Lecture 4

1- Electric Potential

2- Gauss's Law

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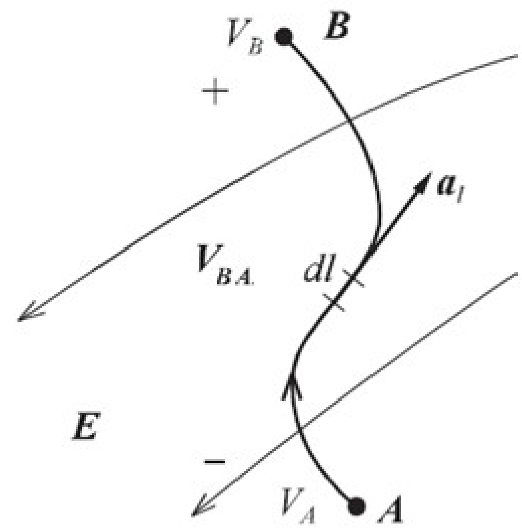
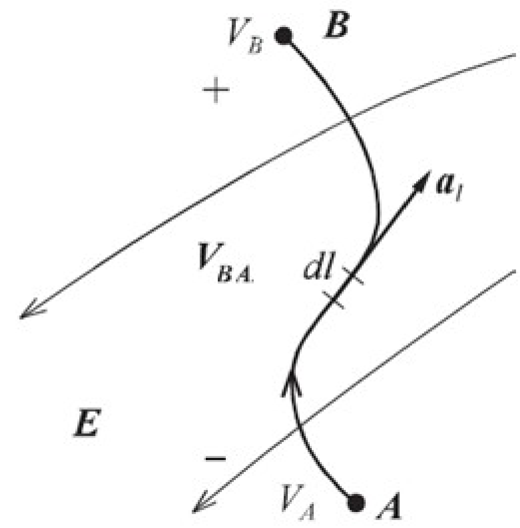
Electric Scalar Potential

- Given that the electric field defines the **force per unit charge** acting on a positive test charge, any attempt to move the test charge against the electric field requires work to be done.



- The *potential difference* between two points in an electric field is defined as **the work per unit charge** performed when moving a positive test charge from one point to the other against the field.

From Coulomb's law, the vector force on a positive point charge in an electric field is given by



$$\boldsymbol{F} = Q\boldsymbol{E}$$

For a differential element of length (dl), the small amount of work done against the field (dW) is defined as

$$dW = -\boldsymbol{F}\cdot d\boldsymbol{l} = -Q\boldsymbol{E}\cdot d\boldsymbol{l}$$

The total amount of work performed (W) is

$$W = \int_A^B dW = -\int_A^B \boldsymbol{F}\cdot d\boldsymbol{l} = -Q\int_A^B \boldsymbol{E}\cdot d\boldsymbol{l} \qquad \text{(J)}$$

The potential difference , W/Q , between A and B is then

$$\boxed{V_{BA} = -\int_A^B \boldsymbol{E}\cdot d\boldsymbol{l} = V_B - V_A}$$

The potential difference

- The potential difference between any two points is independent of the path taken between the points.
- For a closed path (point A = point B), the line integral of the electric field yields the potential difference between a point and itself yielding a value of zero.

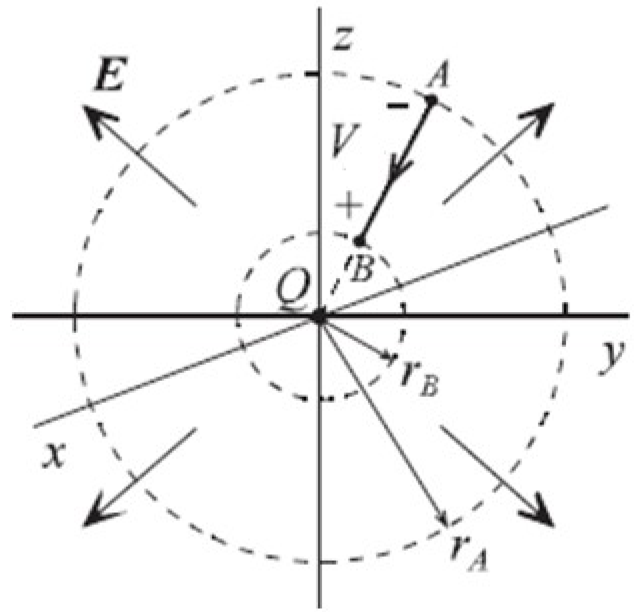
$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

Example 1: Absolute Potential

- Determine the absolute potential in the electric field of a point charge Q located at the coordinate origin.

The electric field of a point charge at the origin is given by

$$\boldsymbol{E} = \frac{Q}{4\pi\epsilon_o r^2}\boldsymbol{a}_r$$



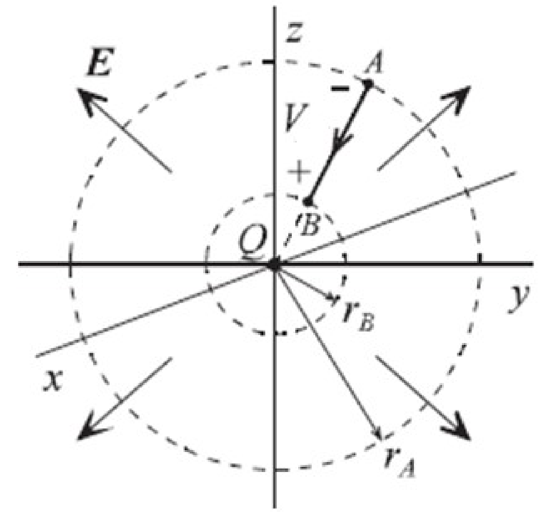
First, we obtain the potential difference between two points A and B

$$V_{BA} = -\int_{A}^{B} \mathbf{E} \cdot d\mathbf{l} = V_B - V_A$$

the vector differential length is

$$d\mathbf{l} = dr\,\mathbf{a}_r$$

which yields



$$V_{BA} = -\int_{r_A}^{r_B} \left(\frac{Q}{4\pi\epsilon_o r^2} \mathbf{a}_r \right) \cdot (dr\,\mathbf{a}_r) = -$$

The absolute potential at point B is found by taking the limit as r_A approaches infinity

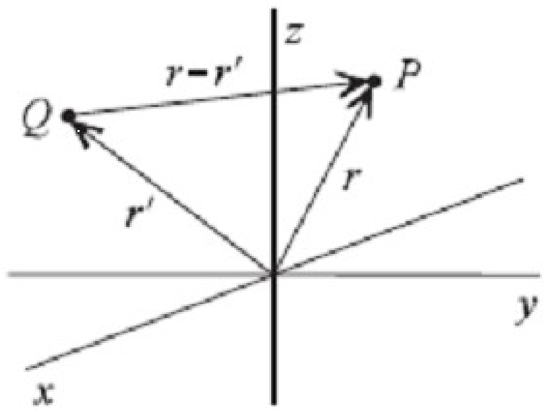
The absolute potential at point B is found by taking tl
approaches infinity.

Absolute Potential of a point charge at an arbitrary location

The absolute potential of a point charge at an arbitrary location is

$$V = \frac{Q}{4\pi\epsilon_o |\boldsymbol{r}-\boldsymbol{r}'|}$$

(Absolute potential for a point charge at an arbitrary location)



The principle of superposition can be applied to determine the potential due to a set of point charges which yields

$$V = \frac{1}{4\pi\epsilon_o} \sum_{k=1}^{N} \frac{Q_k}{|\boldsymbol{r}-\boldsymbol{r}'_k|}$$

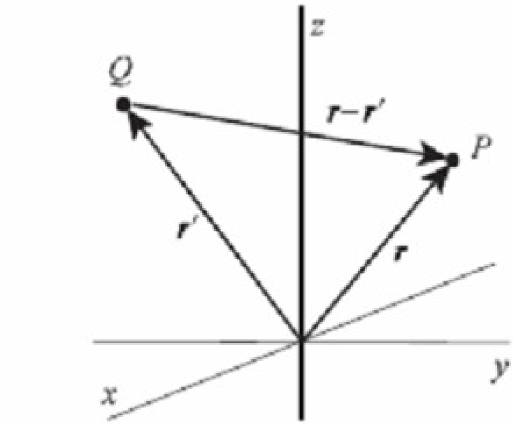
(Absolute potential of a set of point charges)

Potentials of Charge Distributions

<u>Point Charge</u>

$$V = \frac{Q}{4\pi\epsilon_o R}$$

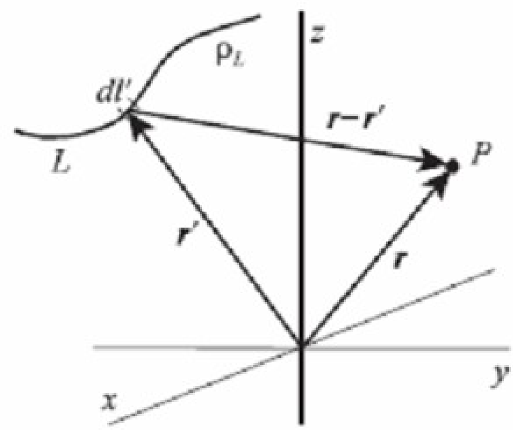
$$R = |\boldsymbol{r} - \boldsymbol{r}'|$$



<u>Line Charge</u> $(\rho_L\, dl' \Leftrightarrow Q)$

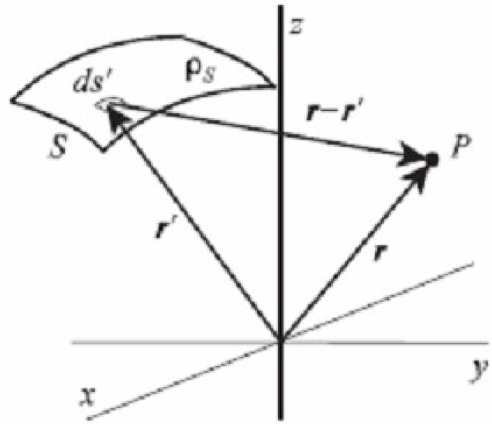
$$dV = \frac{\rho_L\, dl'}{4\pi\epsilon_o R}$$

$$V = \frac{1}{4\pi\epsilon_o}\int_L \frac{\rho_L}{R}\, dl'$$



$$R = |\boldsymbol{r} - \boldsymbol{r}'|$$

Surface Charge $(\rho_S \, ds' \Leftrightarrow Q)$

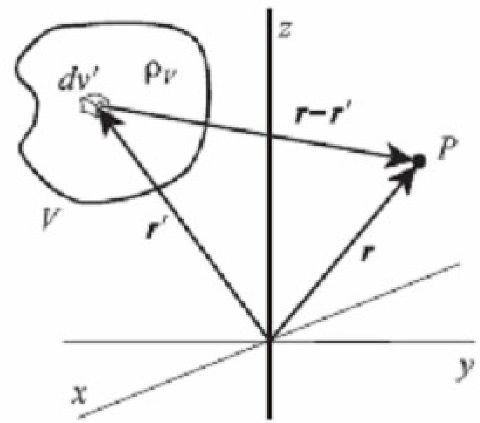


$$dV = \frac{\rho_S \, ds'}{4\pi\epsilon_o R}$$

$$V = \frac{1}{4\pi\epsilon_o} \iint_S \frac{\rho_S}{R} ds'$$

$$R = |\boldsymbol{r} - \boldsymbol{r}'|$$

Volume Charge $(\rho_V \, dv' \Leftrightarrow Q)$



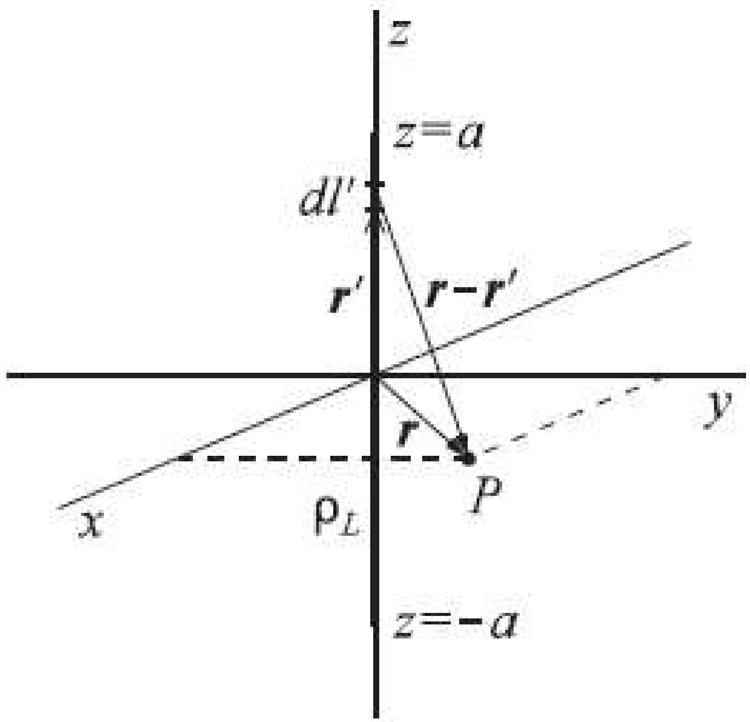
$$dV = \frac{\rho_V \, dv'}{4\pi\epsilon_o R}$$

$$V = \frac{1}{4\pi\epsilon_o} \iiint_V \frac{\rho_V}{R} dv'$$

$$R = |\boldsymbol{r} - \boldsymbol{r}'|$$

Example 2 (Potential due to a line charge)

- Determine the potential in the *x*-*y* plane due to a uniform line charge of length $2a$ laying along the z-axis and centered at the coordinate origin



$$\boxed{V = \frac{1}{4\pi\epsilon_o}\int_L \frac{\rho_L}{R}\,dl'}$$

$$dl' = dz'$$

$$\boldsymbol{r} = \rho\,\boldsymbol{a}_\rho$$

$$\boldsymbol{r}' = z'\boldsymbol{a}_z$$

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}' = \rho\,\boldsymbol{a}_\rho - z'\boldsymbol{a}_z$$

$$R = |\boldsymbol{R}| = \sqrt{z'^2 + \rho^2}$$

$$V = \frac{\rho_L}{4\pi\epsilon_o}\int_{-a}^{a}\frac{dz'}{\sqrt{z'^2+\rho^2}} = (2)\frac{\rho_L}{4\pi\epsilon_o}\int_{0}^{a}\frac{dz'}{\sqrt{z'^2+\rho^2}}$$

Even integrand
Symmetric limits

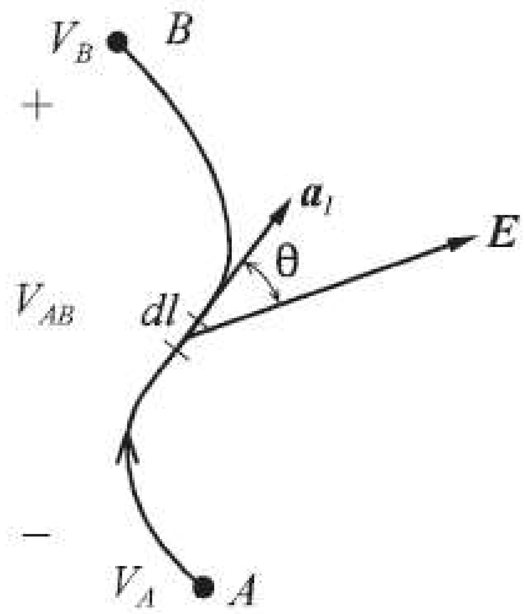
$$\int\frac{dx}{\sqrt{x^2+a^2}} = \ln[x+\sqrt{x^2+a^2}]$$

$$V = \frac{\rho_L}{2\pi\epsilon_o}\ln[z'+\sqrt{z'^2+\rho^2}\,]_0^a = \frac{\rho_L}{2\pi\epsilon_o}\left\{\ln[a+\sqrt{a^2+\rho^2}]-\ln\rho\right\}$$

$$\boxed{V = \frac{\rho_L}{2\pi\epsilon_o}\ln\left[\frac{a+\sqrt{\rho^2+a^2}}{\rho}\right]}$$

(Absolute potential in the x-y plane due to a uniform line charge of length $2a$ lying along the z-axis centered at the coordinate origin)

The Electric Field (revisited)



The potential difference between two points

$$\boxed{V_{BA} = -\int_A^B \mathbf{E} \cdot d\mathbf{l} = V_B - V_A}$$

The incremental change in potential along the integral path is

$$dV = -\mathbf{E} \cdot d\mathbf{l} = -\mathbf{E} \cdot \mathbf{a}_l \, dl = -E\cos\theta \, dl$$

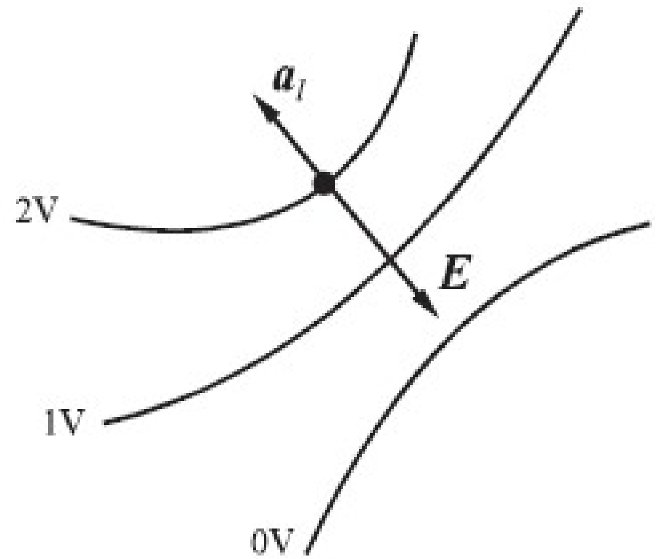
where θ is the angle between the direction of the integral path and the electric field.

$$\frac{dV}{dl} = -E\cos\theta$$

Note that $\left[\frac{dV}{dl}\right]_{\max} = E$ when $\theta = \pi \quad (\cos\theta = -1)$

$$\left[\frac{dV}{dl}\right]_{max} = E \qquad \text{when } \theta = \pi \quad (\cos\theta = -1)$$

This equation shows that the magnitude of the electric field is equal to the maximum space rate of change in the potential. The direction of the electric field is the direction of the maximum decrease in the potential (the electric field always points from a region of higher potential to a region of lower potential).



The electric field can be written in terms of the potential as

$$\boldsymbol{E} = \frac{dV}{dl}(-\boldsymbol{a}_l) = -\frac{dV}{dl}\boldsymbol{a}_l = -\nabla V$$

where the operator "∇ " (del) is the *gradient operator*.

The gradient

- The gradient is a differential operator which operates on a scalar function to yield a vector
  - Its magnitude is the maximum increase per unit distance and
  - Its direction is the direction of the maximum increase

- Since the electric field always points in the direction of decreasing potential, the electric field is **the negative of the gradient of *V*.**

The gradient operator in different coordinate systems

The gradient operator is defined differently in rectangular, cylindrical and spherical coordinates. The electric field in these coordinate systems are

$$\boldsymbol{E} = -\nabla V = -\left[\frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial y}\boldsymbol{a}_y + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right] \qquad \text{(rectangular)}$$

$$= -\left[\frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right] \qquad \text{(cylindrical)}$$

$$= -\left[\frac{\partial V}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi\right] \qquad \text{(spherical)}$$

Example 3:

Given $V(r,\theta,\phi) = \dfrac{10}{r^2}\sin\theta\cos\phi$ (a.) find $E(r,\theta,\phi)$ and (b.) E at $(2,\pi/2,0)$.

(a.)

$$\boldsymbol{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi\right]$$

$$= -\left[10\sin\theta\cos\phi\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\boldsymbol{a}_r + \frac{10}{r^3}\cos\phi\frac{\partial}{\partial\theta}(\sin\theta)\boldsymbol{a}_\theta + \frac{10}{r^3}\frac{\partial}{\partial\phi}(\cos\phi)\boldsymbol{a}_\phi\right]$$

$$= \frac{20}{r^3}\sin\theta\cos\phi\,\boldsymbol{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\boldsymbol{a}_\theta + \frac{10}{r^3}\sin\phi\,\boldsymbol{a}_\phi$$

$$\boldsymbol{E} = -\nabla V = \frac{20}{r^3}\sin\theta\cos\phi\,\boldsymbol{a}_r - \frac{10}{r^3}\cos\theta\cos\phi\,\boldsymbol{a}_\theta + \frac{10}{r^3}\sin\phi\,\boldsymbol{a}_\phi$$

(b.) $$\boldsymbol{E}(2,\pi/2,0) = \frac{20}{8}\boldsymbol{a}_r + 0\,\boldsymbol{a}_\theta + 0\,\boldsymbol{a}_\phi = \frac{20}{8}\boldsymbol{a}_r \ \text{(V/m)}$$

Summary of Electric Field / Potential Relationships

$V = -\int \boldsymbol{E}\cdot d\boldsymbol{l}$ Integrate $\boldsymbol{E}$ to find V

$\boldsymbol{E} = -\nabla V$ Differentiate V to find $\boldsymbol{E}$

Electric Flux Density

The *electric flux density* $\boldsymbol{D}$ in free space is defined as the product of the free space permittivity (ϵ_o) and the electric field ($\boldsymbol{E}$):

$$\boldsymbol{D} = \epsilon_o \boldsymbol{E}$$

Given that the electric field is inversely proportional to the permittivity of the medium, the electric flux density is independent of the medium properties.

Point charge

$$\boldsymbol{E} = \frac{Q}{4\pi\epsilon_o r^2}\boldsymbol{a}_r \longrightarrow \boldsymbol{D} = \frac{Q}{4\pi r^2}\boldsymbol{a}_r$$

Infinite line charge

$$\boldsymbol{E} = \frac{\rho_L}{2\pi\epsilon_o \rho}\boldsymbol{a}_\rho \longrightarrow \boldsymbol{D} = \frac{\rho_L}{2\pi \rho}\boldsymbol{a}_\rho$$

The units on electric flux density are

$$\frac{\mathrm{F}}{\mathrm{m}} \times \frac{\mathrm{V}}{\mathrm{m}} = \frac{\mathrm{C}}{\mathrm{m}^2}$$

The *total electric flux* Ψ passing through a surface *S*

The *total electric flux* (ψ) passing through a surface S is defined as the integral of the normal component of $\boldsymbol{D}$ through the surface.

$$\psi = \iint_S \boldsymbol{D}\cdot d\boldsymbol{s} = \iint_S \boldsymbol{D}\cdot \boldsymbol{a}_n ds = \iint_S D_n ds$$

where $\boldsymbol{a}_n$ is the unit normal to the surface S and D_n is the component of $\boldsymbol{D}$ normal to S. The direction chosen for the unit normal (one of two possible) defines the direction of the total flux.

$\boldsymbol{a}_n$ – outward (total outward flux)

$-\boldsymbol{a}_n$ – inward (total inward flux)

For a closed surface, the total electric flux is

$$\psi = \oint_S \boldsymbol{D}\cdot d\boldsymbol{s}$$

Example 4:

Given a flux density $D = 3xy\hat{a}_x + 4x\hat{a}_y + 4x\hat{a}_z$ mC/m². Find the amount of the outward electric flux passing through the following surface

S: z=0, 0≤x≤5, 0≤y≤3.

Solution

$$\psi = \iint_s \bar{D} \cdot d\bar{S} = \iint_S (3xy\hat{a}_x + 4x\hat{a}_y + 4x\hat{a}_z) \cdot \ dxdy\hat{a}_z$$

$$\psi = \iint_S \cancel{3xy\hat{a}_x \cdot \ dxdy\hat{a}_z}^{\,0} + \iint_S \cancel{4x\hat{a}_y \cdot \ dxdy\hat{a}_z}^{\,0} + \iint_S 4x\hat{a}_z \cdot \ dxdy\hat{a}_z$$

$$\psi = \int_0^5 4x dx \int_0^3 dy$$

$$= (\frac{4}{2}x^2\Big|_0^5)(y\Big|_0^3) = (2 \times 25)(3) = 15 \text{ mC/m}^2$$

Gauss's Law

Gauss's law is one of the set of four *Maxwell's equations* that govern the behavior of electromagnetic fields.

Gauss's Law - The total outward electric flux ψ through any closed surface is equal to the total charge enclosed by the surface.

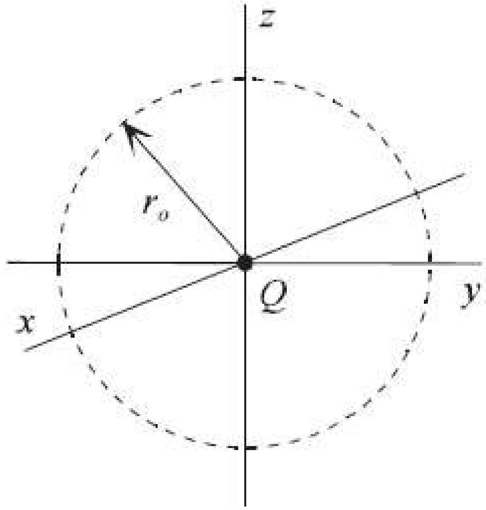
Gauss's law is written in equation form as

$$\psi = \oint_S \boldsymbol{D} \cdot d\boldsymbol{s} = Q_{enclosed} \qquad \text{(Gauss's law)}$$

where $d\boldsymbol{s} = \boldsymbol{a}_n\, ds$ and $\boldsymbol{a}_n$ is the *outward* pointing unit normal to S.

Example 5: (Gauss's law, point charge at origin)

- Given a point charge at the origin, show that Gauss's law is valid on a spherical surface (S) of radius r_o.



$$\psi = \oint_S \boldsymbol{D}\cdot d\boldsymbol{s} = Q_{enclosed} = Q$$

On the spherical surface S of radius r_o, we have

$$\boldsymbol{D}(r=r_o) = \frac{Q}{4\pi r_o^2}\boldsymbol{a}_r \qquad d\boldsymbol{s} = r_o^2 \sin\theta\, d\theta\, d\phi\, \boldsymbol{a}_r$$

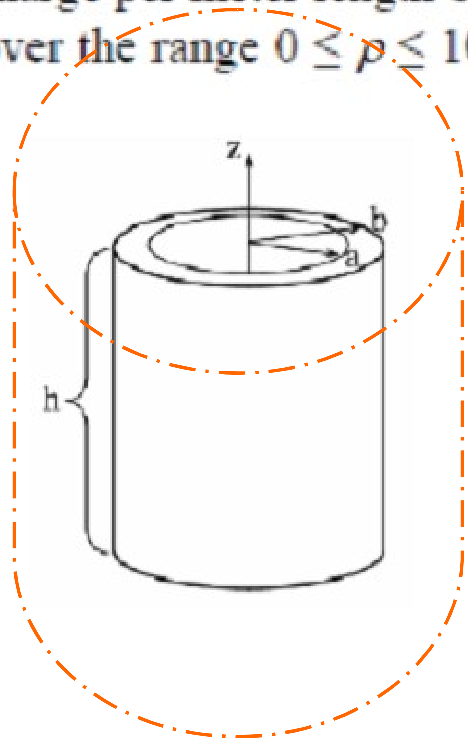
$$\boldsymbol{D}\cdot d\boldsymbol{s} = \left(\frac{Q}{4\pi r_o^2}\boldsymbol{a}_r\right)\cdot\left(r_o^2\sin\theta\, d\theta\, d\phi\, \boldsymbol{a}_r\right) = \frac{Q}{4\pi}\sin\theta\, d\theta\, d\phi$$

$$\psi = \oint_S \boldsymbol{D}\cdot d\boldsymbol{s} = \frac{Q}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\sin\theta\, d\theta\, d\phi = \frac{Q}{4\pi}\left[-\cos\theta\right]_0^{\pi}\left[\phi\right]_0^{2\pi}$$

$$= \frac{Q}{4\pi}(2)(2\pi) = Q \qquad \text{(charge enclosed)}$$

Example 6:

A cylindrical pipe with a 1.00 cm wall thickness and an inner radius of 4.00 cm is centered on the z-axis and has an evenly distributed 3.00 C of charge per meter length of pipe. Plot D_ρ as a function of radial distance from the z-axis over the range $0 \leq \rho \leq 10$ cm.



We need to find D_ρ everywhere; the idea is to relate D_ρ to radial distance, ρ through the charge enclosed by a Gaussian surface

We have

$$Q_{enc} = \oint \mathbf{D} \cdot d\mathbf{S} = \int D_\rho \mathbf{a}_\rho \cdot \rho \, d\phi \, dz \, \mathbf{a}_\rho$$

$$\boxed{Q_{enc} = 2\pi h \rho D_\rho} \longrightarrow (1)$$

* Equation (1) is valid for any Gaussian surface

For the 1st Gaussian surface

GS1 ($\rho < a$): since $Q_{enc} = 0$, $D_\rho = 0$.

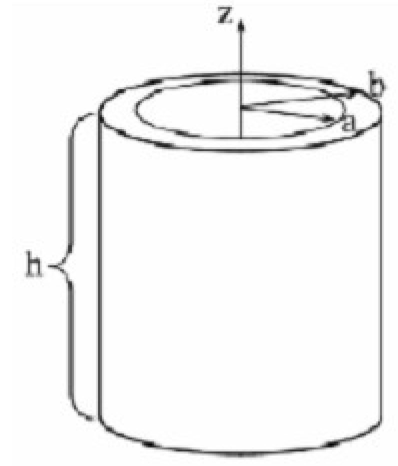
GS2$(a < \rho < b)$:

$$\rho_v = \frac{3h}{\iiint \rho d\rho d\phi dz} = \frac{3}{\pi\left(b^2 - a^2\right)}$$

$$Q_{enc} = \int \rho_v dv$$

$$= \frac{3}{\pi\left(b^2 - a^2\right)} \int_a^{\rho} \rho d\rho \int_0^{2\pi} d\phi \int_0^{h} dz$$

$$= 3h\frac{\left(\rho^2 - a^2\right)}{\left(b^2 - a^2\right)} \longrightarrow \quad (2)$$



So,

Equating (1) and (2) leads to

$$D_\rho = \frac{3h\left(\rho^2 - a^2\right)}{2\pi h\rho\left(b^2 - a^2\right)} = \frac{3}{2\pi\rho}\frac{\left(\rho^2 - a^2\right)}{\left(b^2 - a^2\right)}$$

For the 3rd region outside the pipe

GS3$(\rho > b)$:

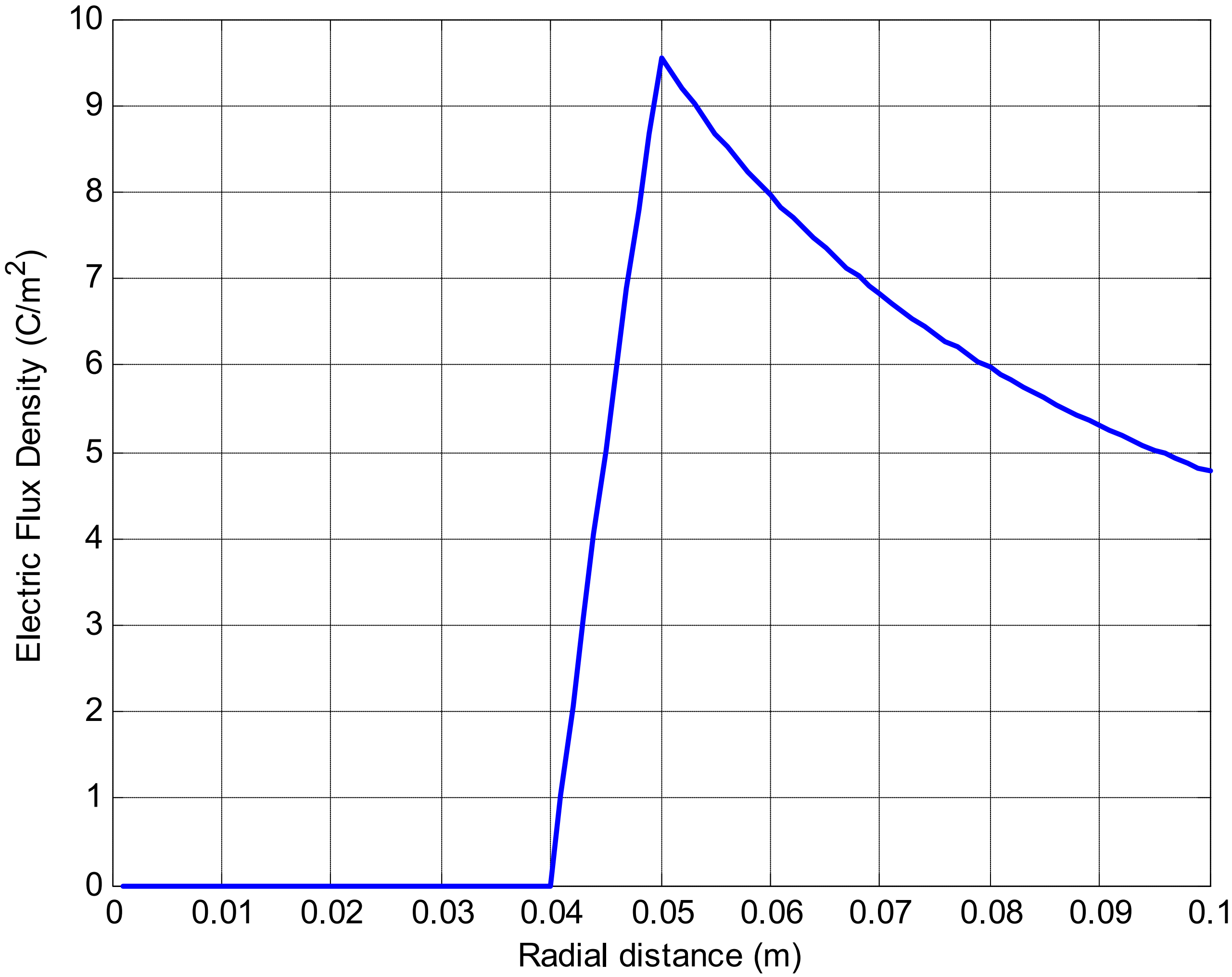
$$Q_{enc} = 3h, \longrightarrow \quad (3)$$

$$D_\rho = \frac{3}{2\pi\rho} \quad \text{for } \rho > b.$$

Matlab program to plot Dρ as a function of radial distance

```matlab

clc;clear;

%   initialize variables
a =.04;                              % Inner radius of pipe (m)
b =.05;                              % outer radius of pipe (m)
maxrad =0.10;                        % max radius for plot (m)
N =150;                              % number of data points
bnd_a = round(N*a/maxrad);           % Number of data pointa between 0 and a
bnd_b = round(N*b/maxrad);           % Number of data pointa between 0 and b
Interval_length = maxrad/N;
for i = 1 :1: bnd_a
    rho(i)= i*Interval_length;
    D(i)=0;
end

for i = bnd_a +1 :1: bnd_b
    rho(i)= i*Interval_length;
    D(i)=(3/(2*pi*rho(i)))*((rho(i)^2-a^2)/(b^2-a^2));
end

for i = bnd_b+1:N
    rho(i) = i*Interval_length;
    D(i)= 3/(2*pi*rho(i));
end

plot(rho,D,'LineWidth',2)
xlabel('Radial distance (m)')
ylabel('Electric Flux Density (C/m^2)')
grid on

```

The plot Dρ as a function of radial distance



Vector Analysis Using Matlab

- Given the points $M(0.1, -0.2, -0.1)$, $N(-0.2, 0.1, 0.3)$ and $P(0.4,0,0.1)$, write a MATLAB program to find:

  a) the vector $\mathbf{R}_{NM}$,

  b) the dot product $\mathbf{R}_{NM} \cdot \mathbf{R}_{PM}$,

  c) the projection of $\mathbf{R}_{NM}$ on $\mathbf{R}_{PM}$ and

  d) the angle between $\mathbf{R}_{NM}$ and $\mathbf{R}_{PM}$.

Vector Analysis Example

```
clc;                                        % Clear the command line
clear;                                      % Remove all previous variables

% Declaration of points' coordinates
O = [0 0 0];                                % The origin
M = [0.1 -0.2 -0.1];                        % Point M
N = [-0.2 0.1 0.3];                         % Point N
P = [0.4 0 0.1];                            % Point P
% Declaration of position vectors
R_MO = M-O;                                 % Vector R_MO
R_NO = N-O;                                 % Vector R_NO
R_PO = P-O;                                 % Vector R_PO

% Calculations of displacement vectors
R_NM = R_MO - R_NO                          % Vector R_NM
Mag_R_NM = norm(R_NM);                      % The magnitude of R_NM
R_PM = R_MO - R_PO;                         % Vector R_PM
Mag_R_PM = norm(R_PM);                      % The magnitude of R_PM

% Dot product calculations
R_PM_dot_R_NM = dot(R_PM,R_NM);             % The dot product of R_PM and R_NM


% Projection of R_NM_ON_R_PM (scalar)
Proj_R_NM_ON_R_PM = (R_PM_dot_R_NM / Mag_R_PM);

%the angle between R_PM and R_NM
COS_theta = R_PM_dot_R_NM /( Mag_R_PM * Mag_R_NM);

Theta = acos(COS_theta)*180/ pi;

fprintf(' The vector R_NM in Cartesian coordinates is < %.3f, %.3f, %.3f >\n', R_NM);
fprintf('\n The dot product of R_PM and R_NM =  %.3f \n', R_PM_dot_R_NM);
fprintf('\n The projection of R_NM_ON_R_PM =  %.3f \n', Proj_R_NM_ON_R_PM);
fprintf('\n The value of theta is %.3f degrees\n', Theta);
```

Matlab returns the following results:

```
R_NM =

    0.3000   -0.3000   -0.4000

 The vector R_NM in Cartesian coordinates is < 0.300, -0.300, -0.400 >

 The dot product of R_PM and R_NM =  0.050

 The projection of R_NM_ON_R_PM =  0.121

 The value of theta is 77.996 degrees
```

Announcements

- Homework 3 is assigned.
- More on Gauss's Law in section 2.7 in your textbook.