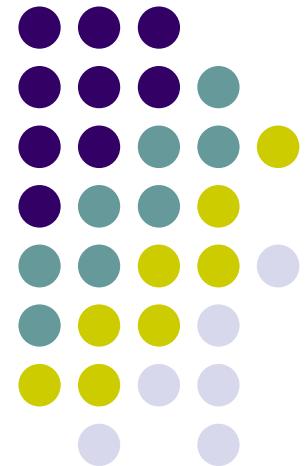


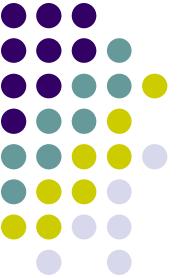
Electrostatic Field Intensity (continued)

and

Introduction to Matlab

Dr. Shady Elashhab





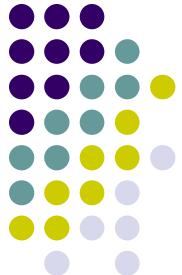
From our last lecture

- Electric Field Intensity due a point charge

$$\mathbf{E} \equiv \frac{\mathbf{F}}{q_t} = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_o |\mathbf{R} - \mathbf{R}'|^3} \quad (\text{N/C})$$

- Electric Field Intensity due to Multiple point charges

$$\mathbf{E} = \frac{\mathbf{F}}{q_t} = \frac{1}{4\pi\epsilon_o} \sum_{k=1}^N q_k \frac{(\mathbf{R} - \mathbf{R}_k')}{|\mathbf{R} - \mathbf{R}_k'|^3}$$



Electric Field Due to Charge Distributions

Each differential element of charge on a line charge ($\rho_l dl'$), a surface charge ($\rho_s ds'$) or a volume charge ($\rho_v dv'$) can be viewed as a point charge. By superposition, the total electric field produced by the overall charge distribution is the vector summation (integration) of the individual contributions due to each differential element. Using the equation for the electric field of a point charge, we can formulate an expression for dE (the incremental vector electric field produced by the given differential element of charge). We then integrate dE over the appropriate line, surface or volume over which the charge is distributed to determine the total electric field E at the field point P .

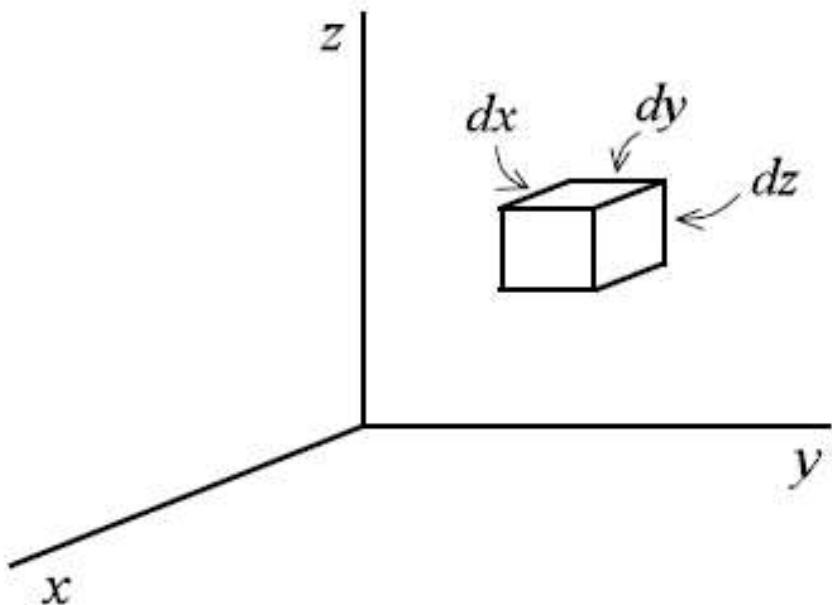


Differential Lengths, Surfaces and Volumes

- In order to evaluate these integrals, we must properly define the differential elements of length, surface and volume in the coordinate system of interest.
- The definition of the proper differential elements of length (dl for line integrals) and area (ds for surface integrals) can be determined directly from the definition of the differential volume (dv for volume integrals) in a particular coordinate system.



1. Differential Elements in Cartesian Coordinates



$$dv = (dx)(dy)(dz)$$

x-constant

$$ds = dy dz$$

y-constant

$$ds = dx dz$$

z-constant

$$ds = dx dy$$

x,y-constant

$$dl = dz$$

x,z-constant

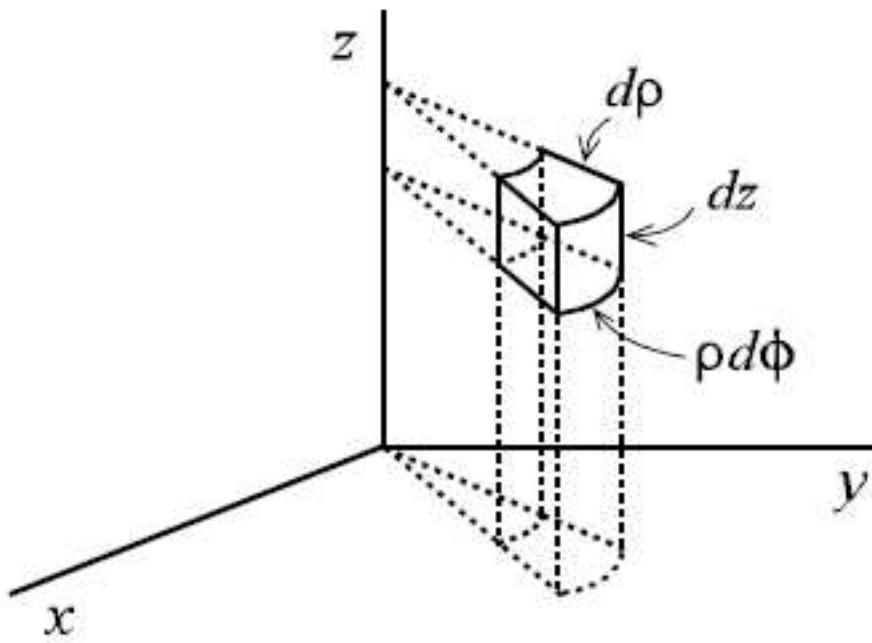
$$dl = dy$$

y,z-constant

$$dl = dx$$



2- Differential Elements in Cylindrical Coordinates



$$dV = (d\rho)(\rho d\phi)(dz)$$

$$= \rho d\rho d\phi dz$$

ρ -constant $ds = \rho_o d\phi dz$

ϕ -constant $ds = d\rho dz$

z -constant $ds = \rho d\rho d\phi$

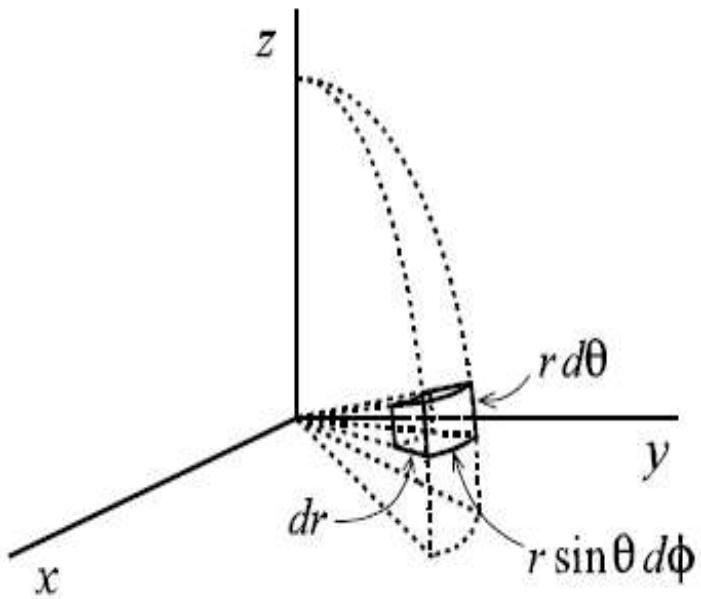
ρ, ϕ -constant $dl = dz$

ρ, z -constant $dl = \rho_o d\phi$

ϕ, z -constant $dl = d\rho$



3- Differential Elements in Spherical Coordinates



$$dv = (dr)(rd\theta)(r\sin\theta d\phi)$$

$$= r^2 \sin\theta dr d\theta d\phi$$

r-constant $ds = r_o^2 \sin\theta d\theta d\phi$

theta-constant $ds = r \sin\theta_o dr d\phi$

phi-constant $ds = r dr d\theta$

r, theta-constant $dl = r_o \sin\theta_o d\phi$

r, phi-constant $dl = r_o d\theta$

theta, phi-constant $dl = dr$

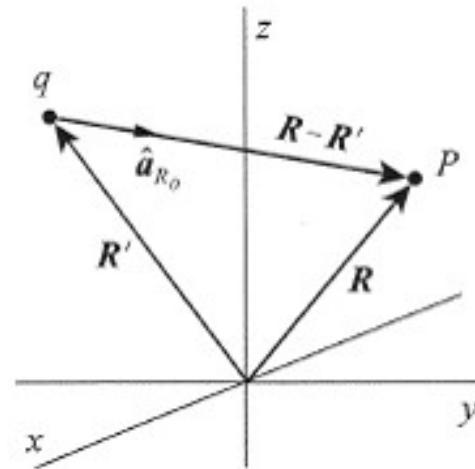


Electric Field Due to Charge Distributions

Point Charge

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 R_o^2} \hat{\mathbf{a}}_{R_o}$$

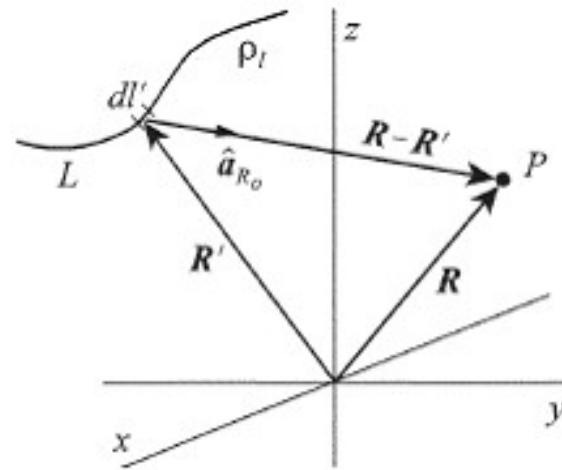
$$R_o = |\mathbf{R} - \mathbf{R}'| \quad \hat{\mathbf{a}}_{R_o} = \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|}$$

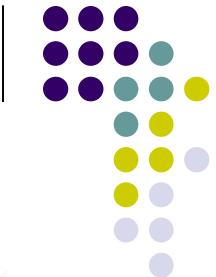


Line Charge ($\rho_l dl' \approx Q$)

$$d\mathbf{E} = \frac{\rho_l dl'}{4\pi\epsilon_0 R_o^2} \hat{\mathbf{a}}_{R_o}$$

$$\mathbf{E} = \int_L d\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_L \frac{\rho_l}{R_o^2} \hat{\mathbf{a}}_{R_o} dl'$$

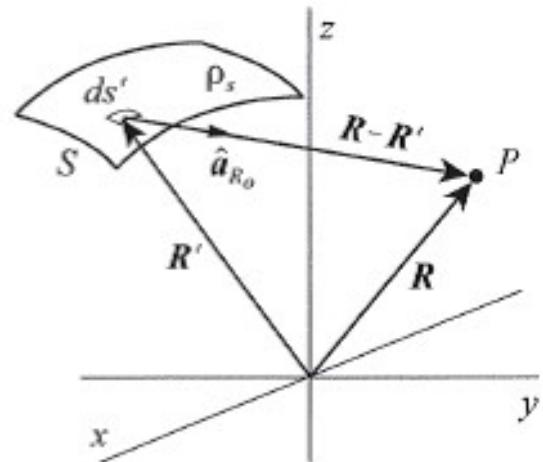




Surface Charge ($\rho_s ds' \rightarrow Q$)

$$d\mathbf{E} = \frac{\rho_s ds'}{4\pi\epsilon_o R_o^2} \hat{\mathbf{a}}_{R_o}$$

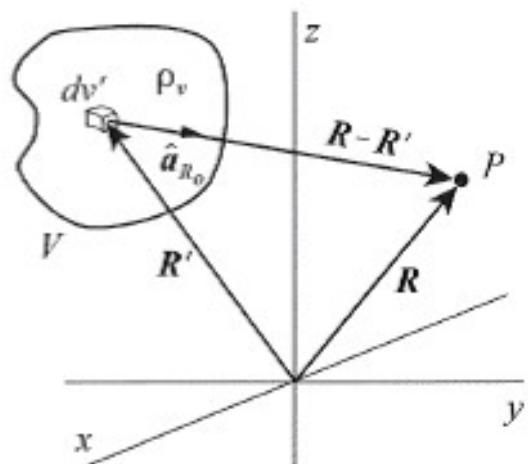
$$\mathbf{E} = \iint_S d\mathbf{E} = \frac{1}{4\pi\epsilon_o} \iint_S \frac{\rho_s}{R_o^2} \hat{\mathbf{a}}_{R_o} ds'$$



Volume Charge ($\rho_v dv' \rightarrow Q$)

$$d\mathbf{E} = \frac{\rho_v dv'}{4\pi\epsilon_o R_o^2} \hat{\mathbf{a}}_{R_o}$$

$$\mathbf{E} = \iiint_V d\mathbf{E} = \frac{1}{4\pi\epsilon_o} \iiint_V \frac{\rho_v}{R_o^2} \hat{\mathbf{a}}_{R_o} dv'$$





Example 1 (E due to a line charge)

Evaluate \mathbf{E} at $P = (x, y, z)$ due to a uniform line charge lying along the z -axis between $(0, 0, z_A)$ and $(0, 0, z_B)$ with $z_B > z_A$.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_L \frac{\rho_l}{R_o^2} \hat{\mathbf{a}}_{R_o} dl'$$

$$\rho_l = \frac{Q}{L} = \frac{Q}{z_B - z_A}$$

$$dl' = dz'$$

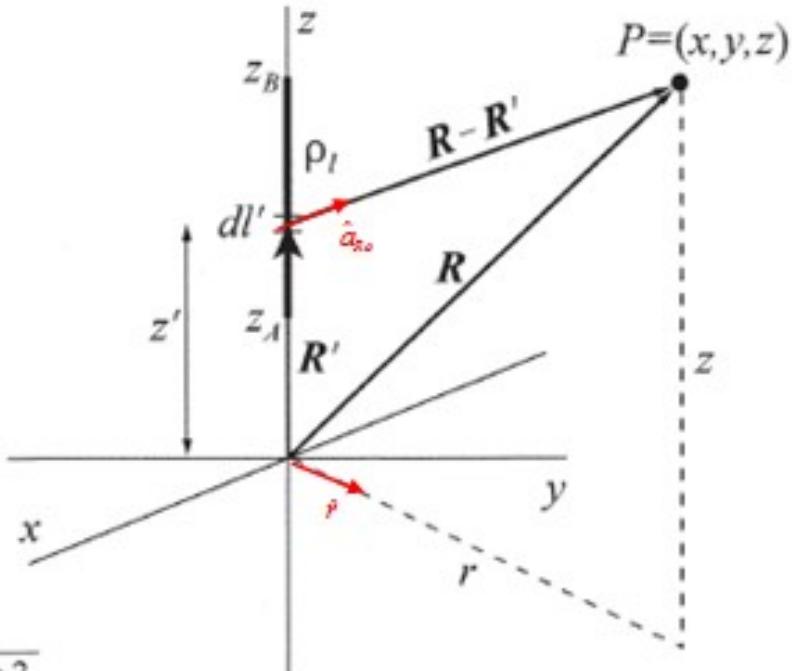
$$\mathbf{R} = x\hat{x} + y\hat{y} + z\hat{z} = r\hat{r} + z\hat{z}$$

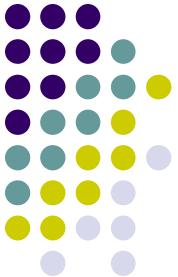
$$\mathbf{R}' = z'\hat{z}$$

$$\mathbf{R} - \mathbf{R}' = r\hat{r} + (z - z')\hat{z}$$

$$R_o = |\mathbf{R} - \mathbf{R}'| = \sqrt{r^2 + (z - z')^2}$$

$$\hat{\mathbf{a}}_{R_o} = \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|} = \frac{r\hat{r} + (z - z')\hat{z}}{\sqrt{r^2 + (z - z')^2}}$$





$$\begin{aligned}
 E &= \frac{\rho_l}{4\pi\epsilon_0} \int_{z_A}^{z_B} \frac{r\hat{r} + (z - z')\hat{z}}{[r^2 + (z - z')^2]^{3/2}} dz' \\
 &= \frac{\rho_l}{4\pi\epsilon_0} \left[r\hat{r} \int_{z_A}^{z_B} \frac{dz'}{[r^2 + (z - z')^2]^{3/2}} + \hat{z} \int_{z_A}^{z_B} \frac{(z - z')}{[r^2 + (z - z')^2]^{3/2}} dz' \right]
 \end{aligned}$$

From Integration Tables

$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2 \sqrt{x^2 + a^2}}$$

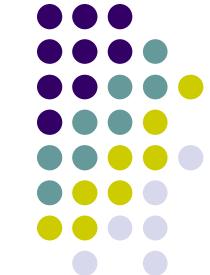
$$\text{ & } \int \frac{x dx}{(a^2 + x^2)^{3/2}} = -\frac{1}{\sqrt{x^2 + a^2}}$$

The integrals in the electric field expression may be evaluated analytically using the following variable transformation:

$$\text{Let } \alpha = z - z' \quad d\alpha = -dz'$$

$$z' = z_A \quad \rightarrow \quad \alpha = z - z_A$$

$$z' = z_B \quad \rightarrow \quad \alpha = z - z_B$$



$$\mathbf{E} = \frac{\rho_l}{4\pi\epsilon_o} \left[-r \hat{\mathbf{r}} \int_{z=z_A}^{z=z_B} \frac{d\alpha}{(r^2 + \alpha^2)^{3/2}} - \hat{\mathbf{z}} \int_{z=z_A}^{z=z_B} \frac{\alpha d\alpha}{(r^2 + \alpha^2)^{3/2}} \right]$$

I_1

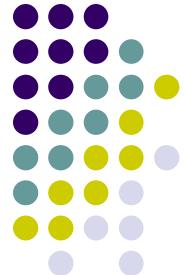
$$\int \frac{dx}{(a^2 + x^2)^{3/2}} = \frac{x}{a^2 \sqrt{x^2 + a^2}}$$

I_2

$$\int \frac{x dx}{(a^2 + x^2)^{3/2}} = -\frac{1}{\sqrt{x^2 + a^2}}$$

$$\mathbf{E} = \frac{\rho_l}{4\pi\epsilon_o} \left\{ -r \hat{\mathbf{r}} \left[\frac{\alpha}{r^2 \sqrt{\alpha^2 + r^2}} \right]_{z=z_A}^{z=z_B} + \hat{\mathbf{z}} \left[\frac{1}{\sqrt{\alpha^2 + r^2}} \right]_{z=z_A}^{z=z_B} \right\}$$

$$\boxed{\mathbf{E} = \frac{\rho_l}{4\pi\epsilon_o r} \left\{ \left[\frac{z - z_A}{\sqrt{r^2 + (z - z_A)^2}} - \frac{z - z_B}{\sqrt{r^2 + (z - z_B)^2}} \right] \hat{\mathbf{r}} + \left[\frac{r}{\sqrt{r^2 + (z - z_B)^2}} - \frac{r}{\sqrt{r^2 + (z - z_A)^2}} \right] \hat{\mathbf{z}} \right\}}$$



Special Case 1: Finite length line charge centered at the origin and test charge in the x-y plane

For the special case of a line charge centered at the coordinate origin ($z_A = -a$, $z_B = a$) with the field point [$P = (x, y, 0)$], the electric field expression reduces to

$$\mathbf{E} = \frac{\rho_l}{4\pi\epsilon_o r} \left\{ \left[\frac{a}{\sqrt{r^2 + a^2}} - \frac{-a}{\sqrt{r^2 + a^2}} \right] \hat{\mathbf{r}} + \left[\frac{r}{\sqrt{r^2 + a^2}} - \frac{r}{\sqrt{r^2 + a^2}} \right] \hat{\mathbf{z}} \right\}$$

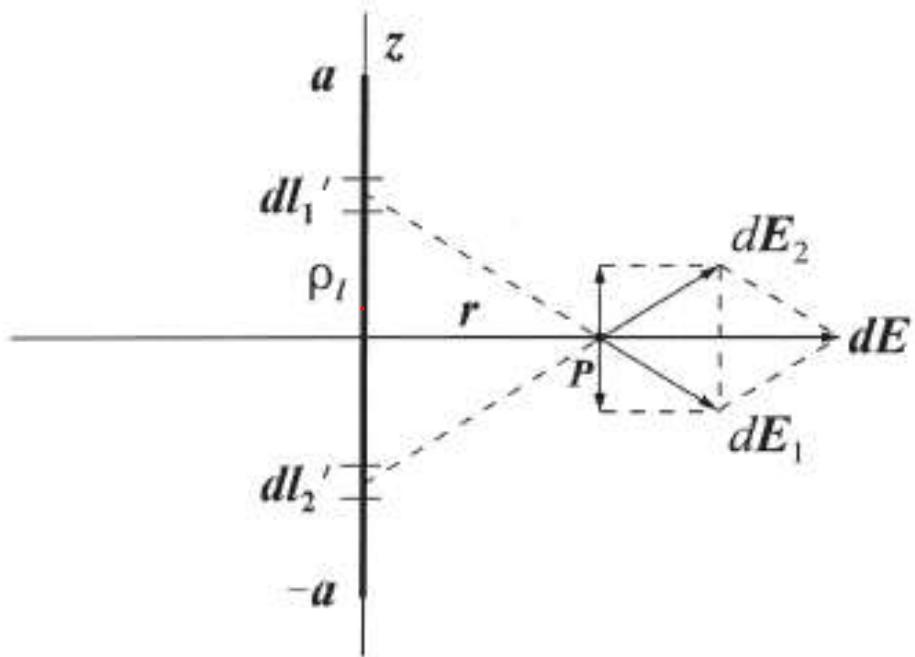
$$\mathbf{E} = \frac{\rho_l a}{2\pi\epsilon_o r \sqrt{r^2 + a^2}} \hat{\mathbf{r}} \quad (\text{E-field in the } x\text{-y plane due to a uniform line charge of length } 2a \text{ centered at the origin})$$

The zero-valued $\hat{\mathbf{z}}$ -component of the electric field in this case is a direct result of the symmetry of the line charge relative to the field point location.



Special Case 1: Symmetry of the finite line charge

For every differential element of charge located above the x - y plane, there exists another differential element of charge an equal distance below the x - y -plane yielding a cancellation of the \hat{z} -components of the electric field contributions.





Special Case 2: Infinite line charge

To determine the electric field of an infinite length line charge, we take the limit of the previous result as a approaches ∞ .

$$\mathbf{E} = \frac{\rho_l}{2\pi\epsilon_0 r} \hat{\mathbf{r}} \lim_{a \rightarrow \infty} \left[\frac{a}{\sqrt{r^2 + a^2}} \right]$$

$$= \frac{\rho_l}{2\pi\epsilon_0 r} \hat{\mathbf{r}} \lim_{a \rightarrow \infty} \left[\frac{1}{\sqrt{(r/a)^2 + 1}} \right]$$

$$\mathbf{E} = \frac{\rho_l}{2\pi\epsilon_0 r} \hat{\mathbf{r}}$$

(E-field due to a uniform line charge of infinite length lying along the z-axis.)

Note that the electric field of the infinite-length uniform line charge is cylindrically symmetric (*line source*). That is, the electric field is independent of ϕ due to the symmetry of the source. The electric field of the infinite-length uniform line charge is also independent of z due to the infinite length of the uniform source.



E due to a uniformly charged disk (radius = a)

Evaluate \mathbf{E} at a point on the z -axis $P = (0, 0, h)$ due to a uniformly charged disk of radius a laying in the x - y plane and centered at the coordinate origin.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \iint_S \frac{\rho_s}{R_o^2} \hat{a}_{R_o} ds' \quad \rho_s = \frac{Q}{A} = \frac{Q}{\pi a^2}$$

$$\mathbf{R} = h\hat{z}$$

$$\mathbf{R}' = r'\hat{r}$$

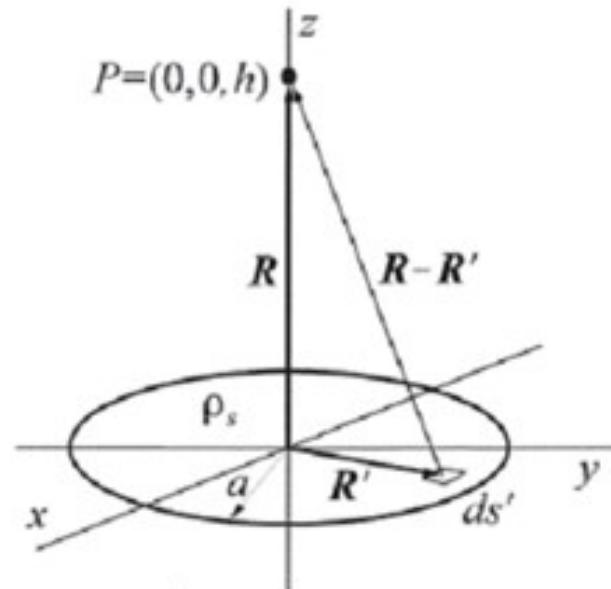
$$\mathbf{R} - \mathbf{R}' = h\hat{z} - r'\hat{r}$$

$$R_o = |\mathbf{R} - \mathbf{R}'| = \sqrt{r'^2 + h^2}$$

$$\hat{a}_{R_o} = \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|} = \frac{-r'\hat{r} + h\hat{z}}{\sqrt{r'^2 + h^2}}$$

$$ds' = r' dr' d\phi'$$

$$\mathbf{E} = \frac{\rho_s}{4\pi\epsilon_0} \int_{\phi'=0}^{2\pi} \int_{r'=0}^a \frac{-r'\hat{r} + h\hat{z}}{(r'^2 + h^2)^{3/2}} r' dr' d\phi'$$





The unit vector \hat{r} , can be transformed into rectangular coordinate unit vectors to simplify the integration.

$$\hat{r} = \cos\phi' \hat{x} + \sin\phi' \hat{y}$$

$$\mathbf{E} = \frac{\rho_s}{4\pi\epsilon_o} \int_{\phi'=0}^{2\pi} \int_{r'=0}^a \frac{-r' \cos\phi' \hat{x} - r' \sin\phi' \hat{y} + h \hat{z}}{(r'^2 + h^2)^{3/2}} r' dr' d\phi'$$

The \hat{x} and \hat{y} -dependent terms in the electric field expression integrate to zero given the sine and cosine integrals with respect to ϕ' over one period.

$$\int_0^{2\pi} \sin\phi' d\phi' = \int_0^{2\pi} \cos\phi' d\phi' = 0$$

The electric field expression reduces to

$$\mathbf{E} = \frac{\rho_s h}{4\pi\epsilon_o} \hat{z} \int_0^{2\pi} d\phi' \int_0^a \frac{r' dr'}{(r'^2 + h^2)^{3/2}} = \frac{\rho_s h}{4\pi\epsilon_o} \hat{z} \left[\phi' \right]_0^{2\pi} \left[\frac{-1}{\sqrt{r'^2 + h^2}} \right]_0^a$$

$$\mathbf{E} = \frac{\rho_s h}{2\epsilon_o} \left[\frac{1}{h} - \frac{1}{\sqrt{a^2 + h^2}} \right] \hat{z}$$

(E-field on the z -axis due to a uniformly charged disk of radius a in the x - y plane centered at the origin, h = height above disk)



Uniformly Charged Disk (radius = ∞)

The electric field produced by an infinite charged sheet can be determined by taking the limit of the charged disk electric field as the disk radius approaches ∞ .

$$E(\text{infinite sheet}) = \lim_{a \rightarrow \infty} [E(\text{disk, radius } = a)]$$

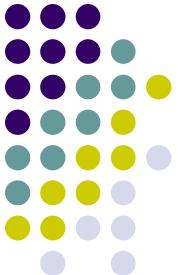
$$E = \frac{\rho_s h}{2\epsilon_o} \hat{z} \lim_{a \rightarrow \infty} \left[\frac{1}{h} - \frac{1}{\sqrt{a^2 + h^2}} \right] = \frac{\rho_s}{2\epsilon_o} \hat{z} \quad (\text{E-field due to a uniformly charged infinite sheet})$$

Note that the electric field of the uniformly charged infinite sheet is uniform (independent of the height h of the field point above the sheet).



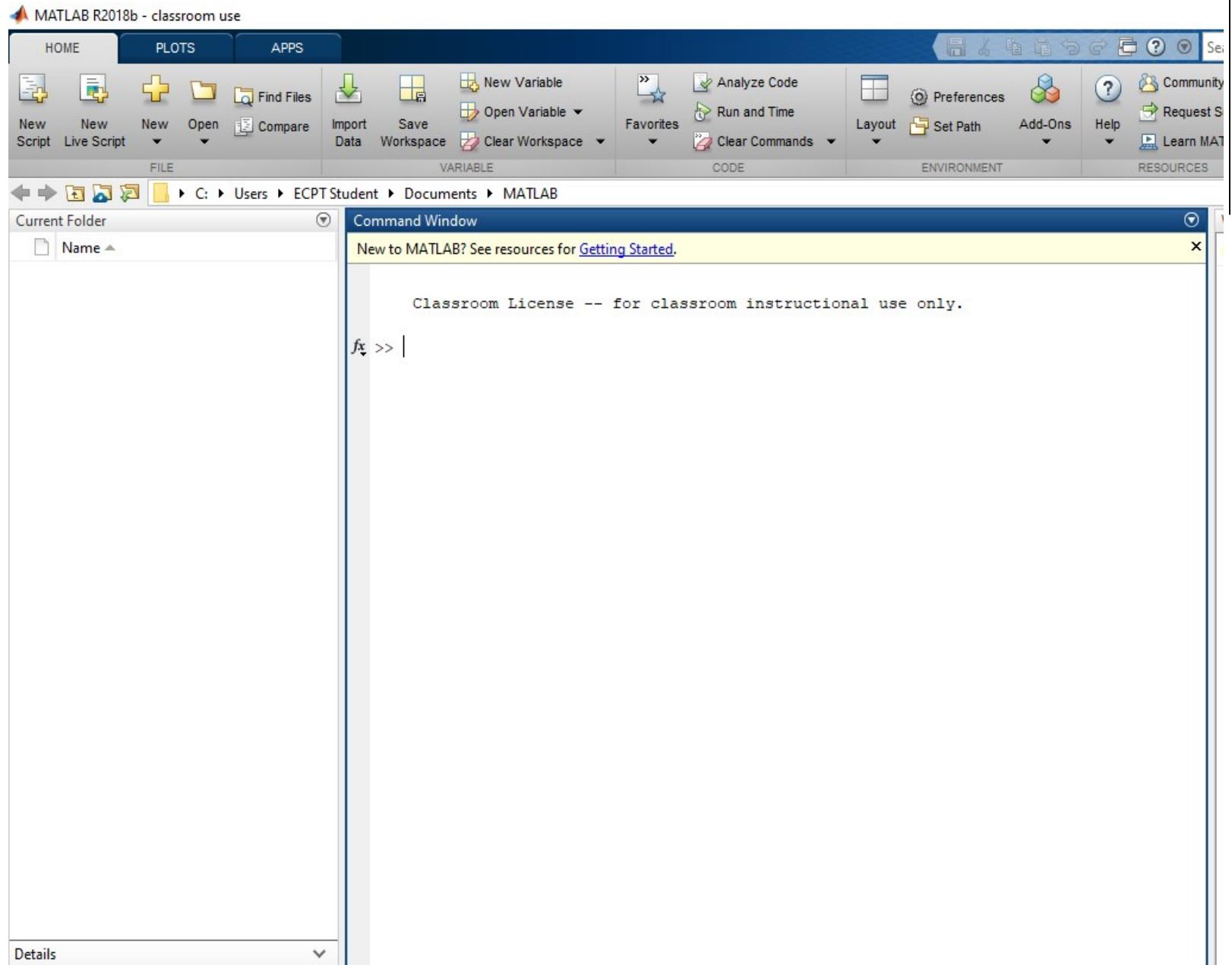
Introduction to MATLAB

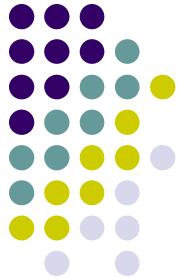
- Provide an introduction and overview of how MATLAB's calculator mode is used to implement interactive computations
 - Learning how real and complex numbers are assigned to variables
 - Learning how vectors and matrices are assigned values using simple assignment, the colon operator, and the `linspace` and `logspace` functions
 - Understanding the priority rules for constructing mathematical expressions
 - Gaining a general understanding of built-in functions and how you can learn more about them with MATLAB's Help facilities
 - Learning how to use vectors to create a simple line plot based on an equation



The MATLAB Environment

- MATLAB uses three primary windows:
 - Command window - used to enter commands and data
 - Graphics window(s) - used to display plots and graphics
 - Edit window - used to create and edit M-files (programs)
- Depending on your computer platform and the version of MATLAB used, these windows may have different looks and feels





Calculator Mode

- The MATLAB command widow can be used as a calculator where you can type in commands line by line. Whenever a calculation is performed, MATLAB will assign the result to the built-in variable ans
- Example:
 - >> 55 - 16
 - ans =
 - 39



MATLAB Variables

- MATLAB allows you to assign values to variable names. This results in the storage of values to memory locations corresponding to the variable name
- MATLAB can store individual values as well as arrays; it can store numerical data and text (which is actually stored numerically as well)
- MATLAB does not require that you pre-initialize a variable; if it does not exist, MATLAB will create it for you
- Note that variable names must start with a letter, though they can contain letters, numbers, and the underscore (`_`) symbol



Scalars

- To assign a single value to a variable, simply type the variable name, the = sign, and the value:
- You can tell MATLAB not to report the result of a calculation by appending the semi-solon (;) to the end of a line. The calculation is still performed
- You can ask MATLAB to report the value stored in a variable by typing its name:

```
>> a  
a =  
4
```



- You can use the complex variable `i` (or `j`) to represent the unit imaginary number
- You can tell MATLAB to report the values back using several different formats using the `format` command. Note that the values are still stored the same way, they are just displayed on the screen differently. Some examples are:
 - `short` - scaled fixed-point format with 5 digits
 - `long` - scaled fixed-point format with 15 digits for double and 7 digits for single
 - `short eng` - engineering format with at least 5 digits and a power that is a multiple of 3 (useful for SI prefixes)



Format Examples

```
>> format short; pi  
ans =  
    3.1416  
>> format long; pi  
ans =  
    3.14159265358979  
>> format short eng; pi  
ans =  
    3.1416e+000  
>> pi*10000  
ans =  
    31.4159e+003
```

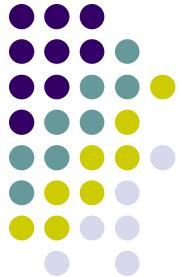
- Note - the format remains the same unless another `format` command is issued



Arrays, Vectors, and Matrices

- Arrays are set off using square brackets [and] in MATLAB
- Entries within a row are separated by **spaces** () or **commas** (,)
- Rows are separated by **semicolons** (;)

```
>> a = [1 2 3 4 5]
a =
    1      2      3      4      5
>> a = [1,2,3,4,5]
a =
    1      2      3      4      5
```



```
>> b = [2;4;6;8;10]
b =
    2
    4
    6
    8
   10
```

- A 2-D array, or matrix, of data is entered row by row, with **spaces** (or **commas**) separating entries within the row and **semicolons** separating the rows:

```
>> A = [1 2 3; 4 5 6; 7 8 9]
A =
    1      2      3
    4      5      6
    7      8      9
```



Accessing Array Entries

- Individual entries within a array can be both read and set using either the *index* of the location in the array or the row and column
- **The index value starts with 1** for the entry in the top left corner of an array and increases down a column - the following shows the indices for a 4-row, 3-column matrix:

1	5	9
2	6	10
3	7	11
4	8	12



- Assuming some matrix C:

C =

2	4	9
3	3	16
3	0	8
10	13	17

- C(2) would report 3
- C(4) would report 10
- C(13) would report an error!
- Entries can also be access using the row and column:
 - C(2, 1) would report 3
 - C(3, 2) would report 0
 - C(5, 1) would report an error!



Array Creation - Colon Operator

- The **colon** operator (`:`) is useful in several contexts. It can be used to create a linearly spaced array of points using the notation

`start:diffval:limit`

- where `start` is the first value in the array, `diffval` is the difference between successive values in the array, and `limit` is the *boundary* for the last value (**though not necessarily the last value**)

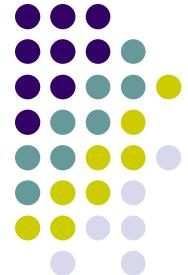
```
>> C = 0:0.2:3

C =
Columns 1 through 8

     0      0.2000      0.4000      0.6000      0.8000      1.0000      1.2000      1.4000

Columns 9 through 16

    1.6000      1.8000      2.0000      2.2000      2.4000      2.6000      2.8000      3.0000
```



Array Creation - linspace

- To create a row vector with a specific number of linearly spaced points between two numbers, use the `linspace` command
- `linspace(x1, x2, n)` will create a linearly spaced array of n points between x_1 and x_2

```
>> E = linspace(0, 1, 6)

E =

    0    0.2000    0.4000    0.6000    0.8000    1.0000

>> F = E'

F =

    0
    0.2000
    0.4000
    0.6000
    0.8000
    1.0000
```



Mathematical Operations

- Mathematical operations in MATLAB can be performed on both scalars and arrays
- The common operators, in order of priority, are:

$^$	Exponentiation	$4^2 = 16$
$-$	Negation (unary operation)	$-8 = -8$
$*$ $/$	Multiplication and Division	$2*pi = 6.2832$ $pi/4 = 0.7854$
\backslash	Left Division	$6\backslash 2 = 0.3333$
$+$ $-$	Addition and Subtraction	$3+5 = 8$ $3-5 = -2$



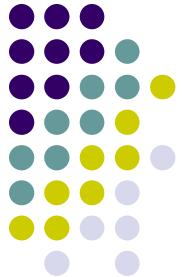
Mathematical Operations

- The order of operations is set first by **parentheses**, then by the default order given above:
 - $y = -4^2$ gives $y = -16$
since the exponentiation happens first due to its higher default priority, but
 - $y = (-4)^2$ gives $y = 16$
since the negation operation on the 4 takes place first



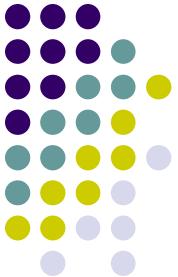
Vector-Matrix Calculations

- MATLAB can also perform operations on vectors and matrices
- The `*` operator for matrices is defined as the *outer product* or what is commonly called “matrix multiplication”
 - The number of columns of the first matrix must match the number of rows in the second matrix
 - The size of the result will have as many rows as the first matrix and as many columns as the second matrix
- The `^` operator for matrices results in the matrix being matrix-multiplied by itself a specified number of times.
 - Note - in this case, the matrix must be square!



Element-by-Element Calculations

- At times, you will want to carry out calculations item by item in a matrix or vector. The MATLAB manual calls these *array operations*. They are also often referred to as *element-by-element* operations
- MATLAB defines `.*` and `./` (note the dots) as the array multiplication and array division operators
 - For array operations, both matrices must be the same size or one of the matrices must be 1×1
- Array exponentiation (raising each element to a corresponding power in another matrix) is performed with `.^`



Graphics

- MATLAB has a powerful suite of built-in graphics functions
- Two of the primary functions are `plot` (for plotting 2-D data) and `plot3` (for plotting 3-D data)
- In addition to the plotting commands, MATLAB allows you to label and annotate your graphs using the `title`, `xlabel`, `ylabel`, and `legend` commands

Example:

**Write a Matlab script to plot free fall velocity of a bungee jumper.
Free fall velocity is given by**

$$v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}} t\right)$$



The program

```
% Matlab script to illustrate 2-D graphing capability

t = [0:0.5:25];      % Create time array

g = 9.81;            % Declare constants
m = 68.1;
cd = 0.25;

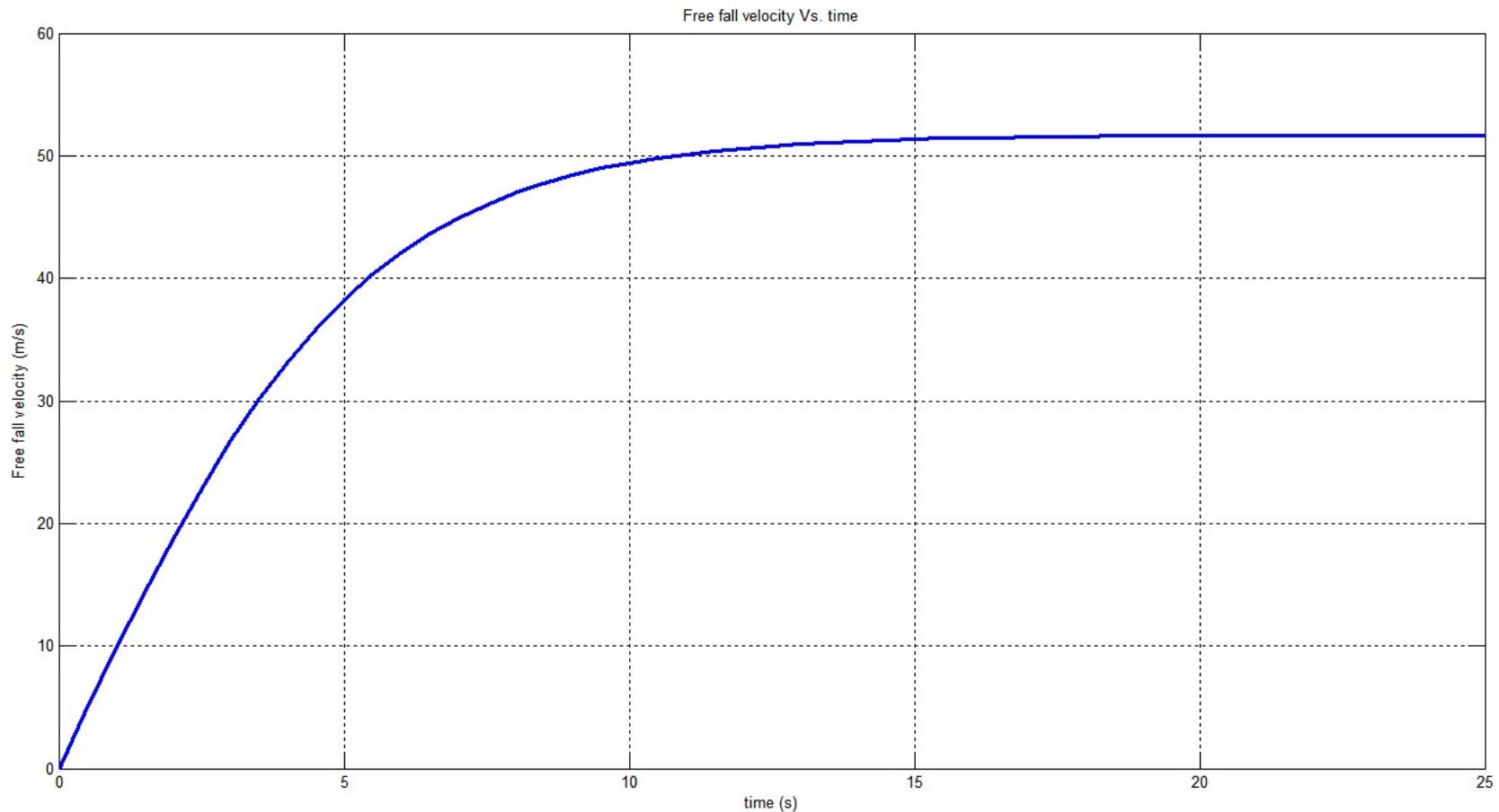
v = sqrt(g*m/cd) * tanh(sqrt(g*cd/m)*t);  % coded equation for free fall velocity

plot(t, v, 'linewidth', 2.5)                  % Graph V Vs. time using a specific line width

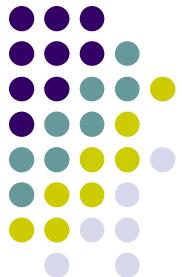
title('Plot of v versus t')                    % Add title ,labels, and grid
xlabel('time (s)')
ylabel('Free fall velocity (m/s)')
grid
```



Matlab returns the following plot



- `subplot(m, n, p)`
 - subplot splits the figure window into an mxn array of small axes and makes the p th one active. Note - the first subplot is at the top left, then the numbering continues across the row. This is different from how elements are numbered within a matrix!



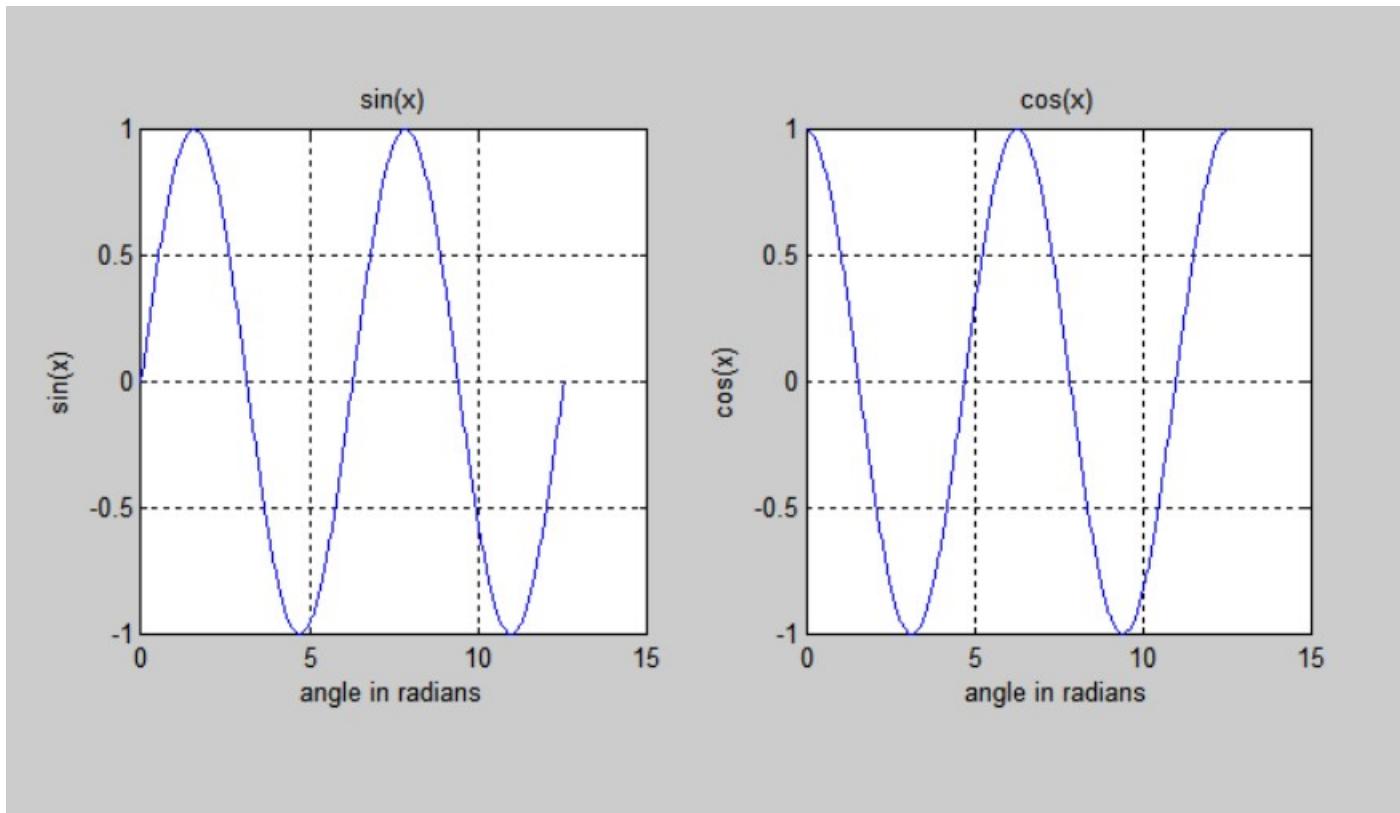
```

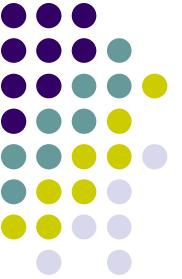
1      x=0:pi/100:4*pi;          % create an array for x in radians
2
3
4      subplot(1,2,1);          % Graph sin(x) in the 1st sub-plotting window
5      plot(x, sin(x));
6      axis square;            % equal length axes
7
8      title('sin(x)');        % Add title ,labels, and grid
9      xlabel('angle in radians');
10     ylabel('sin(x)');
11     grid
12
13     subplot(1,2,2);          % Graph cos(x) in the 2nd sub-plotting window
14     plot(x, cos(x));
15
16     axis square;            % equal length axes
17
18     title('cos(x)');        % Add title ,labels, and grid
19     xlabel('angle in radians');
20     ylabel('cos(x)');
21     grid
22

```



Matlab returns the following plot





Announcements

- Homework 2 is assigned (available on Webcourses)