

Complex Dynamics of The Hyperbolic Tangent of The Logarithm Of One Minus The Square of The Hardy Z Function

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1 Introduction

There are many well known functions such as $Z(t)$, $\xi(t)$, $\eta(t)$ whose roots, or an affine transform of them, coincide with those of the Riemann zeta function $\zeta(t)$. Another function, $X(t)$, shall be introduced which is a composition of a rational meromorphic quartic and the Hardy Z function which has some intriguing properties.

The Hardy Z function has the property that it is known, independently of the Riemann hypothesis, that $Z(t) \in \mathbb{R} \forall t \in \mathbb{R}$, that is, Z is real-valued when t is real.

If one looks at the curves where the real and imaginary parts of $Z(t)$ vanish independently, we find that the boundaries of both sets extend to infinity and do not cross orthogonally, rather, they meet at infinity

If we look at the curves where $\text{Re}(X)$ and $\text{Im}(X)$ vanish we see that they are orthogonal at their intersection points where they intersect at 90 degree angles, at 1 point at the root on the real axis for a total of 5 intersection points.

The bounds the range of the function on the real line in the interval $[-1, -1]$ and this is important because $Z(t)$ is known to grow without bound as t increases. It also has the property that 0 is a fixed-point of \tanh , that is $\tanh(0) = 0$, so that roots of Z are roots of X .

is that the corresponding Riemann surface of X has a topology which features closed lemniscate curves around each root so that compactness theorems and whatnot can be applied since measures on the surface of Y do not escape to infinite like measures on the surfaces of the Hardy Z , Riemann ζ (zeta), Riemann ξ (xi), or Dirichlet η (eta) functions. Also, the surface of X is compact, unlike the others.

1.1 The Schröder Equation

Definition 1. Schröder's equation is functional eigenvalue equation for the composition operator

$$C_h f(x): f(x) \rightarrow h(f(x)) \quad (1)$$

in one independent variable; where a function, $h(x)$, is given and a function, $\Psi(x)$, is sought which satisfies

$$\Psi(h(x)) = s \Psi(x) \quad (2)$$

where $s = \dot{h}(0)$ is the eigenvalue.

1.1.1 Koenig's Linearization Theorem

Let $f_{t_0}(t) = f(t_0 + t)$ be a function such that the fixed-point of interest corresponds to the origin such that $f_{t_0}(0) = 0$ and $f(t_0) = 0$.

Definition 2. A holomorphic function $f(t)$ that is one-to-one is said to be injective in a domain $t \in B \subset \mathbb{C}$ such that $f(z_1) \neq f(z_2)$ when $z_1 \neq z_2$ and is also said to be univalent or conformal. The inverse function $z = f^{-1}(w)$ is then also necessarily conformal in the same domain B .

Theorem 3. (Koenigs Linearization Theorem) If the magnitude (absolute value) of the multiplier $\lambda = \dot{f}(0)$ of a holomorphic map f is not strictly equal to 0 or 1, that is $|\lambda| \notin \{0, 1\}$, then a local holomorphic change of coordinates $w = \phi(z)$, called the Koenig's function, unique up to a scalar multiplication by nonzero constant, exists, having a fixed-point at the origin $\phi(0) = 0$ such that Schröder's equation is true

$$\phi \circ f \circ \phi^{-1} = \lambda w \forall w \in \varepsilon_0 \quad (3)$$

for some neighborhood ε_0 of the origin 0. [7] [7, Theorem 8.2] [6, 2. Koenig's Theorem, Part I.]

1.2 Physical Interpretations of the Cauchy-Riemann Equations

A standard physical interpretation of the Cauchy–Riemann equations

$$\frac{\partial \text{Re}}{\partial x} = \frac{\partial \text{Im}}{\partial y} \quad (4)$$

$$\frac{\partial \text{Re}}{\partial y} = -\frac{\partial \text{Im}}{\partial x} \quad (5)$$

going back to Riemann's work on function theory [2] is that the real part of an analytic function is a velocity potential of incompressible steady-fluid-flow in the plane, and the imaginary part is its stream function.

Suppose that the pair of twice continuously differentiable functions $\{\text{Re}(f(x + iy)), \text{Im}(f(x + iy))\}$ of f satisfies the Cauchy–Riemann equations. We will take the real part Re to be a velocity potential, meaning that we imagine a flow of fluid in the plane such that the velocity vector of the fluid at each point of the plane is equal to the gradient of the real part, defined by

$$\nabla \text{Re} = \frac{\partial \text{Re}(f(x + iy))}{\partial x} + i \frac{\partial \text{Re}(f(x + iy))}{\partial y} \quad (6)$$

By differentiating the Cauchy–Riemann equations a second time, one shows that the real part solves Laplace’s equation:

$$\frac{\partial^2 \operatorname{Re}(f(x+iy))}{\partial x^2} + \frac{\partial^2 \operatorname{Re}(f(x+iy))}{\partial y^2} = 0 \quad (7)$$

That is, the real part of an analytic function is a harmonic which means the it is incompressible since the divergence of its gradient vanishes . The imaginary part also satisfies the Laplace equation, by a similar analysis. The Cauchy–Riemann equations also imply that the dot product

$$\nabla \operatorname{Re} \cdot \nabla \operatorname{Im} = 0 \quad (8)$$

which implies that the gradient of the real part must point along the streamlines $\operatorname{Im} = \text{const}$ of the flow. The $\operatorname{Re} = \text{const}$ curves are the equipotential curves of the flow.

2 The Operator $S_f^a(t) = \tanh\left(\ln\left(1 - \left(\frac{f(t)}{a}\right)^2\right)\right)$

Definition 4. Let the operator which takes a complex analytic function from $\bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ and returns the hyperbolic tangent of the logarithm of one minus the square of that function, divided by a scaling factor a , be defined by

$$S_f^a(t) = S^a(f(t)) = \tanh\left(\ln\left(1 - \left(\frac{f(t)}{a}\right)^2\right)\right) = \frac{\left(1 - \left(\frac{f(t)}{a}\right)^2\right)^2 - 1}{\left(1 - \left(\frac{f(t)}{a}\right)^2\right)^2 + 1} \quad (9)$$

where $f(t) \in \bar{\mathbb{C}} \forall t \in \bar{\mathbb{C}}$ is an analytic function of a single complex variable whose domain is the extended complex plane. If a is not specified then it is assumed to be equal to 1, e.g. $S_f(t) = S_f^1(t)$. When the function $f(t)$ is the identity function $f:t \rightarrow t$ then we have

$$\begin{aligned} S^a(t) &= S_{t \rightarrow t}^a(t) \\ &= \frac{\left(1 - \left(\frac{t}{a}\right)^2\right)^2 - 1}{\left(1 - \left(\frac{t}{a}\right)^2\right)^2 + 1} \\ &= 1 - \frac{2}{1 + \left(1 - \left(\frac{t}{a}\right)^2\right)^2} \end{aligned} \quad (10)$$

which is a quartic, a rational (meromorphic) function of degree 4 from $\bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ with a double-root at the origin.

Theorem 5. The function $S_{t \rightarrow t}(t)$ is in the Hardy class H^2

Proof. Recall that a function is in the Hardy class H^p if

$$\sup_{0 \leq r < 1} \left(\frac{\int_0^{2\pi} |f(re^{ia})|^p da}{2\pi} \right)^{\frac{1}{p}} < \infty$$

then let $p=2$ and note that

$$\sqrt{\frac{1}{2\pi} \int |S(re^{ia})|^2 dx} = \sqrt{\frac{1}{2\pi} \frac{2\pi^4(r^8 - 2r^4 + 8)}{(r^4 - 2)(r^8 + 4)}} \quad (11)$$

is bounded $\forall 0 \leq r < 1$

□

2.1 The Curve $\operatorname{Re}(S(t)) = 0$ is a Bernoullian Lemniscate

Theorem 6. The zero set $\{t: \operatorname{Re}(S(t))=0\}$ of the real part $\operatorname{Re}(S(t))$ of $S(t)$ where $t=x+iy$ is a horizontally oriented lemniscate of Bernoulli at the origin with parameter 2. That is,

$$\{(x, y): \operatorname{Re}(S(x+iy))=0\} = \{(x, y): (x^2+y^2)^2=2(x^2-y^2)\} \quad (12)$$

Proof.

The parametric equations[5] for the lemniscate of Bernoulli with parameter 2 are given by

$$\begin{aligned} x(t) &= \frac{\sqrt{2}\cos(t)}{1+\sin^2(t)} \\ y(t) &= \frac{\sqrt{2}\sin(t)\cos(t)}{1+\sin^2(t)} \end{aligned} \quad (13)$$

Let us combine the coordinate functions $(x(t), y(t)) \in \mathbb{R}^2$ into an equivalent function $z(t) \in \bar{\mathbb{C}}$

$$\begin{aligned} z(t) &= x(t) + iy(t) \\ &= \frac{\sqrt{2}\cos(t)}{1-i\sin(t)} \end{aligned} \quad (14)$$

where it can be shown that

$$\begin{aligned} S(z(t)) &= S_z(t) \\ &= S\left(\frac{\sqrt{2}\cos(t)}{1-i\sin(t)}\right) \\ &= i \frac{32\cos(t)^2\sin(t)}{20\cos(2t)+\cos(4t)-13} \end{aligned} \quad (15)$$

so that

$$\operatorname{Re}(S_z(t))=0 \forall t \in \mathbb{R} \quad (16)$$

and thus $z(t)$ is a geodesic of the real part of S □

2.2 The Curve $\operatorname{Im}(S(t))=0$ is a Conjugate Pair of Rectangular Hyperbolas

Theorem 7. The zero set $\{t: \operatorname{Im}(S(t))=0\}$ of the imaginary part $\operatorname{Im}(S(t))$ of $S(t)$ where $t=x+iy$ is a conjugate pair of rectangular hyperbolas.

$$\{(x, y): \operatorname{Im}(S(x+iy))=0\} = \{(x, y): x^2-y^2=1\} \quad (17)$$

Proof.

The parametric equations[5] for the equilateral (rectangular) hyperbola with unit parameter are given by

$$\begin{aligned} x(t) &= \sec(t) \\ y(t) &= \tan(t) \end{aligned} \quad (18)$$

which are combined into a complex-valued function

$$\begin{aligned} z(t) &= x(t) + iy(t) \\ &= \sec(t) + i \tan(t) \end{aligned} \quad (19)$$

where it can be shown that

$$\begin{aligned} S(z(t)) &= S(\sec(t) + i \tan(t)) \\ &= -\frac{2(\cos(2t)-3)^2}{20\cos(2t)+\cos(4t)-13} \end{aligned} \quad (20)$$

so that

$$\operatorname{Im}(S(z(t)))=0 \forall t \in \mathbb{R} \quad (21)$$

and thus $z(t)$ is a geodesic of the imaginary part of S □

2.3 The Newton Map of $S_f(t)$

Definition 8. Let $N_f(t)$ denote the Newton map of $f(t)$

$$N_f(t) = t - \frac{f(t)}{\dot{f}(t)} \quad (22)$$

Definition 9. The Newton map $N_{S_f}(t)$ of the composition $S_f(t) = S(f(t))$ is a rational meromorphic function of $f(t)$ given by

$$\begin{aligned} N_{S_f}(t) &= t - \frac{S_f(t)}{\dot{S}_f(t)} \\ &= t - \frac{\frac{(1-f(t)^2)^2-1}{(1-f(t)^2)^2+1}}{\frac{8\dot{f}(t)f(t)(f(t)^2-1)}{((f(t)-1)^2(f(t)+1)^2+1)^2}} \\ &= t - \frac{1}{8} \frac{((f(t)-1)^2(1+f(t)^2)^2+1)^2((1-f(t)^2)^2-1)}{f(t)\dot{f}(t)(f(t)^2-1)(1+(1-f(t)^2)^2)} \end{aligned} \quad (23)$$

Theorem 10. The Newton map of S_f transforms superattractive ($\lambda=0$) fixed-points of $N_f(t)$ to geometrically attractive fixed-points of $N_{S_f}(t)$

Proof.

There is geometrically attractive fixed-point at $t=0$ with multiplier equal to

$$\lambda_{N_{S_f}(0)} = \left| 1 - \frac{1 + \lambda_{N_f}(0)}{2} \right| \quad (24)$$

TODO: prove this.. apply S to some other functions and see how it transforms the multipliers of the fixed-points

□

2.3.1 Factoring Out The Double-Root at the Origin of $N_{S_f}(t)$

If $m = m_f(\alpha)$ is the multiplicity of the root of f at the point α then f factorizes as

$$f(x) = (x - \alpha)^m g(x) \quad (25)$$

where $g(\alpha) = 0$.

2.3.2 The Newton Flow

Definition 11. The Newton flow of $S_f(t)$ is defined by the differential equation

$$\dot{z}(t) = \frac{d}{dt} z(t) = -\frac{S_f(z(t))}{\dot{S}_f(z(t))} \quad (26)$$

which is approximated by the relaxed Newton method where the limit of the step size is taken towards zero, it is defined by

$$\begin{aligned} N_{S_f}^h(t) &= t - h \frac{S_f(t)}{\dot{S}_f(t)} \\ &= t - \frac{h \frac{(1-f(t)^2)^2-1}{(1-f(t)^2)^2+1}}{\frac{8\dot{f}(t)f(t)(f(t)^2-1)}{((f(t)-1)^2(f(t)+1)^2+1)^2}} \\ &= t - \frac{h}{8} \frac{((f(t)-1)^2(1+f(t)^2)^2+1)^2((1-f(t)^2)^2-1)}{f(t)\dot{f}(t)(f(t)^2-1)(1+(1-f(t)^2)^2)} \end{aligned} \quad (27)$$

where h is taken to be a small number which is used to approximate the flow $\dot{z}(t)$

2.3.3 The Desingularized Newton Flow

Definition 12. An equivalent flow which is devoid of singularities is given by

$$\dot{z}(t) = -\overline{\dot{S}_f(z(t))} S_f(z(t)) \quad (28)$$

as described in [4].

2.3.4 The Continuous Newton Method for Meromorphic Functions

Define the flow

TODO: read [3]

2.3.5 Linearizing the Newton Map of $S(t)$

3 The Riemann Zeta ζ Function

Definition 13. The Riemann zeta function is defined by

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \forall \operatorname{Re}(s) > 1 \\ &= \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \forall \operatorname{Re}(s) > 0 \end{aligned} \quad (29)$$

and its argument has a representation as

$$S(t) = \frac{1}{\pi} \arg \left(\zeta \left(\frac{1}{2} + it \right) \right) = \frac{1}{\pi} \int_{\frac{1}{2}}^{\infty} \operatorname{Im} \left(\frac{\dot{\zeta}(\sigma + it)}{\zeta(\sigma + it)} \right) d\sigma \forall t \in \mathbb{R} \setminus \{0\} \quad (30)$$

3.1 The Riemann Hypothesis

Conjecture 14. (The Riemann Hypothesis)

Bernhard Riemann conjectured in 1859[1] that

$$\left\{ \operatorname{Re}(t) = \frac{1}{2} : \zeta(t) = 0 \forall t \neq -2n \forall t \in \bar{\mathbb{C}}, n \in \mathbb{N}^+ \right\}$$

or in words, that all of the roots that are not negative even integers where $\zeta(-2n) = 0$ all lie on the **critical line** $\operatorname{Re}(\frac{1}{2})$ in the complex plane such that $\zeta(\sigma + is) = 0$ only when $\sigma = \frac{1}{2}$ where $\mathbb{R}^+ \ni s > 0$.

3.1.1 Lines of Constant Phase and the Riemann Hypothesis

The following theorem is the main result of [8].

Theorem 15.

1. If all lines of constant phase $\arg(\zeta(t)) = kn$ of ζ where $k \in \mathbb{N}$ merge with the critical line
OR
2. all points where $\dot{\zeta}(t)$ vanishes are located on the critical line and the phases of ζ at consecutive zeros of $\dot{\zeta}$ differs by π

then the Riemann Hypothesis (14) is true.

3.2 The Hardy Z Function

Definition 16. (The Gamma and Log Gamma functions)

Let

$$\Gamma(t) = (t-1)! = \int_0^\infty x^{t-1} e^{-x} dx \forall \operatorname{Re}(t) > 0 \quad (31)$$

be the gamma function and

$$\ln \Gamma(t) = \ln(\Gamma(t)) \quad (32)$$

be the principle branch of the logarithm of the Γ function.

3.2.1 The Phase of ζ

The **Riemann – Siegel** theta function $\vartheta(t)$ corresponds to the smooth part of the phase of the zeta function which has a jump discontinuity when t is equal to the imaginary part of a Riemann zero on the critical line.

Definition 17. (The Riemann-Siegel vartheta function)

Let

$$\vartheta(t) = -\frac{i}{2} \left(\ln \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \ln \Gamma\left(\frac{1}{4} - \frac{it}{2}\right) \right) - \frac{\ln(\pi) t}{2} \quad (33)$$

be the the Riemann-Siegel (var)theta function.

Definition 18. (The Hardy Z function)

Let

$$Z(t) = e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right) \quad (34)$$

be the Hardy Z function which has the property that $Z(t)$ is real when t is real, that is, $Z(t) \in \mathbb{R} \forall t \in \mathbb{R}$ independently of the Riemann hypothesis.



Figure 1. Illustration of the relationship between vartheta ϑ and the argument of zeta ζ on the critical line

3.3 The Function $X(t) = (S \circ Z)(t)$

Definition 19. (The X function)

Let $X(t)$ be defined as the composition of the S function with the Hardy Z function

$$X(t) = C_Z(S)(t) = (S \circ Z)(t) = S(Z(t)) = \frac{(1 - Z(t)^2)^2 - 1}{(1 - Z(t)^2)^2 + 1} \quad (35)$$

3.3.1 Integration Along a Curve: A Newton Iteration for the Angle

In order to find an explicit expression for the angle $\theta_m(h)$ in the implicit formula Formula (?) we can use Newton's method

Definition 20. Let the Newton map for the roots of the real part of

$$X(t + he^{ia}) \quad (36)$$

be defined by

$$\begin{aligned} N_{\theta_m}(a_{m,k}; t, h) &= \text{frac} \left(a_{m,k-1} + \tanh \left(\frac{\text{Re}(X(t + he^{ia}))}{\text{Re}(\frac{d}{da} X(t + he^{ia}))} \right) \right)_{a=a_{m,k-1}} \\ &= \text{frac} \left(a_{m,k-1} + \tanh \left(\frac{\text{Re}(X(t + he^{ia}))}{\pi \text{Im}(\dot{X}(t + he^{ia}) h e^{ia})} \right) \right)_{a=a_{m,k-1}} \end{aligned}$$

TODO: this takes a special form, see

where the initial ($k=0$) angle of the first($m=0$) step of length h is $a_{m,0} = \frac{\theta_{m-1}}{\pi}$ where we normalize by π since the variable has domain $[-1, 1]$ (the angle at the previous point) or $a_{0,0} = \frac{3}{4}$ which is -45° for the initial element of the sequence when $m=0$, that is, the (initial)boundary conditions are

$$a_{m,0} = \begin{cases} \frac{3}{4} & m=0 \\ \frac{\theta_{m-1}}{\pi} & m \geq 1 \end{cases} \quad (37)$$

and the corresponding curve is traversed in a positive clockwise direction moving initially into the upper-left quadrant . Let the angle at the m -th step (of length h) be defined as the limit

$$\begin{aligned} \theta_m &= \theta_m(t, h) \in [-\pi, +\pi] \\ &= \pi \lim_{k \rightarrow \infty} N_{\theta_m}(a_{m,k}; t, h) \end{aligned} \quad (38)$$

which is dependent on the basepoint t and radius h , but the when the dependence is not written as $\theta_m(t, h)$ it is still implied unless otherwise noted. The notation $\dot{Y}(t) = \frac{d}{dt}Y(t)$ is the more concise notation for first-derivative.

3.3.2 Roots of $X(t)$ on the Real Line

The critical line of the zeta function $\operatorname{Re}(s) = \frac{1}{2}$ corresponds to the real line $\operatorname{Im}(s) = 0$ of the Z and X functions

4 Linearizing

Definition 21. Let the Newton map of the shifted X function

$$X_n(t) = X(z_n + t) \quad (39)$$

where z_n is the n -th root of the Hardy Z function on the real line, starting with

$$\begin{aligned} z_1 &\cong 14.1347251\dots \\ z_2 &= 21.0220396\dots \\ &\dots \dots \end{aligned}$$

, be denoted

$$N_{X_{z_n}}(t) = t - \frac{X_{z_n}(t)}{\dot{X}_{z_n}(t)} \quad (40)$$

TODO: this has nice symmetric factorized form, see (23)

5 The Spectral Theorem

Theorem 22. The Spectral Theorem

Let $U: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ be unitary then U extends uniquely to a unitary operator on all of $L^2(\mathbb{R}^n)$ and all generalized eigenvalues λ lie on the unit circle $|\lambda| = 1$. The space $L^2(\mathbb{R}^n)$ can be expressed as a direct integral

$$\int_{|\lambda|=1} \mathcal{H}(\lambda) d\mu(\lambda) \quad (41)$$

of Hilbert spaces $\mathcal{H}(\lambda) \subseteq E_\lambda$ so that U sends the function $h \in L^2(\mathbb{R}^n)$ to the function Uh with λ -component

$$(Uh) = \lambda h_\lambda \in \mathcal{H}(\lambda) \quad (42)$$

and its set of generalized eigenvectors forms a complete basis. [?, Theorem 1.3.2]

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