Homework 4

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1.a) $g(x,y) := f(x_0,y_0) + \left[f_x(x_0,y_0),\ f_y(x_0,y_0)\right]\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + \frac{1}{2}\left[x-x_0,\ y-y_0\right]\begin{bmatrix} f_{xx}(x_0,y_0), & f_{xy}(x_0,y_0) \\ f_{xy}(x_0,y_0), & f_{yy}(x_0,y_0) \end{bmatrix}\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$

$1\times 2 \times 2\times 1 \to 1\times 1$ $\qquad$ $1\times 2 \times 2\times 2$ $\qquad$ 1×1 $\qquad$ 1×2 $\qquad$ 2×1

$= f(x_0,y_0) + \left(f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)\right) + \left[\frac{(x-x_0)}{2}f_{xx}(x_0,y_0) + \frac{(y-y_0)}{2}f_{xy}(x_0,y_0),\ \frac{(x-x_0)}{2}f_{xy}(x_0,y_0) + \frac{(y-y_0)}{2}f_{yy}(x_0,y_0)\right]\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$

$= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \frac{(x-x_0)^2}{2}f_{xx}(x_0,y_0) + \left(\frac{(y-y_0)(x-x_0)}{\cancel{2}}f_{xy}(x_0,y_0)\right)\cancel{2} + \frac{(y-y_0)^2}{2}f_{yy}(x_0,y_0)$

$g(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) + \frac{(x-x_0)^2}{2}f_{xx}(x_0,y_0) + (y-y_0)(x-x_0)f_{xy}(x_0,y_0) + \frac{(y-y_0)^2}{2}f_{yy}(x_0,y_0)$

OR

$$\boxed{g(x,y) = A + (x-x_0)B + (y-y_0)C + \frac{(x-x_0)^2}{2}E + (y-y_0)(x-x_0)F + \frac{(y-y_0)^2}{2}G}$$

b) $\nabla g = \langle g_x, g_y, g_z \rangle$

$= \langle f_x + f_{xx}\times(x-x_0) + f_{xy}\times(y-y_0) + f_{xxx}\times\frac{(x-x_0)^2}{2} + f_{xyx}\times(y-y_0)(x-x_0) + f_{yyx}\times\frac{(y-y_0)^2}{2},$

$f_y + f_{xy}\times(x-x_0) + f_{yy}\times(y-y_0) + f_{xxy}\times\frac{(x-x_0)^2}{2} + f_{xyy}\times(y-y_0)(x-x_0) + f_{yyy}\times\frac{(y-y_0)^2}{2} \rangle$

OR

$$\boxed{\begin{aligned}\nabla g = \langle\ & B + (x-x_0)E + (y-y_0)F + \frac{(x-x_0)^2}{2}E_x + (y-y_0)(x-x_0)F_x + \frac{(y-y_0)^2}{2}G_x, \\ & C + (x-x_0)F + (y-y_0)G + \frac{(x-x_0)^2}{2}E_y + (y-y_0)(x-x_0)F_y + \frac{(y-y_0)^2}{2}G_y \rangle\end{aligned}}$$

for $H(g)$ $g_{xx} = E + (x-x_0)E_x + (y-y_0)C_{xx} + \frac{(x-x_0)^2}{2}E_{xx} + (y-y_0)(x-x_0)F_{xx} + \frac{(y-y_0)^2}{2}G_{xx}$

$g_{xy} = F + (x-x_0)B_{xy} + (y-y_0)C_{xy} + \frac{(x-x_0)^2}{2}E_{xy} + (y-y_0)(x-x_0)F_{xy} + \frac{(y-y_0)^2}{2}G_{xy}$

$g_{yy} = G + (x-x_0)B_{yy} + (y-y_0)G_y + \frac{(x-x_0)^2}{2}E_{yy} + (y-y_0)(x-x_0)F_{yy} + \frac{(y-y_0)^2}{2}G_{yy}$

$$H(g)=\begin{bmatrix} E+(x-x_0)E_x+(y-y_0)C_{xx}+\frac{(x-x_0)^2}{2}E_{xx}+(y-y_0)(x-x_0)F_{xx}+\frac{(y-y_0)^2}{2}G_{xx}, & F+(x-x_0)B_{xy}+(y-y_0)C_{xy}+\frac{(x-x_0)^2}{2}E_{xy}+(y-y_0)(x-x_0)F_{xy}+\frac{(y-y_0)^2}{2}G_{xy} \\ F+(x-x_0)B_{xy}+(y-y_0)C_{xy}+\frac{(x-x_0)^2}{2}E_{xy}+(y-y_0)(x-x_0)F_{xy}+\frac{(y-y_0)^2}{2}G_{xy}, & G+(x-x_0)B_{yy}+(y-y_0)G_y+\frac{(x-x_0)^2}{2}E_{yy}+(y-y_0)(x-x_0)F_{yy}+\frac{(y-y_0)^2}{2}G_{yy} \end{bmatrix}$$

$\nabla f(x_0,y_0)=\langle B, C\rangle \rightarrow$ uses the quadratic approximation form @B and C

$f(x_0,y_0)=A \rightarrow$ we see that $g(x_0,y_0)$ uses the quadratic aproximation form @ A

$H(f)=\begin{bmatrix} E & F \\ F & G\end{bmatrix} \rightarrow$ we see the quadratic approximation of E, F, and G in $H(g)$

c) Because this is a quadratic and parabolic it will only have one critical point (min or max).

d) We know the definition of critical point is

$$g_x(x,y)=g_y(x,y)=0$$

$$g(x,y)=A+(x-x_0)B+(y-y_0)C+\frac{(x-x_0)^2}{2}E+(y-y_0)(x-x_0)F+\frac{(y-y_0)^2}{2}G$$

$$g_x(x_1,y_1)=B+(x_1-x_0)E+(y_1-y_0)C_x+\frac{(x_1-x_0)^2}{2}E_x+(y_1-y_0)(x_1-x_0)F_x+\frac{(y_1-y_0)^2}{2}G_x=0$$

Compare to expanded form given

$$\frac{1}{EG-F^2}\begin{bmatrix} G & -F \\ -F & E\end{bmatrix}\begin{bmatrix} B \\ C\end{bmatrix}-\begin{bmatrix} x_0 \\ y_0\end{bmatrix}+\begin{bmatrix} x_1 \\ y_1\end{bmatrix}=0$$

$$\frac{1}{EG-F^2}\begin{bmatrix} GB-FC \\ -FB+CE\end{bmatrix}+\begin{bmatrix} x_1-x_0 \\ y_1-y_0\end{bmatrix}=0$$

$$\begin{bmatrix} \frac{GB}{EG-F^2}-\frac{FC}{EG-F^2} \\ \frac{-FB+CE}{EG-F^2}\end{bmatrix}+\begin{bmatrix} x_1-x_0 \\ y_1-y_0\end{bmatrix}=\begin{bmatrix} 0 \\ 0\end{bmatrix}$$

$$x_1-x_0=\frac{FC-GB}{EG-F^2}$$

$$y_1-y_0=\frac{FB-CE}{EG-F^2}$$

$$g_x(x_1,y_1)=B+(x_1-x_0)E+(y_1-y_0)C_x+\frac{(x_1-x_0)^2}{2}E_x+(y_1-y_0)(x_1-x_0)F_x+\frac{(y_1-y_0)^2}{2}G_x=0$$

$$g_x(x_1, y_1) = B + (x_1 - x_0)E + (y_1 - y_0)C_x + \frac{(x_1-x_0)^2}{2}E_x + (y_1-y_0)(x_1-x_0)F_x + \frac{(y_1-y_0)^2}{2}G_x = 0$$

Compare against

$$0 \stackrel{?}{=} B + \left(\frac{FC-GB}{EG-F^2}\right)E + \left(\frac{FB+CE}{EG-F^2}\right)C_x + \left(\frac{(FC-GB)^2}{2(EG-F^2)^2}\right)E_x + \left(\frac{FC-GB}{EG-F^2}\right)\left(\frac{FB+CE}{EG-F^2}\right)F_x + \frac{(FB+CE)^2}{2(EG-F^2)^2}G_x$$

$$0 \stackrel{?}{=} B + \frac{E(FC-GB) + C_x(FB+CE)\times 2}{EG-F^2 \times 2} + \frac{E_x(FC-GB)^2 + G_x(FB+CE)^2 + 2(FC-GB)(FB+CE)F_x}{2(EG-F^2)^2}$$

$$0 \stackrel{?}{=} B + \frac{2E(FC-GB) + 2C_x(FB+CE) + E_x(F_C-GB)^2 + G_x(FB+CE)^2 + 2(FC-GB)(FB+CE)F_x}{2(EG-F^2)^2}$$

$\hookrightarrow$ this is not a good approach.

Approach 2:

$g(x)$ simplified

$$g_x = 0 = B + E(x-x_0) + F(y-y_0)$$

$$g_y = 0 = C + G(y-y_0) + F(x-x_0)$$

$$\left.\begin{aligned} Ex_1 + Fy_1 &= Ex_0 + Fy_0 - B \\ Fx_1 + Gy_1 &= Gy_0 + Fx_0 - C \end{aligned}\right\} \begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$= \begin{bmatrix} E & F \\ F & G \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} B \\ C \end{bmatrix}$$

$$\underbrace{\cancel{\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}\begin{bmatrix} E & F \\ F & G \end{bmatrix}}}_{I}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \overbrace{\cancel{\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}\begin{bmatrix} E & F \\ F & G \end{bmatrix}}}^{I}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}\begin{bmatrix} B \\ C \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = -\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}\begin{bmatrix} B \\ C \end{bmatrix} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

e)

$$\boxed{\begin{aligned} x_1 &= \frac{(FC-GB)}{(EG-F^2)} + x_0 \\ y_1 &= \frac{(FB-CE)}{EG-F^2} + y_0 \end{aligned}}$$

f) g has a min when $g_x = g_y = 0$

$g_{xx} > 0$

and

$g_{xx}g_{yy} - g_{xy}^2 > 0$

so

$$\boxed{\begin{aligned} &B = C \\ &E > 0 \\ &\text{and} \\ &E \times G - F^2 > 0 \end{aligned}}$$

g) to find x_1 and y_1 we need to find

A $f(x_0, y_0) = \sin(\pi x) + y^2 - y + xy$

B $f_x = y + \pi \cos(\pi x)$

C $f_y = x + 2y - 1$

E $f_{xx} = -\pi^2 \sin(\pi x)$

G $f_{yy} = 2$

F $f_{xy} = 1$

$$x_1 = \frac{(FC - GB)}{(EG - F^2)} + x_0 = \frac{((x+1) \times 1) - 0)}{(0 - 1^2)} + x_0 = 1.5267$$

$$y_1 = \frac{(FB - CE)}{EG - F^2} + y_0 = \frac{((1)(y_0 + \pi \cos(\pi x_0))) - (x_0 + 1)(-\pi^2 \sin(\pi x_0)))}{0 - 1^2} + y_0 = -0.2633$$

$$\boxed{(x_1, y_1) = (1.5267, -0.2633)}$$

by plugging into $g(x, y)$ from 1, $\boxed{g(x, y) = -1.0658}$