Lecture # 8

The Static Magnetic Field (continued)

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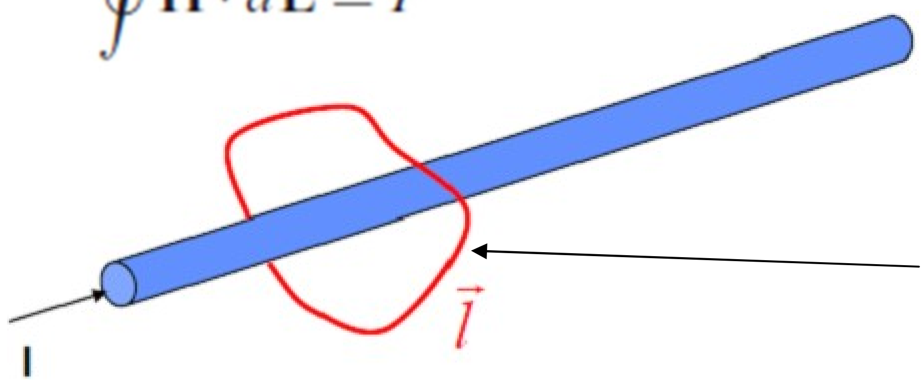
Ampere's Circuital Law

The line integral of the magnetic field about any closed path is equal to the direct current enclosed by the path.

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$



Positive current is in the direction of advancing a right-hand screw

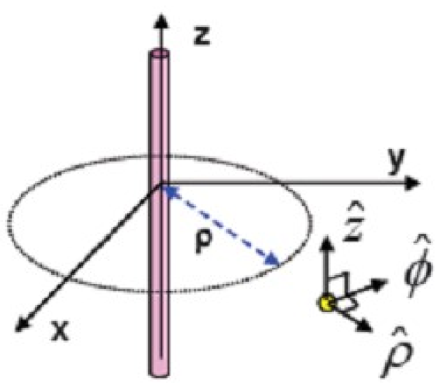
- similar in form to Gauss's law

$$\psi = \oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = Q_{enclosed} = Q$$

Example 1: Using Ampere's Law to find H

Find the magnetic field intensity produced by an **infinitely long and thin wire (filament)** that carries a constant current and lies on the z axis using Ampere's law.

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$



Using the right hand rule to find the direction of **H**

The direction of **H** is $\mathbf{a}_\varphi$ (perpendicular to z-ρ plane)

We can choose any closed path surrounding the current carrying wire , let us use a circle with radius ρ

$$I = \oint \overline{H} \cdot d\overline{L} = \oint H_\varphi \hat{a}_\varphi \cdot dl\, \hat{a}_\varphi$$

$$I = \oint H_\varphi \hat{a}_\varphi \cdot \rho d\varphi\, \hat{a}_\varphi$$

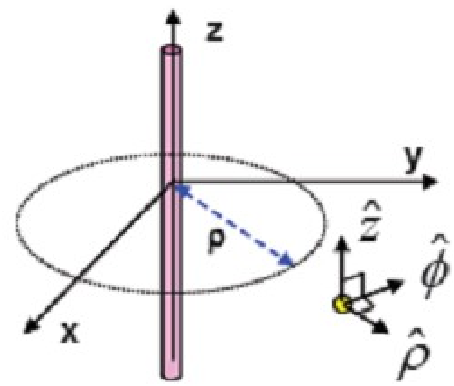
$$I = H_\varphi (2\pi\rho)$$

$$H_\varphi = \frac{I}{2\pi\rho}$$

Same as obtained by

Biot- Savart's law

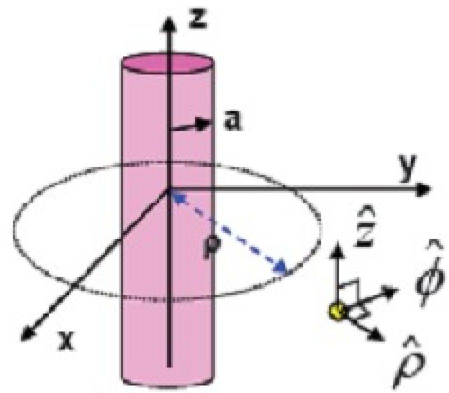
$$\overline{H} = \frac{I}{2\pi\rho} \hat{a}_\varphi$$



The magnetic field intensity is a function of the radial position of the field point (*H* does not change with z or φ)

Example 2: Ampere's Law to find H (infinitely long wire with finite radius)

- The magnetic field does not vary with z or ϕ
- The magnitude of the field is constant at a given radial distance, ρ
- The direction of the field is obtained using the right hand rule ($\boldsymbol{H} = H_\phi \, \boldsymbol{a}_\phi$)
- Choose a circle of radius ρ as the Amperian path ($\boldsymbol{dl} = dl \, \boldsymbol{a}_\phi$)

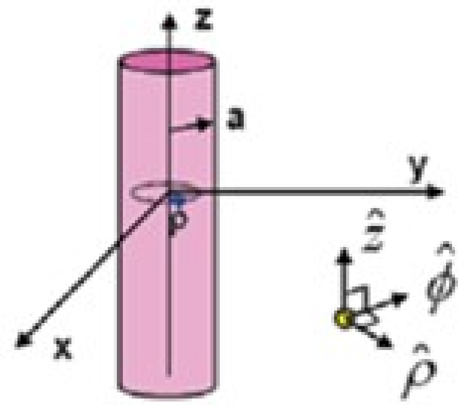


For AP: $\rho > a$

$$I = \oint \bar{H} \cdot d\bar{L} = H_\phi (2\pi\rho)$$

$$\bar{H} = \frac{I}{2\pi\rho} \hat{a}_\phi \quad , \rho > a$$

For AP: $0 < \rho < a$

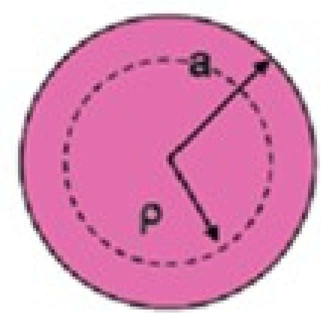


$$I_{inside} = \oint \bar{H} \cdot d\bar{L} = H_\phi (2\pi\rho)$$

$$H_\phi = \frac{I_{inside}}{2\pi\rho}$$

Since,

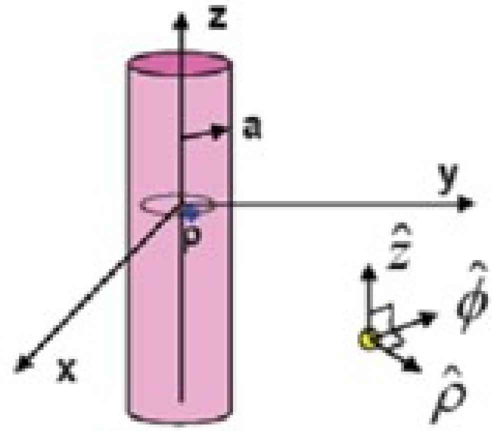
$$I_{inside} = J * (area\ of\ a\ circle) = \frac{I}{\pi a^2}(\pi\rho^2)$$

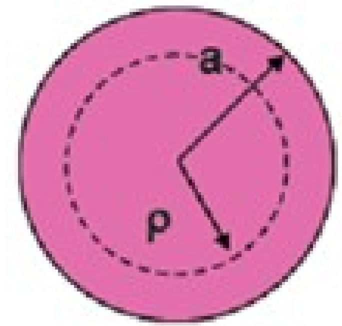


$$\bar{H} = \frac{I\rho}{2\pi a^2}\hat{a}_\phi \quad , \rho < a$$

$$I_{inside} = \oint \overline{H} \cdot d\overline{L} = H_\phi (2\pi\rho)$$

$$\overline{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\hat{a}_\phi & , \rho < a \\ \dfrac{I}{2\pi\rho}\hat{a}_\phi & , \rho > a \end{cases}$$





- **what happens at ρ = a?**

$$\overline{H} = \frac{Ia}{2\pi a^2}\hat{a}_\phi = \frac{I}{2\pi a}\hat{a}_\phi$$

- **so H is continuous here as you move from inside to outside the wire**

The Curl.

- The curl of a vector field is a vector quantity that describes the rotation of the field about a particular point.
- Curl of a vector field **A** is defined as the cross product of the gradient operator and the vector field **A**

$$curl\ \overline{A} = \nabla \times \overline{A} = \begin{vmatrix} \hat{a}x & \hat{a}y & \hat{a}z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

Then evaluate the determinant to find curl **A**

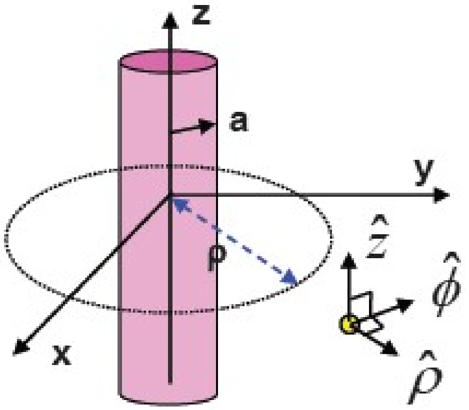
*Curl **H** using Cylindrical and Spherical Coordinates*

$$\boxed{\begin{aligned}\nabla \times \mathbf{H} = &\left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi \\ &+ \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\mathbf{a}_z \quad \text{(cylindrical)}\end{aligned}}$$

$$\boxed{\begin{aligned}\nabla \times \mathbf{H} = &\frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\mathbf{a}_\theta \\ &+ \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\mathbf{a}_\phi \quad \text{(spherical)}\end{aligned}}$$

Example 3:

Consider the case of the wire, we found an expression for **H**, find the curl of **H.**



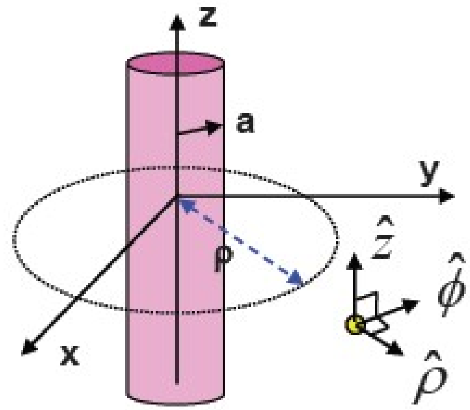
$$\bar{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\hat{a}_\phi & ,\rho < a \\ \dfrac{I}{2\pi\rho}\hat{a}_\phi & ,\rho > a \end{cases}$$

$$\nabla \times \vec{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi}\right)\hat{z}$$

$$= \left(\frac{1}{\rho}\frac{\partial 0}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial 0}{\partial z} - \frac{\partial 0}{\partial \rho}\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{\partial 0}{\partial \phi}\right)\hat{z}$$

$$= \frac{1}{\rho}\left(\frac{\partial(\rho H_\phi)}{\partial \rho}\right)\hat{z}$$

$$\nabla \times \vec{H} = \frac{1}{\rho}\left(\frac{\partial\left(\rho H_{\phi}\right)}{\partial \rho}\right)\hat{z}$$



$$\nabla \times \vec{H} = \begin{cases} \dfrac{1}{\rho}\dfrac{\partial\left(\rho I \dfrac{\rho}{2\pi a^2}\right)}{\partial \rho}\hat{z} & \rho < a \\ \dfrac{1}{\rho}\dfrac{\partial\left(\rho \dfrac{I}{2\pi \rho}\right)}{\partial \rho}\hat{z} & a < \rho \end{cases} = \begin{cases} \dfrac{1}{\rho}\dfrac{I}{2\pi a^2}\dfrac{\partial\left(\rho^2\right)}{\partial \rho}\hat{z} & \rho < a \\ 0 & a < \rho \end{cases}$$

$$\nabla \times \vec{H} = \begin{cases} \dfrac{I}{\pi a^2}\hat{z} & \rho < a \\ 0 & a < \rho \end{cases} = \vec{J}$$

The curl of H is the Current density

Stokes's Theorem

The surface integral of the Curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface,

$$\boxed{\int_s (\nabla \times \overline{H}) \cdot \overline{ds} = \oint \overline{H} \cdot \overline{dl}}$$

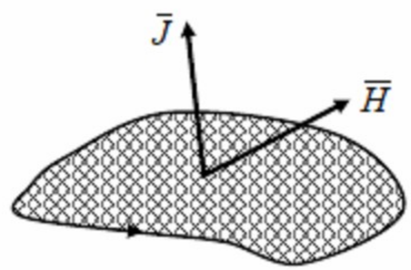
Notice that the directions of d**S** and d**L** are dependent on each other

(they are governed by the right hand rule)

Curl of the Magnetic Field

From Ampere's law;

$$\oint \overline{H} \cdot \overline{dl} = I_{enclosed} = \int_s \overline{J} \cdot \overline{ds}$$



Using Stokes's Theorem

$$\int_s (\nabla \times \overline{H}) \cdot \overline{ds} = \int_s \overline{J} \cdot \overline{ds}$$

Recall that, $\boxed{\nabla \times \overline{H} = \overline{J}}$

For closed surface;

$$\oint_s (\nabla \times \overline{H}) \cdot \overline{ds} = 0 \quad \rightarrow \quad \nabla \times \overline{H} = 0$$

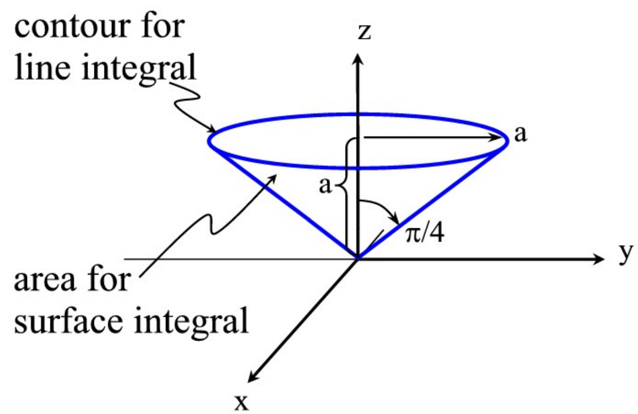
Example 4 (Stokes's Theorem)

$$\int_s (\nabla \times \overline{H}) \cdot \overline{ds} = \oint \overline{H} \cdot \overline{dl}$$

Suppose you have the field $\mathbf{H} = r \cos\theta \ \mathbf{a}_\varphi$ A/m.
Now consider the cone specified by $\theta = \pi/4$,
with height a and circular top of radius a.

1- Evaluate the left side of Stokes's theorem through the $d\mathbf{S} = -dS\mathbf{a}_\theta$ surface.

2- Evaluate the right side of Stokes's theorem by integrating around the loop.



Notice that the directions of d**S** and d**L** are dependent on each other

(they are governed by the right hand rule (i.e. if we choose the direction of dS in $-\mathbf{a}_\theta$ then dL will be in $+\mathbf{a}_\varphi$ (CCW))

$$\boxed{\begin{aligned}\nabla \times \mathbf{H} &= \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial\theta} - \frac{\partial H_\theta}{\partial\phi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\mathbf{a}_\theta \\ &+ \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right)\mathbf{a}_\phi \quad \text{(spherical)}\end{aligned}}$$

Solution

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial\theta} - \frac{\partial H_\theta}{\partial\phi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\mathbf{a}_\theta + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right)\mathbf{a}_\phi \quad \text{(spherical)}$$

Evaluation of the LHS of Stokes's Theorem

$$\int_S (\nabla \times \bar{H}) \cdot \bar{ds}$$

$$\because \quad \bar{H} = r\cos\theta\,\hat{a}_\phi \implies H_\phi = r\cos\theta, \quad H_\theta = 0, \text{ and } H_r = 0$$

$$\therefore \text{Curl } \bar{H} = \nabla \times \bar{H} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi \sin\theta)\right]\hat{a}_r - \frac{1}{r}\left[\frac{\partial}{\partial r}(rH_\phi)\right]\hat{a}_\theta$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\overbrace{\cos\theta}^{f}\,\overbrace{\sin\theta}^{g})\,\hat{a}_r - \frac{1}{r}\frac{\partial}{\partial r}(r^2\cos\theta)\,\hat{a}_\theta$$

$$= \frac{\cancel{r}}{\cancel{r}\sin\theta}(\cos^2\theta - \sin^2\theta)\,\hat{a}_r - \frac{1}{\cancel{r}}(2\cancel{r}\cos\theta)\,\hat{a}_\theta$$

$$\boxed{\nabla \times \bar{H} = \frac{\cos^2\theta - \sin^2\theta}{\sin\theta}\,\hat{a}_r - 2\cos\theta\,\hat{a}_\theta}$$

Therefore, the LHS of Stokes's Theorem

$$= \int_S \nabla \times \bar{H} \cdot \bar{ds} =$$

$$= \int_S \left(\left(\frac{\cos^2\theta - \sin^2\theta}{\sin\theta} \right) \hat{a}_r - 2\cos\theta\, \hat{a}_\theta \right) \cdot \left(-r \sin\theta\, d\phi\, dr \right) \hat{a}_\theta$$

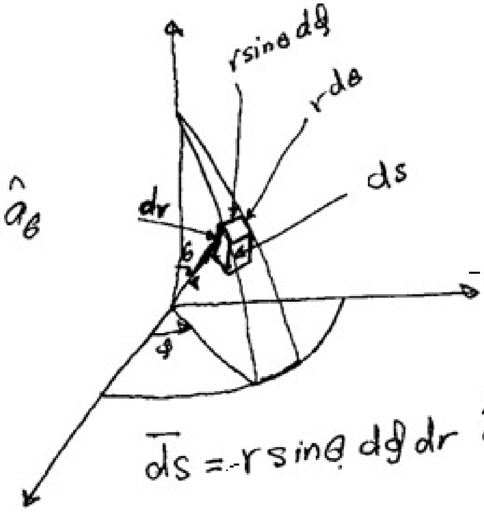
$$= \int_S 2r \cos\theta \sin\theta\, d\phi\, dr$$

$$= \int_{r=0}^{\sqrt{2}a} \int_{\phi=0}^{2\pi} 2r\cos\theta \sin\theta\, d\phi\, dr$$

$$= 2\cos\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{4}\right) \left[\int_0^{\sqrt{2}a} r\, dr \right] \left[\int_0^{2\pi} d\phi \right]$$

$$= \cancel{2} \left[\frac{\sqrt{2}}{2} \right] \left[\frac{\sqrt{2}}{2} \right] \left[\frac{(\sqrt{2}a)^2}{2} \right] \left[2\pi \right] = 2a^2\pi$$

$$\therefore \boxed{\int_S \nabla \times \bar{H} \cdot \bar{ds} = 2a^2\pi} \longrightarrow (1)$$



$$\bar{ds} = -r\sin\theta\, d\phi\, dr\, \hat{a}_\theta$$

where

$0 \leq r \leq \sqrt{2}a$

$\theta = \frac{\pi}{4}$

$0 \leq \phi \leq 2\pi$

Evaluation of the RHS of Stokes's Theorem

$$d\bar{L} = a\,d\phi\,\hat{a}_\phi$$

$$\oint \bar{H} \cdot \bar{dL} = \int_{\phi=0}^{2\pi} \underbrace{r\cos\theta}\,\hat{a}_\phi \cdot \underset{\sim}{a}\,d\phi\,\hat{a}_\phi$$

$$= a r \cos\theta \int_{\phi=0}^{2\pi} d\phi = a r \cos\theta\,(2\pi)$$

$$= a\left(\sqrt{2}\,a\right)\left(\frac{\sqrt{2}}{\cancel{2}}\right)\left(\cancel{2}\pi\right)$$

$$\boxed{\oint \bar{H} \cdot dL = 2a^2\pi} \longrightarrow (2)$$

From equations 1 and 2, both Sides of Stokes's Theorem are equal.

Magnetic Flux Density

- In free space, the magnetic flux density **B** is defined as:

$$\boxed{\mathbf{B} = \mu_0 \mathbf{H}} \quad \text{(free space only)}$$

Where, $\boldsymbol{\mu_0}$ is the permeability of free space (**4π x 10^{-7} H/m**).
and H is the magnetic field intensity

- **B** is measured in Weber per square meter (Wb/m^2) or Tesla (T).

Analogies between electric and magnetic fields

- We have compared the Biot-Savart's law and Coulomb's law, and have seen an analogy between **H** and **E**.
- The relations $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \varepsilon_0 \mathbf{E}$ (**B** is analogous to **D**).
- If **B** is measured in Teslas or Wb/m^2, and the magnetic flux, Φ, is measured in Webers.
- Thus, the magnetic flux is defined as the flux passing through any designated open surface.
- The magnetic flux is given by:

$$\boxed{\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \text{ Wb}}$$

- Our analogy should now remind us of the electric flux, Ψ, measured in coulombs, and of Gauss's law, which states that the total flux passing through any closed surface is equal to the charge enclosed,

$$\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

- The charge Q is the source of the electric flux lines that begin and terminate on positive and negative charge, respectively.
- No such source has ever been discovered for the lines of magnetic flux. Thus, **Gauss's law for the magnetic field is**

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0}$$

- Applying the divergence theorem leads to $\boxed{\nabla \cdot \mathbf{B} = 0}$
  - The divergence of the magnetic flux density is zero for any closed Gaussian surface.

Maxwell's Equations as they apply to static electric fields and steady magnetic fields.

Differential forms

$$\nabla \cdot \mathbf{D} = \rho_v$$
$$\nabla \times \mathbf{E} = 0$$
$$\nabla \times \mathbf{H} = \mathbf{J}$$
$$\nabla \cdot \mathbf{B} = 0$$

Integral forms

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv$$
$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$
$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_S \mathbf{J} \cdot d\mathbf{S}$$
$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Given a vector field **A,** Maxwell's equations can be utilized to determine if it is a valid representation of an electrostatic field, magnetostatic field, or neither.

Announcements

- No homework is assigned today.
- Next lecture: Magnetic Torque, and Circuits