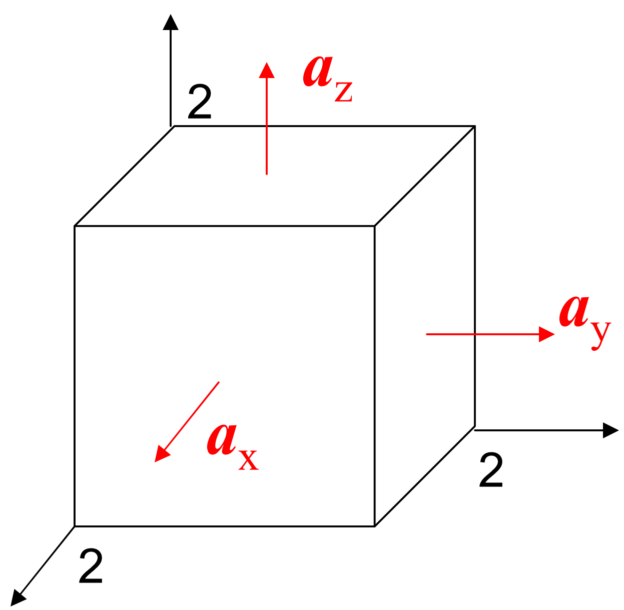
Lecture 5

Gauss's Law and Divergence Theory

Dr. Shady Elashhab

Example 1

Given $\mathbf{D} = 3\boldsymbol{a}_x + 2xy\boldsymbol{a}_y + 8x^2y^3\boldsymbol{a}_z$ C/m^2, find the total outward flux through the surface of a cube with $0 \le x \le 2$ m, $0 \le y \le 2$ m and $0 \le z \le 2$m .



$$\Psi = \oint \mathbf{D} \bullet d\mathbf{S} = Q_{enc}$$

Solution

$$\Psi = \oint \mathbf{D} \bullet d\mathbf{S} = \int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{front}} + \int_{\text{back}} = Q_{enc}$$

$$\int_{top} = \int 8x^2 y^3 \mathbf{a}_z \bullet dxdy\mathbf{a}_z = 8\int_0^2 x^2 dx \int_0^2 y^3 dy = 85.3C$$

$$\int_{bottom} = \int 8x^2 y^3 \mathbf{a}_z \bullet (-dxdy\mathbf{a}_z) = -85.3C$$

$$\int_{left} = \int 2x\, y\big|_{\mathbf{y=0}}\, \mathbf{a}_y \bullet (-dxdz\mathbf{a}_y) = 0$$

$$\int_{right} = \int 2x\, y\big|_{y=2}\, \mathbf{a}_y \bullet dxdz\mathbf{a}_y = 4\int_0^2 xdx \int_0^2 dz = 16C$$

$$\int_{front} = \int 3\mathbf{a}_x \bullet dydz\mathbf{a}_x = 12C$$

$$\int_{back} = \int 3\mathbf{a}_x \bullet (-dydz\mathbf{a}_x) = -12C$$

$$\therefore \Psi = \oint \mathbf{D} \bullet d\mathbf{S} = Q_{enc} = 16C.$$

Divergence and Gauss's law

- So far we have considered cases where the electric flux density, ***D*** is constant on the chosen Gaussian surface and everywhere normal or tangential to the closed surface.

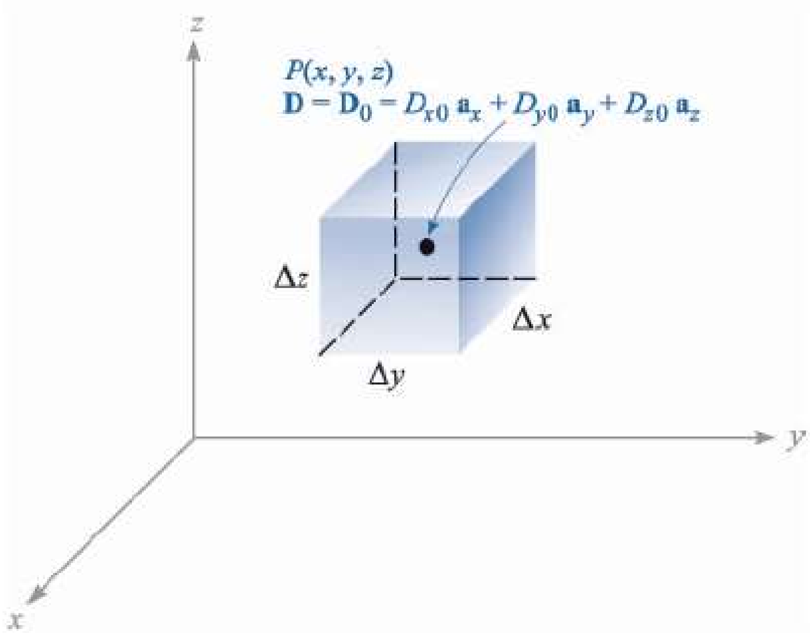
- If the chosen Gaussian surface does not satisfy the aforementioned requirements for ***D*,** the integral in Gauss's law would not be evaluated with ease.

- In order to overcome that problem, one can choose an infinitely small closed surface where ***D*** is almost constant

- The small change in ***D*** can be represented by the first two terms of Taylor's series expansion

  - ***Keep in mind that the results will become more accurate as the volume enclosed by the closed surface approaches zero***

The value of $\mathbf{D}$ at the point P may be expressed in cartesian components, $\mathbf{D}_0 = D_{x0}\mathbf{a}_x + D_{y0}\mathbf{a}_y + D_{z0}\mathbf{a}_z$. We choose as our closed surface the small rectangular box, centered at P, having sides of lengths Δx, Δy, and Δz, and apply Gauss's law,



Gauss's law $\longrightarrow$

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = Q}$$

In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

Consider the front surface:

$$\begin{aligned} \int_{\text{front}} &\doteq \mathbf{D}_{\text{front}} \cdot \Delta \mathbf{S}_{\text{front}} \\ &\doteq \mathbf{D}_{\text{front}} \cdot \Delta y \, \Delta z \, \mathbf{a}_x \\ &\doteq \underline{D_{x,\text{front}}} \Delta y \, \Delta z \end{aligned}$$

The value of $D_{x,\text{front}}$ is not known exactly, it can be approximated by the 1st two terms of Taylor series:

$$\begin{aligned} \underline{D_{x,\text{front}}} &\doteq D_{x0} + \frac{\Delta x}{2} \times \text{ rate of change of } D_x \text{ with } x \\ &\doteq D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \end{aligned}$$

where D_{x0} is the value of D_x at P

$$\boxed{\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \, \Delta z} \qquad (1)$$

Consider now the integral over the back surface,

$$\int_{\text{back}} \doteq \mathbf{D}_{\text{back}} \cdot \Delta \mathbf{S}_{\text{back}}$$

$$\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y \, \Delta z \, \mathbf{a}_x)$$

$$\doteq -D_{x,\text{back}} \Delta y \, \Delta z$$

and

$$D_{x,\text{back}} \doteq D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

giving

$$\boxed{\int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \, \Delta z} \qquad (2)$$

If we combine these two integrals, we have

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

By exactly the same process we find that

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

and

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

and these results may be collected to yield

The differential form of Gauss's law

(*Point from Gauss's Law*)

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$

or

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

$$\text{Charge enclosed in volume, } \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \left(\text{volume } \Delta v \right)$$

$$\text{div } \mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right)$$

Example 2

Find an approximate value for the total charge enclosed in an incremental volume of 10^{-9} m^3 located at the origin, if $\mathbf{D} = e^{-x} \sin y \, \mathbf{a}_x - e^{-x} \cos y \, \mathbf{a}_y + 2z\mathbf{a}_z$ C/m^2.

Solution

$$\text{Charge enclosed in volume}, \Delta v \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \left(\text{volume } \Delta v\right)$$

We first evaluate the three partial derivatives

$$\frac{\partial D_x}{\partial x} = -e^{-x} \sin y$$

$$\frac{\partial D_y}{\partial y} = e^{-x} \sin y$$

$$\frac{\partial D_z}{\partial z} = 2$$

$$\text{Charge enclosed in volume} = 2\Delta v = 2 \text{ nC}.$$

Divergence and Gauss's law

Let us rewrite the differential form of Gauss's law as:

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right) \doteq \frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}$$

which is equivalently written as

$$\boxed{\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right) \doteq \frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\frac{Q}{\Delta v}}$$

This equation contains too much information to discuss all at once, write it as two separate equations,

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right) \doteq \frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v} \longrightarrow \text{(i)}$$

and

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right) = \rho_v \longrightarrow \text{(ii)}$$

The LHS of each equation is a derivative operator called the ***divergence***.
It is denoted as:

$$\boxed{\text{div } \mathbf{D} = \left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)}$$

From (i), the divergence of the vector flux density, **D**, is the outflow of flux from a small closed surface per unit volume as the volume tends to zero

From (ii) $\longrightarrow$ $\boxed{\text{div } \mathbf{D} = \rho_v}$ This is the first of Maxwell's four equations

$$\boxed{\nabla\cdot\mathbf{D} = \rho_v}$$

Divergence of the electric flux density in different coordinate systems

$$\text{div } \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \qquad \text{(cartesian)}$$

$$\text{div } \mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)}$$

$$\text{div } \mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \, D_\theta) + \frac{1}{r \sin\theta}\frac{\partial D_\phi}{\partial \phi} \qquad \text{(spherical)}$$

Divergence Theorem

The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

Proof: from Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

and letting

$$Q = \int_{\mathrm{vol}} \rho_v dv$$

and then replacing ρ_v by its equal,

$$\nabla \cdot \mathbf{D} = \rho_v$$

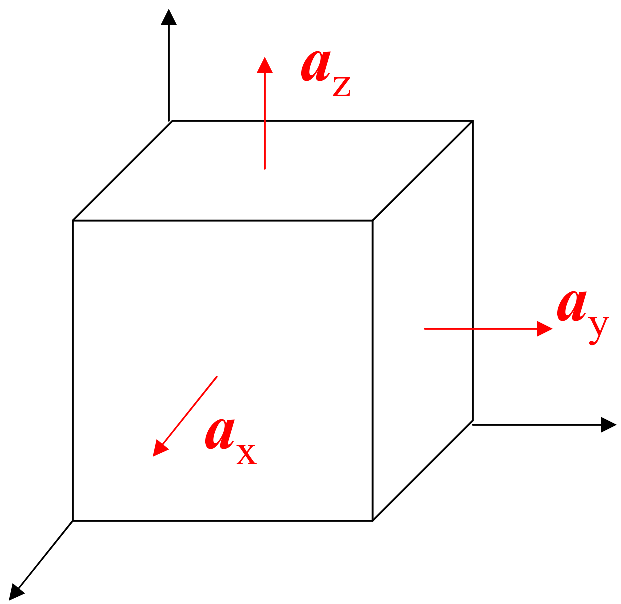
we have
$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\mathrm{vol}} \rho_v dv = \int_{\mathrm{vol}} \nabla \cdot \mathbf{D}\, dv$$

The first and last expressions constitute the divergence theorem,

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\mathrm{vol}} \nabla \cdot \mathbf{D}\, dv}$$

Example 3:

Evaluate both sides of the divergence theorem for the field $\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y$ C/m^2 and the rectangular parellelepiped formed by the planes $x = 0$ and 1, $y = 0$ and 2, and $z = 0$ and 3.



Divergence Theorem:

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv}$$

Solution. Evaluating the surface integral first, we note that **D** is parallel to the surfaces at $z = 0$ and $z = 3$, so $\mathbf{D} \cdot \mathbf{dS} = 0$ there. For the remaining four surfaces we have

The Left side of the divergence theorem

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (\mathbf{D})_{x=0} \cdot (-dy\, dz\, \mathbf{a}_x) + \int_0^3 \int_0^2 (\mathbf{D})_{x=1} \cdot (dy\, dz\, \mathbf{a}_x)$$

$$+ \int_0^3 \int_0^1 (\mathbf{D})_{y=0} \cdot (-dx\, dz\, \mathbf{a}_y) + \int_0^3 \int_0^1 (\mathbf{D})_{y=2} \cdot (dx\, dz\, \mathbf{a}_y)$$

$\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y \text{ C/m}^2$

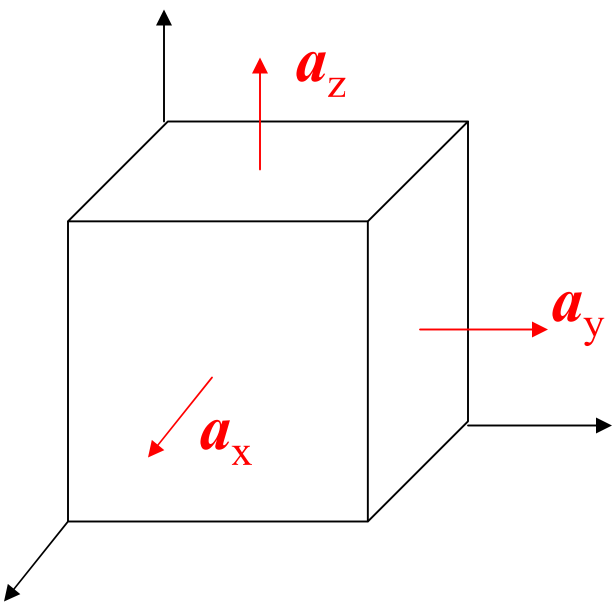
$$= -\cancelto{0}{\int_0^3 \int_0^2 (D_x)_{x=0} dy\, dz} + \int_0^3 \int_0^2 (D_x)_{x=1} dy\, dz$$

$$- \cancel{\int_0^3 \int_0^1 (D_y)_{y=0} dx\, dz} + \cancel{\int_0^3 \int_0^1 (D_y)_{y=2} dx\, dz}$$

However, $(D_x)_{x=0} = 0$, and $(D_y)_{y=0} = (D_y)_{y=2}$, which leaves only

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (D_x)_{x=1} dy\, dz = \int_0^3 \int_0^2 2y\, dy\, dz$$

$$= \int_0^3 4\, dz = 12$$



0

Evaluating the right side of the divergence theorem

Since $$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) = 2y$$

the volume integral becomes

$$\int_{\text{vol}} \nabla \cdot \mathbf{D}\, dv = \int_0^3 \int_0^2 \int_0^1 2y\, dx\, dy\, dz = \int_0^3 \int_0^2 2y\, dy\, dz$$
$$= \int_0^3 4\, dz = 12$$

Example 3:

- Given $\boldsymbol{D} = 3\ \boldsymbol{a_x} + 2xy\ \boldsymbol{a_y} + 8x^2y^3\ \boldsymbol{a_z}$ C/m^2,

  (a) determine the charge density at the point $P(1,1,1)$.

  (b) Find the total flux through the surface of a cube with $0 \leq x \leq 2$m, $0 \leq y \leq 2$m, and $0 \leq z \leq 2$m

(a) determine the charge density at the point $P(1,1,1)$.

$$\boldsymbol{D} = 3\,\boldsymbol{a}_x + 2xy\,\boldsymbol{a}_y + 8x^2y^3\,\boldsymbol{a}_z \text{ C/m}^2$$

$$\rho_v = \nabla \cdot \mathbf{D}$$

$$= \frac{\partial}{\partial y}(2xy) = 2x,$$

Therefore,

$$\rho_v(1,1,1) = 2\frac{C}{m^3}.$$

(b) the total flux through the surface of a cube can be obtained using the Divergence theorem as follows:

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial y}(2xy)$$

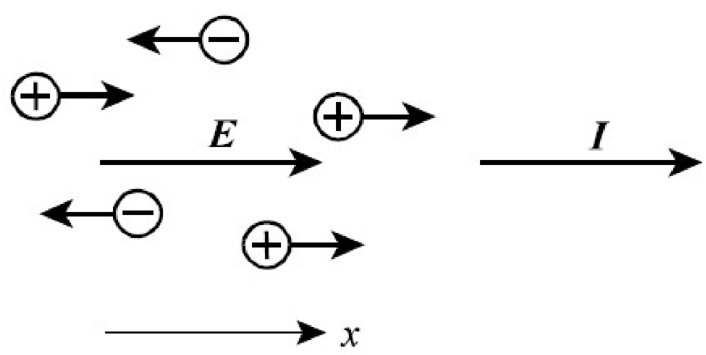
$$= 2x;$$

$$\int \nabla \cdot \mathbf{D}\, dv = 2\int_0^2 x\,dx \int_0^2 dy \int_0^2 dz = 16C.$$

Electric Fields in Material Space

- The charges considered up to this point have been assumed to be stationary and located in free space (vacuum) or air.
- If we place charges within a gas, solid or liquid material, the charge associated with the material atoms will be affected.
- Under the influence of the applied electric field, charges not bound by other forces (free charges) may be set in motion (electric current).

Current (***I***) - net flow of positive charges in a given direction measured in units of Amperes (Ampere = Coulomb/second).



the negative charge moving in the opposite direction constitutes a positive component of the overall current flowing in the $\boldsymbol{a}_x$ direction.

$$\boldsymbol{I} = \frac{+Q}{\text{second}}(\boldsymbol{a}_x) + \frac{-Q}{\text{second}}(-\boldsymbol{a}_x) = (I_+ + I_-)\boldsymbol{a}_x = I\boldsymbol{a}_x$$

There are three distinct types of currents:

(1) *Conduction current* (current in a conductor)

- current in a copper wire.

(2) *Convection current*

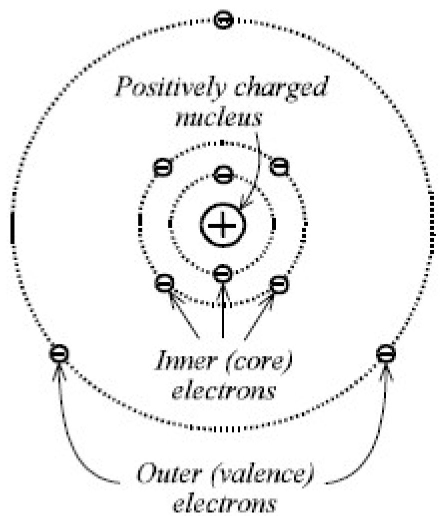
- electron beam in a CRT.

(3) *Displacement current* (time-varying effect to be studied later)

- AC current in a capacitor.

Material Classification Based on Conductivity

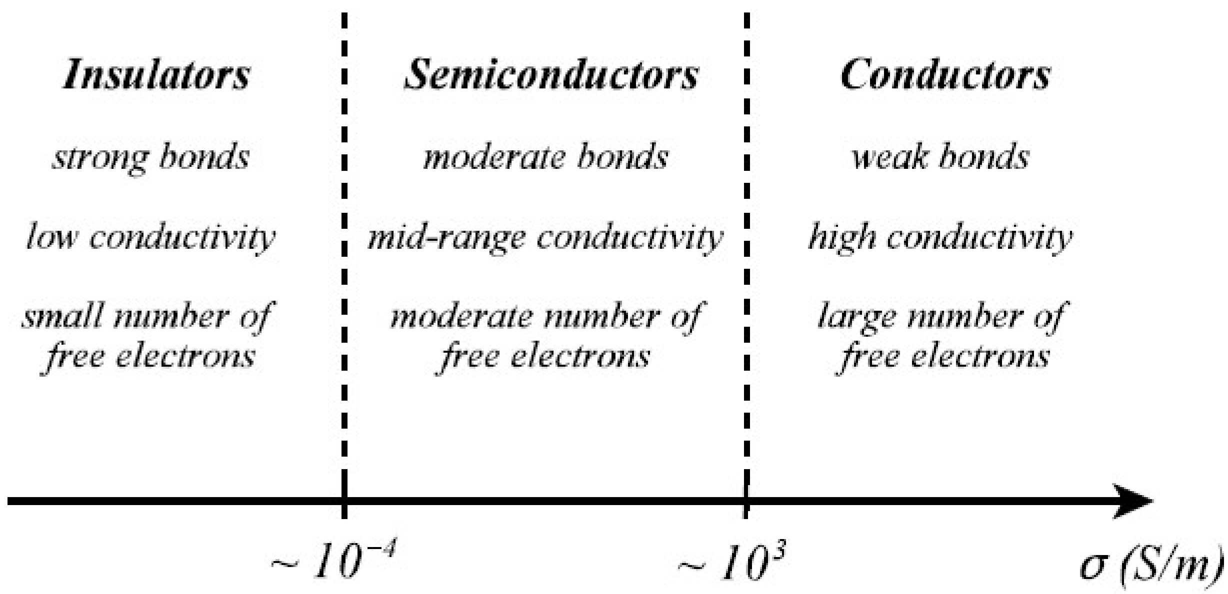
- The *conductivity* σ of a given material is a measure of the ability of the material to conduct current. Conductivity is measured in units of *S*/*m*.
- The reciprocal of conductivity is *resistivity* ($\rho = 1/\sigma$).
- For elements, the structure of the element atom dictates the conductivity of the element.
  - the element conductivity is related to the strength of the bonds between the outer (*valence*) electrons and the atom nucleus.



Positive nucleus charge = Total negative electron charge
Centroid of the nucleus charge - atom center
Centroid of the overall electron charge - atom center

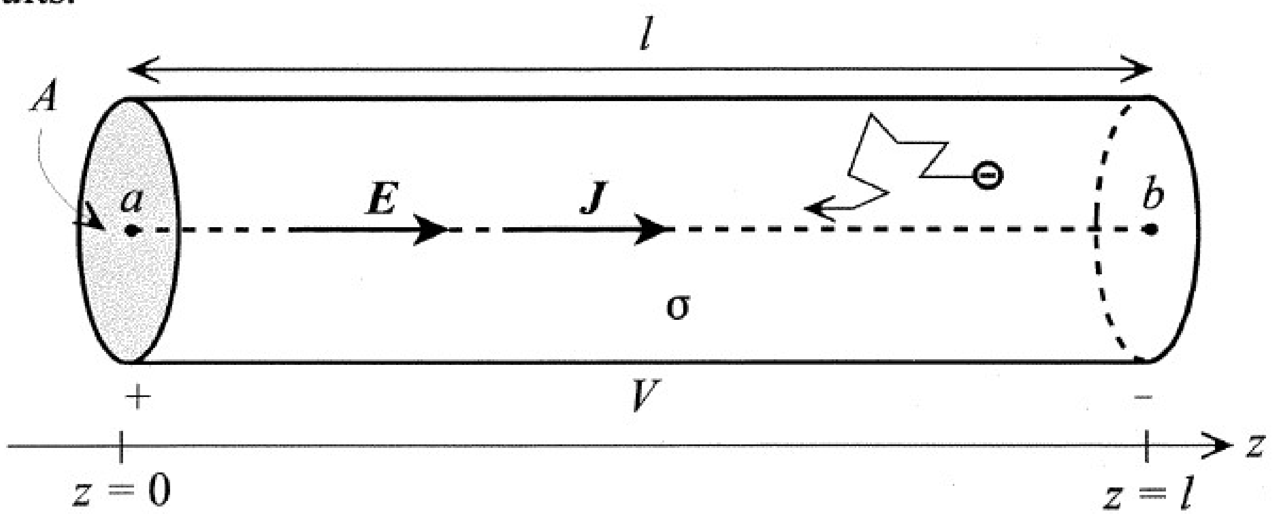
} The atom is electrically neutral. ($\rho_v = 0$, $V = 0$, $E = 0$)

- Under the influence of an electric field, the bond between the valence electron and the atom nucleus may be broken, the electron becomes a *free electron* or *conduction electron*.
- Materials are classified as *conductors*, *insulators*, or *semiconductors* based on the strength of bonds between the valence electrons and the atom nucleus. The stronger the bond between the valence electrons and the nucleus in a particular material, the fewer free electrons are available for conduction.



Conduction current

A simple example of conduction current is the current flowing in a conducting wire. If a voltage V is applied to a cylindrical conductor (conductivity = σ, length = l, cross-sectional area = A), a conduction current results.



The total current $\boldsymbol{I}$ flowing in the conductor is defined in terms of the *current density* $\boldsymbol{J}$ (A/m^2) as

$$I = \iint_S \boldsymbol{J} \cdot d\boldsymbol{s} = \iint_S \boldsymbol{J} \cdot ds\,\hat{\boldsymbol{a}}_n = \iint_S J_n\, ds \qquad \text{(A)}$$

where $\boldsymbol{J}_n$ is the component of the current density normal to the surface .

For the special case when the current density is uniform over the surface *S*, (J_n is constant)

$$I = \iint_S J_n \, ds = J_n A$$

- The conduction current can be defined in terms of the free volume charge density (ρ_V) and the average **drift velocity** (***u***) as follows:

$$\boldsymbol{J} = \rho_V \boldsymbol{u}$$

- The average drift velocity in a conductor may be written as the product of the electric field (***E***) and the conductor *mobility* (μ).

$$\boldsymbol{u} = \mu \boldsymbol{E}$$

Which yields the conduction current density in terms of the electric field:

$$\boxed{\boldsymbol{J} = \rho_V \mu \boldsymbol{E} = \sigma \boldsymbol{E} \qquad \text{Therefore,} \qquad \sigma = \rho_V \mu}$$

If the current density in the conductor is uniform, the corresponding electric field is also uniform ($\boldsymbol{J} = \sigma \boldsymbol{E}$). The voltage between the ends of the wire can be expressed as the line integral of the electric field.

$$V_{ba} = -V = -\int_a^b \boldsymbol{E} \cdot d\boldsymbol{l} = -\int_a^b E\,dz = -E\int_a^b dz = -El$$

Thus, the voltage and the uniform electric field may be written as

$$V = El \qquad\qquad E = \frac{V}{l}$$

The uniform current density is then

$$J = \frac{I}{A} = \sigma E = \sigma \frac{V}{l} \qquad \Rightarrow \qquad V = I\frac{l}{\sigma A} = IR$$

where

$$R = \frac{l}{\sigma A}$$

Resistance of a cylinder (length = l, cross-sectional area = A, conductivity = σ) carrying a uniform current density

If the current density is not uniform, the resistance formula becomes

$$\boxed{R = \frac{V}{I} = \frac{\int \boldsymbol{E} \cdot d\boldsymbol{l}}{\iint \boldsymbol{J} \cdot d\boldsymbol{s}} = \frac{\int \boldsymbol{E} \cdot d\boldsymbol{l}}{\iint \sigma \boldsymbol{E} \cdot d\boldsymbol{s}}}$$

Example 4:

A copper wire ($\sigma = 5.8 \times 10^7$ S/m, $\rho_V = -1.4 \times 10^{10}$ C/m^3, radius = 1 mm, length = 20 cm) carries a current of 1 mA. Assuming a uniform current density, determine

(a.) the wire resistance.
(b.) the current density.
(c.) the electric field within the wire.
(d.) the drift velocity of the electrons in the wire.

Solution

(a.) $$R = \frac{l}{\sigma A} = \frac{0.2}{(5.8 \times 10^7)(\pi)(0.001)^2} = 1.1\,\mathrm{m}\Omega$$

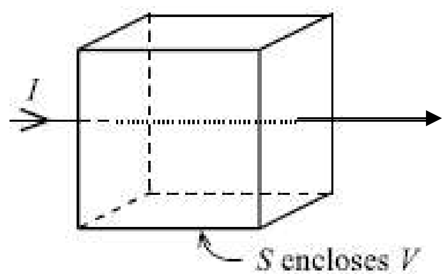
(b.) $$J = \frac{I}{A} = \frac{0.001}{\pi(0.001)^2} = 318\,\mathrm{A/m}^2$$

(c.) $$E = \frac{J}{\sigma} = \frac{318}{5.8 \times 10^7} = 5.5\,\frac{\mu V}{\mathrm{m}}$$

(d.) $$u = \frac{J}{\rho_V} = \frac{318}{1.4 \times 10^{10}} = 22.7\,\mathrm{nm/s}$$

Current Continuity Equation

Current continuity equation defines the basic *conservation of charge* relationship between current and charge. That is, a net current in or out of a given volume must equal the net increase or decrease in the total charge in the volume. If we define a surface S enclosing a volume V, the net current out of the volume (I_{out}) is defined by



$$\boxed{\oint\!\!\oint_S \boldsymbol{J} \cdot d\boldsymbol{s} = I_{out} = -\frac{dQ}{dt}}$$

where $d\boldsymbol{s} = ds\, \boldsymbol{a}_n$ and $\boldsymbol{a}_n$ is the outward normal unit vector.

Current Continuity Equation (differential form)

The previous equation is the integral form of the continuity equation. The differential form of the continuity equation can be found by applying the **divergence theorem** to the surface integral and expressing the total charge in terms of the charge density.

$$\oint\!\!\oint_S \boldsymbol{J}\cdot d\boldsymbol{s} = \iiint_V (\nabla\cdot\boldsymbol{J})\,dv = -\frac{dQ}{dt}$$

$$= -\frac{d}{dt}\iiint_V \rho_v\,dv = -\iiint_V \frac{\partial \rho_v}{\partial t}\,dv$$

Thus,

$$\iiint_V (\nabla\cdot\boldsymbol{J})\,dv = -\iiint_V \frac{\partial \rho_v}{\partial t}\,dv \qquad (\text{valid for any } V)$$

Since the previous equation is valid for any volume V, we equate the integrands

$$\boxed{\nabla\cdot\boldsymbol{J} = -\frac{\partial \rho_v}{\partial t} \qquad (\text{continuity equation})}$$

For DC currents, the charge density does not change with time so that the divergence of the current density is always zero.

$$\boxed{\nabla\cdot \boldsymbol{J} = 0 \qquad \text{(steady current)}}$$

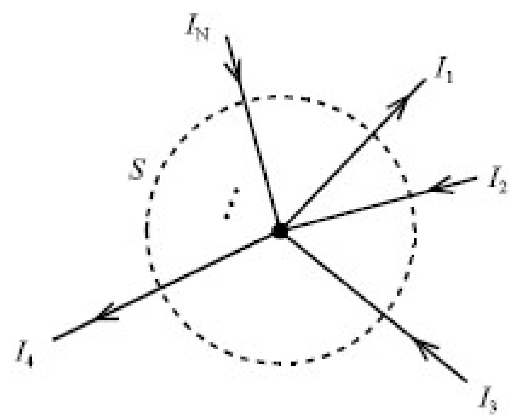
The continuity equation is the basis for *Kirchhoff's current law.*

Given a circuit node connecting a system of N wires (assuming DC currents) enclosed by a spherical surface S, the integral form of the continuity equation gives

$$\oint\!\!\oint_S \boldsymbol{J}\cdot d\boldsymbol{s} = I_{out} = -\frac{dQ}{dt} = 0$$

$$= I_1 - I_2 - I_3 + I_4 + \ldots - I_N$$

$$\text{or} \quad \sum_{n=1}^{N} I_n = 0 \qquad \textit{Kirchoff's current law}$$



The integral form of the continuity equation (and thus Kirchhoff's current law) also holds true for time-varying (AC) currents if we let the surface S shrink to zero around the node.

Announcements

- Homework 2 was graded.
- Homework 4 is assigned.
- Midterm exam will be administered via Zoom on 10/7/2020
  - Exam time is 9:00 to 11:30 AM
  - You are allowed to use the following:
    - Table of Integrals (Available on Webcourses); and
    - Two A4 sheets of handwritten EQUATIONS ONLY;
      - you can write your equations on both sides.
      - No graphs or plots are allowed on your equations sheets.
      - No rubric for solving a given class of problems should be included in your equations sheets.
      - No solution of a given problem should be included.
    - Your engineering calculator.
      - TI Nspire is not allowed