Solution of Midterm

Problem 1 (14 pts)

Given the following two vectors $\boldsymbol{A} = -2\mathbf{a}_x + 6\mathbf{a}_y + 5\mathbf{a}_z$ and $\boldsymbol{B} = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$, find

a) $\boldsymbol{A} \times \boldsymbol{B}$

b) The angle between the two vectors in degrees.

c) The scalar projection of $\boldsymbol{A}$ on $\boldsymbol{B}$.

a) $$\bar{A} \times \bar{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ -2 & 6 & 5 \\ 1 & 2 & 3 \end{vmatrix} = (18-10)\hat{a}_x - (-6-5)\hat{a}_y + (-4-6)\hat{a}_z$$

$$\boxed{\therefore \bar{A} \times \bar{B} = 8\hat{a}_x + 11\hat{a}_y - 10\hat{a}_z}$$

b) $$\cos(\theta_{AB}) = \frac{\bar{A} \cdot \bar{B}}{|\bar{A}|\,|\bar{B}|} = \frac{(-2+12+15)}{\sqrt{65}\,\sqrt{14}} = \frac{25}{30.17} = 0.828$$

$$\therefore \theta_{AB} = \cos^{-1}(0.828)$$

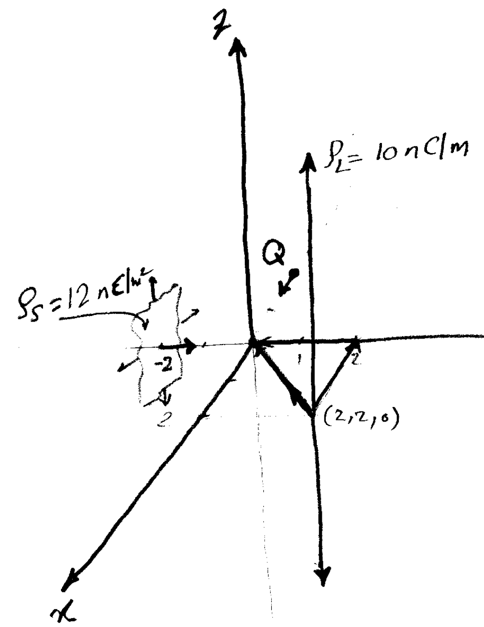
$$\boxed{\theta_{AB} = 34.03^\circ}$$

c) $$A_B = \bar{A} \cdot \hat{a}_B = \frac{\bar{A} \cdot \bar{B}}{|\bar{B}|}$$

$$\boxed{A_B = \frac{25}{\sqrt{14}} = 6.68}$$

Problem 2 (14 pts)

In free space, there is a point charge Q = 8 nC at (-2 m, 0, 0), an infinite line charge ρ_L = 10 nC/m that passes through the point (2 m, 2 m, 0) parallel to the z-axis, and a sheet charge ρ_s = 12 nC/m^2 at y = -2 m. Illustrate the problem graphically and determine ***E*** at the origin.



$$\bar{E}_{tot} = \bar{E}_Q + \bar{E}_L + \bar{E}_s$$

$$\bar{E}_Q = \frac{Q}{4\pi\epsilon_0 |\bar{R}|^2}\hat{a}_R \;, \quad \hat{a}_R = \hat{a}_x$$
$$|R| = 2$$

$$\bar{E}_s = \frac{\rho_s}{2\epsilon_0}\hat{a}_y$$

$$\bar{E}_L = \frac{\rho_L}{2\pi\epsilon_0 |\bar{R}_L|}\hat{a}_R \;, \quad \hat{a}_R = \frac{-2(\hat{a}_x + \hat{a}_y)}{\sqrt{(2)^2+(2)^2}} = -0.707(\hat{a}_x + \hat{a}_y)$$

$$|\bar{R}_L| = \sqrt{8} = 2.828$$

$\therefore$ at the origin

$$\bar{E}_{tot} = \frac{8E^{-9}}{(4\pi\epsilon_0)(2)^2}\hat{a}_x + \frac{12E^{-9}}{2\epsilon_0}\hat{a}_y - 0.707\frac{10E^{-9}}{(2\pi\epsilon_0)(2.828)}(\hat{a}_x + \hat{a}_y) =$$

$$\bar{E}_{tot} = 17.975\,\hat{a}_x + 677.659\,\hat{a}_y - 44.939\,\hat{a}_x - 44.939\,\hat{a}_y$$

$$\boxed{\bar{E}_{tot} = -26.963\,\hat{a}_x + 632.72\,\hat{a}_y} \quad \text{V/m}$$

Problem 3 (14 pts)

The electric flux density is given by $\boldsymbol{D} = 2\rho(z+1)\cos\phi\,\boldsymbol{a_\rho} - \rho(z+1)\sin\phi\,\boldsymbol{a_\phi} + \rho^2\cos\phi\,\boldsymbol{a_z}$ $\mu C/m^2$

a) Find the volume charge density.
b) Calculate the total charge enclosed by the volume $0 < \rho < 2$ m, $0 < \phi < \pi/2$, $0 < z < 4$m
c) Calculate the outward flux passing through the surface S_1 where $\phi = 0$.

a) $$\rho_v = \nabla\cdot\bar{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\bar{D}_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}$$

$$\therefore \rho_v = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(2\rho^2(z+1)\cos\phi\right) + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(-\rho(z+1)\sin\phi\right) + \frac{\partial}{\partial z}\left(\rho^2\cos\phi\right)$$

$$\rho_v = 4(z+1)\cos\phi - (z+1)\cos\phi + 0$$

$$\boxed{\rho_v = 3(z+1)\cos\phi}\ \mu C/m^2$$

b) Using Divergence Thrm to find Q_{enc}

$$Q_{enc} = \iiint_V \rho_v\,dv = \iiint_V 3(z+1)\cos\phi\,\rho\,d\rho\,d\phi\,dz$$

$$= 3\int_0^2 \rho\,d\rho \int_0^4 (z+1)\,dz \int_0^{\pi/2} \cos\phi\,d\phi$$

$$Q_{enc} = (3)(2)\left(\frac{z^2}{2}+z\right)\Big|_0^4 \sin\phi\Big|_0^{\pi/2} = \boxed{72\ \mu C}$$

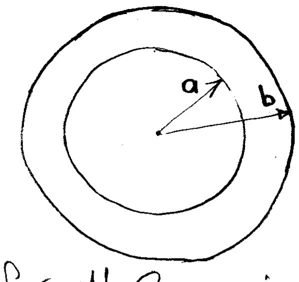
c) $S_1: \phi = 0 \Rightarrow d\bar{S} = d\rho\,dz\,(-\hat{a}_\phi)$

$$\Psi = \iint_S \bar{D}\cdot d\bar{S} = \int_0^2\int_0^4 \rho(z+1)\underbrace{\sin\phi}_{=0}\,d\rho\,dz \Rightarrow \boxed{\Psi = 0\ \mu C}$$

Problem 4 (14 pts)

A volume charge density, $\rho_v = 10\,r$ C/cm^3 is distributed inside a spherical shell of inner radius "*a*" and outer radius "*b*" where $a \le r \le b$. Use Gauss's law to determine the electric field intensity vector every where as a function of the radial distance *r*.

$$\oint_S \bar{D} \cdot \bar{ds} = Q_{enc} \qquad \bar{ds} = 4\pi r^2 \hat{a}_r$$



$$\int_S D\,\hat{a}_r \cdot ds\,\hat{a}_r = Q_{enc}$$

$$\therefore \boxed{Q_{enc} = D\left[4\pi r^2\right]} \Rightarrow \text{for all Gaussian surfaces.}$$

GS1: $0 < r < a \Rightarrow Q_{enc} = 0 \Rightarrow D = 0$, $\boxed{\bar{E} = 0}$

GS2: $a \le r < b$

$$Q_{enc} = \int_0^{2\pi}\int_0^{\pi}\int_a^{r} (10r)(r^2 \sin\theta)\,dr\,d\theta\,d\phi$$

$$= 2\pi \int_0^{\pi} \sin\theta\,d\theta \int_a^{r} 10r^3\,dr$$

$$= 2\pi\,[2]\left[\frac{10r^4}{4}\right]_a^r$$

$$= 10\pi\,(r^4 - a^4)$$

$$\therefore D\left[4\cancel{\pi}r^2\right] = 10\cancel{\pi}\,(r^4 - a^4)$$

$$D = \frac{10\,(r^4 - a^4)}{4} = 2.5\,(r^4 - a^4)$$

$$\therefore \boxed{\bar{E} = \frac{2.5\,(r^4 - a^4)}{\epsilon r^2}\,\hat{a}_r}$$

GS3: $b \le r < \infty$

$$Q_{enc} = \int_0^{2\pi}\int_0^{\pi}\int_a^{b} 10r^3 \sin\theta\,dr\,d\theta\,d\phi$$

$$= 10\pi\,(b^4 - a^4)$$

$$\therefore D = 2.5\,(b^4 - a^4), \quad \boxed{\bar{E} = \frac{2.5\,(b^4 - a^4)}{\epsilon r^2}\,\hat{a}_r}$$

Problem 5 (14 pts)

In free space, the potential field is given by $V = \dfrac{10}{r^2}\sin\theta\cos\varphi$, find:

a) The electric flux density at (2 m, 90˚, 0)
b) The work done to move a 10 μC from point A = (1 m, 30˚, 120˚) to point B = (4 m, 90˚, 60˚).

a) $\bar{D} = \epsilon_0 \bar{E}$

$$\bar{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi\right]$$

$$\bar{E} = \frac{20}{r^3}\sin\theta\cos\phi\,\hat{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\hat{a}_\theta + \frac{10}{r^3}\sin\phi\,\hat{a}_\phi$$

at $(2, \pi/2, 0) \Rightarrow \bar{D} = \epsilon_0 \bar{E}(2, \pi/2, 0)$

$$\bar{D} = \epsilon_0\left[\frac{20}{8}\hat{a}_r - 0\,\hat{a}_\theta + 0\,\hat{a}_\phi\right]$$

$$= 8.854\,E^{-12}\left[\frac{20}{8}\right]\hat{a}_r$$

$$\boxed{\bar{D} = 22.1\ \text{pC/m}^2\ \hat{a}_r}$$

b) $W = Q\,V_{AB}$

$= Q\,V_B - V_A$

$$= 10\mu C\left[\frac{10}{(4)^2}\sin(90^\circ)\cos(60^\circ)\right] - 10\mu C\left[10\sin(30^\circ)\cos(120^\circ)\right]$$

$$\boxed{W = 28.125\ \mu J}$$

End of Exam