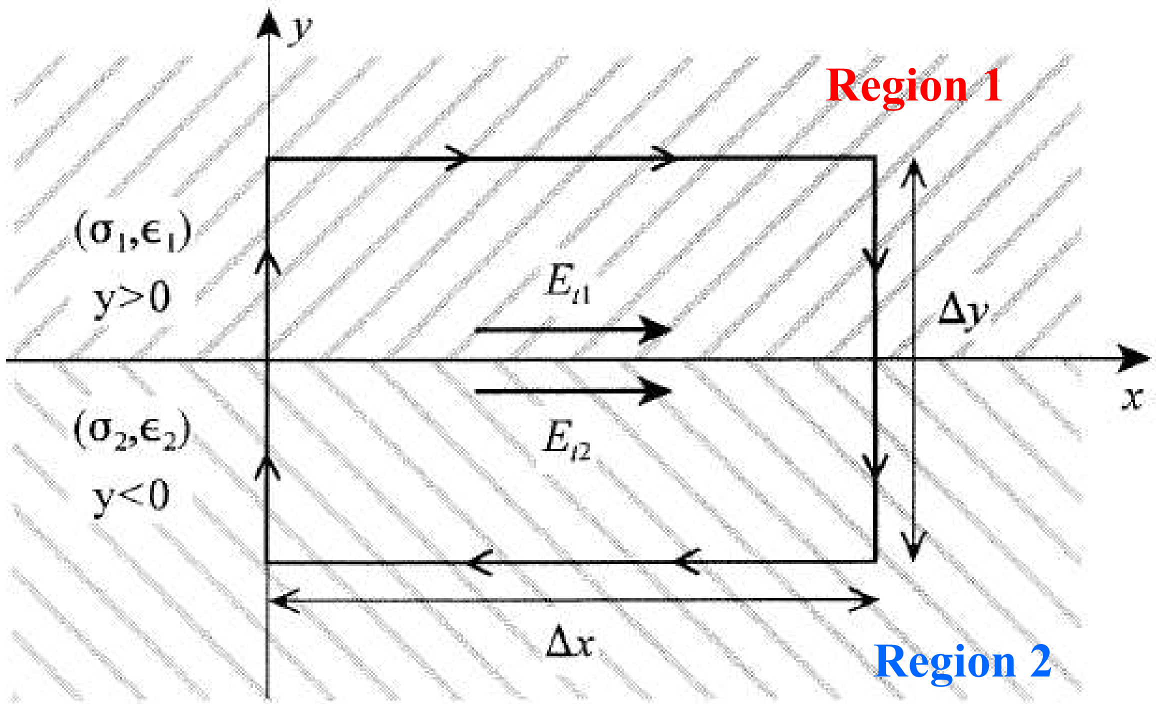
Electric Field Boundary Conditions

A knowledge of the behavior of electric fields at a media interface between distinct materials is necessary to solve many common problems in electromagnetics. The fundamental *boundary conditions* involving electric fields relate the tangential components of electric field and the normal components of electric flux density on either side of the media interface.



Tangential Electric Field, E_t

In order to determine the boundary condition on the tangential electric field at a media interface, we evaluate the line integral of the electric field along a closed incremental path that extends into both regions as shown below.



$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

Let us take the limit of the line integral as Δy tends to zero

$$\lim_{\Delta y \to 0} \oint \boldsymbol{E} \cdot d\boldsymbol{l} = \int_0^{\Delta x} (E_{t1}\hat{\boldsymbol{x}}) \cdot (dx\hat{\boldsymbol{x}}) + \int_{\Delta x}^{0} (E_{t2}\hat{\boldsymbol{x}}) \cdot (-dx)(-\hat{\boldsymbol{x}}) = 0$$

Assuming uniform electric fields over the paths of length Δx

$$E_{t1} \int_0^{\Delta x} dx + E_{t2} \int_{\Delta x}^{0} dx = E_{t1}\,\Delta x - E_{t2}\,\Delta x = 0$$

Dividing this result Δx gives

$$E_{t1} - E_{t2} = 0 \qquad \Rightarrow \qquad \boxed{E_{t1} = E_{t2}}$$

The tangential components of electric field are continuous across a media interface.

Tangential Electric Field, E_t (region 2 is a perfect conductor)

If region 1 is a dielectric and region 2 is a perfect conductor ($\sigma_2 = \infty$), then $E_{t2} = 0$ (the electric field inside a perfect conductor is zero) so that

$$\boxed{E_{t1} = 0}$$

The tangential component of electric field on the surface of a perfect conductor is zero.

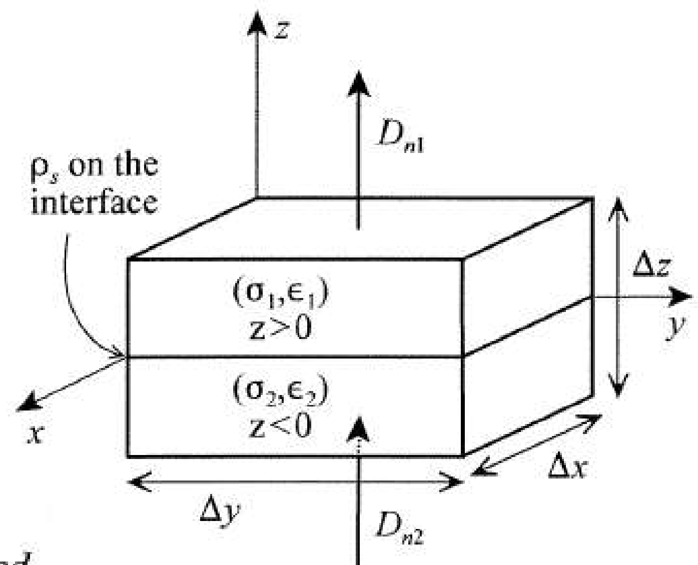
Normal Electric Flux Density

In order to determine the boundary condition on the normal electric flux density at a media interface, we apply Gauss's law to an incremental volume that extends into both regions as shown in the figure. The application of Gauss's law gives

$$\oint \boldsymbol{D}\cdot d\boldsymbol{s} = Q_{enclosed}$$

$$= \int_0^{\Delta y}\int_0^{\Delta x}(D_{n1}\hat{z})\cdot(dx\,dy\,\hat{z}) +$$

$$\int_0^{\Delta y}\int_0^{\Delta x}(D_{n2}\hat{z})\cdot(dx\,dy)(-\hat{z}) = Q_{enclosed}$$



Thus,

$$\oint \mathbf{D}\cdot d\mathbf{s} = \int_0^{\Delta y}\int_0^{\Delta x}(D_{n1}\hat{z})\cdot(dx\,dy\,\hat{z}) + \int_0^{\Delta y}\int_0^{\Delta x}(D_{n2}\hat{z})\cdot(dx\,dy)(-\hat{z}) = Q_{enclosed}$$

The integrals reduce to

$$D_{n1}\int_0^{\Delta y}\int_0^{\Delta x} dx\,dy - D_{n2}\int_0^{\Delta y}\int_0^{\Delta x} dx\,dy = Q_{enclosed}$$

where the electric flux density is assumed to be constant over the upper and lower incremental surfaces. Evaluation of the surface integrals yields

$$D_{1n}\,\Delta x\,\Delta y - D_{2n}\,\Delta x\,\Delta y = Q_{enclosed}$$

Dividing by $\Delta x \Delta y$ gives

$$D_{n1} - D_{n2} = \frac{Q_{enclosed}}{\Delta x\,\Delta y} = \rho_S$$

Assuming uniform surface charge density

$$\boxed{D_{n1} - D_{n2} = \rho_s}$$

The difference in the normal component of electric flux density across the media interface is equal to the charge density on the interface.

On a charge-free interface ($\rho_s = 0$), such that

$$\boxed{D_{n1} = D_{n2}}$$

The normal components of electric flux density are continuous across a charge-free media interface.

If region 1 is a dielectric and region 2 is a perfect conductor ($\sigma_2 = \infty$), then $D_{n2} = 0$ and

$$\boxed{D_{n1} = \rho_s}$$

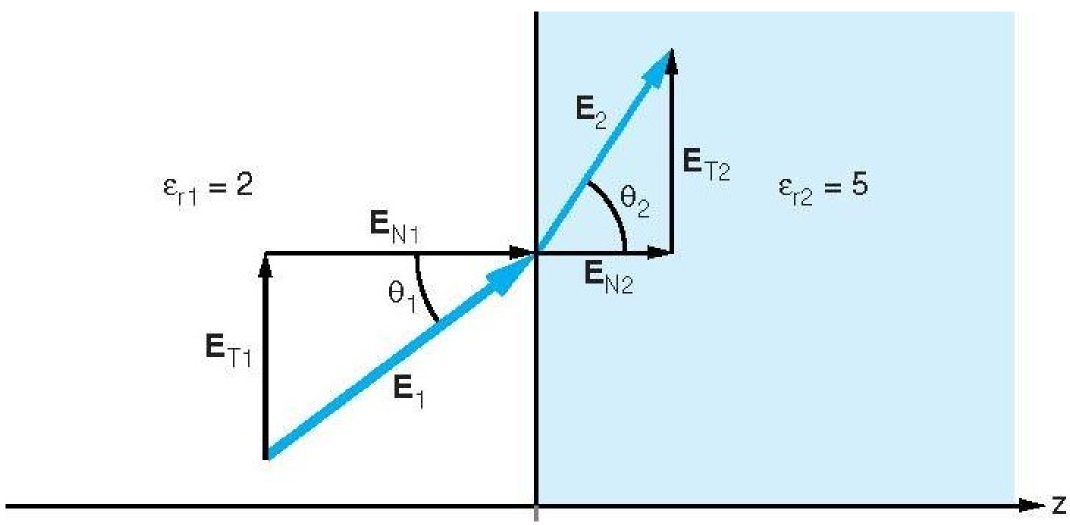
The normal component of electric flux density on the surface of a perfect conductor equals the surface charge density on the conductor.

To make sure we always get the sign correctly, the following general expression should be used

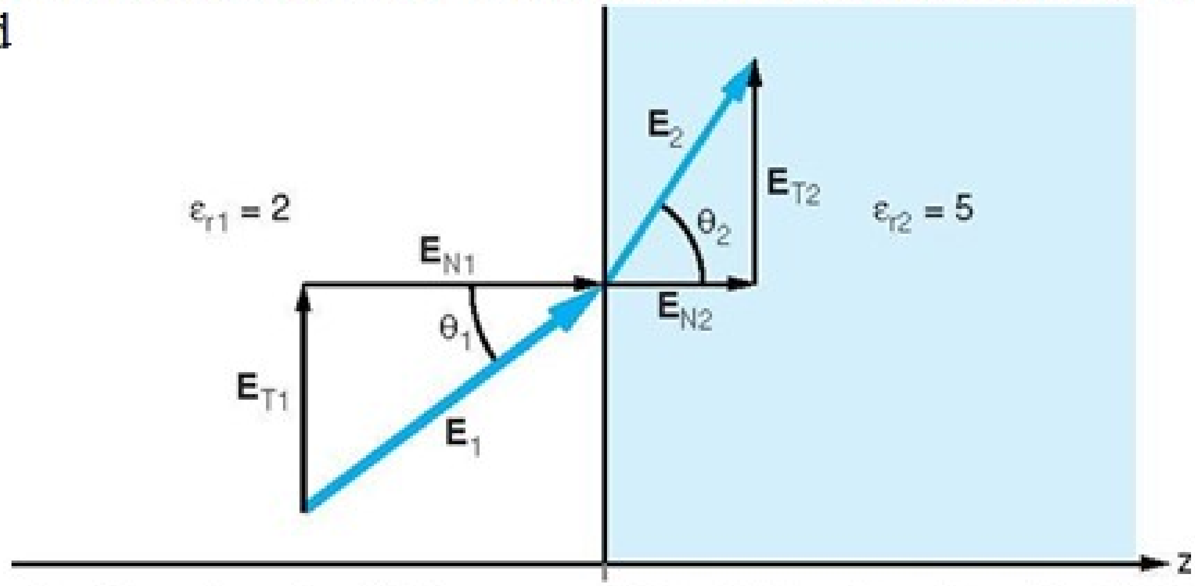
$$\mathbf{a}_{21} \cdot \left(\mathbf{D}_1 - \mathbf{D}_2\right) = \rho_s,$$

Example:

Consider the field $\boldsymbol{E_1} = 3\boldsymbol{a_x} + 4\boldsymbol{a_y} + 5\boldsymbol{a_z}$ that exists in one of a pair of dielectrics as illustrated below. We wish to find the electric field in the other side of the dielectric pair as well as the angle that each electric field makes with the normal direction to the surface. Assume charge free interface.



Procedure for evaluating the fields on both sides of a boundary separating two dielectric med



$\mathbf{E}_1 = 3\mathbf{a}_x + 4\mathbf{a}_y + 5\mathbf{a}_z$ (V/m)

Step 1. $\mathbf{E}_{N1} = 5\mathbf{a}_z$

Step 2. $\mathbf{E}_{T1} = 3\mathbf{a}_x + 4\mathbf{a}_y$

Step 3. $\theta_1 = \tan^{-1}\left(\frac{|\mathbf{E}_{T1}|}{|\mathbf{E}_{N1}|}\right) = 45°$

Step 5. $\mathbf{D}_{N1} = \varepsilon_{r1}\varepsilon_o\mathbf{E}_{N1} = 10\varepsilon_o\mathbf{a}_z$

Step 9. $\mathbf{E}_2 = 3\mathbf{a}_x + 4\mathbf{a}_y + 2\mathbf{a}_z$ (V/m)

Step 7. $\mathbf{E}_{N2} = \mathbf{D}_{N2}/\varepsilon_{r2}\varepsilon_o = 2\mathbf{a}_z$

Step 4. $\mathbf{E}_{T2} = \mathbf{E}_{T1} = 3\mathbf{a}_x + 4\mathbf{a}_y$

Step 8. $\theta_2 = \tan^{-1}\left(\frac{|\mathbf{E}_{T2}|}{|\mathbf{E}_{N2}|}\right) = 68.2°$

Step 6. $\mathbf{D}_{N2} = \mathbf{D}_{N1} = 10\varepsilon_o\mathbf{a}_z$

Matlab program for solving last example

```
clc
clear
% Using the command disp to display prompt messages

disp(' Enter vector quantities in brackets, for example: [x y z]')

epsilon_0 = 8.854*10^-12;        % define permittivity of free space

% prompt user to input variables

epsilon_r1 = input('\n Enter relative permittivity in region 1: ');
epsilon_r2 = input('\n Enter relative permittivity in region 2: ');
a12 = input('\n Enter unit vector from region 1 to region 2: ');
E1 = input('\n Enter electric field intensity vector in region 1: ');

%   perform calculations
En1 = dot(E1,a12)*a12;
Et1 = E1-En1;
Et2 = Et1;
Dn1 = epsilon_0*epsilon_r1*En1;
Dn2 = Dn1;
En2 = Dn2/(epsilon_0*epsilon_r2);
E2 = Et2+En2;

%   calculate the angles
th1_rad = atan(norm(Et1)/norm(En1));
th2_rad = atan(norm(Et2)/norm(En2));

th1_deg = th1_rad*180/pi;
th2_deg = th2_rad*180/pi;

% Display results

fprintf('\n E2 = %0.2f ax + %0.2f ay  + %0.2f az\n', E2(1), E2(2), E2(3));

fprintf('\n Theta_1 = %0.3f degrees\n', th1_deg);
fprintf('\n Theta_2 = %0.3f degrees\n', th2_deg);
```

Command window displaying program inputs and results

```
Enter vector quantities in brackets, for example: [x y z]

Enter relative permittivity in region 1: 2

Enter relative permittivity in region 2: 5

Enter unit vector from region 1 to region 2: [0 0 1]

Enter electric field intensity vector in region 1: [3 4 5]

E2 = 3.00 ax + 4.00 ay  + 2.00 az

Theta_1 = 45.000 degrees

Theta_2 = 68.199 degrees
fx >>
```

Example

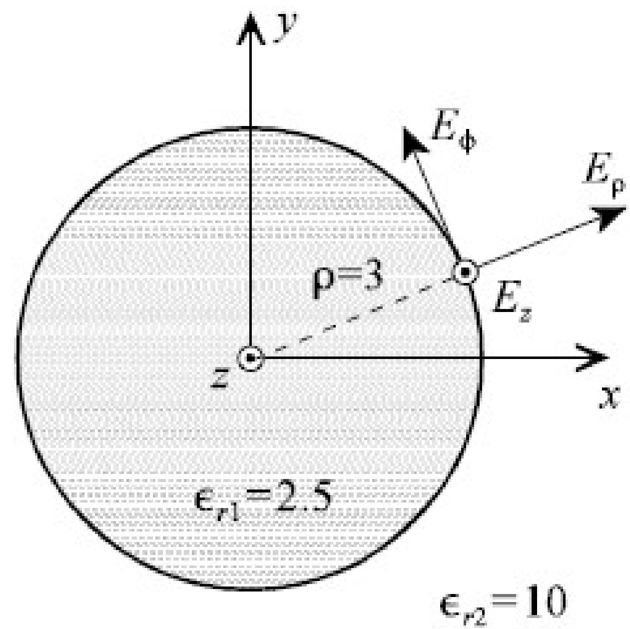
A dielectric cylinder (region 1) of radius $\rho=3$ and permittivity $\epsilon_{r1}=2.5$ is surrounded by another dielectric (region 2) of permittivity $\epsilon_{r2}=10$. Given an electric field inside the cylinder of

$$\boldsymbol{E}_1 = 2\boldsymbol{a}_\rho + 5\boldsymbol{a}_\phi - 4\boldsymbol{a}_z \quad \text{(kV/m)}$$

determine

(a.) $\boldsymbol{P}_1$ and ρ_{vp1}

(b.) $\boldsymbol{E}_2$ and $\boldsymbol{D}_2$ (Assume $\rho_s = 0$)



(a.) $\boldsymbol{P}_1 = \chi_{e1}\epsilon_o\boldsymbol{E}_1 = (\epsilon_{r1}-1)\epsilon_o\boldsymbol{E}$

$= 1.5\epsilon_o(2\boldsymbol{a}_\rho + 5\boldsymbol{a}_\phi + 4\boldsymbol{a}_z)10^3$

$= 26.56\boldsymbol{a}_\rho + 66.41\boldsymbol{a}_\phi + 53.12\boldsymbol{a}_z \quad (\text{nC/m}^2)$

$$\text{div } \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)}$$

$$\rho_{vp1} = -\nabla\cdot\boldsymbol{P}_1 = -\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho P_\rho)$$

$$= -\frac{10^{-9}}{\rho}\frac{\partial}{\partial\rho}(26.56\rho) = -\frac{26.56}{\rho}\ \text{nC/m}^3$$

(b.) $\boldsymbol{E}_{t2} = \boldsymbol{E}_{t1} = 5\boldsymbol{a}_\phi - 4\boldsymbol{a}_z$

$$\boldsymbol{D}_{n2} = \boldsymbol{D}_{n1} \quad \Rightarrow \quad \epsilon_2 \boldsymbol{E}_{n2} = \epsilon_1 \boldsymbol{E}_{n1} \quad \Rightarrow \quad \epsilon_2 E_{n2}\boldsymbol{a}_\rho = \epsilon_1 E_{n1}\boldsymbol{a}_\rho$$

$$E_{n2} = \frac{\epsilon_1}{\epsilon_2} E_{n1} = \frac{2.5\epsilon_o}{10\epsilon_o} E_{n1} = \frac{1}{4} E_{n1}$$

$$= \frac{1}{4}(2) = 0.5$$

$$\boldsymbol{E}_{n2} = 0.5\boldsymbol{a}_\rho \text{ kV/m}$$

$$\boxed{\boldsymbol{E}_2 = \boldsymbol{E}_{n2} + \boldsymbol{E}_{t2} = 0.5\boldsymbol{a}_\rho + 5\boldsymbol{a}_\phi - 4\boldsymbol{a}_z \quad (\text{kV/m})}$$

$$\boldsymbol{D}_2 = \epsilon_2 \boldsymbol{E}_2 = 10\epsilon_o(0.5\boldsymbol{a}_\rho + 5\boldsymbol{a}_\phi - 4\boldsymbol{a}_z)10^3 \quad (\text{C/m}^2)$$

$$\boxed{\boldsymbol{D}_2 = (44.3\boldsymbol{a}_\rho + 442.7\boldsymbol{a}_\phi - 354.2\boldsymbol{a}_z) \quad (\text{pC/m}^2)}$$

Announcements

- Midterm exam will be administered via Zoom on 10/7/2020, it is your responsibility to take the test using a laptop/desktop with reliable internet connection and working webcam and microphone.
  - Exam time is 9:00 to 11:30 AM
  - You are allowed to use the following:
    - Table of Integrals (Available on Webcourses); and
    - Two A4 sheets of handwritten EQUATIONS ONLY;
      - you can write your equations on both sides.
      - No graphs or plots are allowed on your equations sheets.
      - No rubric for solving a given class of problems should be included in your equations sheets.
      - No solution of a given problem should be included.
    - Your engineering calculator.
      - TI Nspire is not allowed