Complex Dynamics of The Hyperbolic Tangent of The Logarithm Of One Minus The Square of The Hardy Z Function

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1 Introduction

There are many well known functions such as $Z(t)$, $\xi(t)$, $\eta(t)$ whose roots, or an affine transform of them, coincide with those of the Riemann zeta function $\zeta(t)$. Another function, $X(t)$, shall be introduced which is a composiotion of a rational meromorphic quartic and the Hardy Z function which has some intriguing properties.

The Hardy Z function has the property that it is known, independently of the Riemann hypothesis, that $Z(t) \in \mathbb{R} \forall t \in \mathbb{R}$, that is, Z is real-valued when t is real.

If one looks at the curves where the real and imaginary parts of $Z(t)$ vanish independently, we find that the boundaries of both sets extend to infinity and do not cross orthogonally, rather, they meet at infinity

If we look at the curves where $\mathrm{Re}(X)$ and $\mathrm{Im}(X)$ vanish we see that they are orthogonal at their intersection points where they intersect at 90 degree angles, at 1 point at the root on the real axis for a total of 5 intersection points.

The bounds the range of the function on the real line in the interval $[-1, -1]$ and this is important because $Z(t)$ is known to grow without bound as t increases. It also has the property that 0 is a fixed-point of $\tanh$, that is $\tanh(0) = 0$, so that roots of Z are roots of X.

is that the corresponding Riemann surface of X has a topology which features closed lemniscate curves around each root so that compactness theoreoms and whatnot can be applied since measures on the surface of Y do not escape to infinite like measures on the surfaces of the Hardy Z, Riemann ζ(zeta), Riemann ξ(xi), or Dirichlet η(eta) functions. Also, the surface of X is compact, unlike the others.

1.1 The Schröder Equation

Definition 1. *Schröder's equation is functional eigenvalue equation for the composition operator*

$$C_h f(x): f(x) \rightarrow h(f(x)) \tag{1}$$

in one independent variable; where a function, $h(x)$, is given and a function, $\Psi(x)$, is sought which satisifes

$$\Psi(h(x)) = s\,\Psi(x) \tag{2}$$

where $s = \dot{h}(0)$ is the eigenvalue.

1.1.1 Koenig's Linearization Theorem

Let $f_{t_0}(t) = f(t_0 + t)$ be a function such that the fixed-point of interest corresponds to the origin such that $f_{t_0}(0) = 0$ and $f(t_0) = 0$.

Definition 2. *A holomorphic function $f(t)$ that is one-to-one is said to be injective in a domain $t \in B \subset \mathbb{C}$ such that $f(z_1) \neq f(z_2)$ when $z_1 \neq z_2$ and is also said to be univalent or conformal. The inverse function $z = f^{-1}(w)$ is then also necessarily conformal in the same domain B.*

Theorem 3. *(**Koenigs Linearization Theorem**) If the magnitude (absolute value) of the multiplier $\lambda = \dot{f}(0)$ of a holomorphic map f is not strictly equal to 0 or 1, that is $|\lambda| \not\subset \{0, 1\}$, then a local holomorphic change of coordinates $w = \phi(z)$, called the Koenig's function, unique up to a scalar multiplication by nonzero constant, exists, having a fixed-point at the origin $\phi(0) = 0$ such that Schröder's equation is true*

$$\phi \circ f \circ \phi^{-1} = \lambda\, w \forall w \in \varepsilon_0 \tag{3}$$

for some neighborhood ε_0 of the origin 0.[9] [9, Theorem 8.2][8, 2. Koenig's Theorem, Part I.]

1.2 The Frobenius-Perron Transfer Operator

Definition 4. *The (Frobenius-Perron)* `transfer operator`*[6, Ch9] is defined as normalized sum over the inverse branches of an iteration function $f \in C^1$*

$$\mathcal{L} f(x) = \sum_{y \in f^{-1}(x)} \frac{f(y)}{|\dot{f}(y)|} \tag{4}$$

which is a linear operator which determines how densities evolve under the action of $f(x)$; The invariant measure of the map is the measure which is unchanged by the action of f and satisifies

$$\mathcal{L}\varphi(x) = \varphi(x) \tag{5}$$

1.3 Physical Interpretations of the Cauchy-Riemann Equations

The physical interpretation[10, 14.2.2 III] of the Cauchy–Riemann equations

$$\frac{\partial \mathrm{Re}}{\partial x}=\frac{\partial \mathrm{Im}}{\partial y} \tag{6}$$

$$\frac{\partial \mathrm{Re}}{\partial y}=-\frac{\partial \mathrm{Im}}{\partial x} \tag{7}$$

going back to Riemann's work on function theory [2] is that the real part of an analytic function f is the velocity potential of an incompressible fluid flow in the complex plane and the its imaginary part is the corresponding stream function.

When the pair of twice continuously differentiable functions $\{\mathrm{Re}(f(x+iy)),\mathrm{Im}(f(x+iy))\}$ of f satisfies the Cauchy–Riemann equations its real part $\mathrm{Re}(f)$ is its velocity potential and the gradient of the real part $\nabla\mathrm{Re}$ is its velocity vector defined by

$$\nabla\mathrm{Re}=\frac{\partial \mathrm{Re}(f(x+iy))}{\partial x}+i\frac{\partial \mathrm{Re}(f(x+iy))}{\partial y} \tag{8}$$

By differentiating the Cauchy–Riemann equations a second time, it is shown that real part solves Laplace's equation:

$$\frac{\partial^2 \mathrm{Re}(f(x+iy))}{\partial x^2}+\frac{\partial^2 \mathrm{Re}(f(x+iy))}{\partial y^2}=0 \tag{9}$$

That is, the real part of an analytic function is harmonic which means that it is incompressible since the divergence of its gradient vanishes and can therefore go no lower. The imaginary part also satisfies the Laplace equation, by a similar analysis. The Cauchy–Riemann equations also imply that the dot product of the gradients of the real and imaginary parts vanishes

$$\nabla\mathrm{Re}\cdot\nabla\mathrm{Im}=0 \tag{10}$$

which indicates that the gradient of the real part must point along the streamlines of the flow where the imaginary part is constant $\mathrm{Im}=\mathrm{const}$ and therefore the curves of constant real part $\mathrm{Re}=\mathrm{const}$ are the corresponding orthogonal equipotential curves.

2 The Operator $S^a_f(t)=\tanh\Big(\ln\Big(1-\big(\frac{f(t)}{a}\big)^2\Big)\Big)$

Definition 5. *Let the operator which takes a complex analytic function from $\bar{\mathbb{C}}\to\bar{\mathbb{C}}$ and returns the hyperbolic tangent of the logarithm of one minus the square of that function, divided by a scaling factor a, be defined by*

$$S^a_f(t)=S^a(f(t))=\tanh\left(\ln\left(1-\left(\frac{f(t)}{a}\right)^2\right)\right)=\frac{\left(1-\left(\frac{f(t)}{a}\right)^2\right)^2-1}{\left(1-\left(\frac{f(t)}{a}\right)^2\right)^2+1} \tag{11}$$

where $f(t)\in\bar{\mathbb{C}}\forall t\in\bar{\mathbb{C}}$ is an analytic function of a single complex variable whose domain is the extended complex plane. If a is not specified then it is assumed to be equal to 1, e.g, $S_f(t)=S^a_f(t)$. When the function $f(t)$ is the identity function $f:t\to t$ then we have

$$\begin{aligned} S^a(t) &= S^a_{t\to t}(t) \\ &= \frac{(1-(\frac{t}{a})^2)^2-1}{(1-(\frac{t}{a})^2)^2+1} \\ &= 1-\frac{2}{1+(1-(\frac{t}{a})^2)^2} \end{aligned} \tag{12}$$

which is a quartic, a rational (meromorphic) function of degree 4 from $\bar{\mathbb{C}}\to\bar{\mathbb{C}}$ with a double-root at the origin.

Theorem 6. *The function* $S_{t\to t}(t)$ *is in the Hardy class* H^2

Proof. *Recall that a function is in the Hardy class* H^p *if*

$$\sup_{0\leqslant r<1}\left(\frac{\int_0^{2\pi}|f(re^{ia})|^p\mathrm{da}}{2\pi}\right)^{\frac{1}{p}}<\infty$$

then let $p=2$ *and note that*

$$\sqrt{\frac{1}{2\pi}\int|S(re^{ia})|^2\mathrm{d}x}=\sqrt{\frac{1}{2\pi}\frac{2\pi^4(r^8-2r^4+8)}{(r^4-2)(r^8+4)}} \tag{13}$$

is bounded $\forall 0\leqslant r<1$ □

2.1 The Curve $\mathrm{Re}(S(t))=0$ is a Bernoullian Lemniscate

Theorem 7. *The zero set* $\{t{:}\mathrm{Re}(S(t))=0\}$ *of the real part* $\mathrm{Re}(S(t))$ *of* $S(t)$ *where* $t=x+iy$ *is a horizontally oriented lemniscate of Bernoulli, also known as a hyperbolic lemniscate[3, 5.3 p.121], at the origin with parameter* 2. *That is,*

$$\{(x,y){:}\,\mathrm{Re}(S(x+iy))=0\}=\{(x,y){:}\,(x^2+y^2)^2=2(x^2-y^2)\} \tag{14}$$

Proof.

The parametric equations[7] for the lemniscate of Bernoulli with parameter 2 are given by

$$\begin{aligned} x(t) &= \frac{\sqrt{2}\cos(t)}{1+\sin^2(t)} \\ y(t) &= \frac{\sqrt{2}\sin(t)\cos(t)}{1+\sin^2(t)} \end{aligned} \tag{15}$$

Let us combine the coordinate functions $(x(t),y(t))\in\mathbb{R}^2$ *into an equivalent function* $z(t)\in\bar{\mathbb{C}}$

$$\begin{aligned} z(t) &= x(t)+iy(t) \\ &= \frac{\sqrt{2}\cos(t)}{1-i\sin(t)} \end{aligned} \tag{16}$$

where it can be shown that

$$\begin{aligned} S(z(t)) &= S_z(t) \\ &= S\left(\frac{\sqrt{2}\cos(t)}{1-i\sin(t)}\right) \\ &= i\frac{32\cos(t)^2\sin(t)}{20\cos(2t)+\cos(4t)-13} \end{aligned} \tag{17}$$

so that

$$\mathrm{Re}(S_z(t))=0\forall t\in\mathbb{R} \tag{18}$$

and thus $z(t)$ *is a geodesic of the real part of* S □

2.2 The Curve $\mathrm{Im}(S(t)) = 0$ is a Conjugate Pair of Rectangular Hyperbolas

Theorem 8. *The zero set* $\{t: \mathrm{Im}(S(t)) = 0\}$ *of the imaginary part* $\mathrm{Im}(S(t))$ *of* $S(t)$ *where* $t = x + iy$ *is a conjugate pair of rectangular hyperbolas.*

$$\{(x, y): \mathrm{Im}(S(x + iy)) = \{(x, y): x^2 - y^2 = 1\}\} \tag{19}$$

Proof.
The parametric equations[7] for the equilaterial (rectangular) hyperbola with unit parameter are given by

$$\begin{aligned} x(t) &= \sec(t) \\ y(t) &= \tan(t) \end{aligned} \tag{20}$$

which are combined into a complex-valued function

$$\begin{aligned} z(t) &= x(t) + iy(t) \\ &= \sec(t) + i\tan(t) \end{aligned} \tag{21}$$

where it can be shown that

$$\begin{aligned} S(z(t)) &= S(\sec(t) + i\tan(t)) \\ &= -\frac{2(\cos(2t) - 3)^2}{20\cos(2t) + \cos(4t) - 13} \end{aligned} \tag{22}$$

so that

$$\mathrm{Im}(S(z(t))) = 0 \forall t \in \mathbb{R} \tag{23}$$

and thus $z(t)$ *is a geodesic of the imaginary part of* S

□

2.3 Newton Maps and Flows of $S_f(t)$

Definition 9. *Let* $N_f(t)$ *denote the Newton map of* $f(t)$

$$N_f(t) = t - \frac{f(t)}{\dot{f}(t)} \tag{24}$$

Definition 10. *The Newton map* $N_{S_f}(t)$ *of the composition* $S_f(t) = S(f(t))$ *is a rational meromorphic function of* $f(t)$ *given by*

$$\begin{aligned} N_{S_f}(t) &= t - \frac{S_f(t)}{\dot{S}_f(t)} \\ &= t - \frac{\frac{(1 - f(t)^2)^2 - 1}{(1 - f(t)^2)^2 + 1}}{\frac{8\dot{f}(t) f(t)(f(t)^2 - 1)}{((f(t) - 1)^2 (f(t) + 1)^2 + 1)^2}} \\ &= t - \frac{1}{8}\frac{((f(t) - 1)^2 (1 + f(t)^2)^2 + 1)^2 ((1 - f(t)^2)^2 - 1)}{f(t)\dot{f}(t)(f(t)^2 - 1)(1 + (1 - f(t)^2)^2)} \end{aligned} \tag{25}$$

Theorem 11. *The Newton map of S_f transforms superattractive $(\lambda = 0)$ fixed-points of $N_f(t)$ to geometrically attractive fixed-points of $N_{S_f}(t)$*

Proof.

There is geometrically attractive fixed-point at $t = 0$ with multiplier equal to

$$\lambda_{N_{S_f}(0)} = \left| 1 - \frac{1 + \lambda_{N_f}(0)}{2} \right| \tag{26}$$

TODO: prove this.. apply S to some other functions and see how it transforms the multipliers of the fixed-points

□

2.3.1 Factoring Out The Double-Root at the Origin of $N_{S_f}(t)$

If $m = m_f(\alpha)$ is the multiplicity of the root of f at the point α then f factorizes as

$$f(x) = (x - \alpha)^m g(x) \tag{27}$$

where $g(\alpha) = 0$.

2.3.2 The Newton Flow

Definition 12. *The* `Newton flow` $\mathcal{N}_S(f)$ *of* $S_f(t)$ *is defined by the differential equation*

$$\dot{z}(t) = \frac{\mathrm{d}}{\mathrm{d}t} z(t) = -\frac{S_f(z(t))}{\dot{S}_f(z(t))} \tag{28}$$

which is approximated by the relaxed Newton method where the limit of the step size is taken towards zero , it is defined by

$$\begin{aligned} \mathcal{N}_S^h(f) &= t - h \frac{S_f(t)}{\dot{S}_f(t)} \\ &= t - \frac{h \frac{(1 - f(t)^2)^2 - 1}{(1 - f(t)^2)^2 + 1}}{\frac{8 \dot{f}(t) f(t) (f(t)^2 - 1)}{((f(t) - 1)^2 (f(t) + 1)^2 + 1)^2}} \\ &= t - \frac{h}{8} \frac{((f(t) - 1)^2 (1 + f(t)^2)^2 + 1)^2 ((1 - f(t)^2)^2 - 1)}{f(t) \dot{f}(t) (f(t)^2 - 1)(1 + (1 - f(t)^2)^2)} \end{aligned} \tag{29}$$

where h is taken to be a small number which is used to approximate the flow $\dot{z}(t)$

The Newton flow $\mathcal{N}(f)$ has the drawback that it is undefined at the critical points of f where the limit diverges to a different direction of the pole depending upon the direction it is approached from. To remedy this situation there exists the desingularized Newton flow for entire functions.

2.3.3 The Desingularized Newton Flow For Entire Functions

Definition 13. *If f is an entire function then an equivalent* `desingularized Newton flow` *which is devoid of singularities at the critical points is given by*

$$\dot{z}(t) = -\overline{\dot{f}(z(t))} f(z(t)) \tag{30}$$

[5].

The function of interest here, S_f, is meromorphic and therefore will be undefined at the critical points of f. To remedy this situation we can apply the continuous Newton method for mermorphic functions which defines an equivalent real holomorphic vector field devoid of any singularities.

2.3.4 The Continuous Desingularized Newton Flow for Meromorphic Functions

Lemma 14. *(Desingularization Lemma) The flow defined by*

$$\bar{\mathcal{N}}(f) = -\frac{\bar{\dot{f}}(z)f(z)}{(1+|f(z)|^4)} \tag{31}$$

is a **`real analytic vector field`** *[4] defined on the whole complex plane* $\mathbb{C}$ *with the properties that*

i. *Trajectories of* $\mathcal{N}$ *are also trajectories of* $\bar{\mathcal{N}}(f)$

ii. *A critical point of* f *is an equilibrium state for* $\bar{\mathcal{N}}(f)$

iii. $\bar{\mathcal{N}}(f) = -\bar{\mathcal{N}}\left(\frac{1}{f}\right)$

2.3.5 The Continuous Newton Flow $\bar{\mathcal{N}}(S_f)$ and Its Approximation $\bar{\mathcal{N}}^h(S_f)$

Apply Lemma 14 to define a real analytic vector field on $\mathbb{C}$

$$\bar{\mathcal{N}}(S_f) = -\frac{\bar{\dot{S}}_f(z)S_f(z)}{(1+|S_f(z)|^4)} \tag{32}$$

which is approximated by a similiarly modified relaxed Newton's method

$$\mathcal{N}^h(S_f) \quad = t - h\frac{\bar{\dot{S}}_f(t)S_f(t)}{(1+|S_f(z)|^4)} \tag{33}$$

where h is accuracy of the solution. TODO: insert some figures

3 The Riemann Zeta ζ Function

Definition 15. *The Riemann zeta function is defined by*

$$\begin{aligned}\zeta(s) \quad &= \sum_{n=1}^{\infty}\frac{1}{n^s}\forall \mathrm{Re}(s)>1 \\ &= \frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}\forall \mathrm{Re}(s)>0\end{aligned} \tag{34}$$

and its argument has a representation as

$$S(t) = \frac{1}{\pi}\arg\left(\zeta\left(\frac{1}{2}+it\right)\right) = \frac{1}{\pi}\int_{\frac{1}{2}}^{\infty}\mathrm{Im}\left(\frac{\dot{\zeta}(\sigma+it)}{\zeta(\sigma+it)}\right)\mathrm{d}\sigma\forall t\in\mathbb{R}\backslash\{0\} \tag{35}$$

3.1 The Riemann Hypothesis

Conjecture 16. *(The Riemann Hypothesis)*
Bernhard Riemann conjectured in 1859[1] that

$$\left\{\mathrm{Re}(t) = \frac{1}{2} \colon \zeta(t) = 0 \forall t \neq -2n \forall t \in \bar{\mathbb{C}}, n \in \mathbb{N}^+\right\}$$

or in words, that all of the roots that are not negative even integers where $\zeta(-2n) = 0$ *all lie on the* `critical line` $\mathrm{Re}\left(\frac{1}{2}\right)$ *in the complex plane such that* $\zeta(\sigma + i s) = 0$ *only when* $\sigma = \frac{1}{2}$ *where* $\mathbb{R}^+ \ni s > 0$.

3.1.1 Lines of Constant Phase and the Riemann Hypothesis

The following theorem is the main result of [11].

Theorem 17.

1. *If all lines of constant phase* $\arg(\zeta(t)) = kn$ *of* ζ *where* $k \in \mathbb{N}$ *merge with the critical line*
   OR
2. *all points where* $\dot{\zeta}(t)$ *vanishes are located on the critical line and the phases of* ζ *at consecutive zeros of* $\dot{\zeta}$ *differs by* π

then the Riemann Hypothesis (16) is true.

3.2 The Hardy Z Function

Definition 18. *(The Gamma and Log Gamma functions)*
Let

$$\Gamma(t) = (t-1)! = \int_0^\infty x^{t-1} e^{-x} \mathrm{d}x \forall \mathrm{Re}(t) > 0 \tag{36}$$

be the gamma function and

$$\ln\Gamma(t) = \ln(\Gamma(t)) \tag{37}$$

be the principle branch of the logarithm of the Γ *function.*

3.2.1 The Phase of ζ

The $\mathrm{Riemann} - \mathrm{Siegel}$ theta function $\vartheta(t)$ corresponds to the smooth part of the phase of the zeta function which has a jump discontinuity when t is equal to the imaginary part of a Riemann zero on the critical line.

Definition 19. *(The Riemann-Siegel vartheta function)*
Let

$$\vartheta(t) = -\frac{i}{2}\left(\ln\Gamma\left(\frac{1}{4} + \frac{i t}{2}\right) - \ln\Gamma\left(\frac{1}{4} - \frac{i t}{2}\right)\right) - \frac{\ln(\pi) t}{2} \tag{38}$$

be the the Riemann-Siegel (var)theta function.

Definition 20. *(The Hardy Z function)*
Let

$$Z(t) = e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + i t\right) \tag{39}$$

be the Hardy Z function which has the property that $Z(t)$ *is real when* t *is real, that is,* $Z(t) \in \mathbb{R} \forall t \in \mathbb{R}$ *independently of the Riemann hypothesis.*

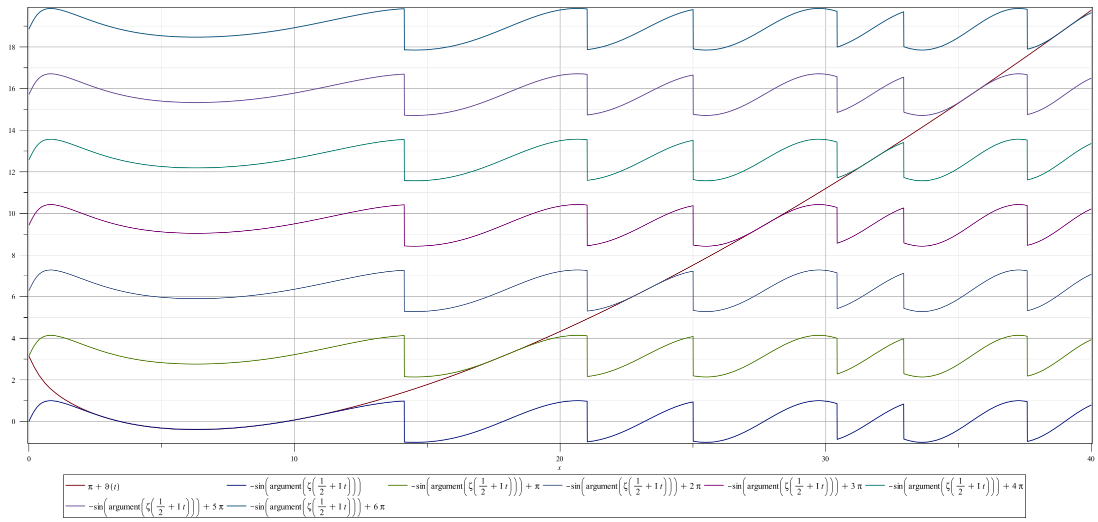

Figure 1. Illustration of the relationship between vartheta ϑ and the argument of zeta ζ on the critical line

3.3 The Function $X(t) = (S \circ Z)(t)$

Definition 21. *(The X function)*
Let $X(t)$ be defined as the composition of the S function with the Hardy Z function

$$X(t) = C_Z(S)(t) = (S \circ Z)(t) = S(Z(t)) = \frac{(1 - Z(t)^2)^2 - 1}{(1 - Z(t)^2)^2 + 1} \tag{40}$$

3.3.1 Integration Along a Curve: A Newton Iteration for the Angle

In order to find an explicit expression for the angle $\theta_m(h)$ in the implicit formula Formula (?) we can use Newton's method

Definition 22. *Let the Newton map for the roots of the real part of*

$$X(t + h e^{ia}) \tag{41}$$

be defined by

$$\begin{aligned} N_{\theta_m}(a_{m,k}; t, h) &= \mathrm{frac}\left(a_{m,k-1} + \tanh\left(\frac{\operatorname{Re}(X(t + h e^{ia}))}{\operatorname{Re}(\frac{\mathrm{d}}{\mathrm{d}a} X(t + h e^{ia}))} \right)_{a = a_{m,k-1}} \right) \\ &= \mathrm{frac}\left(a_{m,k-1} + \tanh\left(\frac{\operatorname{Re}(X(t + h e^{ia}))}{\pi \operatorname{Im}(\dot{X}(t + h e^{ia})\, h e^{ia})} \right)_{a = a_{m,k-1}} \right) \end{aligned}$$

TODO: this takes a special form, see

where the initial $(k=0)$ angle of the first$(m=0)$ step of length h is $a_{m,0}=\frac{\theta_{m-1}}{\pi}$ where we normalize by π since the variable has domain $[-1,1]$ (the angle at the previous point) or $a_{0,0}=\frac{3}{4}$ which is -45 $^{\circ}$ for the initial element of the sequence when $m=0$, that is, the (initial)boundary conditions are

$$a_{m,0} = \begin{cases} \frac{3}{4} & m=0 \\ \frac{\theta_{m-1}}{\pi} & m \geqslant 1 \end{cases} \tag{42}$$

and the corresponding curve is traversed in a positive clockwise direction moving initially into the upper-left quadrant . Let the angle at the m-th step (of length h) be defined as the limit

$$\begin{aligned} \theta_m &= \theta_m(t,h) \in [-\pi, +\pi] \\ &= \pi \lim_{k\to\infty} N_{\theta_m}(a_{m,k}; t, h) \end{aligned} \tag{43}$$

which is dependent on the basepoint t and radius h, but the when the dependence is not written as $\theta_m(t,h)$ it is still implied unless otherwise noted. The notation $\dot{Y}(t) = \frac{\mathrm{d}}{\mathrm{d}t} Y(t)$ is the more concise notation for first-deriative.

3.3.2 Roots of $X(t)$ on the Real Line

The critical line of the zeta function $\mathrm{Re}(s)=\frac{1}{2}$ corresponds to the real line $\mathrm{Im}(s)=0$ of the Z and X functions

4 Linearizing

Definition 23. *Let the Newton map of the shifted X function*

$$X_n(t) = X(z_n + t) \tag{44}$$

where z_n is the n-th root of the Hardy Z function on the real line, starting with

$$\begin{aligned} z_1 &\cong \texttt{14.1347251...} \\ z_2 &= \texttt{21.0220396...} \\ \dots & \quad \dots \end{aligned}$$

, be denoted

$$N_{X_{z_n}}(t) = t - \frac{X_{z_n}(t)}{\dot{X}_{z_n}(t)} \tag{45}$$

TODO: this has nice symmetric factorized form, see (25)

5 Appendix

5.1 The Spectral Theorem

Theorem 24. *The Spectral Theorem*

Let $U: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be unitary then U extends uniquely to a unitary operator on all of $L^2(\mathbb{R}^n)$ and all generalized eigenvalues λ lie on the unit circle $|\lambda|=1$. The space $L^2(\mathbb{R}^n)$ can be expressed as a direct integral

$$\int_{|\lambda|=1} \mathcal{H}(\lambda) d\mu(\lambda) \tag{46}$$

of Hilbert spaces $\mathcal{H}(\lambda) \subseteq E_\lambda$ so that U sends the function $h \in L^2(\mathbb{R}^n)$ to the function Uh with λ-component

$$(Uh) = \lambda h_\lambda \in \mathcal{H}(\lambda) \tag{47}$$

and its set of generized eigenvectors forms a complete basis. [?, Theorem 1.3.2]

Bibliography

[1] Ueber die anzahl der primzahlen unter einer gegebenen grösse. *Monatsberichte der Berliner Akademie*, R1, 1859.
[2] *On Riemann's theory of algebraic functions and their integrals*. Cambridge: Macmillan and Bowes, 1893.
[3] *A Catalog of Special Plane Curves*. Dover Books on Mathematics. Dover Publications, 1972.
[4] Global aspects of the continuous and discrete newton method: a case study. *Acta Applicandae Mathematicae*, 13, 1988.
[5] The continuous, desingularized newton method for meromorphic functions. In *Newton's Method and Dynamical Systems*. Springer, 1988.
[6] *Dynamical Systems II: Ergodic Theory with Applications to Dynamical Systems and Statistical Mechanics*. Springer-Verlag, 1989.
[7] *Introduction to Geometry*. Wiley Classics Library. Wiley, 1989.
[8] Composition operators and schroder's functional equation. *Contemporary Mathematics*, (213), 1998.
[9] *Dynamics in One Complex Variable*. Annals of Mathematics Studies 160. Princeton University Press, 2nd edition, 2006.
[10] *Handbook of Complex Variables*. Birkhauser Boston, 2012.
[11] Equivalent formulations of the riemann hypothesis based on lines of constant phase. *Physica Scripta*, 93(6), 2018.

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