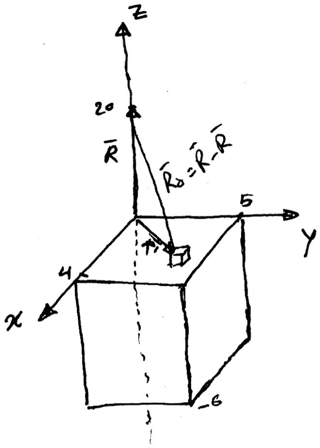
EEL 3470: ELECTROMAGNETIC FIELDS

Solution of Homework 4

Problem 2.26 (Matlab and analytical solutions)

Analytical Solution:



$\rho_v = 40\ nC/m^3$

$0 \le x \le 4$

$0 \le y \le 5$

$-6 \le z \le 0$

$$\bar{R} = 0\,\hat{a}_x + 0\,\hat{a}_y + 20\,\hat{a}_z$$

$$\bar{R}' = x'\,\hat{a}_x + y'\,\hat{a}_y + z'\,\hat{a}_z$$

$$\bar{R}_o = \bar{R} - \bar{R}'$$

$$= -x'\,\hat{a}_x - y'\,\hat{a}_y + (20 - z')\,\hat{a}_z$$

$$|\bar{R}_o| = \sqrt{x'^2 + y'^2 + (20 - z')^2}$$

$$dv' = dx'\,dy'\,dz'$$

$$\hat{a}_{R_o} = \frac{\bar{R}_o}{|\bar{R}_o|}$$

$$\therefore \bar{E} = \frac{\rho_v}{4\pi\varepsilon_o} \iiint_v \frac{\bar{R}_o}{R_o^3}\, dv'$$

$$\therefore \bar{E} = \frac{\rho_v}{4\pi\varepsilon_o}\Bigg[\int_{x'=0}^{4}\int_{y'=0}^{5}\int_{z'=-6}^{0} \frac{-x'}{\left[x'^2 + y'^2 + (20 - z')^2\right]^{3/2}}\,\hat{a}_x\, dx'\,dy'\,dz'$$

$$+ \int_{x'=0}^{4}\int_{y'=0}^{5}\int_{z'=-6}^{0} \frac{-y'}{\left[x'^2 + y'^2 + (20 - z')^2\right]^{1.5}}\,\hat{a}_y\, dx'\,dy'\,dz'$$

$$+ \int_{x'=0}^{4}\int_{y'=0}^{5}\int_{z'=-6}^{0} \frac{(20 - z')}{\left[x'^2 + y'^2 + (20 - z')^2\right]^{1.5}}\,\hat{a}_z\, dx'\,dy'\,dz'\Bigg]$$

$$\boxed{\bar{E} = -6.99\,\hat{a}_x - 8.71\,\hat{a}_y + 79.77\,\hat{a}_z}$$

Another analytical solution:

Another Solution
(Lumped Q at Center)

* ρ_v is constant.
* The test point at $(0, 0, 20)$
* Center of Box at $\overset{\bar{R}'}{(2, 2.5, -3)}$
* total charge at Center of the Box : $\underline{Q = \rho_v V}$

$$Q = 120\,\rho_v = 4.8E^{-6}\,C$$

$\Rightarrow$ Now, the problem becomes, find electric field $\bar{E}(0, 0, 20_m)$ due to $4.8E^{-6}\,C$ at $(2, 2.5, -3)$

$$\bar{E} = \frac{Q}{4\pi\epsilon_0} \frac{(\bar{R} - \bar{R}')}{|\bar{R} - \bar{R}'|^3} \quad N/C$$

$$\bar{R} - \bar{R}' = (-2, -2.5, 23) = \boxed{-2\hat{a}_x - 2.5\hat{a}_y + 23\hat{a}_z}$$

$$|\bar{R} - \bar{R}'| = \sqrt{4 + 6.25 + 529} = \sqrt{539.25} = 23.221$$

$$\bar{E} = \frac{4.8E^{-6}}{4\pi(8.854E^{-12})} \left[\frac{-2\hat{a}_x - 2.5\hat{a}_y + 23\hat{a}_z}{(23.221)^3} \right]$$

$$= 3.445\left[-2\hat{a}_x - 2.5\hat{a}_y + 23\hat{a}_z\right]$$

$$\boxed{\bar{E} = -6.89\,\hat{a}_x - 8.613\,\hat{a}_y + 79.245\,\hat{a}_z} \quad N/C$$

Matlab Solution

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%   Lumped Q at the center of the box
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

X_1 = 0;            % declare and initialize limits for X, Y, and Z
X_2 = 4;
Y_1 = 0;
Y_2 = 5;
Z_1 = -6;
Z_2 = 0;

rho_v = 40e-9;          % declare and initialize volume charge density and Epsilon
eps_0 = 8.854e-12;

% Calculate total volume and total charge
X = norm (X_2 - X_1);
Y = norm (Y_2 - Y_1);
Z = norm (Z_2 - Z_1);

V_total = X*Y*Z;
Q_total = rho_v * V_total;

% calculate electric field intensity
Rprime = [2, 2.5, -3];
R = [0,0,20];

Vect_R_Rprime = R - Rprime;

MagVect = norm (Vect_R_Rprime);


E =(Q_total/(4*pi*eps_0))*(Vect_R_Rprime/MagVect^3);
fprintf('\n E(0,0,20) = %0.3f ax + %0.3f ay  + %0.3f az\n', E(1), E(2), E(3));
```

Command Window

New to MATLAB? Watch this Video, see Demos, or read Getting Started.

```
 E(0,0,20) = -6.890 ax + -8.613 ay  + 79.238 az
>>
```

Another Matlab Solution

```
clc
clear

%   Limits of variables x, y, z that define the total volume.
x_1 = 0;     % initial value of x
x_2 = 4;     % final value of x
y_1 = 0;
y_2 = 5;
z_1= -6;
z_2= 0;
% coordinates of test point
x_t = 0;
y_t = 0;
z_t =  20;

rho_v = 40e-9;     % Volume charge density
N = 10;            % number of intervals for all three variables x, y, z
eo = 8.854e-12;    % permittivity of free space

dx =(x_2 - x_1)/N;
dy =(y_2 - y_1)/N;
dz =(z_2 -z_1)/N;

dQ = rho_v*dx*dy*dz;    % charge within differential volume

% We have 1000 differential elements to find E associated with them and
% accumulate the total E. Need three nested loops


for k = 1:N
    for j = 1:N
        for i = 1:N
            x_cen_dv = x_1+(i-0.5)*dx;            % x coordinate of the center of diff volume
            y_cen_dv = y_1+(j-0.5)*dy;            % y coordinate of the center of diff volume
            z_cen_dv =z_1+(k-0.5)*dz;             % z coordinate of the center of diff volume

            R =[x_t-x_cen_dv, y_t-y_cen_dv, z_t-z_cen_dv];  % displacement vector

            magR = norm(R);
            a_R = R/magR;

            dEi=(dQ/(4*pi*eo*magR^2))*a_R;
            dEx_i(i)=dEi(1);
            dEy_i(i)=dEi(2);
            dEz_i(i)=dEi(3);
        end
        dEx_j(j)=sum(dEx_i);
        dEy_j(j)=sum(dEy_i);
        dEz_j(j)=sum(dEz_i);
    end
    dEx_k(k)= sum(dEx_j);
    dEy_k(k)= sum(dEy_j);
    dEz_k(k)= sum(dEz_j);
end
Etotx = sum(dEx_k);
Etoty = sum(dEy_k);
Etotz = sum(dEz_k);

fprintf('\n E(0,0,20)  = %0.3f ax + %0.3f ay  + %0.3f az\n', Etotx, Etoty, Etotz);
```

Command Window

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```
 E(0,0,20)  = -6.998 ax + -8.710 ay  + 79.767 az
>>
```

P2.38: Determine the charge density at the point P(3.0m,4.0m,0.0) if the electric flux density is given as $\mathbf{D} = xyz\ \mathbf{a}_z$ C/m^2.

Solution

$$\nabla \bullet \mathbf{D} = \frac{\partial D_z}{\partial z} = \frac{\partial (xyz)}{\partial z} = xy = \rho_v.$$

$\rho_v(3,4,0)=(3)(4)=12$ C/m^3.

P2.40: Suppose $\mathbf{D} = 6\rho\cos\phi\ \mathbf{a}_\phi$ C/m^2. (a) Determine the charge density at the point (3m, 90°, -2m). Find the total flux through the surface of a quartered-cylinder defined by $0 \le \rho \le$ 4m, $0 \le \phi \le 90°$, and -4m $\le z \le 0$ by evaluating (b) the left side of the divergence theorem and (c) the right side of the divergence theorem.

(a) $$\left(\nabla \bullet \mathbf{D}\right)_{cylinder} = \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} = \frac{1}{\rho}\frac{\partial (6\rho\cos\phi)}{\partial \phi} == -6\sin\phi.$$

$$\rho_v\left(3, 90^\circ, -2\right) = -6\frac{C}{m^3}.$$

(b) $$\oint \mathbf{D} \bullet d\mathbf{S} = \int_{\phi=0^\circ} + \int_{\phi=90^\circ} + \int_{top} + \int_{bottom} + \int_{outside},$$

note that the top, bottom and outside integrals yield zero since there is no component of D in the these d**S** directions.

$$\int_{\phi=0^\circ} = \int 6\rho\cos\phi\big|_{\phi=0^\circ} \mathbf{a}_\phi \bullet \left(-d\rho dz \mathbf{a}_\phi\right) = -192C$$

$$\int_{\phi=90^\circ} = \int 6\rho\cos\phi\big|_{\phi=90^\circ} \mathbf{a}_\phi \bullet \left(d\rho dz \mathbf{a}_\phi\right) = 0$$

So, $\oint \mathbf{D} \bullet d\mathbf{S} = -192C.$

(c)

$\nabla \bullet \mathbf{D} = -6\sin\phi, \quad dv = \rho d\rho d\phi dz$

$$\int \nabla \bullet \mathbf{D} dv = -6\int_0^{90^\circ} \sin\phi d\phi \int_0^4 \rho d\rho \int_{-4}^0 dz = -192C.$$

P2.41: Suppose $\mathbf{D} = r^2\sin\theta\,\mathbf{a}_r + \sin\theta\cos\phi\,\mathbf{a}_\phi$ C/m^2. (a) Determine the charge density at the point (1.0m, 45°, 90°). Find the total flux through the surface of a volume defined by $0.0 \le r \le 2.0$ m, $0.0° \le \theta \le 90.°$, and $0.0 \le \phi \le 180°$ by evaluating (b) the left side of the divergence theorem and (c) the right side of the divergence theorem.

Solution

The volume is that of a quartered-sphere, as indicated in Figure P2.41.

(a)

$$\nabla\bullet\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi} = 4r\sin\theta - \frac{\sin\phi}{r} = \rho_v,$$

$$\rho_v\left(1, 45^\circ, 90^\circ\right) = 1.83\frac{C}{m^3}$$

(b) $\oint \mathbf{D}\bullet d\mathbf{S} = \int_{\phi=0^\circ} + \int_{\phi=180^\circ} + \int_{\theta=90^\circ} + \int_{r=2}$; note that $\int_{\theta=90^\circ} = 0$ since $D_\theta = 0$.

$$\int_{\phi=0^\circ} = \int \sin\theta\cos\phi\big|_{\phi=0^\circ}\,\mathbf{a}_\phi\bullet\left(-r dr d\theta\,\mathbf{a}_\phi\right) = -2C$$

$$\int_{\phi=180^\circ} = \int \sin\theta\cos\phi\big|_{\phi=180^\circ}\,\mathbf{a}_\phi\bullet r dr d\theta\,\mathbf{a}_\phi = -2C$$

$$\int_{r=2} = \int r^2\sin\theta\,\mathbf{a}_r\bullet r^2\sin\theta d\theta d\phi\,\mathbf{a}_r = r^4\int_0^{\pi/2}\sin^2\theta d\theta\int_0^{\pi} d\phi = 8\pi\int_0^{90^\circ}\left(1-\cos 2\theta\right)d\theta = 4\pi^2 C$$

Summing these terms we have $Q = 4(\pi^2 - 1)$C = 35.5C.

(c)

$$\int\nabla\bullet\mathbf{D}dv = \int\left(4r\sin\theta - \frac{\sin\phi}{r}\right)r^2\sin\theta dr d\theta d\phi$$

$$= 4\int_0^2 r^3 dr\int_0^{\pi/2}\sin^2\theta d\theta\int_0^{\pi} d\phi - \int_0^2 r dr\int_0^{\pi/2}\sin\theta d\theta\int_0^{\pi}\sin\phi d\phi = 4\pi^2 - 4 = 35.5C.$$

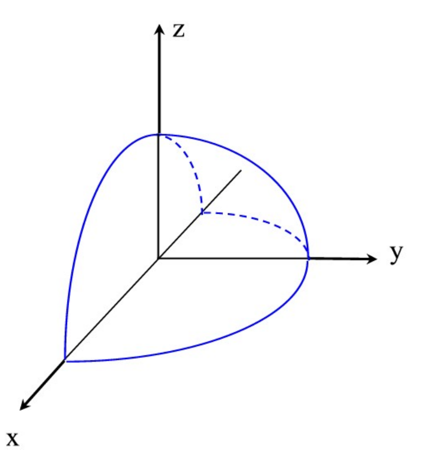

Fig. P2.41

P2.51: Two spherical conductive shells of radius a and b ($b > a$) are separated by a material with conductivity σ. Find an expression for the resistance between the two spheres.

Solution

First we find **E** for $a < r < b$; using Gauss's law, the charge enclosed by a spherical Gaussian surface of radius r, where $a < r < b$ is

$$Q = \oint \mathbf{D} \bullet d\mathbf{S} = \int D_r \mathbf{a}_r \bullet r^2 \sin\theta d\theta d\phi \mathbf{a}_r = 4\pi r^2 D_r;$$

Therefore, the flux density in the region between the two spherical shells of radius a and b is given by

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r; \text{Thus}$$

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a}_r$$

Now we find V_{ab}:

$$V_{ab} = -\int_b^a \mathbf{E} \bullet d\mathbf{L} = -\int_b^a \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a}_r \bullet dr\mathbf{a}_r$$

$$= \frac{-Q}{4\pi\varepsilon_o}\int_b^a \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_o}\frac{1}{r}\bigg|_b^a = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right).$$

Now we find I:

$$I = \int \mathbf{J} \bullet d\mathbf{S} = \sigma \int \mathbf{E} \bullet d\mathbf{S} = \sigma \int \frac{Q}{4\pi\varepsilon_o}\frac{1}{r^2}\mathbf{a}_r \bullet r^2 \sin\theta d\theta d\phi \mathbf{a}_r$$

$$= \frac{\sigma Q}{4\pi\varepsilon_o}\int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{\sigma Q}{\varepsilon_o}.$$

Finally,

$$R = \frac{V_{ab}}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)$$