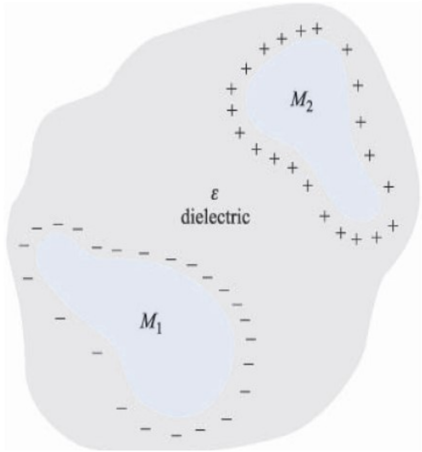
Lecture 7

1. The Electric Capacitor
2. Static Magnetic Fields

Dr. Shady Elashhab

Capacitance

- Consider two conductors embedded in a dielectric.
- Conductor M_2 carries a total positive charge Q, and M_1 carries an equal magnitude of negative charge.
- There are no other charges present, and the total charge of the system is zero.
- The charge is carried on the conductors surfaces as surface charge densities.
- The electric field is normal to the conductor surface
- The electric field is directed from M_2 (+Q) to M_1 (-Q), and M_2 is thus at higher potential.



Capacitance

- Work must be done to move a positive charge from M_1 to M_2 against the direction of the electric field.
- Let us denote the potential difference between M_2 and M_1 as $V_0 = V_2 - V_1$

The capacitance of this two-conductor system is defined as the ratio of the magnitude of the total charge on either conductor to the magnitude of the potential difference between conductors,

$$\boxed{C = \frac{Q}{V_0}} \Longleftrightarrow \boxed{C = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{S}}{-\int_-^+ \mathbf{E} \cdot d\mathbf{L}}}$$

Capacitance

$$C = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{S}}{-\int_-^+ \mathbf{E} \cdot d\mathbf{L}}$$

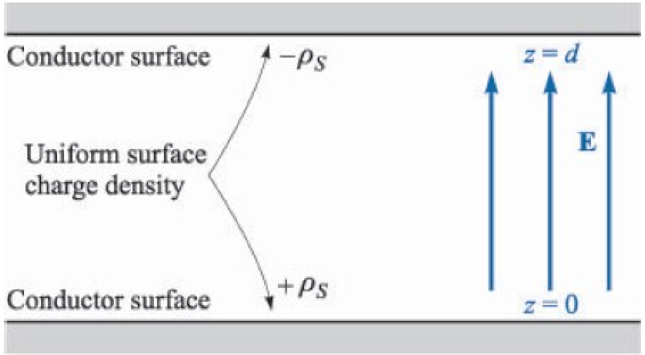
- In general terms,
  - we determine Q by a surface integral of the surface charge density over the positive conductor's area, and
  - we find V_0 by moving a unit positive charge from the negative to the positive surface i.e. against the electric field.

- The capacitance is independent of the potential and total charge, for their ratio is constant.
  - If the charge density is increased by a factor of N, the electric flux density, ***D*** and electric field intensity, ***E*** would be increased by a factor of N. It follows that the potential difference also would increase by the same factor.
  - Thus, the capacitance is a function only of the physical dimensions of the system of conductors and of the permittivity of the dielectric.

The Parallel Plate Capacitor

$$C = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{S}}{-\int_-^+ \mathbf{E} \cdot d\mathbf{L}}$$

- The conductors are two identical parallel plates with surface area S separated by a distance d (S >> d).
- The plates are separated by a dielectric material and a uniform surface charge density (its magnitude is ρ_S) that exists on each conductor
- Choosing the lower conducting plate at z = 0, (ρ_S) and the upper one at z = d, ($-\rho_S$),



- We found that the electric field due to a uniformly charged infinite sheet, in free space, can be obtained from

$$\mathbf{E} = \frac{\rho_S}{2\epsilon_0} \mathbf{a}_N$$

Thus, within the capacitor, $\mathbf{E}_+ = \frac{\rho_S}{2\epsilon}\mathbf{a}_z \quad \mathbf{E}_- = \frac{\rho_S}{2\epsilon}\mathbf{a}_z$

Therefore, the total E within the capacitor

$$\boxed{\mathbf{E} = \frac{\rho_S}{\epsilon}\mathbf{a}_z}$$

The potential difference

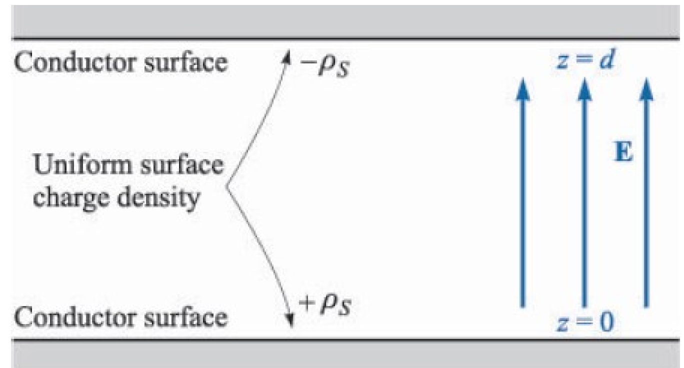
$$V_0 = -\int_{\text{upper}}^{\text{lower}} \mathbf{E} \cdot d\mathbf{L} = -\int_d^0 \frac{\rho_S}{\epsilon} dz = \boxed{\frac{\rho_S}{\epsilon} d}$$

The magnitude of the total charge on each plate is $\boxed{Q = \rho_S S}$

The capacitance of a parallel plate capacitor is $\boxed{C = \frac{Q}{V_0} = \frac{\epsilon S}{d}}$

The Parallel Plate Capacitor

- The capacitance is independent of the potential and total charge, for their ratio is constant.



- The capacitance is a function only of the physical dimensions of the system of conductors and of the permittivity of the dielectric.

$$\boxed{C = \frac{Q}{V_0} = \frac{\epsilon S}{d}}$$

Example 1:

Calculate the capacitance of a parallel plate capacitor having a mica dielectric, $\epsilon_R = 6$, a plate area of $10\,\text{in}^2$, and a separation of 0.01 in.

Solution. We may find that

$$S = 10 \times 0.0254^2 = 6.45 \times 10^{-3} \quad \text{m}^2$$
$$d = 0.01 \times 0.0254 = 2.54 \times 10^{-4} \quad \text{m}$$

$$C = \frac{\epsilon S}{d}$$

and therefore

$$C = \frac{6 \times 8.854 \times 10^{-12} \times 6.45 \times 10^{-3}}{2.54 \times 10^{-4}} = 1.349 \quad \text{nF}$$

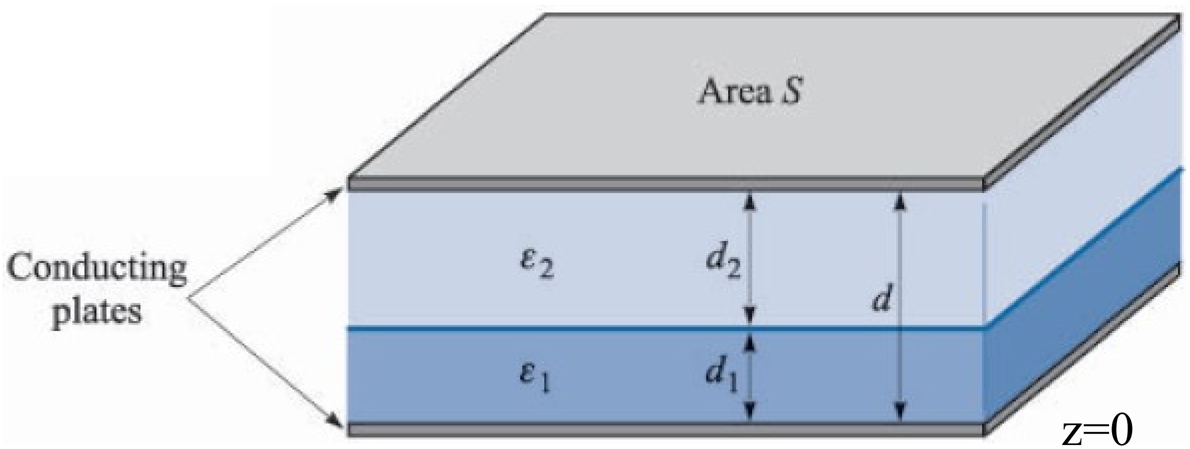
The Total Energy Stored in a Parallel Plate Capacitor

$$\boxed{C = \frac{Q}{V_0} = \frac{\epsilon S}{d}}$$

$$W_E = \tfrac{1}{2} C\, V_0^2$$

- The equation for the total energy stored in a parallel plate capacitor indicates that:
  - the energy stored in a capacitor with a fixed potential difference across it increases as the dielectric constant of the medium increases.

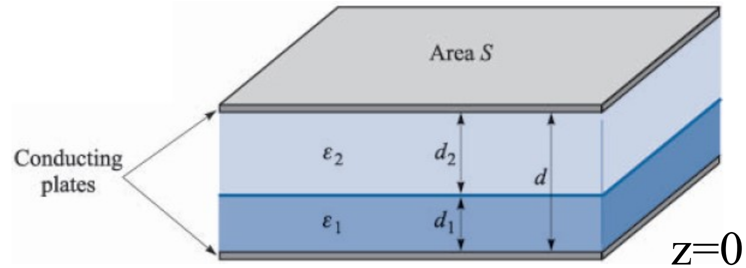
Example 2: *Parallel Plate Capacitor containing two dielectrics*



- Consider a parallel plate capacitor of area S and spacing d, with the usual assumption that d is small compared to the linear dimensions of the plates.
- The space between the two conducting plates is filled with two different dielectrics of permittivity ε_1 and ε_2 , and thicknesses d_1 and d_2 , respectively..

- Assume a uniform surface charge ρ_s C/m^2 on lower plate and $-\rho_s$ C/m^2 on the upper plate, which leads to the uniform field,

$$\overline{E} = \begin{cases} \dfrac{\rho_s}{\varepsilon_1}\hat{a}_z & 0 < z < d_1 \\ \dfrac{\rho_s}{\varepsilon_2}\hat{a}_z & d_1 < z < d \end{cases}$$



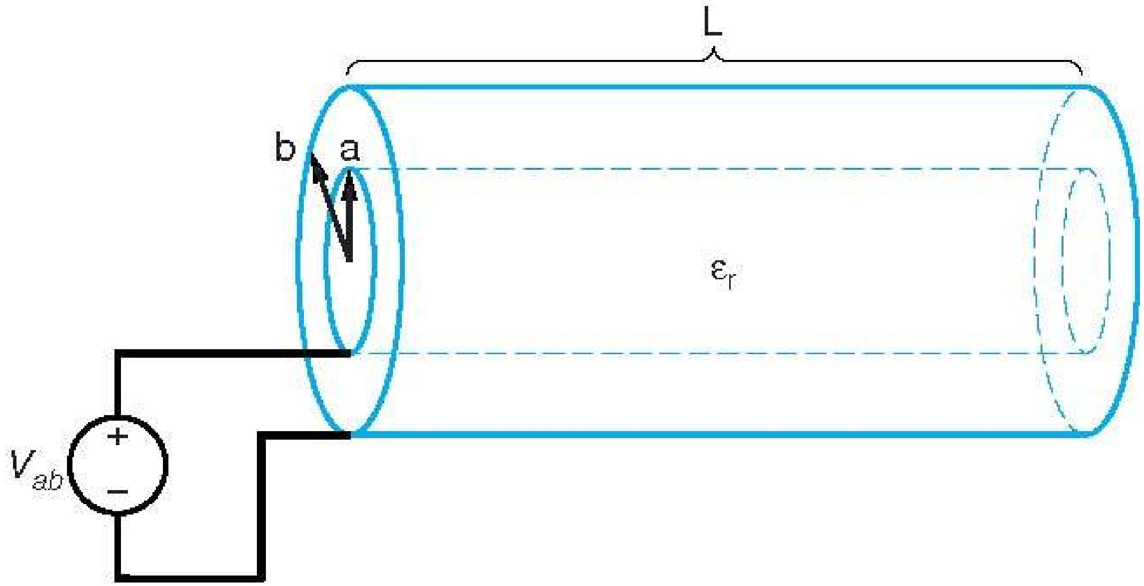
The potential difference is:

$$V_o = -\int_{upper}^{lower} \overline{E}\cdot\overline{dl} = -\int_{d}^{d_1}\frac{\rho_s}{\varepsilon_2}dz - \int_{d_1}^{0}\frac{\rho_s}{\varepsilon_1}dz = \frac{\rho_s}{\varepsilon_2}d_2 + \frac{\rho_s}{\varepsilon_1}d_1$$

The surface area of each plate is "S" whose linear dimensions are much greater than the separation "d". since $Q = \rho_s S$, then

$$\boxed{C = \frac{Q}{V_o} = \frac{S}{\frac{d_2}{\varepsilon_2} + \frac{d_1}{\varepsilon_1}}} \longrightarrow C = \frac{1}{\dfrac{d_1}{\varepsilon_1 S} + \dfrac{d_2}{\varepsilon_2 S}} \qquad \begin{cases} C_1 = \dfrac{\varepsilon_1 S}{d_1} \\ C_2 = \dfrac{\varepsilon_2 S}{d_2} \end{cases}$$

<u>Example 3: *The Coaxial* capacitor</u>



- Consider a coaxial cable or coaxial capacitor of inner radius a, outer radius b, and length L.
- The inner surface carries a uniform charge per unit length $+\rho_l$ C/m and the outer one carries $-\rho_l$ C/m.
  - Find the Capacitance and the total energy stored in the capacitor

The electric field in the medium between the two cylindrical surfaces is given by:

$$\overline{E} = \frac{\rho_l}{2\pi\varepsilon r}\hat{a}_r$$

The potential difference

$$V_o = -\int_b^a \frac{\rho_l}{2\pi\varepsilon r}dr = \frac{\rho_l}{2\pi\varepsilon}\ln(b/a)$$

Thus,

$$C = \frac{Q}{V_o} = \frac{\rho_l L}{\frac{\rho_l}{2\pi\varepsilon}\ln(b/a)}$$

The capacitance of a Coaxial capacitor is

$$\boxed{C = \frac{2\pi\varepsilon L}{\ln(b/a)}}$$

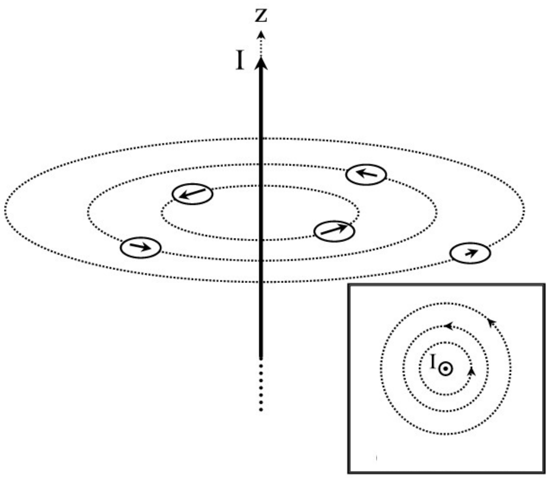
and

$$W_E = \tfrac{1}{2}C\,V_0^2$$

Magnetostatics – Oersted's Experiment

Magnetism and electricity were considered distinct phenomena until 1820 when ***Hans Christian Oersted*** (1777-1851) conducted an experiment that showed a compass needle deflecting when in proximity to a current carrying wire.

Oersted's experiment



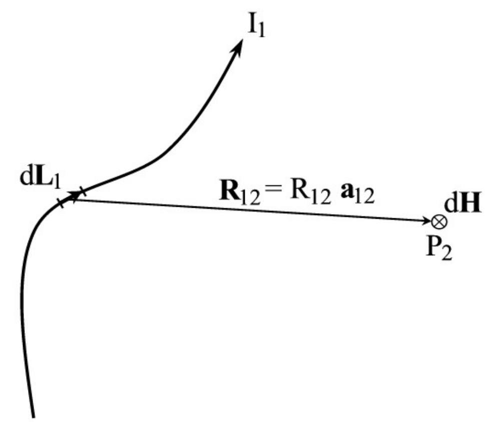
It was observed that moving away from the source of current, the field grows weaker.

Magnetostatics – Biot-Savart's Law

Jean Baptiste Biot (1774-1862) and Felix Savart (1791-1841) arrived at a mathematical relation between the magnetic field intensity and current.

The *Law of Biot-Savart* is

$$\boxed{d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{12}}{4\pi R_{12}^2}} \qquad \text{(A/m)}$$



To get the total magnetic field resulting from a current, one can sum the contributions from each segment by integrating

$$\boxed{\mathbf{H} = \int \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}} \qquad \text{(A/m)}$$

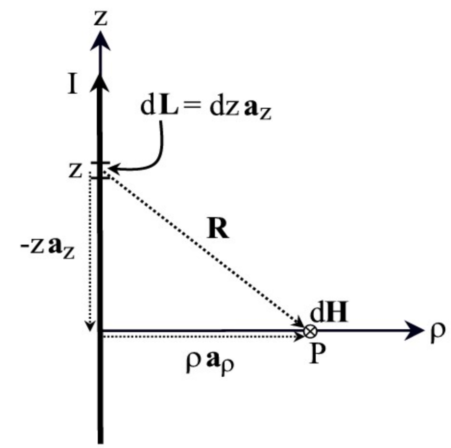
$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{12}}{4\pi\varepsilon R_{12}^2}$$

Note: Biot-Savart's law is analogous to Coulomb's law for the electric field resulting from a differential charge element.

Magnetostatics – An Infinite Line current

Example 4: Consider an infinite length line along the z-axis conducting constant current I in the $+\mathbf{a}_z$ direction. We want to find the magnetic field at a point P, at distance ρ from the line along the ρ axis.

So we consider a differential length, dl, at distance z from the origin.



The Biot-Savart Law

$$\mathbf{H} = \int \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

$I d\mathbf{L}$ is simply $I dz\mathbf{a}_z$, and the vector from the source to the test point is

$$R\mathbf{a}_\mathbf{R} = -z\mathbf{a}_\mathbf{z} + \rho\mathbf{a}_\rho$$

The Biot-Savart Law becomes

$$\mathbf{H} = \int_{-\infty}^{\infty} \frac{I dz\mathbf{a}_\mathbf{z} \times \left(-z\mathbf{a}_\mathbf{z} + \rho\mathbf{a}_\rho\right)}{4\pi\left(z^2 + \rho^2\right)^{3/2}}.$$

Pulling the constants to the left of the integral and realizing that $\mathbf{a}_z \times \mathbf{a}_z = 0$ and $\mathbf{a}_z \times \mathbf{a}_\rho = \mathbf{a}_\phi$, we have

$$\mathbf{H} = \frac{I\rho}{4\pi}\int_{-\infty}^{\infty}\frac{dz}{\left(z^2+\rho^2\right)^{3/2}}\ \mathbf{a}_\phi$$

The integral can be evaluated using the following formula:

$$\boxed{\int\frac{dx}{\left(x^2+a^2\right)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}}$$

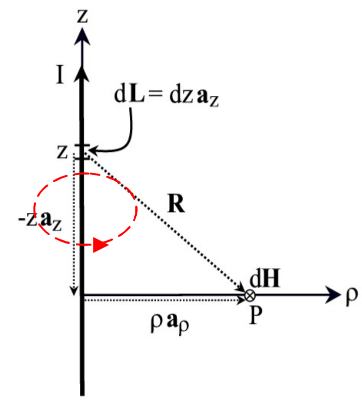
$$\int_{-\tau}^{\tau}\frac{dz}{\left(\rho^2+z^2\right)^{3/2}} = \left.\frac{z}{\rho^2\sqrt{\rho^2+z^2}}\right|_{-\tau}^{\tau} = \left(\frac{\tau}{\rho^2\sqrt{\rho^2+\tau^2}}\right) - \left(\frac{-\tau}{\rho^2\sqrt{\rho^2+\left(-\tau\right)^2}}\right)$$

$$\left(\frac{\tau}{\rho^2\sqrt{\rho^2+\tau^2}}\right) + \left(\frac{\tau}{\rho^2\sqrt{\rho^2+\tau^2}}\right) = \frac{2\tau}{\rho^2\sqrt{\rho^2+\tau^2}}$$

When the limit $\tau \to \infty$

$$\int_{-\infty}^{\infty}\frac{dz}{\left(z^2+\rho^2\right)^{3/2}} = \frac{2}{\rho^2\sqrt{\left(\frac{\rho}{\tau}\right)^2+1}} = \frac{2}{\rho^2\sqrt{\left(\frac{\rho}{\infty}\right)^2+1}} = \frac{2}{\rho^2}$$

$$\mathbf{H} = \frac{I\rho \mathbf{a}_\phi}{4\pi}\int_{-\infty}^{\infty}\frac{dz}{\left(z^2+\rho^2\right)^{3/2}}$$



An infinite length line of current

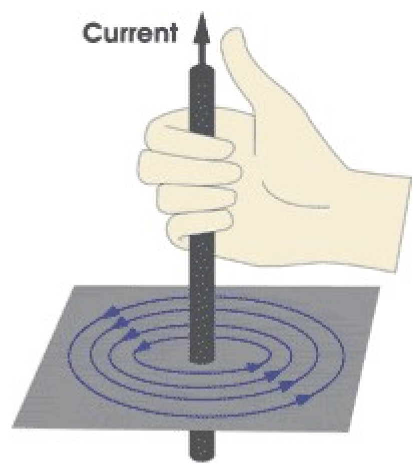
Substituting for $\displaystyle\int_{-\infty}^{\infty}\frac{dz}{\left(z^2+\rho^2\right)^{3/2}} = \frac{2}{\rho^2}$

We find the magnetic field intensity resulting from an infinite length line of current is

$$\boxed{\mathbf{H} = \frac{I\mathbf{a}_\phi}{2\pi\rho}} \Rightarrow \mathbf{H} \propto \frac{1}{\rho}\,\mathbf{a}_\phi$$

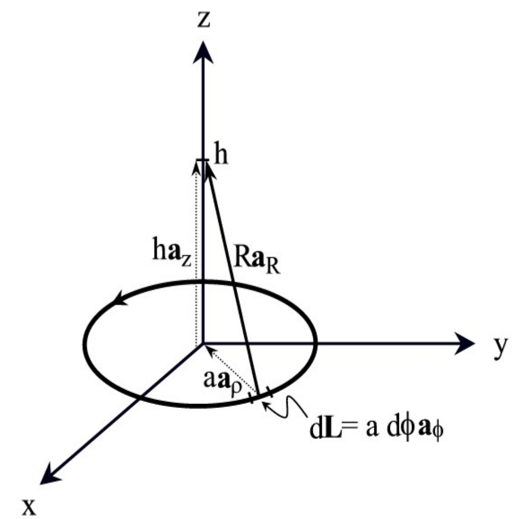
Direction: The direction of the magnetic field can be found using the right hand rule.

Magnitude: The magnitude of the magnetic field is inversely proportional to radial distance.



Magnetostatics – A Ring of Current

Example 5: Let us now consider a ring of current with radius "a" lying in the x-y plane with a current I in the $+\mathbf{a}_z$ direction.
The objective is to find an expression for the field at an arbitrary point P, at a height h on the z-axis.



The *Biot-Savart Law* $$\mathbf{H} = \int \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

The differential segment $dL = ad\phi a_\phi$

The vector drawn from the source to the test point is

$$\mathbf{R} = \mathrm{R}\mathbf{a}_\mathbf{R} = h\mathbf{a}_z - a\mathbf{a}_\rho$$

Magnitude: $\mathrm{R} = \sqrt{h^2 + a^2}$, Unit Vector: $\mathbf{a}_\mathbf{R} = \left(h\mathbf{a}_z - a\mathbf{a}_\rho\right)/\mathrm{R}$

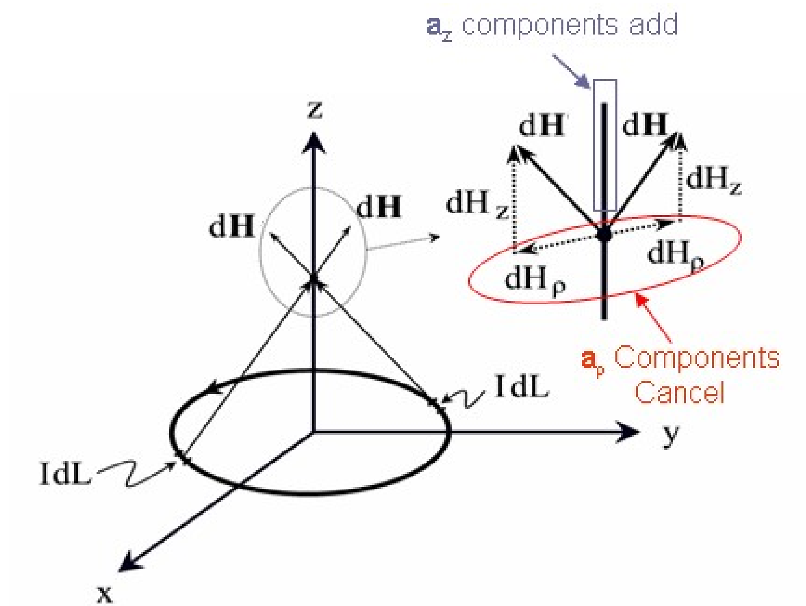
The Biot-Savart Law can be written as

$$\mathbf{H} = \int_{\phi=0}^{2\pi} \frac{Iad\phi\mathbf{a}_\phi \times \left(h\mathbf{a}_z - a\mathbf{a}_\rho\right)}{4\pi\left(h^2 + a^2\right)^{3/2}} = \frac{Ia}{4\pi}\int_{\phi=0}^{2\pi} \frac{d\phi\mathbf{a}_\phi \times \left(h\mathbf{a}_z - a\mathbf{a}_\rho\right)}{\left(h^2 + a^2\right)^{3/2}}$$

$$\mathbf{H} = \frac{Ia}{4\pi}\int_{\phi=0}^{2\pi}\frac{d\phi\mathbf{a}_\phi \times \left(h\mathbf{a}_z - a\mathbf{a}_\rho\right)}{\left(h^2+a^2\right)^{3/2}} = \frac{Ia}{4\pi}\int_{\phi=0}^{2\pi}\frac{d\phi\left(h\mathbf{a}_\rho + a\mathbf{a}_z\right)}{\left(h^2+a^2\right)^{3/2}}$$

A particular differential current element will give a field with an $\mathbf{a}_\rho$ component (from $\mathbf{a}_\phi \times \mathbf{a}_z$) and an $\mathbf{a}_z$ component (from $\mathbf{a}_\phi \times -\mathbf{a}_\rho$).

We can further simplify the previous expression for **H** by considering the symmetry of the problem

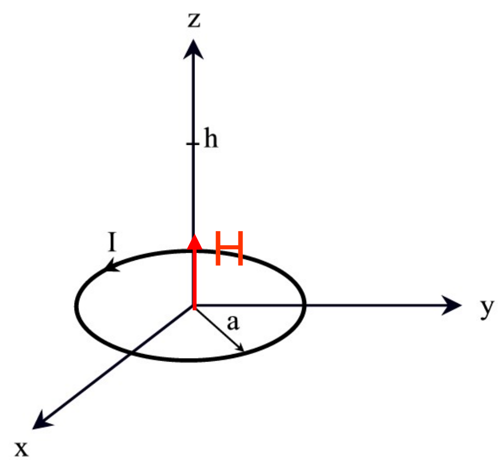


- Considering the field from a differential current element on the opposite side of the ring, it is apparent that the radial components cancel while the a_z components add.

$$\mathbf{H} = \frac{Ia^2 \mathbf{a}_z}{4\pi\left(h^2 + a^2\right)^{3/2}} \int_0^{2\pi} d\phi \quad \Rightarrow \quad \boxed{\mathbf{H} = \frac{Ia^2}{2\left(h^2 + a^2\right)^{3/2}} \mathbf{a}_z}$$

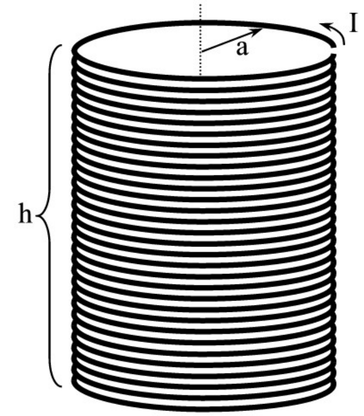
At h = 0, the center of the loop, this equation reduces to

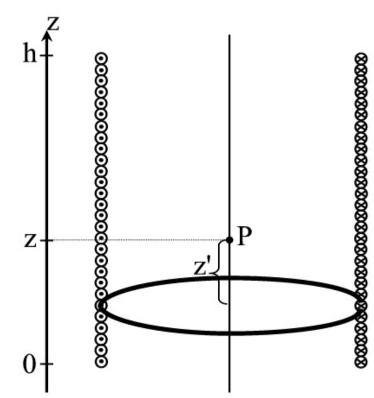
$$\boxed{\mathbf{H} = \frac{I}{2a} \mathbf{a}_z}$$



Magnetic Field Intensity for a Solenoid

A solenoid





$$\boxed{\mathbf{H} = \frac{Ia^2}{2\left(h^2 + a^2\right)^{3/2}}\mathbf{a}_z}$$

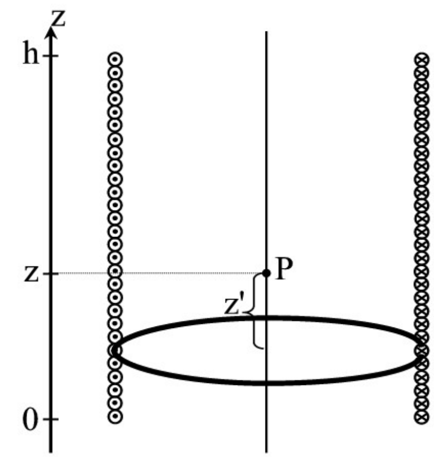
- *Solenoids* (inductors) are many turns of insulated wire coiled in the shape of a cylinder.
- Suppose the solenoid has a length *h*, a radius *a*, and is made up of N turns of current carrying wire.
- For tight wrapping, we can consider the solenoid to be made up of N loops of current carrying wire.
- To find the magnetic field intensity from a single loop at a point P along the axis of the solenoid, we have
  - The differential amount of field resulting from a differential amount of current in one loop is given by

$$d\mathbf{H}_P = \frac{dIa^2}{2\left(z'^2 + a^2\right)^{3/2}}\mathbf{a}_z$$

- The differential current element in one loop can be written as a function of the number of loops and the length of the solenoid as follows

$$dI = \frac{N}{h} I dz'$$

Fixing the point P where the field is desired, z' will range from –z to h-z, or



$$\mathbf{H} = \int_{-z}^{h-z} \frac{NIa^2 dz'}{2h\left(z'^2 + a^2\right)^{3/2}} \mathbf{a}_z = \frac{NIa^2}{2h} \int_{-z}^{h-z} \frac{dz'}{\left(z'^2 + a^2\right)^{3/2}} \mathbf{a}_z .$$

$$\int \frac{dx}{\left(x^2 + a^2\right)^{3/2}} = \frac{x}{a^2 \sqrt{x^2 + a^2}}$$

$$\boxed{\mathbf{H} = \frac{NI}{2h} \left[\frac{h - z}{\sqrt{\left(h - z\right)^2 + a^2}} + \frac{z}{\sqrt{z^2 + a^2}} \right] \mathbf{a}_z}$$

At the center of the solenoid ($z = h/2$), with the assumption that the length is considerably larger than the loop radius ($h >> a$), the equation reduces to

$$\mathbf{H} = \frac{NI}{h} \mathbf{a}_z$$

Announcements

- Midterm has been graded (Avg. 69%), check Webcourses.
  - ..\..\New folder\Midterm_Solution__EEL3470_Fall20.pdf
- Homework 5 is assigned.