Introductory Microeconomics Notes

April 2024

Abstract

Disclaimer: These notes are entirely interpretations of resources (and definitely contains mistakes) to which it is greatly indebted: lecture slides and lectures by Ian Crawford and Sanjay Jain; *Intermediate Microeconomics with Calculus* by Hal Varian; *Advanced Microeconomic Theory* by Jehle and Reny

Contents

1 Introduction to Economics; Specialisation and Trade; Technological Progress

1.1 Introduction to Economics

Definition. Opportunity Cost: The net benefit of the next best alternative forgone

Corollary. Cost includes both explicit (accounting) cost and implicit (opportunity) cost

1.2 Specialisation and Trade

Definition. Absolute Advantage: refers to the ability of an agent (individual/firm/country) to produce a greater quantity of something than competitors, *using the same amount of resources.*

Definition. Comparative Advantage: A country is said to have comparative advantage in some arbitrary good X if and only if it has a lower opportunity cost in producing that good.

If an appropriate terms of trade (the amount of good traded for another, or 'relative price') were then chosen, both countries could end up with more of both goods are specialisation and free trade than they each had before trade. Country A may still benefit from free trade even though it is assumed to be technologically inferior to Country B in the production of everything (aboluste disadvantage in both products).

Definition. Production Possibility Frontier: The production possibility set embodies the feasible alternatives. The boundary of the production possibility set is the PPF.

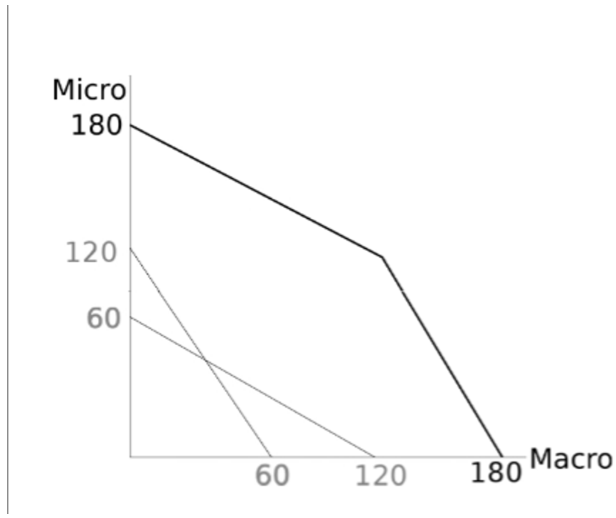
- At any point strictly inside the production possibility set, it is possible to increase the production of everything
- On the boundary this is no longer possible; so the boundary corresponds to output combinations which are technically efficient

The slope of the PPF is called the **marginal rate of transformation**. It reflects the opportunity cost as it describes what must be given up in order to acquire more of something else. Typically concave toward the origin to reflect diminishing marginal returns.

This gives us insight into benefits of trade and the existence of firms since there is heterogeneity in productive capacity and opportunity costs.

Result of Specialisation

We draw the joint PPF.



Example. Imagine a scenario where Anne can give 120 Micro Tutorials or 60 Macro Tutorials in 60 hours. We compute her opportunity cost of Macro tutorials as

$$\frac{\partial Mi}{\partial Ma} = -2$$

That is, for every units increase in Macro tutorials, she needs to give up 2 units of Micro tutorials. Conversely, Bob can give 120 Macro Tutorials or 60 Micro Tutorial in 60h. His OC is

$$\frac{\partial Mi}{\partial Ma} = -\frac{1}{2}$$

Bob has a comparative advantage in Macro lectures while Anne has a comparative advantage in Micro lectures.

Note: the Opportunity cost of X is the variable that one is differentiating with respect to.

If they specialise, there is an outward shift in both their PPFs. Intuition: If they both dedicate all their time to Micro, they will produce 180 Micro. If they were to increase the amount of time of Macro, the person to do so is Bob since he has the CA there. Bob dedicates all his time to Macro while Anne to Micro, yielding a mutually beneficial optimal output of $\{120, 120\}$ (utility given by indifference curve is higher).

Holds for absolute advantage. Trade can be beneficial even when one party is less efficient than the other.

Sources of comparative advantage

1. Mobility of factors (not fixed factors)
   - Can more easily substitute productive processes for good B with good A's (less opportunity cost)
   - Financial capital is highly mobile, followed by labour (pending immigration flows), and lastly natural resources
2. Legal system which enforces property rights and contracts
3. Fixed factors give particular regions a comparative advantage in the production of some kind of goods and not in others

Trade Theory

- What it can explain

1. Suggests that some countries with the right weather and large land mass should export agricultural goods
2. Suggests that complex technical goods should be created in developed nations with highly educated workforces
3. Suggests that natural resources should be exported by less developed nations

- What it cannot explain
  1. Suggests that there should be more trade between developed and underdeveloped nations than between developed and other developed nations (intra-industry trade)
  2. But not the case as most trade is between developed nations — intra-industry trade
  3. Productive heterogeneity provides only the incentive to trade — needs people to have notice the potential and ability to act on it — politics plays a big role

1.3 Technological Progress and Production

Definition. Technology: (the process of) converting inputs into outputs.

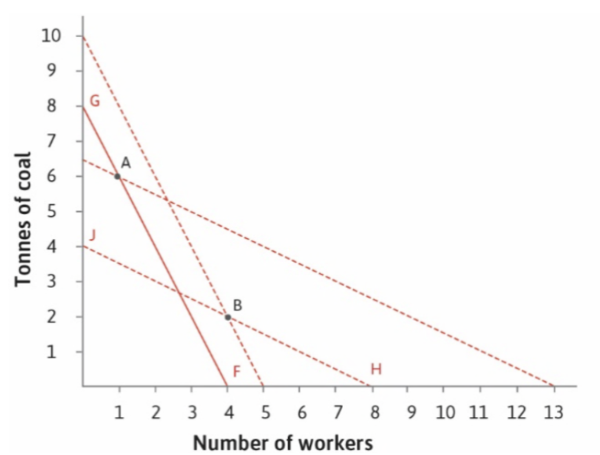
Definition. Dominated/Non-dominated technologies: Technologies that are able to use less of both (or less of one and equal of another input) are dominated by other technologies. So we only choose between non-dominated technologies.

Definition. Iso-cost lines: combination of inputs that give the same total cost

$$C = wL + pR$$
$$R = \frac{C}{p} - \frac{w}{p}L$$

So $\frac{w}{p}$ is the relative price of the two inputs.

How relative prices changes impacts how one chooses technologies/production processes. Consider



Initially, the technology in use was labour intensive during the Industrial Revolution. But the increase in wages relative to price of coal leads to a steeper slope of the iso-cost line, implying that more energy intensive technologies were chosen.

2 Exchange: Demand & Supply; Comparative Statics; Surplus and Welfare

2.1 Demand

Definition. Willingness to Pay: the value of an extra unit acquired (such that it is the price the buyer is willing and able to purchase the next unit at). WTP is heterogeneous along the dimension of different people and along the dimension of quantity.

Definition. Reservation Price: the amount the consumer is willing to pay for the (infinitesimally small) first unit of the good.
Example (Demand Curve with no finite reservation price). Suppose that the individual's demand curve is

$$\ln q = c + d\ln p, \quad \text{where } c > 0, d < 0$$

Then,

$$\ln p = -\frac{c}{d} + \frac{1}{d}\ln q$$
$$p = e^{-\frac{c}{d}} q^{\frac{1}{d}}$$

As $q \to 0$ then $p \to \infty$, so the consumer has no finite reservation price.

The horizontal intercept indicates that the consumer has a satiation point for the good/maximum valuation, since further units would have negative value.
Marginal value of a good determines the slope:

- Diminishing value: $\frac{dWTP}{dq} < 0$
- Increasing value: $\frac{dWTP}{dq} > 0$

WTP schedule for discrete goods is characterised by 'steps': price is constant for $q \in [x, x+1)$. Continuous schedule is a function: as long as it is strictly monotonic, we have a unique relationship between price and quantity. The demand curve is thus well-defined.

Definition. Demand Curve: The demand curve is quantity as a function of price

$$q = q(p) = WTP^{-1}(p)$$

Definition. Inverse Demand Curve: The inverse demand curve is price as a function of quantity

$$p = p(q) = WTP(p)$$

Definition. Consumer Surplus: Consumer Surplus is the difference between the maximum price the consumer is willing and able to pay and the price he actually pays. We compute it by

$$CS = \int_0^q p(q)dq$$

From individual demand to Market Demand: Let $x_i^1(p_1, p_2, m_i)$ represent consumer i's demand function for good 1 and $x_i^2(p_1, p_2, m_i)$ represent the same consumer's demand for good 2. Then the market demand for good 1 is the sum of all these individual demands over all consumers (and for good 2 too).

$$X^1(p_1, p_2, m_1, \dots, m_n) = \sum_{i=1}^{n} x_i^1(p_1, p_2, m_i)$$

Note: Horizontal summation of individual demand curves (to keep price fixed).

Definition. (Price) Elasticity of Demand: Percentage change in quantity due to a percentage change in price s.t.

$$\epsilon = \frac{\%\Delta q}{\%\Delta p}$$

Derivation.

$$\begin{aligned}
\epsilon &= \lim_{h \to 0} \frac{\frac{q(p+h)-q(p)}{q(p)}}{\frac{p+h-p}{p}} \\
&= \lim_{h \to 0} \frac{q(p+h)-q(p)}{h} \frac{p}{q(p)} \\
&= \frac{dq}{dp}\frac{p}{q}
\end{aligned}$$

$$\text{At } q = 0, \lim_{q \to 0} \frac{p}{q} = \infty, \epsilon \to -\infty$$

$$\text{At } p = 0, \lim_{p \to 0} \frac{p}{q} = 0, \epsilon \to 0$$

When the demand curve is perfectly horizontal, the demand is perfectly elastic since the price change is always 0 and thus $\epsilon = -\infty$
When the demand curve is perfectly vertical, the demand is perfectly inelastic since the quantity change is always 0 and thus $\epsilon = 0$.

Alternative mathematical formalisation: the price elasticity of demand is simply the derivative of a logarithmic function where the y-axis is ln$y$ and the x-axis is ln$x$
Derivation.

$$\epsilon = \frac{d\text{ln}y}{d\text{ln}x} = \frac{dy}{dx}\frac{x}{y}$$

$$f(x) = \text{ln}x \implies f'(x) = \frac{d\text{ln}x}{dx} = \frac{1}{x}$$

$$d\text{ln}x = \frac{dx}{x}$$

$$d\text{ln}y = \frac{dy}{y}$$

Example (Demand with constant price elasticity). Let us assume $\bar{\epsilon}$ is some fixed value,

$$\bar{\epsilon} = \frac{p}{q}\frac{dq}{dp}$$

$$\frac{\bar{\epsilon}}{p} = \frac{1}{q}\frac{dq}{dp}$$

$$\int \frac{\bar{\epsilon}}{p} dp = \int \frac{1}{q} dq$$

$$\bar{\epsilon}\text{ln}p + C = \text{ln}q$$

$$q = e^{(\bar{\epsilon}\text{ln}p + C)}$$

Given that $\bar{\epsilon}^C$ is some arbitrary constant, where $A = \bar{\epsilon}^C$, we have

$$q = Ap^{\bar{\epsilon}}$$

Relationship between elasticity and revenue of the producer. We can take the definition of revenue

$$R = pq(p)$$

To see the change in revenue due to a change in price we have

$$\frac{dR}{dp} = q(p) + pq'(p)$$
$$\frac{dR}{dp} = q(p)\left[1 + \frac{p}{q(p)}q'(p)\right]$$
$$\frac{dR}{dp} = q(p)[1 + \epsilon(p)]$$

From this we can see that

- When $|\epsilon| > 1$, $\frac{dR}{dp} < 0$
- When $|\epsilon| < 1$, $\frac{dR}{dp} > 0$

Marginal revenue due to a change in quantity. This time, we take price as a function of quantity,

$$\frac{dR}{dq} = p(q) + p'(q)q$$
$$\frac{dR}{dq} = p(q)\left[1 + q\frac{p'(q)}{p(q)}\right]$$

By the inverse function theorem, if a function is monotonic and differentiable,

$$p'(q) = \frac{1}{q'(p)}$$
$$\frac{dR}{dq} = p(q)\left[1 + \frac{1}{\epsilon(q)}\right]$$

2.2 Supply

Definition. Willingness to Accept: the value of an extra unit parted with (such that it is the price the seller is willing and able to sell the next unit at).

Reservation price is simply $p = WTA(0)$

Definition. Producer Surplus: the difference between the minimum price the producer is willing and able to part the good at. We compute it as

$$PS = \int_0^q p(q)dq$$

Definition. (Price) Elasticity of Supply): Price Elasticity of Supply is percentage change in quantity for a percentage increase in price, s.t.

$$\eta = \frac{\%\Delta q}{\%\Delta p}$$

Derivation.

$$\begin{aligned}\eta &= \lim_{h\to 0} \frac{\frac{q(p+h)-q(p)}{q(p)}}{\frac{p+h-p}{p}} \\ &= \lim_{h\to 0} \frac{q(p+h)-q(p)}{h}\frac{p}{q(p)} \\ &= \frac{dq}{dp}\frac{p}{q}\end{aligned}$$

$$\text{At } q = 0, \lim_{q\to 0}\frac{p}{q} = \infty, \eta \to \infty$$

When the supply curve is perfectly horizontal, the supply is perfectly elastic since the price change is always 0 and thus $\eta \to \infty$; when the supply curve is perfectly verticle, the supply is perfectly price inelastic since the quantity change is always 0 and thus $\eta = 0$.

2.3 Equilibrium and market interventions

Suppose we have linear supply and demand curves

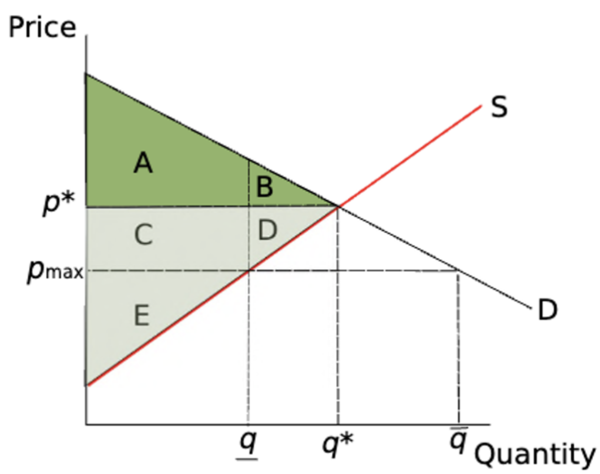
$$D(p) = a - bp$$
$$S(p) = c + dp$$

Solving, we yield the equilibrium

$$D(p) = S(p)$$
$$a - bp = c + dp$$
$$p^* = \frac{ad + bc}{b + d}$$

Taxation and market intervention

1. **Price Ceiling:** Setting a maximum price in the market. Often justified on distributional grounds, because it is often welfare improving for a particular groups of consumers.



At the market clearing price

- CS = A + B
- PS = C + D + E

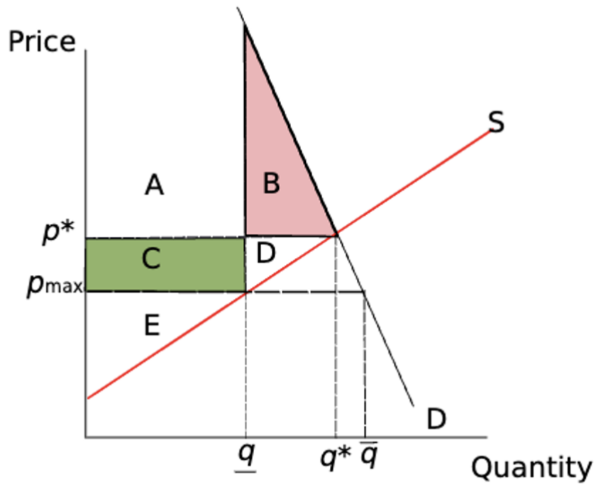
Under the new price control the price falls

- CS' = A + C
- ΔCS = C - B

The area C represents a transfer from producers (who now get less per unit on all of the units sold) to consumers (who make an additional surplus on all the units bought).
The area B is a loss of consumer surplus representing units which are no longer offered for sale by the

producers → Consumers who were buying these units suffer a loss.

Effects on consumers: Consumer gains depends on whether C > B or vice versa, which depends on price elasticity of demand.
If demand is very inelastic s.t. $|\epsilon| < |eta|$, then the loss of surplus caused by the drop in supply (B) outweights the gain due to the lower price paid for units which are supplied (C): Consumers incur a loss.



Intuition: Consumers want this product at any price and would prefer to pay any price rather than to go without. Price ceiling therefore hurts them overall.
Otherwise, if $|\epsilon| > |\eta|$ the consumers would gain.

Effects on Producers: Under the new price control,

- PS' = E
- ΔPS = -(C+D)

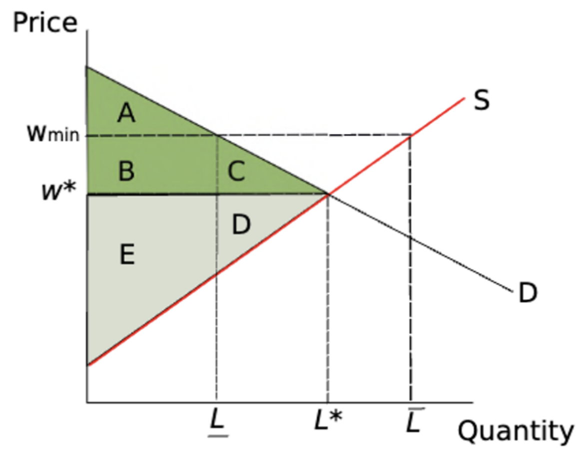
Area D is a loss of producer surplus representing units which are no longer offered for sale by the producers. So clearly, producers lose overall.

Overall welfare loss: Area C is simply a redistribution towards consumers. Overall loss of surplus is incurred as

$$DWL = \Delta CS + \Delta PS = -(B + D)$$

Reflects loss of voluntary and mutually advantageous trades which would otherwise have taken place

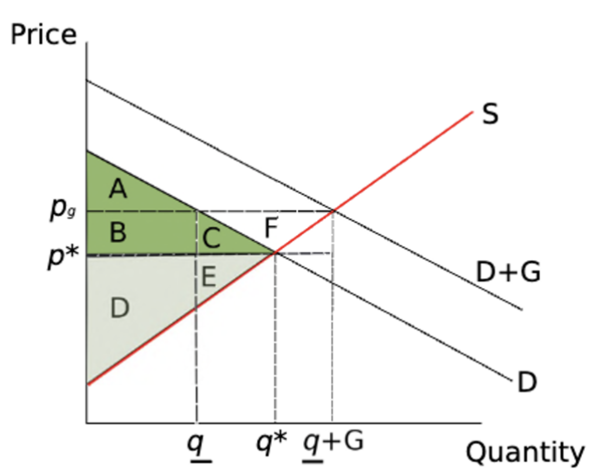
2. **Price Floors:** Minimum wages.
For labour markets: producers are individual workers who are supplying their time and labour consumers are firms who are buying time and their labour.



The market clearing wage w^* is associated with an amount of labour being traded equal to L^* and consumer surplus = A+B+C and producer surplus = D+E.
New CS = A and New PS = B + E.
$\Delta CS = -(B + C), \Delta PS = B - D$
There is a transfer B from consumers (firms) to suppliers (workers) and this benefits the workers who continue to be employed.
But there is a loss of both PS and CS due to the reduction in trades (C and D) from workers who are no longer hired at the new higher wage.
Effectiveness of the minimum wage

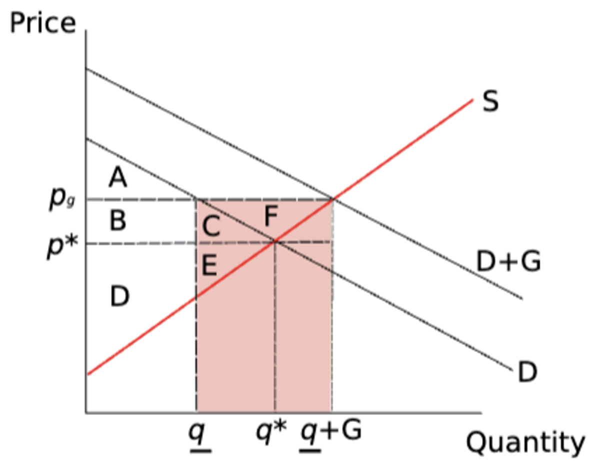
- Whether the increase in surplus for workers who keep their jobs (B) outweighs the loss in surplus associated with lost employment (D) depends upon economic fundamentals: price elasticity
- If ($|\eta| < |\epsilon|$), then the workers would like to work whatever the wage, the loss in employment is not compensated for by the wage increase (D>B); Minimum wage in this case is not beneficial

3. **Price Support:** Government becoming a player in the market themselves to support the value of the goods. For instance, occurred when governments around the world stepped into the financial markets to buy up the financial assets of banks which were in trouble.



CS = A + B + C, PS = D + E.
Government enters the market and buys up to G units whatever the price and the new demand curve

is D + G.
New equilibrium price is now raised to p_g and trade volumes also rise to $\underline{q} + G$.
Effects on consumers: Government is competing against private buyers and that is bad for those buyers.
To measure the effect, we look at the private sector's demand curve and establish that $\Delta CS = -B - C$.
B reflects the effects of the higher price on the private consumers who stay in the market, and C measures the loss associated with consumers who exit.
Effects on producers: Producers are happy as both B and C are transferred to firms and $\Delta PS = B + C + F$
Cost to government:



Cost is equal to the cost of units purchased s.t.

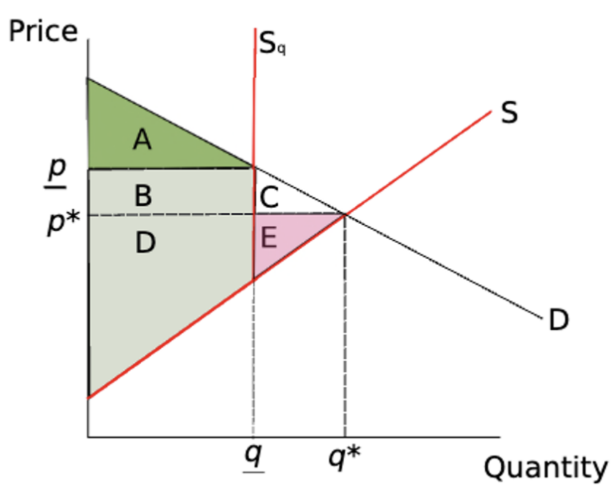
$$\text{Government cost} = p_g G$$

So total cost to society is the sum

$$\Delta CS + \Delta PS + \text{Government Cost} = F - p_g G$$

Much less costly to just give firms the money (equal to B+C+F) instead of trying to support the price; More obvious too but may not be politically expedient or may break state aid rules

4. **Quotas:**

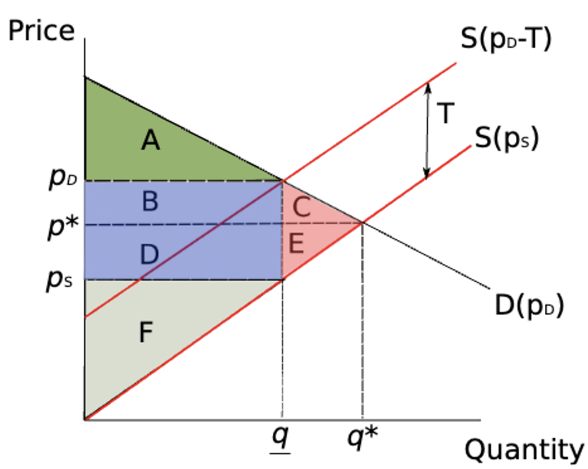


Cuts off supply at the quota $\underline{q}$ (the supply curve becomes S_q) and raises the price from the competitive level up to $\underline{p}$.
We have $\Delta CS = -(B+C), \Delta PS = B - E, DWL = -(C+E)$.
As usual, B is a straight transfer from consumers to firms.
Whether the firms in the aggregate gain or not depends on the elasticity of supply (whether B > E).
If supply is inelastic, then the rise in price would lead to a less than proportional fall in quantity → so E will expand less and B will contract less → higher chance that B > E

5. **Taxes:** When a commodity is subject to a tax, the tax forms a wedge between the price the consumer pays p_D and the price which the supplier receives p_S.
   Two types of taxes:

   - Specific duties of T pounds per unit sold
   $$p_D = p_S + T$$
   - Ad valorem taxes of t percent
   $$p_D = (1+t)p_S$$

   Consumers make their decisions on the basis of p_D and the suppliers on the basis of p_S.
   Definition. Formal Incidence: The formal incidence of a tax is upon the entity which is responsible for remitting the tax.
   Definition. Effective Incidence: The effective incidence of a tax is upon the entity which bears the economic cost i.e. the relative burden of the tax.
   Case 1 (Formal incidence on Supplier): The formal incidence of both specific duties and ad valorem taxes is typically on the supplier — they have to collect and remit it to the government.
   So we think of this as the consumer paying p_D and the supplier receiving this, then remitting T to the government and netting $p_D - T$. Hence, the equilibrium condition is
   $$D(p_D) = S(p_S = p_D - T)$$
   Intuition is even though the equilibrium would be at p_D as per the graph, the producer would only receive $p_S = p_D - T$. So the producer would only receive $p_D - T$ at every price level the consumer is willing to pay.



   Analysing effects:

   - Without the tax:
   Demand curve is $D(p_D)$ and supply curve is $S(p_S)$, and equilibrium is when $p_D = p_S$.

Market clearing price is p^*.
CS = A + B + C.
PS = D + E + F

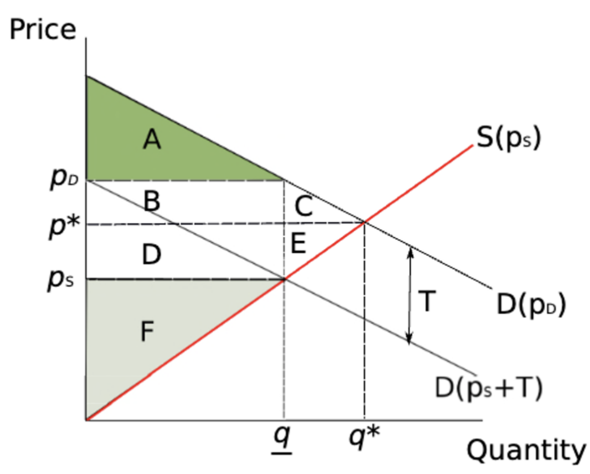
- With the tax:
  Suppliers will only supply $S(p_D - T)$ so the supply is lower and the curve shifts left (graphically actually an upwards shift because the y-variable is subtracted).
  $\Delta CS = -(B + E), \Delta PS = -(D + E)$.
  Government revenue is $T \times \underline{q}$ or $(p_D - p_S)\underline{q}$ which is B + D.
  Thus, $DWL = \Delta CS + \Delta PS + Revenue = -(C + E)$

Effective incidence relates to how the DWL is shared. Since C falls on consumers and E falls on producers

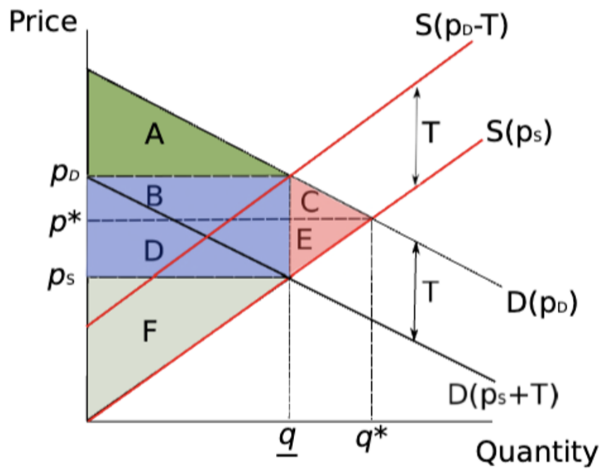
- Consumer's share
$$\frac{C}{C+E}$$
- Producer's share
$$\frac{E}{C+E}$$

Graphically, the effective incidence is related to the ratio of the deadweight loss which in turn is related to the elasticities
$$\frac{DWL_C}{DWL_D} = \frac{\eta}{|\epsilon|}$$

Case 2 (Formal incidence on Consumer): Formal incidence levied on consumers and they have to remit the duty to the government.
Think about the consumer paying whatever the supplier wants plus the tax on top of it i.e. $p_S + T$ since they have to pay taxes to the government directly based on what they consume s.t.
$$D(p_D = p_S + T) = S(p_S)$$



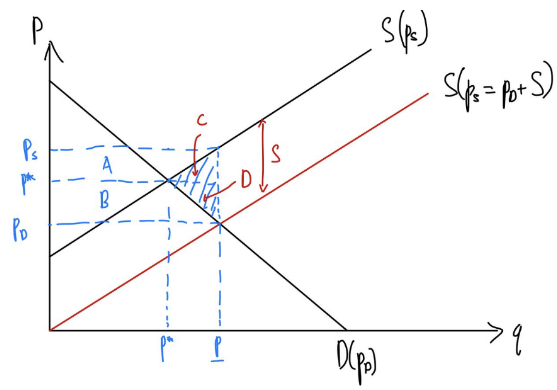
With the tax the buyer will only demand $D(p_S + T)$ at every price level so the demand is lower and the curve shifts left (or graphically translates in the negative y-direction by T units).
Equilibrium in terms of the price the supplier receives is at p_S and the volume traded drops to $\underline{q}$
$$\Delta CS = -(B + E), \Delta PS = -(D + E)$$
Government revenue is $T\underline{q}$ or $(p_D - p_S)\underline{q}$ which is B + D.
Thus $DWL = \Delta CS + \Delta PS + Revenue = -(C + E)$.
Case 3 (Tax on consumer and supplier):



Entire analysis is indentical regardless of formal incidence. In particular, the effective incidence is completely unchanged

6. **Subsidies:** Reverse of taxation
Case 1 (Subsidies levied on producer): We can think of this as the producer receiving p_D from the consumer and then an additional S subsidy from the government. So the supply function is transformed into

$$S(p_S = p_D + S)$$



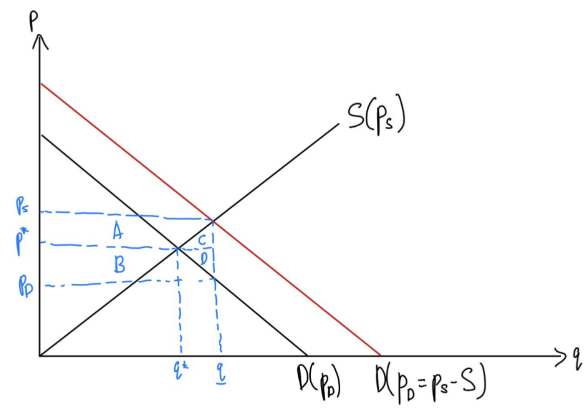
$$\Delta CS = B$$
$$\Delta PS = A$$
$$DWL = C + D$$

The government loss or expenditure to provide the subsidy is given by $G = A + B + C + D$,
Given that the consumer and producer only receive $A + B$ in total, the deadweight loss incurred is $C + D$.
The effective incidence measured as a share of total loss is

- Consumers:
$$\frac{D}{C + D}$$
- Producers:
$$\frac{C}{C + D}$$

Case 2 (Subsidies levied on Consumer): We can think of this as the consumer paying p_S for the subsidised good, and then the government paying them S. So the transformed demand function is

$$D(p_D = p_S - S)$$



We see that the government subsidy comes in the form of

$$G = p_S \underline{q} = A + B + C + D$$

The producer surplus rises by A and the consumer surplus rises by B

$$\therefore DWL = C + D$$

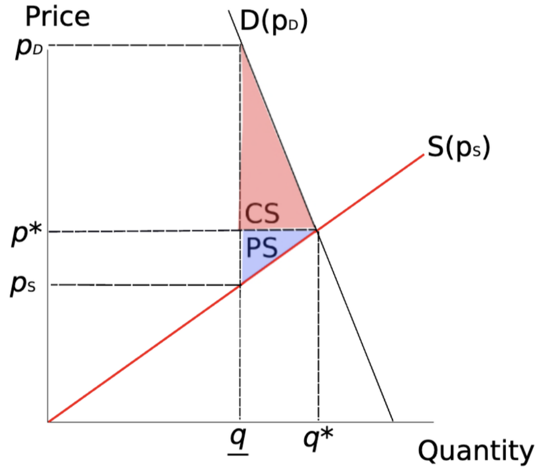
Effective incidence

- Consumer:
$$\frac{D}{C + D}$$
- Producer:
$$\frac{C}{C + D}$$

Relationship between elasticities and welfare effects of taxation/subsidies
We want to use this relationship
$$\frac{DWL_C}{DWL_P} = \frac{\eta}{|\epsilon|}$$

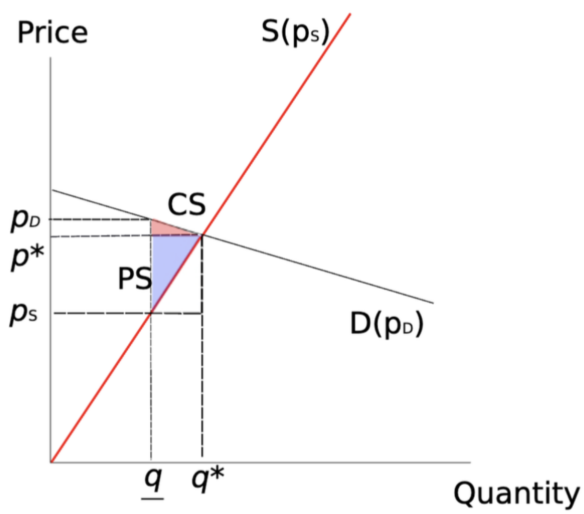
We will prove this below.
Taxation

- $|\epsilon| > \eta \implies DWL_C < DWL_P$.
Intuition:



When demand is very inelastic, the consumers are less willing to change their behaviour with respect to the rise in price so they end up bearing much of the burden of the tax, by buying more at a higher price. The suppliers on the other hand, given their relatively more elastic supply curve, are more willing to cut supply such that the burden of the tax on them is lower

- $\eta > |\epsilon| \implies DWL_C > DWL_P$.
  Intuition:



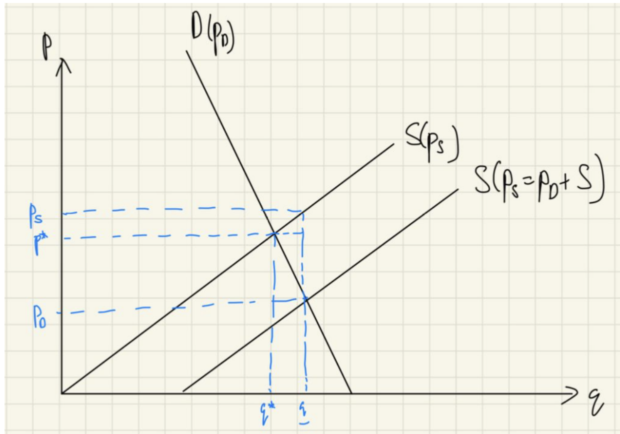
When the supply is inelastic relative to the demand, the suppliers are less willing to change their behaviour through cutting production with respect to the fall in price, so they end up bearing more of the burden by producing more at a much lower price. The buyers were more willing to change their behaviour through cutting consumption, and thus bought less at the higher price.

Subsidies

- $|\epsilon| > \eta \implies DWL_C < DWL_P$.

- $\eta > |\epsilon| \implies DWL_C > DWL_P$

Intuition:



When the demand is much more inelastic than the supply curve, the buyers are less willing to change their consumption. Thus, their consumption increases by very little, implying that they receive most of the fall in price induced by the subsidy. The suppliers are more willing to increase their production with respect to the increase in price, so price need not increase so much since they are more willing to produce more.

Remember that the price change is the only thing that varies, the quantity change is fixed. So if a consumer is less willing to buy more, and the producer is more willing to produce more, then for a fixed quantity change, the government would have to "give" more to the consumer (price fall) than the producer (price rise) to induce the same rise in quantity.

Conversely, when supply is more inelastic than demand, the suppliers are less willing to change their production. So their production increases by very little. Consumers are more willing to change their consumption so their production increases more. But note that since the quantity change for both are equal, this means that the suppliers have to receive a larger share/incidence of the subsidy than the consumer to change their production/consumption by the same amount.

Proof of the relationship.
Proposition:

$$\frac{DWL_C}{DWL_P} = \frac{\eta}{|\epsilon|}$$

The above relationship holds for any linear demand and supply curves.
The proof is relatively informal.
Proof. First establish that the ratio of the incidences always equals the ratio of the deadweight loss for some linear demand and supply curves.
Then, realise that for some linear demand/supply curves

$$\frac{DWL_C}{DWL_P} = \frac{\frac{dp}{dq}_C dq_C}{\frac{dp}{dq}_P dq_P}$$

This is because (1) the DWL triangles are created by a change in quantity (due to shifts in demand/supply) and (2) by the geometric properties of similar triangles, the ratio in area is dependent on the ratio of the triangle's heights. To measure each height, we have the rate of change of each function multiplied the actual change in quantity.
Since $dq_C = dq_P$, we have

$$\frac{DWL_C}{DWL_P} = \frac{\frac{dp}{dq}_C}{\frac{dp}{dq}_P} = \frac{dq}{dp}_P \frac{dp}{dq}_C$$

Note that for point elasticities (at the same point)

$$\frac{\eta}{|\epsilon|} = \frac{dp}{dq}_C \frac{dq}{dp}_P$$

Hence, the claim holds ■

2.4 Worked Examples

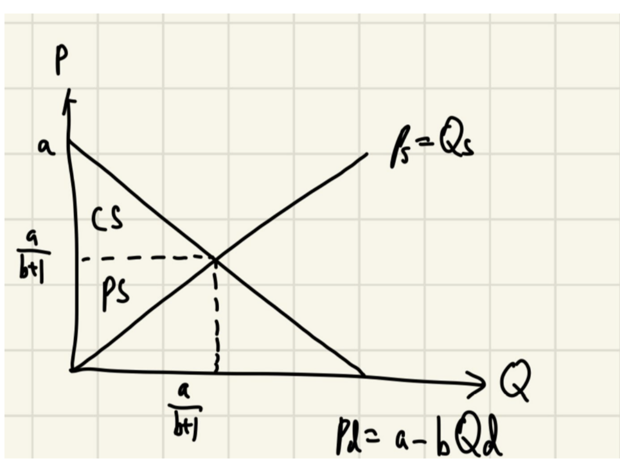
2021-2022 Trinity Paper for Question on taxation.
The inverse demand for an industry's product is $p_d = a - bQ_d$ where $a > 0, b > 0$ and Q_d is the quantity demanded at price p_d. The good is produced by a perfectly competitive industry with the inverse supply function $p_s = Q_s$, where p_s is the price received by producers and Q_s is the total supply.
(a) Calculate the equilibrium price and quantity. What are the values of the price elasticities of demand and supply at the equilibrium? **Solution:**
Equilibrium

$$p = a - bp$$
$$p(1+b) = a$$
$$p = \frac{a}{b+1}, Q = \frac{a}{b+1}$$

Elasticities

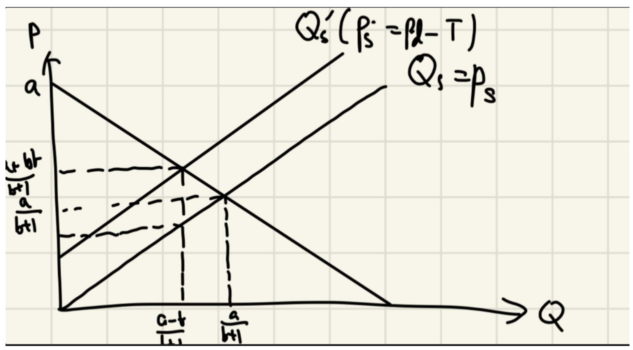
$$\epsilon = \frac{P_d}{Q_d}\frac{dQ_d}{dP_d}$$
$$= \frac{a-1}{b}$$
$$\eta = \frac{P_s}{Q_s}\frac{dQ_s}{dP_s}$$
$$= 1$$

(b) What are the values of consumer and producer surplus as the equilibrium? Illustrate the equilibrium price and quantity, and consumer and producer surplus, in a diagram



$$CS = 0.5\left(\frac{a}{b+1}\right)\left(a - \frac{a}{b+1}\right)$$
$$= \frac{a^2 b}{2(b+1)^2}$$
$$PS = \frac{1}{2}\left(\frac{a}{b+1}\right)^2$$

The government imposes a tax t on each unit of output produced by the firms. Assume that $t < a$.
(c) Calculate the price that consumers pay, the price that producers receive and the equilibrium quantity when the tax is imposed. Comment on the effects of the tax on the consumer and producer prices.



$$Q_s = p_s$$

Recall that $p_s = p_d - t$, so $Q_s = p_s$ does not change. Thus,

$$p_d = a - b(p_d - t)$$
$$p_d = \frac{a+bt}{b+1}$$

Solving for Q_d,

$$\frac{a+bt}{b+1} = a - bQ_d$$
$$Q_d = \frac{a-t}{b+1}$$

Since $Q_d = Q_s$,

$$p_s = \frac{a-t}{b+1}$$

Since $\frac{bt}{b+1} > \frac{t}{b+1}$, the extent to which consumer price increased is larger than the extent to which producer price decreased.

From now on, suppose that $a = 12, b = 1$.
(d) What is the value of the tax that maximises tax revenue? What are the resulting values of the tax revenue and of the price consumers pay?

$$TR = \left[\frac{a+bt}{b+1} - \frac{a-t}{b+1}\right]\left(\frac{a-t}{b+1}\right)$$
$$= \frac{at - t^2}{b+1}$$

Maximising TR we have,

$$\frac{dTR}{dt} = \frac{1}{b+1}(a-2t) = 0$$
$$\implies a - 2t = 0$$
$$\implies t = \$6$$

Calculating resultants,

$$TR = \$18,\ p_d = \$9$$

A government advisor recommends that the tax in part (d) should not be used and instead the government should nationalise the industry, operate it as a profit-maximising monopoly and take the monopoly profits as government revenue.
(e) What price will the monopoly set, and what will its profits be? (You may assume, in line with the competitive supply function, that the monopolist's marginal cost is Q and the total cost is $\frac{1}{2}Q^2$ with there being no fixed costs.) Should the government adopt this recommendation instead of the tax that maximises tax revenue?
Monopolist will set output at $MR = MC$

$$p'(Q)Q + p(Q) = Q$$
$$a - 2bQ = Q$$
$$Q = 4$$
$$p = \$8$$
$$\pi = \$24$$

Government should adopt this recommendation instead since: (i) The consumer surplus is greater in this case since the monopoly produces more at a lower price; (ii) The government revenue is higher in this case than through taxation.

3 Production

3.1 Modelling the Process

Definition. Cobb-Douglas Production Function:

$$y = K^a L^b$$

where L denotes labour and K denotes capital.
Some common properties

1. If no inputs are used, no output is produced: no vertical intercept where input = 0 but output > 0
2. Property of inaction: producing nothing is always feasible
   - (Input, Output) = (0,0)
   - Time horizon of production possibilities matters here: the property of inaction may not hold in the short run due to irrevocable contracts that incur sunk cost; so must produce something as firm already committed to using at least $\bar{x}$ of input in the short run.
3. Long run production set implies all inputs are freely variable. Production technology exhibits decreasing returns to scale when it is the case that any feasible input-output combination can be scaled down and remain feasible.
   This implies that inaction is possible
4. Inada conditions: Given a continuously differentiable function $f : X \to Y$ where $X = \{x : x \in \mathbb{R}^n_+\}$ and $Y = \{y : y \in \mathrm{R}_+\}$, then

- $\lim_{x_i \to 0} \partial f(\mathbf{x})/\partial x_i = +\infty$. Interpretation: the effect of the first unit of input x_i has the largest effect
- $\lim_{x_i \to +\infty} \partial f(\mathbf{x})/\partial x_i = 0$. Interpretation: the effect of one additional unit of input x_i is 0 when approaching the use of infinite units of x_i

Note: Monotonic transformations of the production function such that

$$v(y(K,L)) = a\ln K + b\ln L$$

do not represent the same production function/technology. This is because production functions are not ordinal (representations of input combinations) in nature, unlike utility functions.

Definition. Returns to Scale: Returns to scale for some productive process can be determined by the homogeneity of functions. Consider some scalar λ s.t.

$$f(\lambda K, \lambda L) = \lambda^n f(K,L)$$

- If the function is homogeneous to degree $n = 1$, then constant returns to scale. Alternatively, exponents $a + b = 1$
- If the function is homogeneous to degree $n < 1$, then diminishing returns to scale. Alternatively, $a + b < 1$
- If the function is homogeneous to degree $n > 1$, then increasing returns to scale. Alternatively, $a+b > 1$

Convexity of the production set is often assumed. Consider (K_1, L_1, y_1) and (K_2, L_2, y_2) as feasible input-output combinations, then so is the weighted average feasible s.t.

$$\lambda(K_1, L_1, y_1) + (1-\lambda)(K_2, L_2, y_2), \forall \lambda \in \mathbb{R}^+$$

Definition. Marginal Products: The change in output for an increase in one unit of input

- The Marginal Product of Capital is given by
$$MPK = \frac{\partial y}{\partial K}$$
- The Marginal Product of Labour is given by
$$MPL = \frac{\partial y}{\partial L}$$

Definition. Marginal rate of technical substitution: The amount of input 2 one must give up for each increase in input one such that total output remains the same.
Derivation. We consider the total derivative:

$$dy = \frac{\partial y}{\partial K} dK + \frac{\partial y}{\partial L} dL$$

By construction, $\partial y = 0$, then

$$MRTS_{LK} = \frac{dK}{dL} = -\frac{MPL}{MPK}$$

In the case of perfect substitutes, the iso-quants are linearly sloped s.t. for some constant a and b,

$$MRTS_{LK} = -\frac{b}{a}$$

Perfect complement case: Occurs when the production process is such that trade-offs are impossible and capital and labour must be used in fixed proportions.

Measuring marginal returns to capital/labour: we take the first-derivatives s.t.

$$\frac{\partial y}{\partial K} > 0, \frac{\partial y}{\partial L} > 0$$

Diminishing Marginal returns depends on the second-order derivative

$$\frac{\partial^2 y}{\partial K^2} < 0, \frac{\partial^2 y}{\partial L^2} < 0$$

3.2 Modelling Costs

Iso-cost line represents the combinations of inputs such that cost remains fixed. Generally,

$$c = wL + rK$$

As total cost changes holding relative prices fixed, the isocost line moves parallel to itself; slope changes when relative price changes.

Cost minimisation: To minimise cost we have two methods
Note: Check for boundary optimums OR linear isoquant curves (usually perfect substitutes) because the gradient at the point of intersection between isoquant and isocost curves are different
Langrangian method. We have the following minimisation problem

$$\min_{K,L}(wL + rK) \text{ s.t. } y = K^a L^b$$

We then construct the Lagrangian

$$\mathcal{L}(K, L, \lambda) = wL + rK - \lambda(K^a L^b - y)$$

Solve for the FOCs:

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda a K^{a-1} L^b = 0$$
$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda b K^a L^{b-1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = y - K^a L^b = 0$$

We yield the **tangency condition**

$$\frac{w}{r} = \frac{bK}{aL}$$
$$K = \frac{aw}{br}L, \quad L = \frac{rb}{wa}K$$

Thus, substituting into the isoquant (constraint) to obtain the demands for each factor to yield the **feasibility condition**

$$y = \left(\frac{aw}{br}L\right)^a L^b$$
$$y = \left(\frac{aw}{br}\right)^a L^{a+b}$$
$$L = \left(\frac{br}{wa}\right)^{\frac{a}{a+b}} y^{\frac{1}{a+b}}$$
$$K = \left(\frac{wa}{rb}\right)^{\frac{b}{a+b}} y^{\frac{1}{a+b}}$$

Solving for the λ we realise that it equals the marginal cost of the firm.

Tangency method. Basically you take the derivatives of the two curves (the MRTS and the relative prices)

$$MRTS_{KL} = -\frac{MP_L}{MP_K} = -\frac{w}{r}$$

And then you substitute your result back into the constraint function.
Intuition:
When

$$\frac{MPL}{MPK} > \frac{w}{r}$$

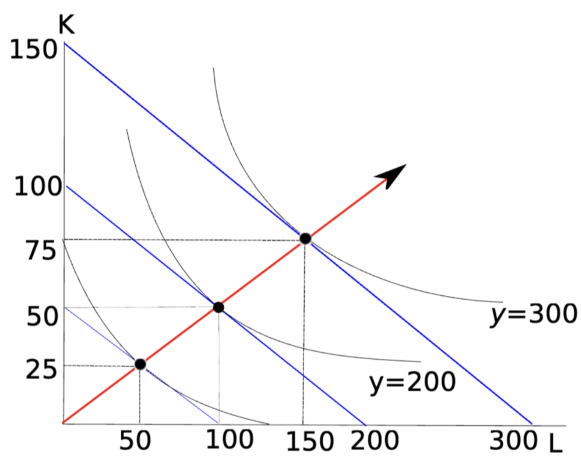
This means that your relative marginal output of labour is greater than the relative price of labour for each increase in labour-input → this means that for \$1 of more labour in place of an existing \$1 of capital, you can produce more. This implies that you can maintain your current output with less than \$1 of labour and thus can maintain output at a lower cost.

When

$$\frac{MP_L}{MP_K} < \frac{w}{r} \implies \frac{MP_K}{MP_L} > \frac{r}{w}$$

This means that your relative marginal output of labour is lower than the relative price of labour for each increase in labour-input → this means that for \$1 of more capital in place of an existing \$1 of labour, you can produce more. This implies that you can maintain your current output with less than \$1 of capital and thus maintain output at a lower cost (or that you are spending too much money on labour). ■

Definition. Output expansion path: The locus of all efficient combination of inputs for any possible level of output.



Need not be a straight line but it will be for the Cobb-Douglas function since $\forall y \in \mathbb{R}^+, K = \frac{wa}{rb}L$.
We compute this by substituting the cost-minimising factor demands in terms of y into the cost function.

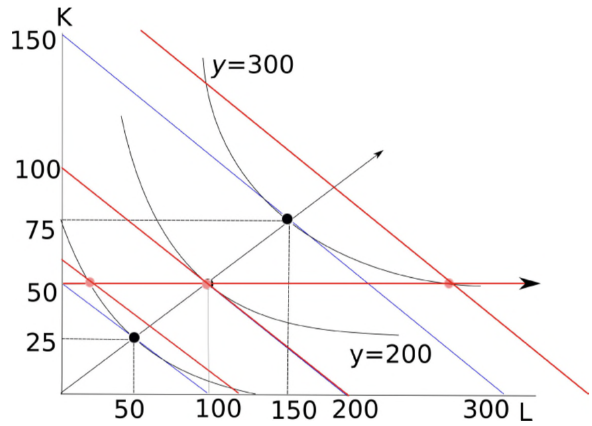
Example (Tutorial 3 Q4(d)). Suppose $F(K, L) = \sqrt{K} + \sqrt{L}$. Find the minimum cost of producing the good as a function of y, w and r. Discuss briefly how it demands on y.
Minimum cost function:

$$C = r\left(\frac{w}{r+w}y\right)^2 + w\left(\frac{r}{w+r}y\right)^2 \tag{1}$$

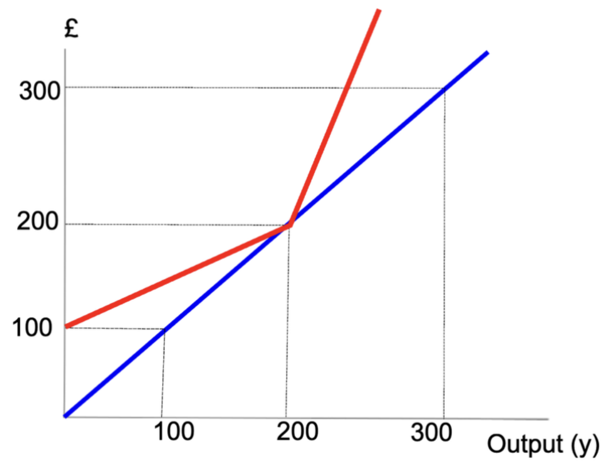
$$= y^2\left[\left(\frac{w}{w+r}\right)^2 r + \left(\frac{r}{w+r}\right)^2 w\right] \tag{2}$$

The minimum cost varies positively with y and the output expansion path increases at an increasing rate. So minimum cost rises faster than the rise in output.

The short run, when some factor of production is fixed: Output expansion path is horizontal linear as the firm can only change one factor of production.



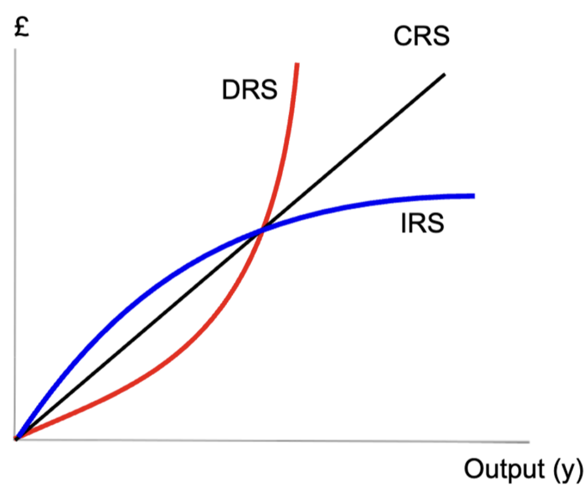
The long run cost will only intersect the short run cost at $y = 200$ and then will diverge from there.



Vertical intercept is the fixed cost even when output is 0 since one fixes the level of capital used.

Relationship between cost curves and returns to scale. We have a cost function

$$c = c(y)$$



Intuition: If the cost function is convex by increasing at an increasing rate, then it indicates decreasing returns to scale as each \$1 of input yields increasingly less amount of output. Conversely, concave functions indicates increasing returns to scale. Note: Convexity/Concavity determined by second derivative.
Consider a Cobb-Douglas Function and a cost function

$$y = K^a L^b$$
$$c = wL + rK$$

We know that the demand functions are

$$L = \left(\frac{br}{wa}\right)^{\frac{a}{a+b}} y^{\frac{1}{a+b}}$$

$$K = \left(\frac{wa}{rb}\right)^{\frac{b}{a+b}} y^{\frac{1}{a+b}}$$

Therefore,

$$c = w\left[y^{\frac{1}{a+b}}\left(\frac{rb}{wb}\right)^{\frac{a}{a+b}}\right] + r\left[y^{\frac{1}{a+b}}\left(\frac{wa}{rb}\right)^{\frac{b}{a+b}}\right]$$

Hence,

$$c = Ay^{\frac{1}{a+b}}$$

Returns to scale

- If $a + b = 1$, CRS
  $\partial^2 c/\partial y^2 = 0$
- If $a + b > 1$, IRS
  $\partial^2 c/\partial y^2 < 0$
- If $a + b < 1$, DRS
  $\partial^2 c/\partial y^2 > 0$

Definition. Average Cost: Average cost is cost per unit produced.

$$TC(y) = c(y) \implies AC(y) = \frac{c(y)}{y}$$

Definition. Marginal Cost: the increase in cost for each additional unit produced (the rate of change of total cost with respect to quantity)

$$MC(y) = c'(y)$$

Marginal cost intersects AC at the minimum point

- When $MC < AC$, every new unit adds a cost that is lower than the average cost, thus pulling the average cost down
- When $MC > AC$, every new unit adds a cost that is higher than the average cost, thus pulling the average cost up

Proof.

$$\min AC(y) = AC'(c) = \frac{c'(y)y - c(y)}{y^2} = 0$$
$$c'(y) = \frac{c(y)}{y}$$
$$MC(y) = c'(y)$$
$$MC(y) = \min AC(y)$$

Graphically, the area below the MC curve at each instantaneous point represents the actual cost of the corresponding good (after all the effect of increasing production by 1 unit on total cost is the cost of that good itself).

Definition. Fixed Costs: Costs which do not vary with output F(only for short run).

Definition. Variable Costs: Costs which do vary with output $c_v(y)$.

Definition. Total Costs: Fixed plus variable cost

$$c(y) = F + c_v(y)$$

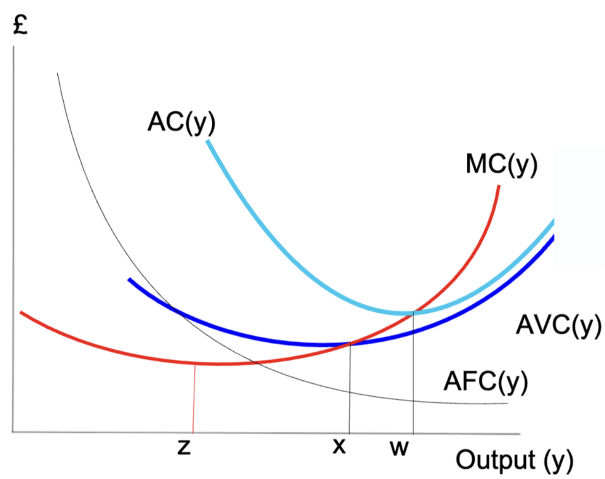
Definition. Average Variable Cost: Variable cost divided by quantity

$$AVC(y) = \frac{c_v(y)}{y}$$

Definition. Average Fixed Cost: Fixed cost divided by quantity

$$AFC(y) = \frac{F}{y}$$

Graphical illustration of the cost functions



Note a few things:

- MC intersects AC at minimum point
- At high levels of output, AFC is falling and becomes negligible

$$\lim_{y\to\infty} AFC(y) = \frac{F}{y} = 0$$

Therefore, $AVC(y) \approx AC(y)$ as $y \to \infty$

Proposition (Long run and short run costs). You cannot achieve greater efficiency by controlling a subset of factors since

$$AC(y) \geq AVC(y)$$

Claim.

$$\frac{c(y)}{y} \geq \frac{c_v(y)}{y}, \forall y \in \mathbb{R}^+$$

Proof. By the definition of total cost

$$c(y) = c_v(y) + F$$

since $\forall y \in \mathbb{R}^+$, i.e. $y \geq 0$

$$c(y) = c_v(y_+F, F \geq 0 \implies c(y) > c_v(y)$$

$$\frac{c(y)}{y} \geq \frac{c_v(y)}{y}$$

Since Short run cost entails that our average cost includes some sort of fixed cost and our long run cost entails that we only have variable cost, immediately we conclude

$$LAC(y) \leq SAC(y) \quad \blacksquare$$

3.3 Modelling Competitive Supply

Profit maximisation for a competitive firm (Note: p is exogenously determined since firms are price-takers)

$$\pi(y) = py - c(y)$$
$$\max \pi(y) = p - c'(y) = 0$$
$$p = c'(y)$$

Second order condition, by construction (since maximisation)

$$\pi''(y) = -c''(y) \leq 0 \implies c''(y) \geq 0$$

Your marginal cost should be increasing. So the marginal cost is essentially the firm's supply curve since the firm will price equal to the marginal cost and thus produce accordingly.

Relationship between Producer surplus and profits (Short-Run)

- Producer Surplus = Revenue - Variable Cost
- Profit = Revenue - Variable costs - fixed costs
- Profit = Producer Surplus - Fixed Costs

In the Long Run, Profit = Producer Surplus
Short Run Shut down conditions. In the short run, the firm would still persist if

$$AC(y) > p \geq AVC(y)$$

But if,

$$AVC(y) > p$$

the firm will just shut down.
Derivation. In the short run, the profit equation is

$$\pi(y) = py - c_v(y) - F$$

If it produces nothing, the firm must pay

$$\pi(0) = -F$$

So, it is not worth producing anything (shut down) if $\pi(0) > \pi(y), \forall y$. Thus

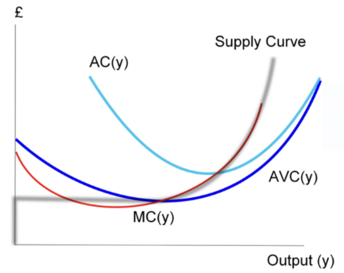
$$-F > py - c_v(y) - F$$

Rearranging this, we yield the shut down condition in the short run

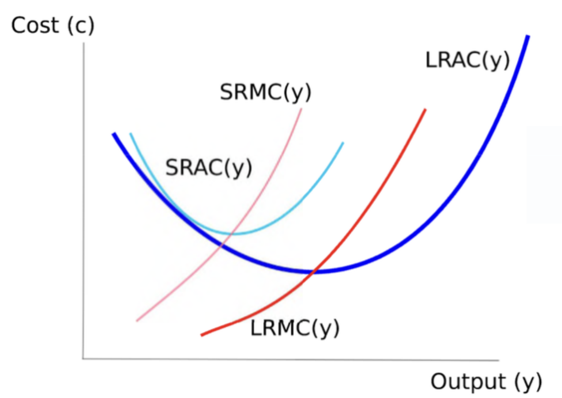
$$c_v(y)/y > p \implies AVC(y) > p$$

The supply curve is therefore

$$\text{Set} \begin{cases} y \text{ such that } p = c'(y) \text{ and } c''(y) > 0 & \text{if } p \geq AVC(y) \\ y = 0 & \text{otherwise} \end{cases}$$



Long Run Shut Down conditions: Simply $AC > p$, since $AC = AVC$. Long run and short run costs can be illustrated by:



Clearly, short run costs are higher than long run costs due to fixed factors as established.

Competitive Long Run equilibrium conditions: Since

- Exit is when
$$p < \frac{c(y)}{y} \implies py - c(y) < 0 \implies \pi(y) < 0$$
- Entry is when $p > \frac{c(y)}{y} \implies py - c(y) > 0 \implies \pi(y) > 0$

We conclude that the long run equilibrium is when

$$p = \frac{c(y)}{y} \implies py - c(y) = 0 \implies \pi(y) = 0$$

All firms make zero economic profits and will not exit since current situation is at least as good as the net benefit of the next best alternative forgone. Otherwise, they will exit because they can do better elsewhere (by definition), since negative economic profits are inclusive of opportunity costs.

Definition. Industry Supply: Industry supply is the horizontal summation of supply curves where

$$S(p_i) = \sum_{i=1}^{n} S_i(p_i)$$

4 Consumption

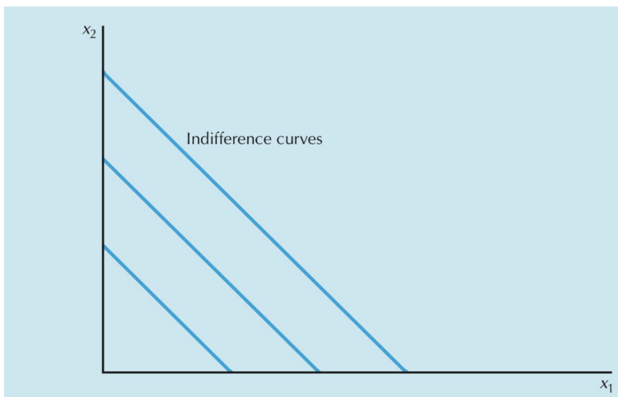
4.1 Preferences

Definition. Indifference Curves: represents the different bundles of goods that a consumer is indifferent to (because they provide similar levels of utility).

Proposition. Indifference curves representing distinct levels of preferences cannot cross.
Proof. Let us choose three bundles of goods X, Y, Z such that X lies on one indifference curve, Y lies only on the other indifference curves, and Z lies in the intersection between both indifference curves. By construction, the indifference curves represent distinct levels of preferences, so either $X \succ Y$ or $Y \succ X$. By construction, $X \sim Z$ and $Z \sim Y$. By transitivity, $X \sim Y$. But this contradicts the prior claim. ■

Constructing indifference curve

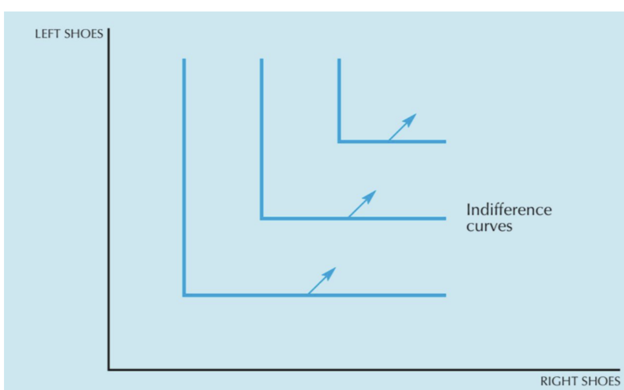
1. **Perfect Substitutes**



Perfect substitutes. The consumer only cares about the total number of pencils, not about their colors. Thus the indifference curves are straight lines with a slope of -1.

   If the consumer is willing to substitute one good for another at a constant rate. So the slope need not just be -1, but just some constant $-a$

2. **Perfect Complements**



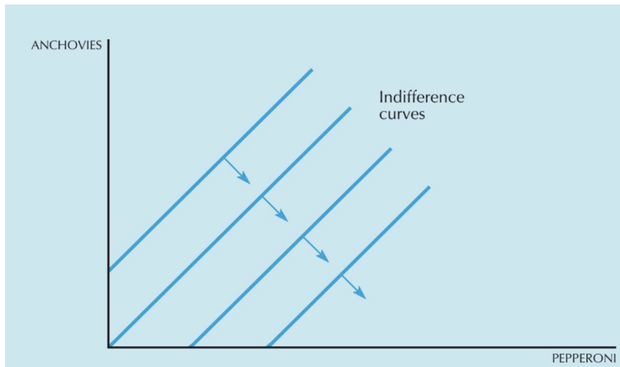
Perfect complements. The consumer always wants to consume the goods in fixed proportions to each other. Thus the indifference curves are L-shaped.

Goods that are always consumed together. Represented by the function

$$u(x_1, x_2) = \min\{x_1, x_2\}$$

So utility will not increase if you consume more of one good without increasing your consumption of another

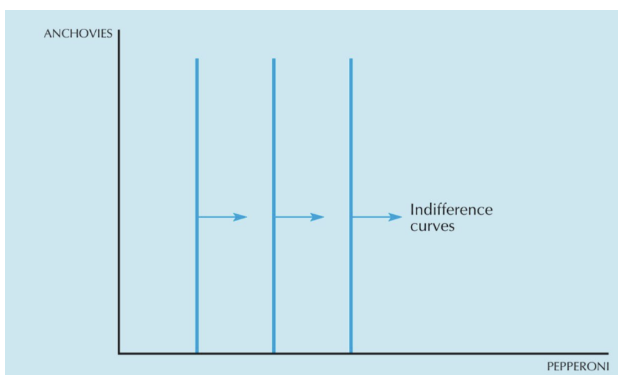
3. **Bads**



Bads. Here anchovies are a "bad," and pepperoni is a "good" for this consumer. Thus the indifference curves have a positive slope.

If you increase the quantity of 'bads', you need to increase the quantity of the other good to keep utility constant

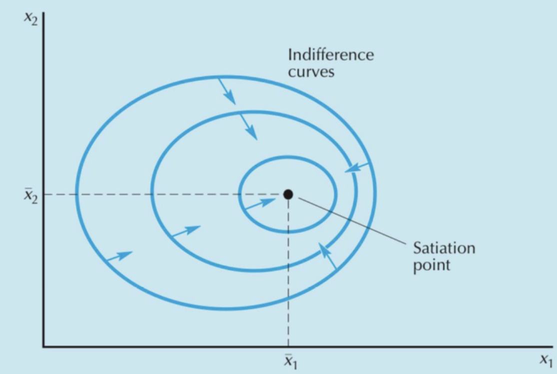
4. **Neutral Goods**



A neutral good. The consumer likes pepperoni but is neutral about anchovies, so the indifference curves are vertical lines.

Utility is invariant with the quantity of the neutral good

5. **Satiation point**

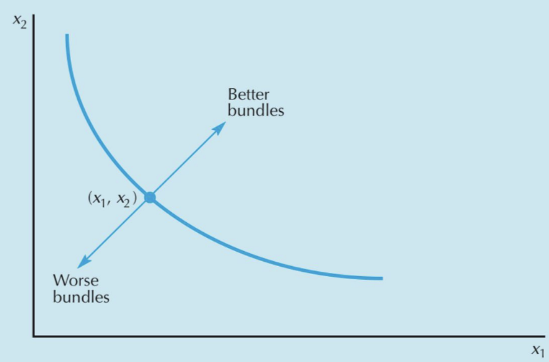


Satiated preferences. The bundle $(\overline{x}_1, \overline{x}_2)$ is the satiation point or bliss point, and the indifference curves surround this point.

Overall, the best bundle for the consumer. The closer the consumer is to the satiation point is, the better off he is. $(\bar{x}_1, \bar{x}_2)$ is considered the satiation point. People may have too much of something such that increasing quantity actually reduces utility.

Properties of Well-behaved indifference curves.

1. Property 1: Monotonicity of preferences



Monotonic preferences. More of both goods is a better bundle for this consumer; less of both goods represents a worse bundle.

If (x_1, x_2) is a bundle of goods and (y_1, y_2) is a bundle of goods with at least as much of both goods and more of one, then $(y_1, y_2) succ (x_1, x_2)$.
Monotonicity implies that indifference curves have a negative slope. If we start from (x_1, x_2) and we move up and right, we reach a better position; if we move down and left we reach a worse position. We can only reach a position of indifference if we move up and left or down and right i.e. negative slope

2. Property 2: Convexity of preferences.
For $t \in [0, 1]$ and $(x_1, x_2) \sim (y_1, y_2)$,

$$(tx_1 + (1-t)y_1, tx_2 + (1-t)y_2) \succeq (x_1, x_2)$$

Strict convexity implies that the weighted averages of two indifferent bundles is strictly preferred to the two extreme bundles; Convex preferences may have flat spots but strictly convex preferences must have rounded indifference curves. Preferences for two goods that are perfect substitutes are convex but not strictly convex

4.2 The Budget Constraint

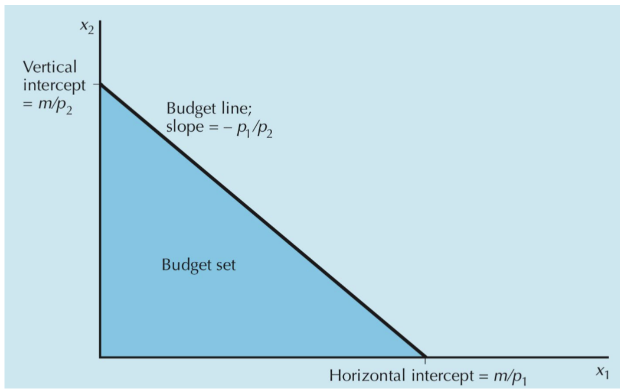
Without loss of generality, consider the prices of two goods as (p_1, p_2) and the amount of money one has to spend as m. Our budget constraint is

$$p_1x_1 + p_2x_2 \leq m$$

Our two good assumption is reasonable because we often consider x_2 as a composite good i.e. the money we can spend on everything else.

Definition. Budget Set: the set of bundles that cost exactly m :

$$p_1x_1 + p_2x_2 = m$$



Rearranging the budget line equation, we yield:

$$x_2 = \frac{m}{p_2} - \frac{p_1}{p_2}x_1$$

Tells us how many units of good 2 the consumer needs to consume in order to satisfy the budget constraint if she is consuming x_1 units of good 1.

The slope of the budget line measures the rate at which the market is willing to substitute good 1 for good 2

Proof. Suppose the consumer is increasing her consumption of good 1 by dx_1. Let dx_2 be some change in good 2. Then we have

$$p_1x_1 + p_2x_2 = m \tag{3}$$
$$p_1(x_1 + dx_1) + p_2(x_2 + dx_2) = m \tag{4}$$

Subtracting (1) from (2):

$$p_1dx_1 + p_2dx_2 = 0$$
$$\frac{dx_2}{dx_1} = -\frac{p_1}{p_2}$$

We can normalise variables to make them redundant

- Income:

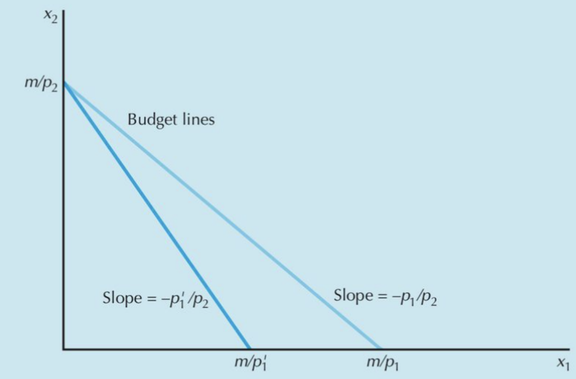
$$p_1x_1 + p_2x_2 = m$$
$$\frac{p_1}{m}x_1 + \frac{p_2}{m}x_2 = 1$$

- Price of good 2:

$$\frac{p_1}{p_2}x_1 + x_2 = \frac{m}{p_2}$$

Changes in the Budget Constraint:

1. **Increase in income:** a rise in m would cause a parallel shift outwards for the budget constraint/line
2. **Changes in price of one good:** A change in relative prices would change the slope of the curve and its respective intercept.



Increasing price. If good 1 becomes more expensive, the budget line becomes steeper.

Alternatively, for a fall in p_2, the vertical intercept will fall and the slope becomes gentler

3. Change in price of both goods: Suppose our original budget line is

$$p_1x_1 + p_2x_2 = m$$

Multiply both prices by t:

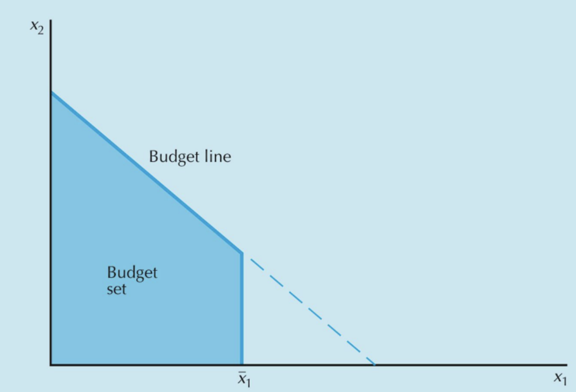
$$tp_1x_1 + tp_2x_2 = m$$

Therefore,

$$p_1x_1 + p_2x_2 = \frac{m}{t}$$

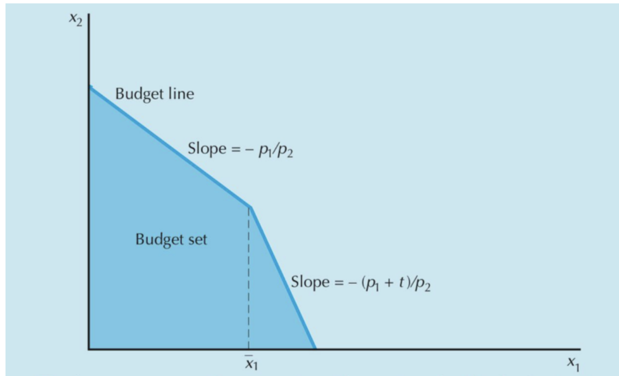
In effect, the budget line shifts inwards. If prices change by different degrees, you divide the whole equation by the lowest multiplier and then assume that it is a price change for the new budget line (parallel shift then pivotal shift)

4. **Governmental policy**
   (a) Rationing constraints: Suppose the government sets a rationing constraints on good 1 s.t. $x_1 \leq \bar{x}_1$.



Budget set with rationing. If good 1 is rationed, the section of the budget set beyond the rationed quantity will be lopped off.

(b) Quantity taxes: pay the government a certain amount per unit consumed. Changes price from p_1 to $p_1 + t$. Budget line gets steeper for good 1 taxation. Can also tax consumption greater than some quantity $\bar{x}_1$.



Taxing consumption greater than $\bar{x}_1$. In this budget set the consumer must pay a tax only on the consumption of good 1 that is in excess of $\bar{x}_1$, so the budget line becomes steeper to the right of $\bar{x}_1$.

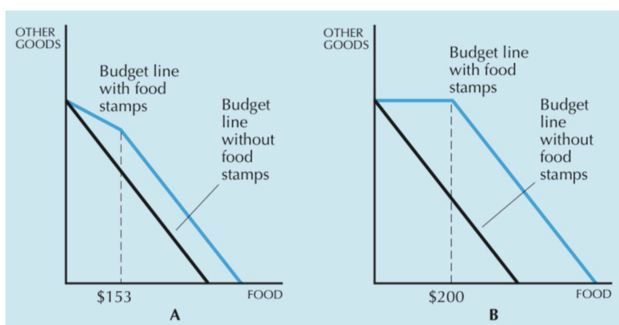
(c) Ad valorem tax: tax on the price of the good rather than quantity (% terms). Change of price from p_1 to $(1 + t)p_1$, so budget line is effectively steeper too

(d) Quantity subsidy: per unit subsidy. Changes of price from p_1 to $p_1 - s$; gentler slope and the horizontal intercept moves outwards.

(e) Value subsidy: per value subsidy. Change of price from p_1 to $(1-\sigma)p_1$; gentler slope with outward movement of the horizontal intercept.

(f) Lump sum tax: government just takes a fixed amount of money regardless of consumption. Income just falls

(g) Food stamps: giving some quantity of goods



Food stamps. How the budget line is affected by the Food Stamp program. Part A shows the pre-1979 program and part B the post-1979 program.

First graph shows impact of ad valorem subsidy on a certain value of goods; second graph shows impact when the government just gives the food directly

Definition. Marginal Rate of Substitution: The amount of good 2 the consumer must give up for each unit increase in consuming good 1 s.t. utility remains constant.
Derivation. Similar to MRTS. By the total derivative, we have

$$dy = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2$$

By construction, $\partial y = 0$, then

$$MRS = \frac{dx_2}{x_1} = -\frac{MU_1}{MU_2}$$

MRS for different type of goods

- Perfect substitutes: MRS = -1
- Neutrals: MRS is everywhere infinite
- Perfectly complements: MRS is zero or infinite

Convex preferences exhibit a diminishing marginal rate of substitution: for each unit of good 1 one gives up, one adds less and less of good 2 to sustain their utility (i.e. good 2 is increasingly worth less as quantity increases).

4.3 Utility

Definition. Utility function: A real-valued function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is called a utility function representing the preference relation $\succeq$, if for all vectors $\mathbf{x}^0, \mathbf{x}^1 \in \mathbb{R}^n_+$, $u(\mathbf{x}^0) \geq u(\mathbf{x}^1) \iff \mathbf{x}^0 \succeq \mathbf{x}^1$.
Less formally/More trivially, a utility function is a way of assigning a number to every possible consumption bundle such that more preferred bundles get assigned larger numbers than less-preferred bundles.

Note: A utility function is ordinal in nature such that it only considers the ranking of bundles and not the intensity of preferences (cardinal utility). Thus, interval size of utility is ignored.

This assumption of ordinal utility implies that there is no unique way to assign utilities to bundles of goods. Consider: If $u(x_1, x_2)$ represents a way to assign utility numbers to bundles (x_1, x_2), then by multiplying $u(x_1, x_2)$ by some arbitrary constant $a > 0$ s.t. $v(x_1, x_2) = 2u(x_1, x_2)$, then $v(x_1, x_2)$ is just as good a way to assign utilities.

Definition. Monotonic functions: A monotonic transformation is a way of transforming one set of numbers into another set of numbers in a way that preserves the order of numbers. We represent a monotonic transformation by a function $f(u)$ that transforms each u into a corresponding $f(u)$, in a way that preserves the order of numbers s.t. if $u_1 > u_2$, then $f(u_1) > f(u_2)$. Its derivative is given by

$$f'(u) = \lim_{\widehat{u} \to u} \frac{f(\widehat{u}) - f(u)}{\widehat{u} - u} > 0$$

For a monotonic transformation, $f(\widehat{u}) - f(u)$ always has the same sign as $\widehat{u} - u$, so its first order derivative is always positive Monotonic functions can be achieved by (i) multiplication by some positive constant; (ii) addition of some arbitrary constant; (iii) raising to power of some arbitrary constant.
Implication of monotonicity: If $f(u)$ is any monotonic transformation of a utility function that represents some particular preferences, then $f(u(x_1, x_2))$ is also a utility function that represents those same preferences.
Informal Proof. To say that $u(x_1, x_2)$ represents some particular preferences means that $u(x_1, x_2) > u(y_1, y_2)$ iff $(x_1, x_2) \succ (y_1, y_2)$. But if $f(u)$ is a monotonic transformation, then $u(x_1, x_2) > u(xy_1, y_2)$ iff $f(u(x_1, x_2)) > f(u(y_1, y_2))$.
Therefore, $f(u(x_1, x_2)) > f(u(y_1, y_2))$ iff $(x_1, x_2) \succ (y_1, y_2)$. So the function $f(u)$ represents the preferences the same way as the original utility function $u(x_1, x_2)$. ■

Utility functions for specific types of preferences.

- Perfect Substitutes

$$u(x_1, x_2) = ax_1 + bx_2$$

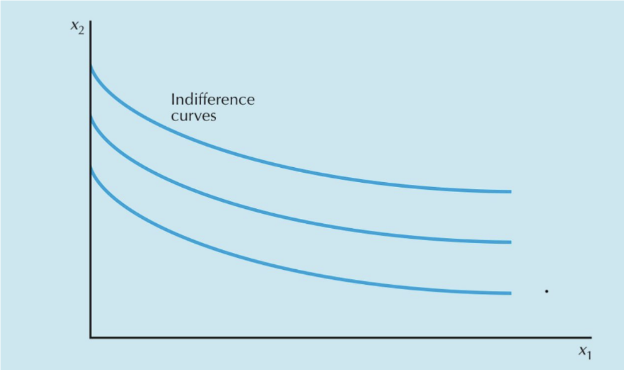
  yields a constant slope of $-\frac{a}{b}$ when $u(x_1, x_2) = k$.

- Perfect complements

$$u(x_1, x_2) = \min\{ax_1, bx_2\}$$

  Constants a, b indicate the proportions in which the goods are to be consumed

- Quasilinear preferences: Indifference curves that are vertical translates of one another.



Quasilinear preferences. Each indifference curve is a vertically shifted version of a single indifference curve.

Usually takes the form

$$x_2 = k - v(x_1)$$

Utility function is

$$u(x_1, x_2) = v(x_1) + x_2$$

So the utility function is linear in good 2 but possibly non-linear in good 1 such as $v(x_1) = \ln x_1$

Definition. Cobb-Douglas Preferences: Convex monotonic indifference curves that are well-behaved.

$$u(x_1, x_2) = x_1^a x_2^b$$

Utility functions representing Cobb-Douglas preferences can be monotonically transformed. Consider

$$u(x_1, x_2) = \ln(x_1^c x_2^d) = c\ln x_1 + d\ln x_2$$

Further, suppose $v(x_1, x_2) = x_1^c x_2^d$ and we raise utility to the power of $\dfrac{1}{c+d}$. We define a new number $a \equiv \dfrac{c}{c+d}$ We yield

$$v(x_1, x_2) = x_1^{\frac{c}{c+d}} x_2^{\frac{d}{c+d}}$$
$$v(x_1, x_2) = x_1^a x_2^{1-a}$$

This means that we can always take a monotonic transformation of the Cobb-Douglass utility function that make the exponents sum to 1.

Definition. Marginal Utility: Consumer's utility change for an increase in the unit of good 1.

$$MU_1 = \lim_{\Delta x_1 \to 0} \frac{u(x_1 + \Delta x_1, x_2) - u(x_1, x_2)}{\Delta x_1} = \frac{\partial u(x_1, x_2)}{\partial x_1}$$

By parity of reasoning for good 2, to calculate a change in utility we can take the total derivative since we multiply the change in consumption for each good by the marginal utility of each good s.t.

$$dU = MU_1 dx_1 + MU_2 dx_2$$

Definition. Marginal Rate of Substitution: Measures the slope of indifference at a given bundle of goods; the rate at which a consumer is willing to substitute good 2 for good 1 to maintain the same amount of utility.
Derivation. By the total derivative we have

$$du = \frac{\partial u(x_1, x_2)}{\partial x_1} dx_1 + \frac{\partial u(x_1, x_2)}{\partial x_2} dx_2 = 0$$

$$\frac{dx_2}{dx_1} = -\frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2}$$

So,

$$MRS = -\frac{MU_1}{MU_2}$$

4.4 Demand

Definition. Demand: We write demand functions as depending on both price and income

$$x_1(p_1, p_2, m) \text{ and } x_2(p_1, p_2, m)$$

Deriving Demand Functions: Equivalent to finding the optimal choices for each good in the bundle. Note: boundary optimums may occur for linear, quasilinear indifference curves and perfect complements.

First method: Tangency method. We take the MRS to equal to slope of the budget line s.t.

$$\frac{MU_1}{MU_2} = \frac{p_1}{p_2}$$

From this, you substitute either x_1 or x_2 into the constraint to yield the demand for one good, and you solve for the other good.
Economic Intuition: MRS is the rate of change at which the consumer is willing to stay put (not adjust consumption). If the market is offering a rate of exchange to the consumer of $-\frac{p_1}{p_2}$, if you give up one unit of good 1, you can buy $\frac{p_1}{p_2}$ units of good 2. If the consumer is at a consumption bundle where he is willing to stay put, it must be one where the MRS is equal to this rate of exchange.
We will use this example to show the property of a quasilinear utility function.
Example. Consider the maximisation problem

$$\max_{x_1, x_2 \in \mathbb{R}} u = \ln x_1 + x_2 \text{ s.t. } p_1 x_1 + p_2 x_2 = m$$

Note that since $x_2 = 0$ is now possible, we have to check for a boundary optimum. The marginal utilities are

$$MU_1 = \frac{\partial u}{\partial x_1} = \frac{1}{x_1}$$

$$MU_2 = \frac{\partial u}{\partial x_2} = 1$$

We obtain by the tangency conditions:

$$MRS_{x_1, x_2} = -\frac{1}{x_1} = -\frac{p_1}{p_2}$$

$$p_1 x_1 + p_2 x_2 = m$$

Solving for Marshallian demand obtains

$$x_1 = \frac{p_2}{p_1}$$

$$x_2 = \frac{m}{p_2} - 1$$

To consider the boundary optimum, we note that $x_2 > 0 \implies \frac{m}{p_2} > 1$. If $\frac{m}{p_2} < 1$, then $x_2 < 0$. But negative demand is not possible so we know that $x_2 = 0$. So the final solution is

$$x_1 = \frac{p_2}{p_1}, x_2 = \frac{m}{p_2} - 1 \text{ for } \frac{m}{p_2} > 1$$
$$x_1 = \frac{p_2}{p_1}, x_2 = 0 \text{ for } \frac{m}{p_2} \leq 1$$

Second Method: Lagrangian multiplier. Trivial.
We will use this example to show a property of Cobb-Douglas preferences:
Example. Let us assume the consumer has the following maximisation problem

$$\max_{x_1, x_2 \in \mathbb{R}} u = a\text{ln}x_1 + (1-a)\text{ln}x_2 \text{ s.t. } p_1x_1 + p_2x_2 = m$$

Constructing the Lagrangian we yield

$$\mathcal{L}(x_1, x_2, \lambda) = a\text{ln}x_1 + (1-a)\text{ln}x_2 - \lambda(p_1x_1 + p_2x_2 - m)$$

FOCs are

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{a}{x_1} - \lambda p_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{(1-a)}{x_2} - \lambda p_2 = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = p_1x_1 + p_2x_2 - m = 0$$

Solving, we yield

$$\frac{a/x_1}{(1-a)/x_2} = \frac{p_1}{p_2}$$
$$p_1x_1 + p_2x_2 = m$$

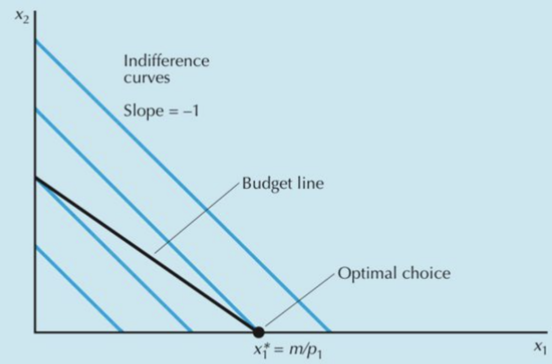
We obtain the demand functions

$$x_1 = a\frac{m}{p_1}$$
$$x_2 = (1-a)\frac{m}{p_2}$$

The Marshallian or uncompensated demand curves tell us the optimal choices of x_1 and x_2 as a function of both income and prices for any and all combinations of income and prices. **Note: for Cobb-Douglas preferences, the demand is a function only of their own price and how much of their income they spend is weighted according to their exponents**.

Optimal Choices/Demand functions for certain goods

1. Perfect Substitutes



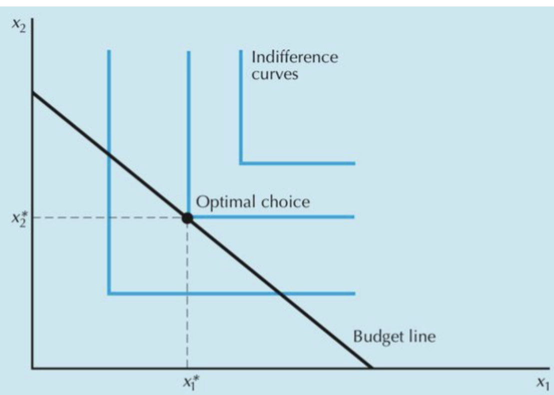
Optimal choice with perfect substitutes. If the goods are perfect substitutes, the optimal choice will usually be on the boundary.

If $p_2 > p_1$, then the slope of the budget line is flatter than the slope of the indifference curves. In this case, the optimal choice would be to spend all his money on good 1. If $p_1 > p_2$, then the slope of the budget line is steeper than the slope of the indifference curves. The optimal choice would be to spend all his money on good 1.
The demand function for good 1 is

$$x_1 = \begin{cases} m/p_1, & \text{when } p_1 < p_2 \\ \text{any number between 0 and } m/p_1, & \text{when } p_1 = p_2 \\ 0, & \text{when } p_1 > p_2 \end{cases}$$

Just says if perfect substitutes, consumer would buy the cheaper good only.

2. Perfect complements



Optimal choice with perfect complements. If the goods are perfect complements, the quantities demanded will always lie on the diagonal since the optimal choice occurs where x_1 equals x_2.

Optimal choice must always lie on the diagonal because the consumer must buy equal amounts of both goods (in terms of the ratio of usage). We know that the consumer is purchasing the same amount of good 1 and 2 regardless of prices. This amount is x. Budget constraint:

$$p_1 x + p_2 x = m$$

Solving for x gives us the optimal choices of goods 1 and 2:

$$x_1 = x_2 = x = \frac{m}{p_1 + p_2}$$

It is as if the consumer is just spending all his money on a single good with the price $p_1 + p_2$

3. Neutral Goods and Bads: The consumer spends all the money on the good she likes and does not purchase any of the neutral/Bad good, regardless of the price. Demand function is

$$x_1 = \frac{m}{p_1}$$
$$x_2 = 0$$

Elasticities of demand:

1. Price elasticity of demand:

$$\epsilon = \frac{\partial x_1}{\partial p_1}\frac{p_1}{x_1}$$

Derivation completed in prior chapter

2. Cross price elasticity of demand:

$$\epsilon_c = \frac{\partial x_1}{\partial p_2}\frac{p_2}{x_1}$$

Derivation. Let us define cross price elasticity of demand as measuring

$$\epsilon = \frac{\%\Delta \text{ in quantity}}{\%\Delta \text{ in other's price}}$$

Let us represent quantity of a function of its own price, other's price and income such that $x_1 = x_1(p_1, p_2, m)$. Then,

$$\epsilon = \frac{[x_1(p_1, p_2 + h, m) - x_1(p_1, p_2, m)]/x_1(p_1, p_2, m)}{[p_2 + h - p_2]/p_2}$$

Now we want to find the price elasticity at some instantaneous point. Using the definition of the partial derivative we have.

- Complements: a negative cross price elasticity.
  Intuition: if there is a rise in price in another good (which leads to a fall in demand for that good) and the demand for your good falls, it means that the demands for both goods are positively related and are thus complements
- Substitutes: a positive cross price elasticity.
  Intuition: if there is a rise in price in another good (which leads to a fall in demand for that good) and the demand for your good rises, it means that the demands for both goods are inversely related. Thus, they are substitutes as they fulfil the same need

3. Income elasticity of demand:

$$y = \frac{\partial x_1}{\partial m}\frac{m}{x_1}$$

Derivation. Let us define income elasticity of demand as measuring

$$\epsilon = \frac{\%\Delta \text{ in quantity}}{\%\Delta \text{ in income}}$$

Let us represent quantity of a function of its own price, other's price and income such that $x_1 = x_1(p_1, p_2, m)$. Then,

$$\epsilon = \frac{[x_1(p_1, p_2, m+h) - x_1(p_1, p_2, m)]/x_1(p_1, p_2, m)}{[m+h-m]/m}$$

Now we want to find the price elasticity at some instantaneous point. Using the definition of the partial derivative we have

$$\epsilon = \lim_{h \to 0} \frac{x_1(p_1, p_2, m+h) - x_1(p_1, p_2, m)}{h} \frac{m}{x_1(p_1, p_2, m)} \implies \epsilon = \frac{\partial x_1}{\partial m} \frac{m}{x_1}$$

Classifying goods according to income elasticity

- Normal goods: positive income elasticity
- Necessities: income elasticity between 0 and 1
- Luxuries: income elasticity greater than 1
- Inferior goods: negative income elasticity

Note: Elasticities vary along the demand curve depending on the slope

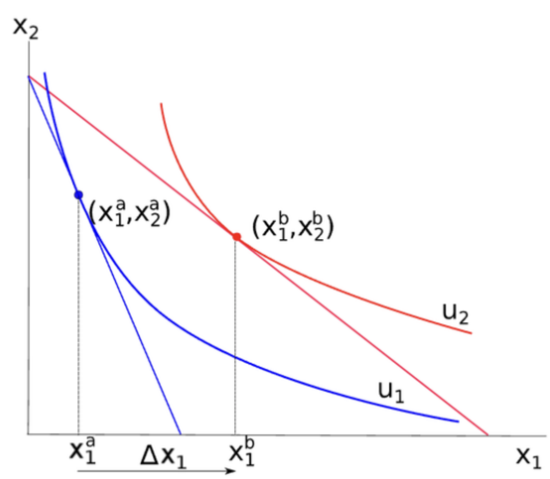
4.5 Slutsky equation: Effects of Price Change

Definition. Slutsky Equation: The Slutsky equation relates the uncompensated effect of a price change to the substitution effect and income effect.

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial x_1^s}{\partial p_1} - \frac{\partial x_1}{\partial m} x_1$$

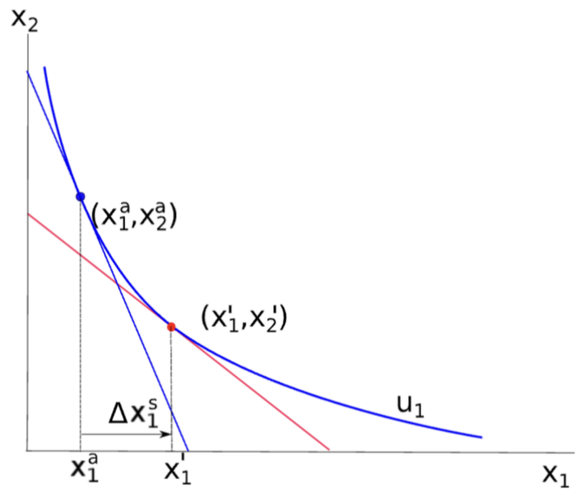
Graphical illustration:

1. Uncompensated effect:



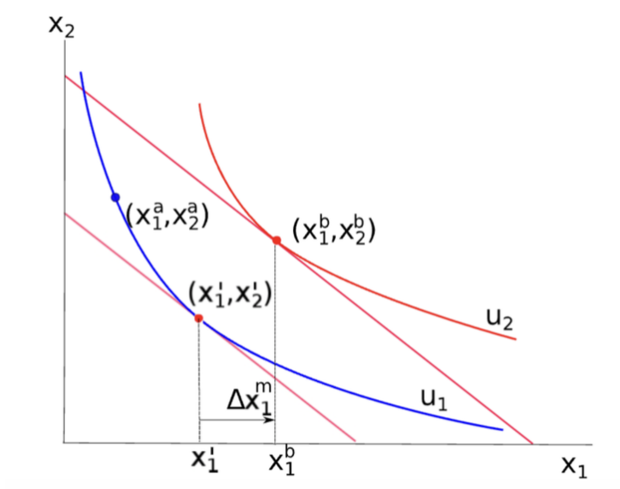
   Suppose the price of x_1 falls. The budget line pivots and the consumer can now buy (x_1^b, x_2^b) now. Overall change is Δx_1 in demand for x_1.

2. Substitution effect: Decomposing the price effect



We consider the effect of a relative price change on demand while holding utility constant. Geometrically, this is a rotation of the budget constraint around the indifference curve. The induced hypothetical change in demand is from (x_1^a, x_2^a) to (x_1', x_2') with the change in demand being Δx_1^s.
Note that in order to allow for such a pivotal shift, under the context of a fall in relative prices, we would have to 'take away' some income such that the consumer stays on the same indifference curve/utility level.

3. Income effect: Now we need to consider the effect of a change in purchasing power on the demand for good 1, holding the prices constant at their new level.



Geometrically, this is a parallel shift of the budget constraint. The induced hypothetical change in demand is from (x_1', x_2') to (x_1^b, x_2^b) and the actual change in demand is Δx_1^m.
This is because at the lower price level, they would be able to buy more and thus have greater 'income' or purchasing power. So this is the income effect.

Overall change in demand is therefore

$$\Delta x_1 = \Delta x_1^s + \Delta x_1^m$$

We can express this with respect to the change in price that caused it which is

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} + \frac{\Delta x_1^m}{\Delta p_1}$$

But the last term, $\dfrac{\Delta x_1^m}{\Delta p_1}$ is not appropriate since it is the effect of a hypothetical income change rather than price.
Instead we can say that the change in income/purchasing power due to a change in price can be approximated as

$$\Delta m \approx -x_1 \Delta p_1$$

For instance, if a consumer was buying 10 units of x_1 and price falls by \$1, then the consumer would have saved \$10 and have \$10 more in 'income' which increases their purchasing power.
Substituting the approximation we now have

$$\frac{\Delta x_1}{\Delta p_1} \approx \frac{\Delta x_1^s}{\Delta p_1} - \frac{\Delta x_1^m}{\Delta m} x_1$$

By taking infinitesimal change, we yield the derivative Slutsky equation.

The Slutsky equation applies to cross price and income elasticity as well

1. Cross price elasticity:
$$\frac{\partial x_1}{\partial p_2} = \frac{\partial x_1^s}{\partial p_2} - \frac{\partial x_1}{\partial m} x_2$$

2. Income elasticity:
$$\frac{\partial x_2}{\partial p_1} = \frac{\partial x_2^s}{\partial p_1} - \frac{\partial x_2}{\partial m} x_1$$

Since $\frac{\partial x_1^s}{\partial p_1}$ refers to the slope of the compensated/Hicksian demand curve,

$$\frac{\partial x_1}{\partial p_1} \neq \frac{\partial x_1^s}{\partial p_1}$$

Slutsky equation with endowment: When the consumer has an endowment $\sqsupseteq_\infty$ which is invariant with p_1, the Slutsky equation simply replaces x_1 with the quantity of goods affected by demand $(x_1 - \sqsupseteq_1)$. Thus,

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial x_1^s}{\partial p_1} - \frac{\partial x_1}{\partial m}(x_1 - \sqsupseteq_\infty)$$

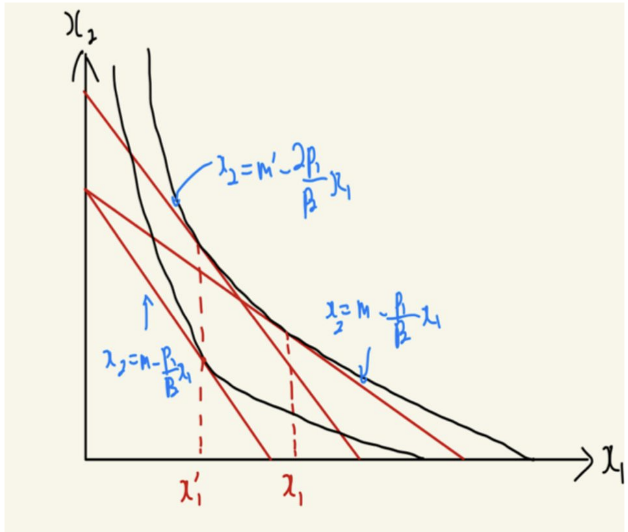
Proposition. The substitution effect $\frac{\partial x_1^s}{\partial p_1} \leq 0$. Proof is in appendix.

Slutsky equation values for different types of goods

1. Normal good: Price increase leads to a fall in demand for good
$$\frac{\partial x_1^s}{\partial p_1} \leq 0, \frac{\partial x_1}{\partial m} x_1 \geq 0 \implies \frac{\partial x_1}{\partial p_1} \leq 0$$

2. Inferior good: Price increase has an indeterminate effect for good
$$\frac{\partial x_1^s}{\partial p_1} \leq 0, \frac{\partial x_1}{\partial m} x_1 \geq 0 \implies \frac{\partial x_1}{\partial p_1} \; =?$$

Some graphical representations



This is the case where the price of good 1, p_1 doubles. Note a few things.

1. Since the price of good 1 doubles, to maintain the same utility, one pivots the budget line around the indifference curve, effectively giving one more income m', represented by the x_2 intercept.

2. Since with the new prices [mistake is that the slope of the inward curve should be $\frac{2p_1}{p_2}$] the income is effectively lower, there is an inward PARALLEL shift of the compensated budget line such that the x_2 intercept (representing the income with respect to the quantity of x_2 since p_2 is unchanged) of both the original budget line and the final budget line are the same.

3. This is equivalent to the change in the budget line as a result of the uncompensated change in demand (pivotal shift around the x_2 intercept)

4.6 Worked Examples

Tutorial sheet 4 Q8 (On the Slutsky equation directly): The demand function corresponding to a Cobb-Douglas Utility function is

$$x_1 = \beta \frac{m}{p_1}$$

where $\beta > 0$ is a constant. Find the substitution and income effects of a price change.
Solution: Simply follow the formulas and dfifferentiate accordingly.
Income effect:

$$x_1 \frac{\partial x_1}{\partial m} = \frac{\beta x_1}{p_1}$$

Substitution effect: We cannot find the derivative of the budget line holding utility constant (finding the derivative of the Hicksian demand curve is out of syllabus). So we use the Slutsky equation for this. By the Slutsky equation we have

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial x_1^s}{\partial p_1} - \frac{\partial x_1}{\partial m} x_1$$

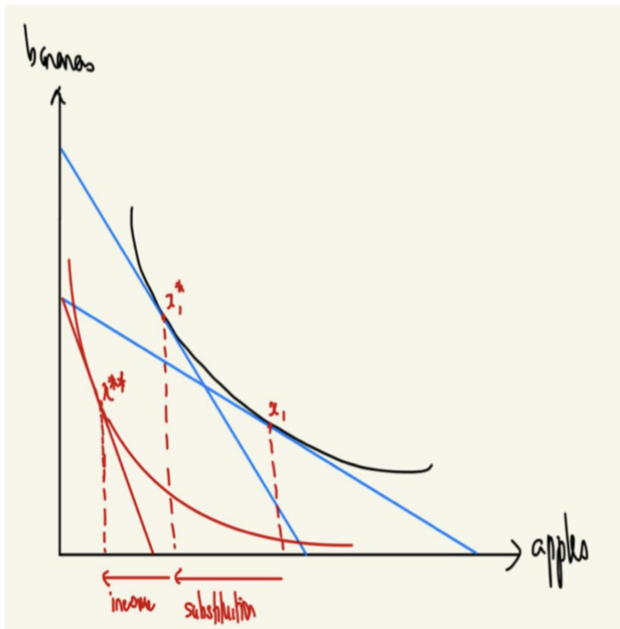
$$\frac{\partial x_1^s}{\partial p_1} = \frac{\partial x_1}{\partial m} x_1 + \frac{\partial x_1}{\partial p_1}$$

$$\frac{\partial x_1^s}{\partial p_1} = -\frac{\beta m}{p_1^2} + \frac{x_1 \beta}{p_1}$$

$$= \frac{\beta^2 m - m\beta}{p_1^2} = \frac{m\beta(\beta - 1)}{p_1^2}$$

Note that we substituted $x_1 = \beta \dfrac{m}{p_1}$ into the equation to yield the above.

Tutorial Sheet 4 Q7 (On using the Slutsky equation to examine welfare effects): An individual has a fixed income and buys two goods, apples and bananas
(a) Explain, using a diagram, how the effect of a rise in the price of apples can be decomposed into a substitution effect and an income effect.



With a rise in price of apples, the relative price of apples increases, thus making the slope of the budget line steeper. Assuming we hold utility constant, the pure substitution effect as a result in the rise of the price of apples would lead to a pivotal shift of the budget line around the indifference curve. So, consumption of apples falls from x to x^*, since to maintain the same amount of utility, the consumer would consume less apples and more bananas, hence substituting his consumption of apples with bananas. Next, given that the price of apples rises, the purchasing power of the consumer falls, since for some non-zero quantity of apple bought, he can only buy less apples than he did originally if he bought the same amount of bananas. So there is a parallel inward shift of the budget line, shifting the optimal consumption of apples from x^* to x^{**}. This reflects the income effect of a rise in price of apples.

(b) Under what conditions would a rise in the price of apples lead to an increase in this individual's demand for apples.

When the apple is a Giffen Good, where the good is so inferior that the negative income effect outweighs the substitution effect. Since these goods are essential, the consumers are willing to pay more for them if their purchasing power decreases. Since the price increase leads to a reduction in their purchasing power, even more expensive and higher quality goods are further out of reach, thus they increase their consumption of these goods as income increases.

Suppose now that the individual has an endowment of 3 apples and 1 banana, and has no income other than that from selling her endowment. The price of bananas is always one. Initially the price of apples is 1 and she consumes 2 apples and 2 bananas.

(c) Would a rise in the price of apples to 2 a make the individual worse off or better off.

It would make her better off because now she can consume 2 apples and 3 bananas after selling 1 apple. Given that her preferences are well behaved in that indifference curves do not cross, (2 apples, 3 bananas) $\succ$ (2 apples, 2 bananas), implying that she would gain more utility.

(d) Would a fall in the price of apples to $\frac{1}{2}$ make her worse off (compared to when apple price was 1).

Indeterminate. We cannot claim that (3 apples, 1 bananas) $\succeq$ (2 apples, 2 bananas) or vice versa. In other words, she cannot obtain a combination of goods that at least maintains her previous consumption bundle.

Tutorial sheet 4 Q9 (Quasilinear preferences): Suppose that a consumer's preferences for two goods, x_1 and x_2, can be represented by the utility function

$$u = \ln(x_1) + x_2$$

Denote the prices of these goods by p_1 and p_2 respectively.
(a) Derive the expressions for the consumer's Marshallian demands.

The consumer's maximisation problem is

$$\max_{x_1,x_2\in\mathbb{R}} (\ln x_1 + x_2) \text{ s.t. } p_1x_2 + p_2x_2 = m$$

The Lagrangian is

$$\mathcal{L}(x_1, x_2, \lambda) = \ln x_1 + x_2 - \lambda(p_1x_1 + p_2x_2 - m)$$

FOCs

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1} - \lambda p_1 = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = 1 - \lambda p_2 = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = m - p_1x_1 - p_2x_2 = 0$$

Solving,

$$\lambda = \frac{p_1}{x_1} = \frac{1}{p_2}$$

$$x_1 = \frac{p_2}{p_1},\ x_2 = \frac{m}{p_2} - 1$$

(b) Show that for the compensated (substitution) effect of a change in its own price is the same as the uncompensated effect. Why is this?
Using the Slutsky equation we have

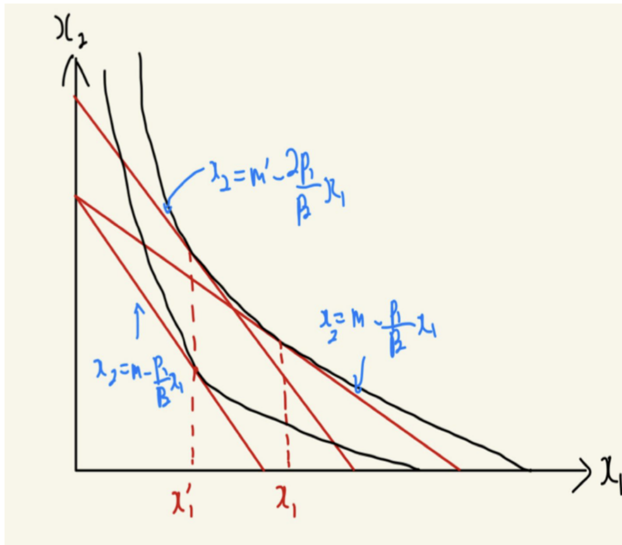
$$\frac{\partial x_1}{\partial p_1} = \frac{\partial x_1^s}{\partial p_1} - \frac{\partial x_1}{\partial m}x_1$$

Note that $\frac{\partial x_1}{\partial m} = 0$. So,

$$\frac{\partial x_1}{\partial p_1} = \frac{\partial x_1^s}{\partial p_1}$$

The reason for this is because the demand for x_1 is invariant with income and only dependent on p_1 and p_2. This is because the utility function is described by a quasilinear indifference curve. Hence a fall in utility would be represented by a vertical downward translation of indifference curves. Since the income effect is graphically described by a downward translation of the post-substitution effect budget constraint, there would be no shift in the demand for good 1 as a result of the falling purchasing power.

(c) Describe the welfare effects of a doubling of the price of good 1, and draw and briefly explain an appropriate diagram to illustrate these effects



When the price of good 1 doubles, holding utility constant, the consumer would consume less of good 1 and more of good 2, reducing consumption from x_1 to x_1'. As the price of good 1 increases, the purchasing power of the consumer consequently falls for the original income, thereby being able to consume less of good 2 as well. This is because given the quasilinear utility function, from part (b), we realise that the substitution effect on good 1 is the same as that on the uncompensated effect. Hence the consumption of good 1 and good 2 falls, the consumer's utility worsens.

5 Game Theory and Institutions, Public Goods and Externalities

5.1 Game Theory

Characterisation. Matrix Game: Usually used to represent single shot, simultaneous games (rather than sequential games which require backward induction)

Definition. Strategies: A strategy is a complete contingent plan of action. It specifies the player's action in every possible distinguishable circumstance whether or not the situation actually arises.

Characterisation. Extensive Form Games: Represented by a game tree and solved using backward induction → at each decision node we find the subgame perfect nash equilibria, starting from the final decision node in the game tree.

Definition. Nash Equilibrium: Nash equilibrium is a situation such that no player has an incentive to change his/her strategy given what the other player is doing (no incentive to unilaterally deviate from his chosen strategy).
Or equivalently, a Nash equilibrium is a situation in which each player is adopting a strategy which is a best response to the strategy actually played by his opponent.

So there can be multiple Nash equilibrium (the case where the payoffs are highest for both players even though that outcome may never come to fruition).

Matching Pennies Game: A game without a pure Nash Equilibrium

	Heads	Tails
Heads	+1*, -1	-1, +1*
Tails	-1, +1*	+1*, -1

Here there is no pure Nash equilibrium; if we allow players to pick their strategies stochastically then there is a mixed Nash equilibrium. Depends on the player's belief on what the other player would do (50-50 probability of choosing heads or tails).

Definition. Strictly Dominated Strategy: A strategy that always delivers a worse outcome than an alternative strategy, regardless of what strategy the opponent chooses.

Definition. Strictly Dominant Strategy: A strategy that always delivers a better outcome than any alternative strategy, regardless of what strategy the opponent chooses.

Prisoner's Dilemma: Classical Game that mutual defection leads to worst outcome.

	Keep quiet	Confess
Keep Quiet	2, 2	0, 3
Confess	3, 0	1, 1

Clearly, if B chooses to keep quiet, A would confess because $3 > 2$; if B chooses to confess, A chooses to confess because $1 > 0$; symmetric for B. Confessing is thus the (strictly) dominant strategy and the nash equilibrium: confessing as a strategy is better strictly than any other strategy no matter what the other player plays.
Characteristics of the Game:

1. Perfect information
2. Simultaneous/single-shot

The prisoner's dilemma game contrasts with Smith's invisible hand in guiding the market to pareto efficiency. The outcome {Keep quiet, Kepp quiet} pareto dominates {Confess, Confess}

Definition. Pareto-Dominant: an outcome is Pareto dominant if and only if some number of individuals are better off without anyone being made worse off.

Stag Hunt Game: Trust dilemma or common interest game. Describes a conflict between safety and social cooperation.
Context: 2 hunters enter a range with lots of hares and a single stag.

- Stags can only be caught if both work together
- Two hunters must decide separately, and without the other knowing, whether to hunt the stag or hare
- If both hunt hares, they will both get half of all hares
- If one hunts the stags and the other hares, the stag-hunter will get nothing while the hare-hunter gets all the hares

- If both hunt the stag, each gets half a stag which is worth more than all the hares

Situation is a useful analogy for many kinds of social cooperations such as international agreements on climate change in which one has to give up autonomy and cooperate towards a more ambitious goal.

	Stag	Hare
Stag	3*, 3*	0, 2
Hare	2, 0	1*, 1*

Unlike in PD, in which the players could effectively ignore what the other person was doing, the stag hunt illustrates a situation in which each player's strategy is entirely a function of what the other player does.
Shows a situation in which there are 2 NE: {Stag, Stag} and {Hare, Hare} → no incentive to unilaterally deviate.
{Stag, Stag} clearly dominates the other but since there is no pre-game coordination, the optimal strategy against a hare-hunter is to hunt hare as well (risk) → to conclude we need mixed strategies.
The stag hunt game can be a revised prisoner's dilemma where the prisoners are friends and care about what happens to each other (part of u_2 is included in u_1).

Dictator Games: The situation where Player 1 decides what to offer both player 1 and player 2 — player 2 can only choose to accept or reject the offer. The intuition is therefore that player 1 must maximise his utility while offering player 2 a payoff that is at least as good as what he would receive otherwise (if there is such a scenario) OR maximise his utility by varying the payoff of the other player.

Example (Tutorial Sheet 5 Q5). Player 1 must decide how to split up a stake between Player 1 and 2. Player 1 has a utility function

$$u_1 = x_1 - \frac{\alpha}{2}\left[\sum_{j=1}^{2}(x_j - \bar{x})^2\right]$$

or similarly expressed as

$$u_1 = x_1 - \frac{\alpha}{4}(x_1 - x_2)^2$$

For the homo economicus egoistical case, the inequality aversion parameter $\alpha = 0$.
The offer to Player 2 would be derived as follows:
Player 1's maximisation problem is

$$\max_{x_1, x_2}(u_1) \text{ s.t. } x_1 + x_2 = 1$$

Substituting the rearranged constraint $x_1 = 1 - x_2$ into the objective function, we obtain

$$u_1(x_2) = 1 - x_2 - \frac{\alpha}{4}(1 - 2x_2)^2$$

Solving for the FOCs

$$u_1'(x_2) = -1 + \alpha(1 - 2x_2) = 0$$
$$x_2 = \frac{1}{2} - \frac{1}{2\alpha}$$

Does not make sense for player 1 to offer player 2 anything more than half the stake, since player 1 ought to maximise his own payoff while reducing inequality between players 1 and 2. Given that anything more than hald (call this x_1^*) would be strictly worse than the $1 - x_1^*$ counterpart since the disutility from inequality would be the same, but utility would be less since $1 - x_1 > x_1$ when $x_1 < \frac{1}{2}$. What this shows is that as the inequality aversion parameter increases, x_2 will tend to $\frac{1}{2}$.

Characterisation. Repeated Games: Differing from one-shot games because now players have to consider dynamic reactions of the opponents. Weigh short run gains from defection vs long costs of reduced propensity for cooperation.

Retaliation: Could could retaliate for a player's previous defection; simple strategy would be tit-for-tat where the first action would be to cooperate and thereafter simply follow what the other player did in the preceding round (Cooperate if opponent does that and defect if opponent does that).

Trouble with Tit-for-Tat is that if P1 chooses D instead of C by mistake (and P2 chose C), then this would lead to an alternating sequence of C,D and D,C strategies, reducing payoffs for both. Forgiveness is needed to restore cooperation.
For dynamic games where players interact repeatedly, cooperation may emerge as the Nash equilibria (for infinitely repeated games).

For finitely repeated games, it by backward induction, since the final stage of the game is equivalent to a one-shot game, always in one's incentive to defect (assuming PD). Thus, the second last stage, knowing the last stage's outcome, would be the same as the final stage of the game, resulting in mutual defection. This logic applies consistently.

Credible threats: your strategy needs to be credible in the sense that it is your best option to employ such a strategy. Uncredible if the opponent knows that your retaliation would make you worse off as well and thus you would never play it

5.2 Public Goods

Definition. Market Failure: 3 conditions to consider a market to work

1. They allocate resources to those who value them the most (and can afford them)
2. All gains from trade are exploited
3. Firms operate at minimum cost

The market then is technically and allocatively efficient; perfectly competitive markets are necessarily pareto efficient.

Definition. Public Goods: Public goods have externalities which are non-rivalrous and non-excludable. i.e. Whatever one person consumes, everyone can consume and no one can be prevented from doing so (consuming 1 unit of good 1 does not mean that there are less units of good 1 available for everyone else).
Examples: National Defence, clean air etc; different people cannot consume different amounts so they have to come to a collective decision on how much to have.

Fundamental problem: individual optimisation $\neq$ Pareto optimal outcome.
Imagine the consumer wants to maximise $u_1(x_1, g)$ subject to what they can afford and their contributions $p_x x_1 + p_g g_1 = m_1$ and $g_1 + g_2 = g$.

This requires $MRS^1_{x_1,g} = -\frac{p_g}{p_x}$.

Suppose that the prices are \$1 each so the relative prices are 1 and that the MRS's are $|MRS^i_{x_i,g}| = 1/1$ for $i = 1, 2$.

Both individuals are willing to accept $1 worth of public goods in returns for a loss of 1 of the private good. This shows that the situation is pareto improvable.

If we cut back expenditure of private goods by \$1 each, we save \$2. If we spend \$1 on public good, since it is non-excludable and non-rivalrous, both players would receive the same amount of utility as if they kept the private goods. But now, society has \$1 more to spend to improve welfare for both players without making anyone worse off. This implies greater utility overall.

Thus, to achieve the Pareto optimal outcome, we need to think how to maximise society's welfare by merging utility and maximising along societal preferences.

Standard Public Goods Game: Suppose that we have two people who have preferences over a private good x and a public good g given by

$$u_1(x_1, g) \text{ and } u_2(x_2, g)$$

They contribute g_1 and g_2 respectively to the provision of the public good and so the total provision g is the sum of

$$g = g_1 + g_2$$

1. Individual optimal choice:
   Consider person 1. She has to choose $\{x_1, g_1\}$. Bt her utility depends, in part, on g_2. If 2 chooses to contribute more, she gets the benefit. If 2 is expected to provide enough to satisfy 1 then 1 will free ride. Consider 1's problem: to choose the amount of the private good x_1 and her contribution to the public good g_1 according to her preferences given what she can afford and the way the public good is produced

   $$\max_{x_1, g_1} u_1(x_1, g) \text{ s.t. } p_x x_1 + p_g g_1 = m_1 \text{ and } g = g_1 + g_2$$

   We can take the second constraint and substitute it into the objective function

   $$\max_{x_1, g_1} u_1(x_1, g_1 + g_2) \text{ s.t. } p_x x_1 + p_g g_1 = m_1$$

   Solving using the method of the Lagrangian multiplier

   $$\mathcal{L}(x_1, g_1, \lambda) = u_1(x_1, g_1 + g_2) - \lambda[p_x x_1 + p_g g_1 - m_1]$$

   FOCs:

   $$\mathcal{L}_{\S\infty} = \frac{\partial u_1}{\partial x_1} - \lambda p_x = 0$$
   $$\mathcal{L}_{\}\infty} = \frac{\partial u_1}{\partial g_1} - \lambda p_g = 0$$
   $$\mathcal{L}_\lambda = p_x x_1 + p_g g_1 - m_1 = 0$$

   Taking the ratio of $\mathcal{L}_{\S\infty}$ and $\mathcal{L}_{\}\infty}$ gives the usual MRS relative price conditions

   $$\frac{\frac{\partial u_1}{\partial x_1}}{\frac{\partial u_1}{\partial g_1}} = |MRS^1_{x_1, g_1}| = \frac{p_x}{p_g}$$

This describes the trade-off between her private consumption x_1 and her contribution to the public good g_1. We are interested in the trade off with respect to the consumption of the public good g. Recall that $u(x_1, g_1 + g_2)$. The marginal utility of g_1 is (by the chain rule)

$$\frac{\partial u_1}{\partial g_1} = \frac{\partial u_1}{\partial (g_1 + g_2)} \frac{\partial (g_1 + g_2)}{\partial g_1}$$

And since $g_1 + g_2 = g$, we have

$$\frac{\partial u_1}{\partial (g_1 + g_2)} = \frac{\partial u_1}{\partial g}$$

and

$$\frac{\partial (g_1 + g_2)}{\partial g_1} = 1$$

So

$$\frac{\partial u_1}{\partial g_1} = \frac{\partial u_1}{\partial g}$$

This implies that she gets the same benefit from a change in the level of provision of the public good whether she pays for it or someone else does. It is this which gives rise to the free-rider problem (since she incurs a cost by contributing)

$$|MRS^1_{x_1,g_1}| = \frac{p_x}{p_g}$$

	Contribute	Free-ride
Contribute	$[u_1 - \frac{c}{2}]$, $[u_2 - \frac{c}{2}]$	$[u_1 - c]$, $[u_2]$
Free-ride	$[u_1]$, $[u_2 - c]$	0,0

Note there is no free rider problem unless $\frac{c}{2} < u_i < c$ because this condition then makes it such that the dominant strategy and Nash equilibrium would be Free ride, Free ride and outcome would be no provision of public good {0,0}. This is the Nash Condition.

2. Socially Optimal Choice:
To internalise the externality, here we merge the people and choose the quantities of private goods and the overall level of the provision of the public good $\{x_1, x_2, g\}$ according to social preferences

$$W(u_1(x_1, g), u_2(x_2, g))$$

where W is some increasing function of the individuals' preferences. We then choose x_1, x_2, g to optimise social welfare, subject to the way that the public good is provided

$$g = g_1 + g_2$$

and the social budget constraint

$$p_x(x_1 + x_2) + p_g g = m_1 + m_2$$

This must give us a Pareto optimum because if we are maximising joint welfare it will be impossible to find a Pareto improvement. Here is the Langrangian

$$\mathcal{L}(x_1, x_2, g, \lambda) = W(u_1(x_1, g), u_2(x_2, g)) - \lambda[p_x(x_1 + x_2) + p_g g - m_1 - m_2]$$

Here, the FOCs are (by chain rule)

$$\mathcal{L}_{\S\infty} = \frac{\partial W}{\partial u_1}\frac{\partial u_1}{\partial x_1} - \lambda p_x = 0$$

$$\mathcal{L}_{\S\in} = \frac{\partial W}{\partial u_2}\frac{\partial u_2}{\partial x_2} - \lambda p_x = 0$$

$$\mathcal{L}_{\}} = \frac{\partial W}{\partial u_1}\frac{\partial u_1}{\partial g} + \frac{\partial W}{\partial u_2}\frac{\partial u_2}{\partial g} - \lambda p_g = 0$$

$$\mathcal{L}_{\lambda} = p_x(x_1 + x_2) + p_g g - m_1 - m_2 = 0$$

Take $\mathcal{L}_{\S\infty}$

$$\frac{\partial W}{\partial u_1}\frac{\partial u_1}{\partial x_1} - \lambda p_x = 0$$

and rearrange

$$\frac{\partial W}{\partial u_1} = \lambda \frac{p_x}{\frac{\partial u_1}{\partial x_1}}$$

Similarly

$$\frac{\partial W}{\partial u_2} = \lambda \frac{p_x}{\frac{\partial u_2}{\partial x_2}}$$

Here is the $\mathcal{L}_{\}}$ condition:

$$\left[\frac{\partial W}{\partial u_1}\right]\frac{\partial u_1}{\partial g} + \left[\frac{\partial W}{\partial u_2}\right]\frac{\partial u_2}{\partial g} = \lambda p_g$$

Sub in

$$\left[\lambda \frac{p_x}{\frac{\partial u_1}{\partial x_1}}\right]\frac{\partial u_1}{\partial g} + \left[\lambda \frac{p_x}{\frac{\partial u_2}{\partial x_2}}\right]\frac{\partial u_2}{\partial g} = \lambda p_g$$

Simplify

$$\left[\frac{\frac{\partial u_1}{\partial g}}{\frac{\partial u_1}{\partial x_1}}\right] + \left[\frac{\frac{\partial u_2}{\partial g}}{\frac{\partial u_2}{\partial x_2}}\right] = \frac{p_g}{p_x}$$

Which is

$$|MRS^1_{x_1,g}| + |MRS^2_{x_2,g}| = \frac{p_g}{p_x}$$

This characterises the socially-optimal level of provision. This is the Samuelson condition.

5.3 Externalities

Type of Externalities

1. **Production Externalities:** A production externality is a situation in which the production function of a firm is affected by the actions of another party.
   Consider a steel mill produces two outputs (steel s, and pollution x). Because the marginal cost of producing pollution is negative (i.e. abatement activities cost money; not polluting is costly or pollution decreases the cost of the firm), the firm has the cost equations

$$\frac{\partial c(s, x)}{\partial s} > 0$$

$$\frac{\partial c(s, x)}{\partial x} \leq 0$$

A fishery downstream has the cost function $k(f,x)$ where f is the output of fish with the cost function $k(f,x)$ with

$$\frac{\partial k(f,x)}{\partial f} > 0$$

$$\frac{k(f,x)}{\partial x} > 0$$

Marginal cost of pollution is positive since the pollution in the river raises the fishery's cost (in the sense that for each \$1 spent to catch fish, they now catch less fish due to a rise in pollution).
The firm's profit maximisation problems are

$$\max_{s,x} p_s s - c(s,x)$$

$$\max_{f} p_f f - k(f,x)$$

The first order conditions with respect to s, x, f are

$$p_s - \frac{\partial c(s,x)}{\partial s} = 0 \Rightarrow p_s = \frac{\partial c(s,x)}{\partial s}$$

$$-\frac{\partial c(s,x)}{\partial x} = 0 \Rightarrow 0 = \frac{\partial c(s,x)}{\partial x}$$

$$p_f - \frac{\partial k(f,x)}{\partial f} = 0 \Rightarrow p_f = \frac{\partial k(f,x)}{\partial f}$$

These conditions say that (because we have assumed a perfectly competitive situation in the steel and fish markets) that price equals marginal costs.
But notice that there is no market for pollution: so the price of pollution is zero and the condition for optimal production of pollution for the steel mill is the pollute until the marginal cost is 0.
The Coase Theorem. The Coase Theorem claims that when trade in an externality is possible and that there are no transaction costs, bargaining will lead to an efficient outcome regardless of the initial allocation of property rights (presupposes the existence of property rights).
Obstacles to bargaining (such as transaction costs and asymmetric power between agents) or poorly defined property rights can prevent or reduce the efficiency of Coasian bargaining.
The main problem for Production Externalities is that there is no market for these externalities.

Solution: To work out the Pareto efficient situation we need to take into account social utility and preferences by merging the firms; this implies that there is no more externality because there is only one entity. This internalises the externality.
The merged firms profit maximisation problem is

$$\max_{s,f,x} p_s s + p_f f - c(s,x) - k(f,x)$$

Solving for FOCs, we yield

$$p_s - \frac{\partial c(s,x)}{\partial s} = 0$$

$$-\frac{\partial c(s,x)}{\partial x} - \frac{\partial k(f,x)}{\partial x} = 0$$

$$p_f - \frac{\partial k(f,x)}{\partial f} = 0$$

The first and third conditions are as per the individual situation.
The second condition however seems to suggest "equate the marginal cost of pollution abatement to the marginal benefit of a reduction in pollution to the fishery" → the marginal benefit is indicated by the negative value of the marginal cost of increased pollution i.e. the fishery would be willing to pay the steel mill money to prevent them from producing as long as the marginal benefit of reduction in pollution is greater than or equal to the marginal cost of pollution abatement

$$\frac{\partial c(s,x)}{\partial x} = -\frac{\partial k(f,x)}{\partial x}$$

To achieve this, we develop a market through tradeable permits (conferring property rights on either party).
Suppose we confer a property right on the fishery for a clean river, but also allow the fishery to sell the right to pollute. Let q be the unit price of pollution. Now the cost of buying pollution permits is added to the mill's cost so then the steel mill's problem is

$$\max_{s,x} p_s s - qx - c(s,x)$$

The fishery's problem is also changed as selling permits is a secondary source of income

$$\max_{f,x} p_f f + qx - k(f,x)$$

The first order conditions for steel mill are

$$p_s = \frac{\partial c(s,x)}{\partial s}$$

$$q = -\frac{\partial c(s,x)}{\partial x}$$

Intuition: the maximal amount the steel mill is willing to pay for the permits equal to how much it costs them to reduce pollution at each instantaneous x (marginal cost of reducing pollution essentially).
For the fishery, they are

$$p_f = \frac{\partial k(f,x)}{\partial f}$$

$$q = \frac{\partial k(f,x)}{\partial x}$$

Intuition: the minimal amount the fishery is willing to accept for the permits is equal to the marginal cost of pollution onto them.
So, the price of pollution is equal to the social cost of pollution and that yields

$$-\frac{\partial c(s,x)}{\partial x} = \frac{\partial k(f,x)}{\partial x}$$

Problems with Tradeable permits

(a) Often require international agreements and negotiations in setting up these agreements are often prone to hold up problems
(b) Not guarantee that the economically efficient level of pollution is necessarily zero; can even be quite high (so may not be environmentally optimal)

2. **Consumption Externalities:** Consumption externalities arise when one person's consumption directly enters another person's preferences. It is quite a general idea which encompasses nice things like concern for inequality and other's utility etc and also things like envy and jealousy.
We tend to assume that individuals only consider their own preferences and not others. But experimental evidence clearly indicates that most individuals have preferences over, not just their own, but

also over other people's returns.
Let x_1 denote what player 1 receives and x_2 denote what player 2 receives. Under consumption externalities, Player 1's utility function is thus

$$u_1(x_1, x_2)$$

Example of consumption externality where other people's choices directly enter your pay-off function. Players can have preferences over their own pay-off, overall payoff and the distribution of payoffs. Preferences can be used to model other-regarding behaviour:

$$u(x_i, x_j) = \alpha x_1 - \beta \max\{0, x_j - x_i\}^2 - y \max\{0, x_i - x_j\}^2$$

5.4 Worked Examples

Tutorial Sheet 5 Q5 (Free rider problem for k players): Group of k group individuals deciding whether to (C) or not to (N) contribute 1 unit of their wealth to a public good. Amount of public good produced is equal to the aggregate contributions of all members of the group and produces a benefit of b per unit per person.

Assuming $0.5 < b < 1$ and $k = 2$, the Nash Equilibrium is {N,N} and it is not pareto efficient.

1/2	C	N
C	2b-1, 2b-1	b-1, b
N	b, b-1	0, 0

Since $b > 2b - 1 > 0 > b - 1$, this implies that the dominant strategy is to not contribute. However, if $b < 0.5$, then $b > 0 > 2b - 1 > b - 1$. So the N,N outcome would be Pareto efficient. If $b > 1$, then $2b - 1 > b > b - 1 > 0$, then the Nash equilibrium and Dominant strategy would be {C,C} achieving the Pareto efficient outcome.

To derive conditions for Pareto-inefficiency for a generic k-player game, we need two necessary conditions

1. All players would choose N (so it is a Nash Equilibrium)
2. Outcome $\{N_1, N_2, \ldots, N_k\}$ is pareto inefficient.

We let the number of people defecting be $n \in [0, k-1]$. My payoff function would thus be (for each strategy)

$$\pi_C = (k - n)b - 1$$
$$\pi_N = (k - 1 - n)b$$

My condition to choose N for every n is when $\pi_N > \pi_C$ s.t.

$$(k - 1 - n)b > (k - n)b - 1$$
$$b < 1$$

Since all payoffs are symmetric for each player, when $b < 1$, all players would choose N. The value of b for which the outcome of everyone choosing N is inefficient is when the C,C outcome is better than the {N,N} outcome. So,

$$kb - 1 > 0$$
$$b > \frac{1}{k}$$

Thus, for any k, when $\frac{1}{k} < b < 1$, the public goods game would result in a Pareto inefficient outcome.

6 Market Failure: Firm Power

6.1 Monopoly and Monopsony

Performance of Monopoly:

1. Production level: The monopoly produces at

$$MC(y) = MR(y)$$

Consider the firm's maximisation problem

$$\max_y \pi(y) = R(y) - c(y)$$

The competitive firm is modelled by

$$\max_y \pi(y) = py - c(y)$$

But the monopoly is a price setter, so we endogenise the price as a function of the firm's output

$$\max_y \pi(y) = p(y)y - c(y)$$

MC = MR condition derivation

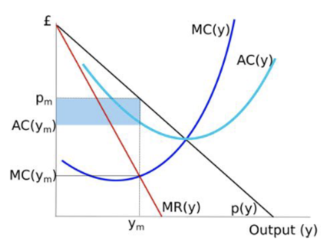
$$\pi'(y) = p(y) + yp'(y) - c'(y) = 0$$
$$p(y) + yp'(y) = c'(y)$$

Therefore,

$$MR(y) = p(y) + yp'(y)$$

Second term involves the slope of the inverse demand function: this reflects that marginal changes in output decisions will affect the WTP in the market
Since $p'(y) < 0$,

$$MR(y) < p(y)$$



- MR lies below the demand curve (it doesn't have to be linear unless the revenue function is quadratic)
- The output at which $MR = MC$ is y_m. Note that $p_m = p(y_m) > MC(y_m)$
- As usual we can read off the price from the demand curve and the costs from the AC curve. The profits are as indicated.

The MC = MR condition can be re-written as the Lerner Index (relates price/begin elasticity of demand with price and marginal cost)

$$\text{Lerner Index} = \frac{p - MC(y)}{p} = \frac{1}{|\epsilon(y)|}$$

which represents the proportional markup of price over the marginal cost.

2. Pricing level: $p(y) > MC(y)$. Given the Lerner Index,

$$\frac{1}{|\epsilon(y)|} > 0 \implies p(y) - MC(y) > 0$$

So the monopolist prices above the marginal cost – but this is limited by the demand curve. When the demand curve is extremely inelastic, the mark-up is at its biggest.
Proposition. Monopoly price is always greater or equal to perfectly competitive price. Alternatively, Monopoly output is always lower than or equal to perfectly competitive output.
Proof. Assume for contradiction that $y_m > y_c$. This implies that $p(y_m) < p(y_c)$ since $p(y)$ is decreasing in y.
Given that $p(y_c) = c'(y_c)$ and $p(y_m) = c'(y_m) - p'(y_m)y_m$, we can conclude that since

$$p'(y_m) < 0, y_m > 0 \implies -p'(y_m)y_m > 0$$

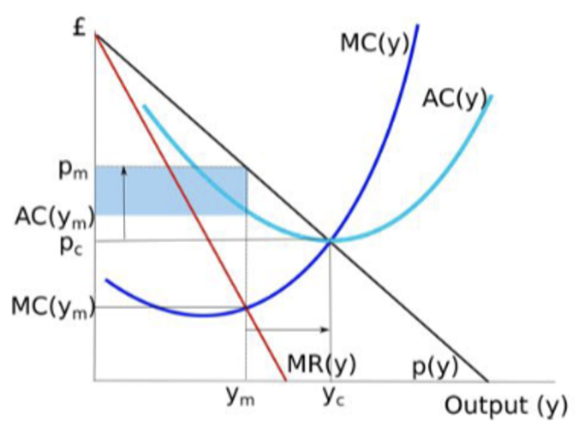
and since $c'(y)$ is increasing in y and $y_m > y_c$

$$c'(y_m) > c'(y_c) \implies p(y_m) > p(y_c)$$

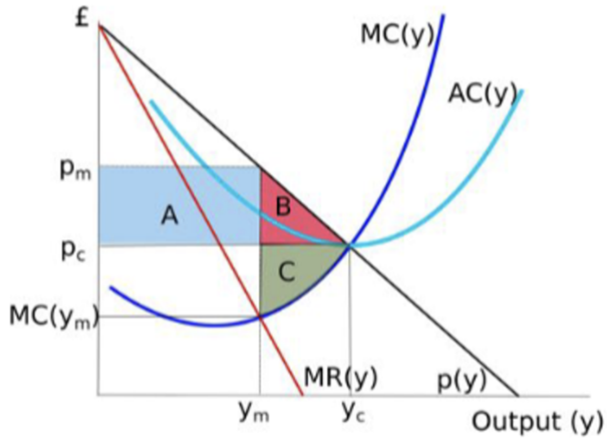
But this contradicts the implication of the assumption. Thus, we can conclude

$$y_c \geq y_m \implies p(y_m) \geq p(y_c)$$

3. Inefficiency of monopoly: Compared to the competitive outcome (p_c, y_c), the monopolist charges more and produces less because it is not being forced by the competition to produce at the most efficient scale



A is consumer surplus which is transferred across to the monopolist; B is lost consumer surplus and C is lost producer surplus



$$
\begin{aligned}
\Delta PS &= A - C \\
\Delta CS &= -(A + B) \\
DWL = \Delta PS + \Delta CS &= B + C
\end{aligned}
$$

Sources of Monopoly Power:

1. Legal BTEs: Granted by government through patents or through a government franchise (local rail franchises, new drugs (granted patents that provide a monopoly for a period of time)). Copyright law also confers a monopoly for a supposedly limited period of time
2. An economy of scale which is large relative to the size of demand. If the average cost when a single firm serves the entire market is lower than when two or more firms serve the market, a monopoly can be the result. Best examples often relate to net-work based industries like telecommunications, electricity distribution etc
3. Control of an essential or sufficiently valuable input to the production process.
   (a) IP or technology that confers a cost advantage or natural resource
   (b) Persistent allegations that Microsoft kept secret some of the "application program interfaces" used by Office as a means of excluding rivals: When software is run by a computer operating system, it needs to be designed to work well with the operating system
4. Differentiated Product: This creates some curvature in the indifference curves between you and your computer's products $\rightarrow$ finite elasticity

Key feature of the analysis of monopoly (power) is that $p'(y) < 0$ (equivalent to $|\epsilon(y)| < \infty$. If you have some finite elasticity to work with you set $p > MC$.

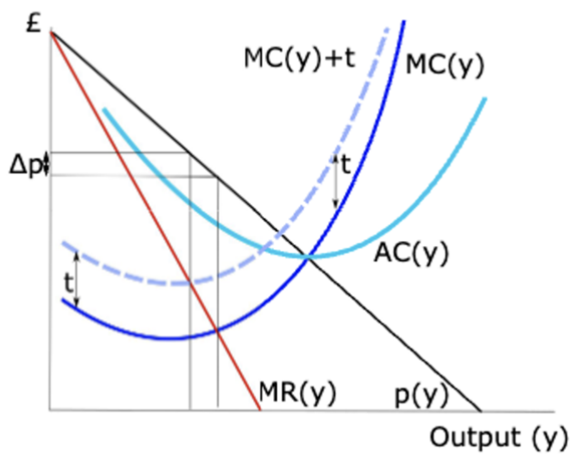
Assumptions of Monopoly that lead to each aspect of market performance

- Profit Maximisation: So the firm sets price at profit maximising $MC = MR$
- Barrier to Entry: So rival firms cannot enter into market to compete its supernormal profit away. Thus, $p_m > AC$
- Sole firm. So its demand curve is downward sloping as it is the enter market's demand curve. It is thus a price setter.

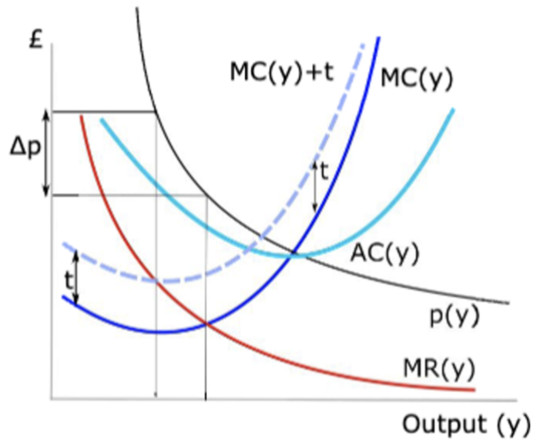
Assumptions of perfectly competitive firm that lead to each aspect of market perfect

- Profit maximisation: So the firm sets $p^* = MC$. Any lower and it would be losing money. The firm's maximisation problem is
$$\max(py - c(y))$$
Note that unlike the monopoly, p is an exogenous variable and is not affected by the choice variable y. So the profit maximising condition is since the price is treated as some arbitrary constant, not as a function of quantity
$$p^* = c'(y)$$
- No BTE: So the firm sets $p^* = AC$. Since if $p^* > AC$, then firms would enter the market to compete away its profits.
$p^* = AC = MC$ is the market clearing price since $p(y)$ is market demand and at $p^* = MC$, everyone who values the product more than or equal to the cost of producing the product would have bought it from the firms.

Taxation on Monopoly: One feature of the analysis of commodity taxes in the context of perfectly competitive markets was that the extent to which a tax could be passed on to consumers was, at worst, 100%. But markets served by a monopolist can be more than 100% tax passed on to consumers by adding their own mark-up to the tax → depending on the shape of the demand curve.



Here, there is a specific duty of t charge: the formal incidence is on the firm so the tax-inclusive marginal cost goes up by a fixed amount (t) as the firm has to remit t on every unit sold. The MC curve moves up by t.
Comparing the pre-tax and post-tax price/output, we note that the amount the price has increased is less than the size of the tax. The price has been shifted by less than 100% of the tax.
But for demand curves that are convex to the origin (sometimes constant elasticity)



The MR is non-linear too as a consequence; in this case, the amount by which the price changes exceed the amount of the tax.

Price Discrimination. The monopoly is producing where $p_m > MC_m$.
This implies that there is a consumer who would be willing to pay a price of $p_c \leq p \leq p_m$ for a marginal unit of the good beyond x_m who is not buying any at the moment. If the firm sold a unit to this person for p, then it would still be the case that the price exceeded marginal cost — so it's still worth doing (while charging the rest of the consumers p_m). This means that both the firm and the buyer would be better off — a Pareto improvement. This is true for all sales between y_m and y_c.
Monopolies are inefficient — do not efficiently exploit all possible Pareto gains.
They would like to sell at a lower price to marginal consumers whilst continuing to charge a high price to everyone else. Thus, they need to find a mechanism to prevent arbitrage (such that they can simultaneously sell different prices to different groups of consumers). They also need to be able to identify the marginal consumers.
Different forms of price discrimination are characterised by how much information the firm has about consumers.

3rd Degree Price Discrimination. Occurs when a firm can identify different groups of consumers and sell to each of them at different prices.
The firm's maximisation problem is

$$\max_{y_1,\ldots,y_G} p_1(y_1)y_1 + \cdots + p_G(y_G)y_G - c(y_1 + \cdots + y_G)$$

- $p_g(y_g)y_g$ is sales revenue from the g-th group
- $c(y_1 + \cdots + y_G)$ is the total cost of production which depends on the total output (they have one production plant)

First Order Conditions for Maximisation Problem

$$p_g(y_g) + y_g p_g'(y_g) = c'(y) \text{ for each } g \text{ group}$$
$$MR_g = MC \text{ for each } g \text{ group}$$

Implication for behaviour: the firm should set marginal revenue equal across all groups. But MR is closely related to the elasticity of demand.
Using the expression we saw earlier we can rewrite $MR_g = MC$ as

$$p_g(y_g)\left[1 - \frac{1}{|\epsilon_g(y_g)|}\right] = MC$$

This implies that for any pair of markets (or all of them)

$$p_i(y_i)\left[1-\frac{1}{|\epsilon_i(y_i)|}\right]=p_j(y_j)\left[1-\frac{1}{|\epsilon_j(y_j)|}\right]$$

Therefore, more elastic implies lower prices sold to that specific group of marginal consumers

$$|\epsilon_i(y_i)|>|\epsilon_j(y_j)| \implies p_i<p_j$$

In the limit, if firms can identify individuals and prevent arbitrage, then they could in principle practice perfect price discrimination by charging every individual a personalised price.

- Simply along the demand curve
- Online retailers are beginning to move in this direction and financial serves firms have been doing this for some time
- Note that perfect price discrimination is perfectly Pareto efficient: everyone is being charged a price exactly equal to their willingness to pay and are trading voluntarily; it is simply that the firm is extracting all of the consumer surplus and transferring it to itself. No DWL

Definition. Monopsony: Sole buyer that exploits its position to influence the market price of its inputs.

Performance of Monopsony:

1. Production level: $MC=MR$
   Profit maximisation problem faced by the monopsonist is to choose the input x

   $$\max_x py(x)-w(x)x$$

   where one assumes the firm is a price-taker in the market for its output y. Differentiate with respect to the choice variable x and use first order conditions

   $$py'(x)-w(x)-xw'(x)=0$$

   The first order condition is just $MC=MR$ again:

   $$py'(x)=w(x)+xw'(x)$$

   The condition essentially says 'hire workers until the extra revenue generated by the last worker just balances the cost of hiring that worker'.
   The term $py'(x)$ measures the revenue generated by using a marginal unit of the input x.
   This is termed the marginal revenue product to distinguish it from the output/sales oriented version.
   $MRP(x)=py'(x)$ is a function of the amount of input used. Its shape depends on returns to scale.

   - With constant returns to scale it is flat
   - With diminishing marginal returns to scale, it is downwards-sloping

2. Analysing the MC curve: the MC has two terms

   $$w(x)+xw'(x)$$

   - The first order effect is $w(x)$ which is the value of the inverse supply curve (e.g. the wage)
   - The second order effect is the effect of a change in demand for the input on input prices; The effect of hiring a marginal worker on wages

   In a competitive factor market the supply curve the firm faces is flat. It can hire as much labour as it likes at the going wage rate.
   But when the firm is the only buyer of labour in the market, it faces an upward sloping supply curve.

- If it wants more labour it will need to pay more and so drive the wage up (since it is the only consumer and thus affects the market singlehandedly)
- If it hires less, it will drive wages down

Using the definition of the supply elasticity of the factor $\eta(x)$ gives

$$MC = w(x)\left[1 + \frac{1}{|\eta(x)|}\right]$$

Since supply curves tend to slope upwards, $\eta(x) > 0$. This implies that the marginal cost of labour (at the profit maximising optimum) will be higher than the market wage: $MC(x) > w(x)$ i.e. the marginal cost curve lies above the (inverse supply curve).
Derivation of the MC equation.

$$MC = w(x) + xw'(x)$$

If we take the supply curve to be $x(w)$, the elasticity of supply is

$$\eta = x'(w)\frac{w}{x}$$

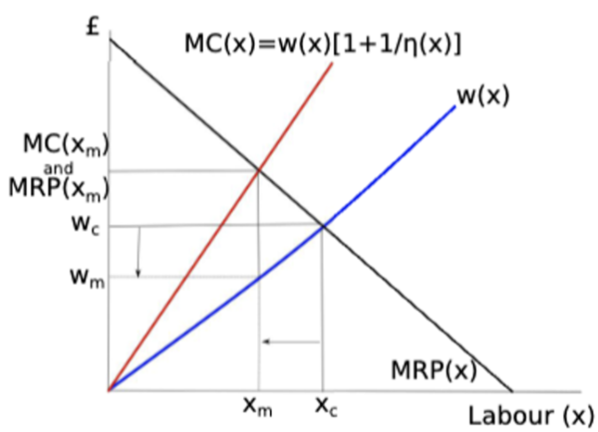
By the inverse function theorem,

$$\eta = \frac{1}{w'(x)}\frac{w}{x}w'(x) = \frac{1}{\eta}\frac{w}{x}$$

Substituting into the marginal cost

$$MC = w(x) + \frac{w(x)}{\eta}$$

$$MC = w(x)\left[1 + \frac{1}{|\eta(x)|}\right]$$



This illustrates what is going on for the DRS case

- The firm wants to hire labour at the level at which $MRP(x) = MC(x)$. This is x_m
- At x_m, the wage can be read off the supply curve. It is w_m

If the firm was buying labour competitively, its $MC(x)$ would just be the inverse supply curve. It would hire more workers and pay them more. As it is the firm exploits workers by not even paying them their marginal revenue product (the value of their contribution to revenue).

Mark-up equation for Monopsony:

$$\frac{MC - w(x)}{w(x)} = \frac{1}{\eta(x)}$$

Workers are more likely to get exploited when their bargaining power is weak due to a low elasticity of supply. When their outside options are poor (thus supply is inelastic). When markets are failing due to monopsony power, a minimum wage can correct DWL's rather than create them.

6.2 Models of Duopoly

We consider a duopoly with the following characteristics

- Homogenous product
- Two player – more can be added as long as an individual firm has the power to affect the price
- Perfect and symmetric information + Centrainty (can be relaxed)
- Firm's payoff (profit) depends on strategies adopted by both themselves and their opponents

Cournot Competition (quantity):
Quantity competition: 2 firms set quantities simultaneously. Each firm has to forecast what the other will do, and given this conjecture, set its output to be profit maximising.

Deriving Output and Price in Cournot Competition. Let firm 1 conjecture that firm 2 will produce y_2 units of output. Thus, the conjectured industry production would be

$$y = y_1 + y_2$$

As usual, firm 1 will produce at $MC_1 = MR_1$ to maximise profits given firm 2's production. Firm 1's revenue is

$$R_1 = p(y)y_1, \text{ where } y = y_1 + y_2$$

Marginal Revenue is thus,

$$MR_1 = p(y) + p'(y)y_1$$

- $p(y)$ is the market price per unit
- $p'(y)y_1$ captures the effect of the firm's output decision on the aggregate WTP ($p'(y)$ is negative if aggregate demand is downward-slopping)

Firm 1's best response is derived from the profit maximising condition:

$$p(y) + p'(y)y_1 = MC_1$$

Since this depends on the conjectured and thus arbitrary constant/exogenously determined y_2, by varying y_2, we can compute firm 1's best response for every possible y_2 in terms of its own output y_1. Like Prisoner's Dilemma but we have a continuum of actions.
This set of $\{y_1, y_2\}$ best response pairs represents firm 1's reaction function and describes its strategy — a complete, contingent plan of action.
We can thus regard a rearrangement of $MR_1 = MC_1$ as the reaction function where

$$y_1 = r_1(y_2)$$

Since payoffs are symmetric (assuming homogenous marginal cost across firms), the treatment of firm 2 is exactly the same

$$y_2 = r_2(y_1)$$

Nash equilibrium situation because each firm's strategy is a best response to the chosen strategy of the other firm. There is no incentive to unilaterally deviate from.

So the Nash equilibrium must be a combination of outputs $\{y_1^*, y_2^*\}$ such that

- the optimal output for firm 1, assuming that firm 2 produces y_2^*, is y_1^*
- the optimal output for firm 2, assuming that firm 1 produces y_1^*, is y_2^*

$$y_1^* = r_1(y_2^*)$$
$$y_2^* = r_2(y_1^*)$$

Both players are playing best responses – any deviation would not be profitable (Nash Equilibrium).

Mark-up Equation: We rearrange the $MR_i = MC_i$ reaction function condition into a more useful form:

$$\frac{p - MC_i}{p} = \frac{s_i}{|\epsilon|}$$

where s_i denotes market share for firm i and $|\epsilon|$ denotes the absolute value of the elasticity of market demand.
LHS is the firm's mark-up of price over marginal cost of production.
We observe a few things from the mark-up equation

- As MC falls, ceteris paribus, the firm's market share rises
- As demand becomes more elastic, ceteris paribus, the firm's market share rises. Because while you make more money on each unit sold, you sell fewer units
- We can think of the term $\frac{s_i}{|\epsilon|}$ as the effective price elasticity facing the firm.
  The smaller the firm is, the more elastic the demand curve it effective faces and its ability to raise price above marginal cost is curtailed.
- In an industry in which both firms have the same technology

  $$\frac{p - MC_1}{p} = \frac{p - MC_2}{p} \Rightarrow s_1 = s_2$$

  As more technologically similar firms enter this industry, $s_i \to 0$ and $p \to MC$. So, we tend towards the competitive equilibrium

Derivation of the Mark-up equation. Take the $MC = MR$ condition for a monopolist

$$p(y) + yp'(y) = c'(y) = MC$$

If we take the demand curve $y(p)$ the elasticity of demand is

$$\epsilon = y'(p)\frac{p}{y}$$

The Inverse Function Theorem says that, for a differentiable, monotonic function of a single variable $y = f(x)$ and its inverse $x = g(y)$, it is the case that

$$f'(x) = \frac{1}{g'(y)}$$

Applying this to the market demand curve and inverse demand curve, we have

$$y'(p) = \frac{1}{p'(y)}$$

So the elasticity of demand can be written as

$$\epsilon = \frac{1}{p'(y)}\frac{p}{y}$$

Rearranging,

$$p'(y) = \frac{1}{y}\frac{p}{\epsilon}$$

Substitute back into the $MC = MR$ condition to obtain

$$p + \frac{p}{\epsilon} = MC$$

Then rearrange for the mark up equation

$$\begin{aligned}
p - MC &= -\frac{p}{\epsilon} \\
\frac{p - MC}{p} &= -\frac{1}{\epsilon} \\
\frac{p - MC}{p} &= \frac{1}{|\epsilon|}, \text{ since } \epsilon < 0
\end{aligned}$$

Oligopoly variation: For an oligopolist (Firm i), the equivalent of the $MR = MC$ condition is

$$p + y_i p'(y) = MC_i$$

As before, we have the rearrangement of the elasticity

$$p'(y) = \frac{1}{y}\frac{p}{\epsilon}$$

Substituting into the $MC = MR$ condition gives

$$p + y_i \left[\frac{1}{y}\frac{p}{\epsilon}\right] = MC_i$$

Let $s_i = \frac{y_i}{y}$ denote market share. Then,

$$\begin{aligned}
p + s_i \frac{p}{\epsilon} &= MC_i \\
p - MC_i &= -s_i \frac{p}{\epsilon} \\
\frac{p - MC_i}{p} &= \frac{s_i}{|\epsilon|}
\end{aligned}$$

Bertrand Competition (price): Nash equilibrium is $p = MC$.
Conditions

- Condition 1: Market for a homogenous product: it must be the case that if there are two firms in the market and the market is in equilibrium, then they must be charging the same price
- No price above the marginal cost can be a stable equilibrium: the firm would lose its sale and thus have an incentive to cut their price again
- No price below can be profit maximising

Formal Derivation. Let us assume that the market demand is $D(p)$ and each firm faces constant marginal cost c.
Each firm's maximisation problem is

$$\max_{p_i \in \mathbb{R}} \pi_i = (p - c)D_i(p_i)$$

Now, let us assume there are two firms in the market. The demand conditions are

$$D_i(p) = \begin{cases} D(p_i), & \text{if } p_i < p_j \\ \dfrac{D(p)}{2}, & \text{if } p_i = p_j \\ 0, & \text{if } p_i > p_j \end{cases}$$

In order to derive the best response for firm i, let p_m be the monopoly price that maximises total industry profit. This highlights the incentive for firms to 'undercut' rival firms. As if the rival firm sets price at p_m, firm i can reduce its price by the smallest currency unit $\epsilon > 0$, to capture the entire market demand $D(p)$.

So the firm's reaction function is

$$R_i(p_j) = \begin{cases} p_m, & \text{if } p_j \geq p_m \\ p_j - \epsilon, & \text{if } c < p_j < p_m \\ c, & \text{if } p_j \leq c \end{cases}$$

6.3 Firm Power: Measures of Firm Power, Collusion, Mergers

Firms in both models have incentive to collude

- In the Cournot Model, they can behave like a monopolist and maximise joint profits and then divide up the spoils
- In the Bertrand Model, they can agree to charge the same price but set price above marginal cost

Cournot competition and collusion. In the Cournot Model, both firms, acting together, would jointly solve

$$\max_{y_1, y_2} p(y_1, y_2)[y_1 + y_2] - [c_1(y_1) + c_2(y_2)]$$

This is a similar problem of how a competitive firm with several plants should allocate its production across those plants. Difference is that the price is endogenous — in that it depends on total output.
Problem is identical to the one faced by a monopolist with two production plants.

Our FOCs are

$$p(y) + \frac{\partial p(y)}{\partial y_1} y = \frac{\partial c_1(y_1)}{\partial y_1}$$

$$p(y) + \frac{\partial p(y)}{\partial y_2} y = \frac{\partial c_2(y_2)}{\partial y_2}$$

Some illustrations on why we can assume

$$\frac{\partial p(y)}{\partial y_1} = \frac{\partial p(y)}{\partial y_2}$$

- In the linear case where $p = 50 - 2y \implies \frac{\partial p(y)}{\partial y_1} = \frac{\partial p(y)}{\partial y_2} = -2$
- In the nonlinear case where $p = 50 - \sqrt{y} \implies \frac{\partial p(y)}{\partial y_1} = \frac{\partial p(y)}{\partial y_2} = 0.5(y_1 + y_2)^{\frac{1}{2}}$

Implies that the firms must split out the market such that their marginal costs of production are the same. After all, since we are taking their behaviour as one firm, the total output y^* must be the same across both equations.
Solution: Solve for y_1 in terms of y_2, then substitute the equation into $y = y_1 + y_2$; after which you make y_2 the subject such that the production level is some function of the industry output. Repeat for y_1.

$$c_1'(y_1^*) = c_2'(y_2^*)$$

By colluding, firms can boost their industry profits by restricting industry output and then optimally allocating those profits among themselves. The problem is that such an arrangement is not robust to cheating or defection.

Problem of cheating in collusion.
Consider the FOCs and Firm 1. The collusive optimum is

$$p(y^*) + p'(y^*)y^* = c_1'(y_1^*)$$

Sub in $y^* = y_1^* + y_2^*$,

$$p(y^*) + p'(y^*)(y_1^* + y_2^*) = c_1'(y_1^*)$$

Rearranging for Firm 1,

$$p(y^*) + p'(y^*)y_1^* - c_1'(y_1^*) = -p'(y^*)y_2^*$$

Since $p'(y^*) < 0 \Rightarrow -p'(y^*) > 0$. Thus,

$$p(y^*) + p'(y^*)y_1^* - c_1'(y_1^*) > 0$$

This equivalent to the marginal profit function of Firm 1:

$$\pi_1(y_1^*) = p(y^*)y_1^* - c_1(y_1^*)$$
$$\pi_1'(y_1^*) = p(y^*) + p'(y^*)y_1^* - c_1'(y_1^*)$$

This implies that $\pi_1'(y_1^*) > 0$, meaning that if Firm 1 produces a little more, it increases its profits. Firm 1 can make more profits by breaking its quota insofar as the other firm sticks to the collusive optimum.

Solving the problem of cheating.
How can cartels police collusive behaviour? A credible threat to respond by increasing one's own output would work but since it will hurt you too it would be better to find a way to make collusion a Nash equilibrium.

Nonetheless, such a response is a dynamic response (involves sequential/repeated games): collusion cannot arise as a Nash equilibrium in one-shot (simultaneous) models.

Mergers

1. Horizontal mergers: Mergers between firms which complete to supply a particular market.
   Generally considered to be bad for consumers. The intuition for this effect comes from the Cournot mark-up equation
   $$\frac{p - MC_i}{p} = \frac{s_i}{|\epsilon|}$$
   Shows that as the market share grows, so does the markup.
   Monopoly represents the limit where $s_i = 1$

2. Vertical mergers: Between firms at different points along the supply chain.
   Less problematic and can even be beneficial for consumers especially when the two firms involved already enjoy a degree of market power within their own markets. Looked upon more benignly by Competition authorities than horizontal mergers (which are generally thought to be detrimental to consumers prima facie).
   Schematically, suppose that a particular industry looks like this
   $$\text{Firm 1} \underset{supplies}{\Longrightarrow} \text{Firm 2} \underset{supplies}{\Longrightarrow} \text{Consumers}$$
   If both firms are monopolists (or indeed exercise any degree of market power), then consumers are doubly exploited: double markup problem.
   The downstream monopolist (Firm 2) produces even less output then it would under the simple monopoly case for two reasons

- For the normal monopoly reason, it restricts output and produces where $MR = MC$
- Its input is now more expensive than it would otherwise be (since the firm 1 is a monopoly)

Intuitively, we can see that if the firms merge then Firm 1 has to start being nice to Firm 2 and the consumer then has only one source of market power to deal with. So merging the two firms may increase output and reduce the final price.

Indicators: HHI Index
If we take the mark-up equation and sum it over all firms in the industry weighted by their market shares (so that big firms count for more of the industry), we get

$$\sum_{i=0}^{i=N} s_i \frac{p - MC}{p} = \frac{\sum_{i=1}^{i=N} s_i^2}{|\epsilon|} = \frac{HHI}{|\epsilon|}$$

HHI is the Hirschman-Herfindahl Index

$$\sum_{i=1}^{i=N} s_i^2 = HHI$$

If all firms have the same market share then $s_i = 1/N$, then the HHI is also $1/N$.
This is why antitrust economists will sometimes use HHI^{-1} as a proxy for the number of firms and describe an industry with two and a half firms → an HHI of 0.4.
Since the mark-up reflects the deviation from competition (where the mark-up is zero), the HHI provides a measure of how large a deviation from competition is present in an industry.
A large HHI means the industry looks like a monopoly while a small HHI looks like perfect competition, holding constant the elasticity of demand.
Advantages of the measure

- No need to know firm's marginal cost since it is private information → only need market share and an estimate of the elasticity of demand
- Used widely in finance to measure portfolio diversification → related to the effective number of positions held in a portfolio

7 Market Failure: Asymmetric Information, Externalities, Role of the Government

7.1 Asymmetric Information

7.2 Externalities

7.3 Role of the Government

8 Appendix

8.1 Existence of a utility function

Theorem 1.1 Existence of a Real-Valued Function Representing the Preference Relation $\succeq$:
If a binary relation $\succeq$ is complete, transitive, continuous, and strictly monotonic, there exists a continuous real-valued function, $u : \mathbb{R}^n_+ \to \mathbb{R}$, which represents $\succeq$.
Proof. Let the relation $\succeq$ be complete, transitive, continuous and strictly monotonic. Let $\mathbf{e} \equiv (1, \dots, 1) \in \mathbb{R}^n_+$ be a vector of ones, and consider the mapping $u : \mathbb{R}^n_+ \to \mathbb{R}$ defined so that the following condition is satisfied:

$$u(\mathbf{x})\mathbf{e} \sim \mathbf{x} \tag{5}$$

This says that 'take any $\mathbf{x}$ in the domain $\mathbb{R}^n_+$ and assign it to the number $u(\mathbf{x})$ s.t. the bundle, $u(\mathbf{x})\mathbf{e}$, with $u(\mathbf{x})$ units of every commodity is ranked indifferent to $\mathbf{x}$. The proof requires us to answer two questions:

(i) does there always exist a number $u(\mathbf{x})$ satisfying (5)? (ii) is it uniquely determined, so that $u(\mathbf{x})$ is a well-defined function?

Answering the first question, fix $\mathbf{x} \in \mathbb{R}^n_+$ and consider the following two subsets of real numbers

$$A \equiv \{t \geq 0 | t\mathbf{e} \succeq x\}$$
$$B \equiv \{t \geq 0 | t\mathbf{e} \preceq x\}$$

Note that if $t^* \in A \cap B$, then $t^*\mathbf{e} \sim \mathbf{x}$, so that setting $u(\mathbf{x}) = t^*$ would satisfy (1). Thus, we need to show that $A \cap B$ is guaranteed to be non-empty.
Since the continuity of $\succeq$ implies that both A and B are both closed in $\mathbb{R}_+$. Also, by strict monotonicity, $t \in A$ implies $t' \in A$ for all $t' \geq t$. Consequently, A must be a closed interval in the form of $[\underline{t}, \infty)$. Similarly, strict monotonicity and the closedness of B in $\mathbb{R}_+$ imply that B must be a closed interval of the form $[0, \bar{t}]$. Now that for any $t \geq 0$, the completeness of $\succeq$ implies that either $t\mathbf{e} \succeq \mathbf{x}$ or $t\mathbf{e} \preceq \mathbf{x}$, that is, $t \in A \cup B$. But this means that $\mathbb{R}_+ = A \cup B = [0, \bar{t}] \cup [\underline{t}, \infty)$. We conclude $\underline{t} \leq \bar{t}$, so that $A \cap B \neq \emptyset$.

We now turn to the second question to show that there is only one number $t \geq 0$ s.t. $t\mathbf{e} \sim \mathbf{x}$. But this follows easily because if $t_1\mathbf{e} \sim \mathbf{x}$ and $t_2\mathbf{e} \sim \mathbf{x}$, then by transitivity, $t_1\mathbf{e} \sim t_2\mathbf{e}$. So $t_1 = t_2$.

We conclude that for every $\mathbf{x} \in \mathbb{R}^n_+$, there is exactly one number, $u(\mathbf{x})$, such that (1) is satisfied. Having constructed a utility function assigning each bundle in X a number, we show next that this utility function represents the preferences $\succeq$.

Consider the two bundles $\mathbf{x}^1$ and $\mathbf{x}^2$ and their corresponding utility numbers $u(\mathbf{x}^1)$ and $u(\mathbf{x}^2)$, which by definition satisfy $u(\mathbf{x}^1) \sim \mathbf{x}^1$ and $u(\mathbf{x}^2) \sim \mathbf{x}^2$. Then we have the following:

$$\mathbf{x}^1 \succeq \mathbf{x}^2 \tag{6}$$
$$\iff u(\mathbf{x}^1)\mathbf{e} \sim \mathbf{x}^1 \succeq \mathbf{x}^2 \sim u(\mathbf{x}^2)\mathbf{e} \tag{7}$$
$$\iff u(\mathbf{x}^1)\mathbf{e} \succeq u(\mathbf{x}^2)\mathbf{e} \tag{8}$$
$$\iff u(\mathbf{x}^1) \geq u(\mathbf{x}^2) \tag{9}$$

Here, (6) $\iff$ (7) follows by definition of u; (7) $\iff$ (8) follows by transitivity of $succeq$, the transitivity of $\sim$ and the definition of u; and (8) $\iff$ (9) follows from the strict monotonicity of $\succeq$. Together, (6) through (9) imply that (6) $\iff$ (9), so $\mathbf{x}^1 \succeq \mathbf{x}^2$ if and only if $u(\mathbf{x}^1) \geq u(\mathbf{x^2})$, as we sought to show.

It remains to show that the utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ representing $\succeq$ is continuous. To show that for some subset D of $\mathbb{R}^n$, the continuity of $f : D \to \mathbb{R}^n$ is equivalent to showing that for every open ball B in $\mathbb{R}^n, f^{-1}(B)$ is open in D. So, it suffices to show that the inverse image under u of every open ball in $\mathbb{R}$ is open in $\mathbb{R}^n_+$. Because open balls in $\mathbb{R}$ are merely open intervals, this is equivalent to showing that $u^{-1}((a, b))$ is open in $\mathbb{R}^n_+$ for every $a < b$. Now,

$$\begin{aligned} u^-1((a,b)) &= \{\mathbf{x} \in \mathbb{R}^n_+ | a < u(\mathbf{x}) < b\} \\ &= \{\mathbf{x} \in \mathbb{R}^n_+ | a\mathbf{e} \succ u(\mathbf{x})\mathbf{e} \succ b\mathbf{e}\} \\ &= \{\mathbf{x} \in \mathbb{R}^n_+ | a\mathbf{x} \succ \mathbf{x} \succ b\mathbf{e}\} \end{aligned}$$

The first equality follows from the definition of the inverse image; the second from the monotonicity of $\succeq$; and the third from $u(\mathbf{x})\mathbf{e} \sim \mathbf{x}$. Rewriting the last set on the right hand side gives

$$u^{-1}((a,b)) = \succ (a\mathbf{e}) \bigcap \prec (b\mathbf{e}) \tag{10}$$

By the continuity of $\succeq$, the sets $\succeq (a\mathbf{e})$ and $\succeq (b\mathbf{e})$ are closed in $X = \mathbb{R}^n_+$. Consequently, the two sets on the right-hand side of (6), being the complements of these closed sets, are open in $\mathbb{R}^n_+$. Therefore, $u^{-1}((a, b))$, beung the intersection of two open sets in $\mathbb{R}^n_+$. is itself open in $\mathbb{R}^n_+$. ■

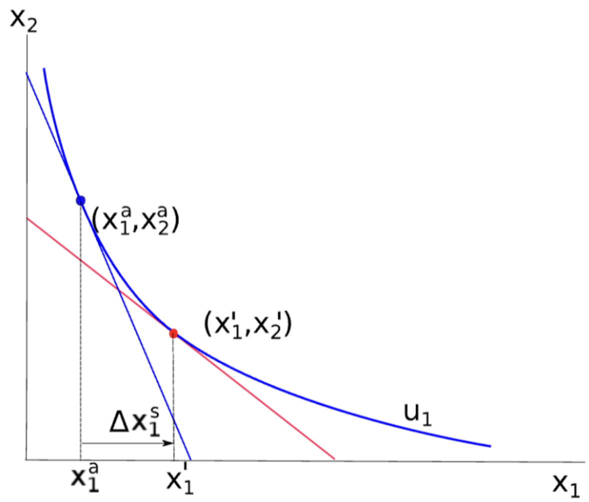
8.2 Finding the Hicksian Demand function

Essentially the expenditure minimisation problem:

$$\min(p_1x_1 + p_2x_2) \text{ s.t. } x_1^a x_2^{1-a} = \bar{U}$$

8.3 Non-positivity of the substitution effect

Claim. The substitution effect is always negative or zero.
Proof.



We can see that both bundles of goods lie above one another's budget line. So we can construct 2 inequalities

$$p_1^a x_1^a + p_2^a x_2^a \leq p_1^a x_1' + p_2^a x_2'$$
$$p_1' x_1^a + p_2' x_2^a \geq p_1' x_1' + p_2' x_2'$$

Adding the two inequalities we have

$$(p_1' - p_1^a)(x_1' - x_1^a) + (p_2' - p_2^a)(x_2' - x_2^a) \leq 0$$

Since p_2 did not change, $p_2' - p_2^a = 0$. Thus,

$$(p_1' - p_1^a)(x_1' - x_1^a) \leq 0 \implies \Delta p_1 \Delta x_1^s \leq 0$$

This says that to hold utility constant, the change in price and demand need to have opposite signs so

$$\frac{\Delta x_1^s}{\Delta p_1} \leq 0 \implies \frac{\partial x_1^s}{\partial p_1} \leq 0$$

8.4 Existence Theorem for Nash Equilibria

8.5 Stackelberg Competition

Essentially sequential Cournot Competition. Price function (price a function of total industry output): $P(q_1 + q_2)$ where 1 is the leader while 2 represents the follower.
Model is solved by backwards induction: the leader considers what the best response of the follower is: how it will respond once it has observed the quantity of the leader. The leader then picks a quantity that maximises its payoff, anticipating the predicted response of the follower.
Solution. To solve for the subgame perfect Nash Equilibrium (SPNE), we calculate the best response

function of the follower.
The follower's maximisation problem is

$$\max_{q_2} \pi_2 = P(q_1 + q_2)q_2 - C_2(q_2)$$

So, find the profit maximising value of q_2 given some exogenously determined q_1.

$$\frac{\partial \pi_2}{\partial q_2} = \frac{\partial P(q_1 + q_2)}{\partial q_2} q_2 + P(q_1 + q_2) - \frac{\partial C_2(q_2)}{\partial q_2} = 0$$

Firm 2's reaction function is expressed as

$$q_2(q_1) = r_2(q_1)$$

Now, the leader takes this value of q_2 into account and substitutes it into his profit equation and solves for q_1 which maximises profits:

$$\pi_1 = P(q_1 + q_2(q_1))q_1 - C_1(q_1)$$

$$\frac{\partial \pi_1}{\partial q_1} = \frac{\partial P(q_1 + q_2)}{\partial q_2(q_1)} \frac{\partial q_2(q_1)}{\partial q_1} q_1 + \frac{\partial P(q_1 + q_2)}{\partial q_1} q_1 + P(q_1 + q_2(q_1)) - \frac{\partial C_1(q_1)}{\partial q_1} = 0$$

From this, we substitute q_1 back into the follower's reaction function