A Tutorial in Topological Data Analysis

Fall 2018 MAA NCS

Matthew Zabka

Mathematics and Computer Science
Southwest Minnesota State University

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Please clone or download the following folder:

https://github.com/MatthewZabka/MAA-NCS18.git

Overview

A short review

As we saw yesterday:

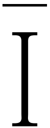
- Topology provides many descriptors about the 'shape' of a space.
- One of these descriptors – homology – is computable, which makes it particularly useful in applications.
- Given a topological space T the n-th homology group $H_n(T)$ – in very basic terms – has a generator for each n-dimensional hole in T.
- So the rank of the n-th homology group, called the n-th **Betti number**, denoted

  $$\beta_n := \text{rank}(H_n(T)),$$

  is the number of n-dimensional holes in T.
- β_1 counts the number of holes in T.
- β_2 counts the number of voids in T ...

A short review

- The n-th Betti number tells us about the number of n-dimensional holes in a space.
    - What is a 0-th dimensional hole?
    - β_0 counts the number of connected components in a space.
    - What is β_0 of the following space?



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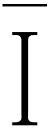

  - $\beta_0 = 2$

A short review

- The n-th Betti number tells us about the number of n-dimensional holes in a space.
  - β_1 counts the number of ‘holes’ in a space.
  - You can think of a hole as something through which you can stick you arm.
  - What are β_0 and β_1 of the following space?

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  - $\beta_0 = 3$ and $\beta_1 = 2$

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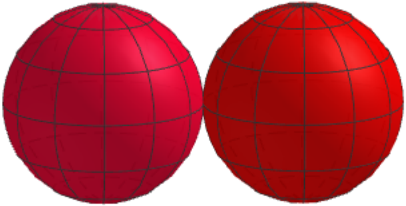
  - β_2 counts the number of voids.
  - What is β_2 of the following space?



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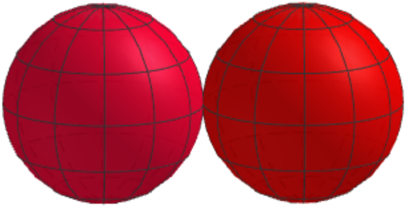

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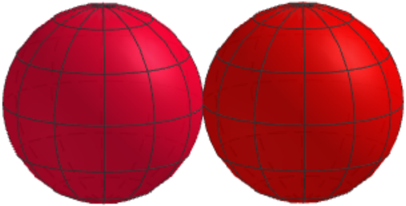

  - $\beta_2 = 2$
  - What are β_0 and β_1?
  - $\beta_0 = 1$ and $\beta_1 = 0$.

- Recall: Given a set of points X and a parameter r, we can create a formal simplicial complex.
- Let the n-simplex $\{x_0, x_1, \ldots, x_n\}$ exist if and only if $d(x_i, x_j) < r$ for all $0 \leq i, j \leq n$.
- We can visualize this process using InteractiveJPDwB.

- InteractiveJPDwB [3] lets one visualize the generated simplicial complex for data in $\mathbb{R}^2$ and different values of r.
- In other words, it demonstrates 0-th and 1-st degree persistence homology via barcodes.
- Thanks to Michael Catanzaro, you can easily install this program onto your computer.
- You can download this program from:

  https://github.com/MatthewZabka/MAA-NCS18.git

Discuss with your groupmates!

https://github.com/MatthewZabka/MAA-NCS18.git

- What is the minimum number of points required so that $\beta_1 = 1$ for some value of r?
- What is the minimum number of points required so that, for some r, we have $\beta_0 = 3$ and $\beta_1 = 2$?
- What is the largest degree of homology that is geometrically feasible in $\mathbb{R}^2$?
- What is the minimum number of points required to have $\beta_2 = 1$? In what dimension must the points lie?
- What is the minimum number of points required to have $\beta_n = 1$? In what dimension must the points lie?

Discussion

Persistent Homology in Dimension 0 (Clustering)

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- Increase r. Create an edge between two points whenever the distance between them is less than r. This creates a graph.
- The graph defines a simplicial complex via Vietoris-Rips.
- If a topological property (like a Betti number) *persist* over a large range of r, we can conclude something about the structure of the data.
- In this case, we should expect to see $\beta_0 = 2$ over a large range of r.

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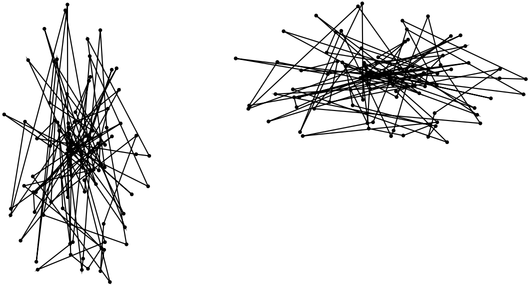

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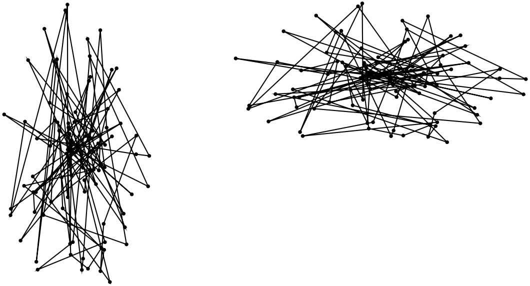

- A graph similar to this should persist over a large range of r.
- How could we see the clusters if these data did not lie in $\mathbb{R}^2$?

RIPSER

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- We shall first look at RIPSER[1].
- Developed by Ulrich Bauer, RIPSER is a very fast C++ program for computing Vietoris-Rips persistence barcodes.
- Let's try this on our data with two clusters.

RIPSER

- Go to github to get the data. These data are stored as a point cloud.

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- Input `data1.txt` into RIPSER:

  https://live.ripser.org/

- Suppose we have data that lie on the circle.

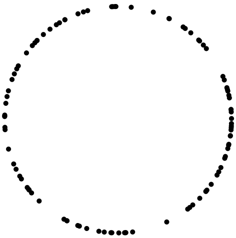

Figure: An unrealistic example `data2.txt`, where data lie perfectly on S^1.

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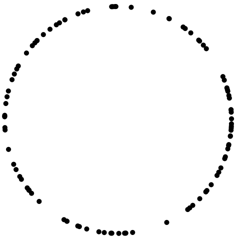

Figure: An unrealistic example `data2.txt`, where data lie perfectly on S^1.

- Input `data2.txt` into RIPSER.
- Wait . . . data are never this nice.

- Suppose we have data that **almost** lie on the circle.

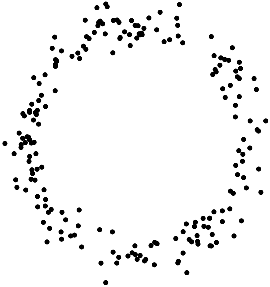

Figure: A slightly more realistic example `data3.txt`.

Persistent Homology

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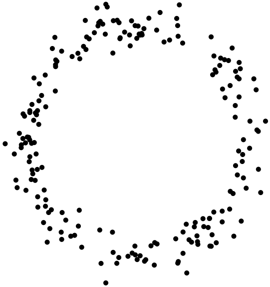


Figure: A slightly more realistic example `data3.txt`.

- Why is this also not realistic or impressive?

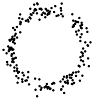


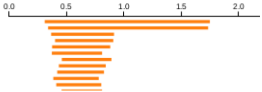
Figure: `data3.txt`.

- Now it is your turn!
- Input `data3.txt` into RIPSER. Confirm that one generator for H_0 and one generator for H_1 persist.
- Try inputting `data4.txt` into RIPSER up to distance 2. What can you say about the data's shape?
- Try some real data! `data5.txt` (up to distance 150) and `data6.txt` (up to distance 2)
- Data are located in the `data` folder that you have already downloaded!

Discussion

Combining Persistence and Other Methods

- Suppose we had a barcode in dimension 1 that looked as follows:



- What are are possibilities for the manifold on which the data lie?

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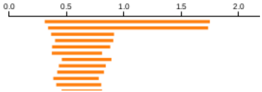

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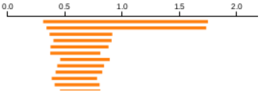
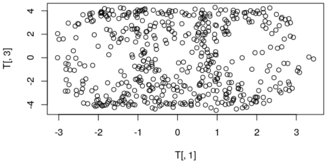


- Suppose we perform PCA and get the following projection



- What does PCA suggest about the dimension on the manifold?
- What does this suggest about the space on which the data lie?

RIVET

- Let’s think about how this could go wrong and how we could fix it!

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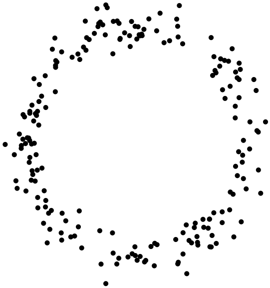
- Suppose that, instead of data that look like this:



- Let's think about how this could go wrong and how we could fix it!
- We had data that looked like this:

RIVET

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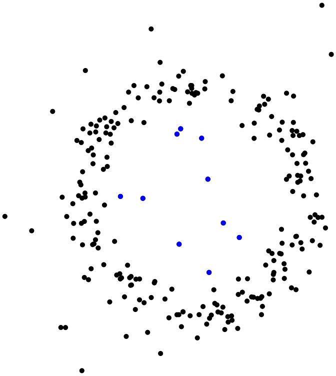
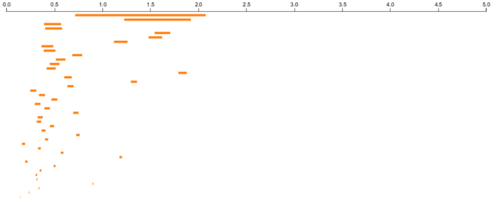
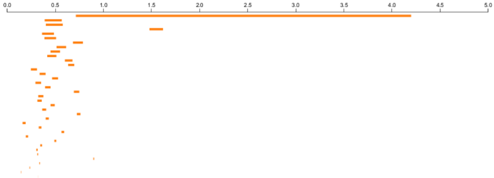
- The blue points – noise – will make it hard to see the generator of H_1.



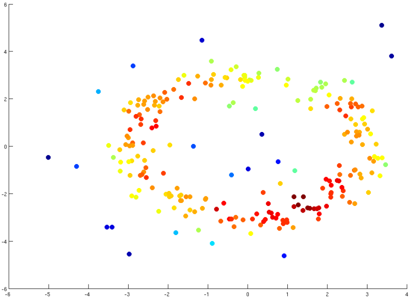
RIVET

The first barcode includes the entire data set. The second barcode eliminates the blue 'noise'.





- One way to deal with such situations: consider the density of the points.



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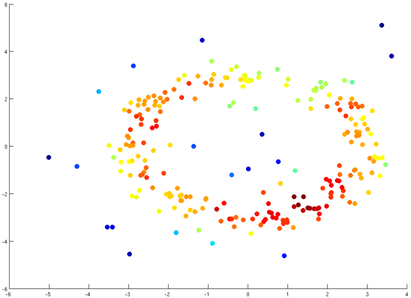

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- Letting the density threshold vary results in **multi-parameter persistence**!

RIVET

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RIVET

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- Installation is not so simple – let us go through it together!

[1] U. Bauer.
Ripser: a lean C++ code for the computation of vietoris-rips persistence barcodes.
https://github. com/Ripser/ripser, 2017.

[2] M. Lesnick and M. Wright.
Interactive visualization of 2-d persistence modules.
arXiv preprint arXiv:1512.00180, 2015.
http://rivet.online/.

[3] L. Wolcott.
InteractiveJPDwB: an interative program for persistence homology.
https://github.com/lukewolcott/InteractiveJPDwB, 2016.