Available online at www.sciencedirect.com
ScienceDirect
International Journal of Engineering Science 46 (2008) 654–676
International Journal of Engineering Science
www.elsevier.com/locate/ijengsci

Two-dimensional models for the combined bending and stretching of plates and shells based on three-dimensional linear elasticity

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Received 4 January 2008; accepted 21 January 2008
Available online 7 March 2008

Abstract

Models for plates and shells derived from three-dimensional linear elasticity, based on a thickness-wise expansion of the strain energy of a thin body, are described. These involve the small thickness explicitly and accommodate combined bending and stretching in a single framework. Physically motivated local constraints on the through-thickness variation of the displacement field, required for consistency with the exact theory, are introduced. When incorporated into the energy functional, these yield an expression for the two-dimensional strain energy density that includes non-standard two-dimensional strain gradient effects.

Keywords: Plates; Shells; Linear elasticity; Strain gradients

> Extreme rigour in the analysis of physical problems, we are inclined to believe, may easily lead to rigor mortis. ...Flexible bodies like thin shells require a flexible approach. ...In a difficult physical problem like shell theory there is both room and a need for intuition and imagination in addition to mathematical analysis.
> Koiter [1].

1. Introduction

The derivation of an accurate two-dimensional model of plates and shells from three-dimensional elasticity theory has been a problem of recurring interest over the course of the history of solid mechanics [2–11]. In recent years, the problem has been addressed through the use of sophisticated tools such as the method of Gamma convergence, aimed at generating the two-dimensional variational problem in the limit of small

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0020-7225/$ - see front matter
doi:10.1016/j.ijengsci.2008.01.015

thickness, or methods of asymptotic analysis, in which leading-order (in thickness) weak forms of the equations are obtained via a process of formal expansion. A discussion of the current state of the subject is summarized concisely in the recent work of Ciarlet [12], which contains an extensive bibliography. To date, these methods have yielded sharp results for pure membrane behavior and for inextensional bending, but not for the practically important intermediate case in which bending and stretching occur simultaneously. In particular, neither method has generated a single model containing the small thickness explicitly, in contrast to the situation for classical theories. Such a structure is to be expected in any model that accommodates bending and stretching simultaneously. The limitations of models derived from Gamma convergence or asymptotic expansions of the weak forms of the equilibrium equations are inherent in these methods. For example, Gamma convergence entails passage to the limit of zero thickness, and so generates only the leading-order model appropriate for the considered loads and boundary conditions. The asymptotic approach is similar in that it requires successive terms in the expansion to be negligibly small in comparison to the preceding terms. Accordingly, as remarked in [13], these methods, despite their limited success, effectively separate the scales associated with membrane and bending behavior to a greater extent than desired. An important exception in this regard is the model of Koiter [14,15], which Ciarlet [12] advocates as the best practical solution to the problem of specifying the equations holding in the interior of the shell in the case of small strains with possibly finite displacements. This despite the fact that Koiter's theory is not a limit model in the sense of Gamma convergence or asymptotic expansions.

The present paper is concerned with the derivation of models for combined bending and stretching in the presence of edge conditions. The approach followed is based on a systematic small thickness expansion of the exact three-dimensional strain energy density of the plate or shell in which the thickness figures explicitly. This may be truncated at any desired level. The equilibrium equations for the truncated model are the Euler equations of the associated energy integral. Membrane effects are associated with the order h problem, where h is the thickness, and corrections, associated with bending and additional non-standard effects, emerge at order h^3 if the midsurface is a plane of symmetry of the material properties. Our model does not suffer from the overly restrictive separation of scales that characterizes the limit models discussed above. Nevertheless, in any particular problem it is in principle necessary to use its predictions to evaluate the leading-order terms not retained to ensure that those appearing explicitly are dominant.

Also considered is a locally constrained variant of the model motivated by comparison with the equations of three-dimensional elasticity. The resulting strain energy density for the midsurface contains non-standard two-dimensional strain gradient terms in addition to the usual stretching and bending energies. These are suppressed in classical plate theory. This finding is not unexpected. Indeed, it is known that such terms are present in the thickness-wise expansions of the equilibrium energies of a thin plate in plane stress or generalized plane stress [16, Arts. 310, 303, 304]. The extra terms are associated with a singular perturbation. Accordingly, their effect may be expected to be confined to a thin layer adjoining an edge, necessary to accommodate assigned or reactive force and moment distributions.

The development of plate theory is described at length in Section 2. In Section 3 we adapt the ideas developed for plates to shells. While the basic concepts are preserved, some reconsideration is necessary to accommodate the differential geometry of the shell. To hold the discussion to a tractable level, and to make contact with a large class of applications, we base our development on the classical linear theory of elasticity. However, our methods easily generalize to non-linear elasticity.

Standard notation is used throughout. Thus, bold face is used for vectors and tensors and indices are used to denote their components. Latin indices take values in {1, 2, 3}; Greek in {1, 2}. The latter are associated with surface coordinates and associated vector and tensor components. A dot between bold symbols is used to denote the standard inner product. Thus, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are second-order tensors, then $\mathbf{A}_1 \cdot \mathbf{A}_2 = \mathrm{tr}(\mathbf{A}_1\mathbf{A}_2^t)$, where $\mathrm{tr}(\cdot)$ is the trace and the superscript $(^t)$ is used to denote the transpose. The norm of a tensor $\mathbf{A}$ is $|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$. The linear operator $\mathrm{Sym}(\cdot)$ delivers the symmetric part of its second-order tensor argument. The notation $\otimes$ identifies the standard tensor product of vectors. If $\mathbf{C}$ is a fourth-order tensor, then $\mathbf{C}[\mathbf{A}]$ is the second-order tensor with orthogonal components $C_{ijkl}A_{kl}$. Finally, we use symbols such as Div and D to denote the three-dimensional divergence and gradient operators, while div and ∇ are reserved for their two-dimensional counterparts. Thus, for example, $\mathrm{Div}\,\mathbf{A} = A_{ij,j}\mathbf{e}_i$ and $\mathrm{div}\,\mathbf{A} = A_{i\alpha,\alpha}\mathbf{e}_i$, where $\{\mathbf{e}_i\}$ is an orthonormal basis and subscripts preceded by commas are used to denote partial derivatives with respect to Cartesian coordinates. Adjustments to these notational conventions to accommodate curved surfaces parametrized by curvilinear coordinates are introduced as needed.

The three-dimensional equation of equilibrium without body force is

$$\operatorname{Div}\mathbf{P} = \mathbf{0}, \tag{1}$$

where

$$\mathbf{P} = \mathbf{C}[\mathbf{H}], \tag{2}$$

is the linear approximation to the Piola stress in the absence of residual stress, $\mathbf{H} = D\mathbf{u}$ is the displacement gradient, $\mathbf{u}(\mathbf{x})$ is the displacement field and $\mathbf{C}$ is the fourth-order tensor of elastic moduli. We require the latter to possess the minor symmetries

$$\mathbf{A}_1 \cdot \mathbf{C}[\mathbf{A}_2] = \mathbf{A}_1^t \cdot \mathbf{C}[\mathbf{A}_2], \quad \mathbf{A}_1 \cdot \mathbf{C}[\mathbf{A}_2] = \mathbf{A}_1 \cdot \mathbf{C}[\mathbf{A}_2^t], \tag{3}$$

and the major symmetry

$$\mathbf{A}_1 \cdot \mathbf{C}[\mathbf{A}_2] = \mathbf{A}_2 \cdot \mathbf{C}[\mathbf{A}_1], \tag{4}$$

for all second-order tensors $\mathbf{A}_1, \mathbf{A}_2$. These restrictions in turn ensure that

$$\mathbf{P} = U_{\mathbf{H}}, \tag{5}$$

where

$$U(\mathbf{H};\mathbf{x}) = \frac{1}{2}\mathbf{H} \cdot \mathbf{C}(\mathbf{x})[\mathbf{H}], \tag{6}$$

is the quadratic-order approximation to the strain energy per unit volume of R in which explicit dependence on $\mathbf{x} \in R$ is present if the material is non-uniform. Such dependence occurs through the moduli. In this work, we take the moduli to be independent of $\mathbf{x}$ and thus restrict attention to uniform materials.

In the classical theory without residual stress the energy is assumed to be a positive-definite function of the infinitesimal strain $\operatorname{Sym}\mathbf{H}$; the minor symmetries of $\mathbf{C}$ imply that U vanishes if $\mathbf{H}$ is skew. Accordingly, $\operatorname{Sym}\mathbf{H}$ may be replaced by $\mathbf{H}$ in the inequality expressing positive-definiteness. This in turn implies that the elastic moduli satisfy the strong-ellipticity condition

$$\mathbf{v} \otimes \mathbf{w} \cdot \mathbf{C}[\mathbf{v} \otimes \mathbf{w}] > 0 \quad \text{for all } \mathbf{v} \otimes \mathbf{w} \neq \mathbf{0}. \tag{7}$$

It is easy to show that the symmetric part of $\mathbf{v} \otimes \mathbf{w}$ vanishes if and only if $\mathbf{v} \otimes \mathbf{w}$ vanishes, so that (7) is meaningful.

In the case of isotropy relative to the undeformed placement of the body, the components of $\mathbf{C}$ with respect to an orthonormal basis $\{\mathbf{e}_i\}$ are

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{8}$$

where δ_{ij} is the Kronecker delta and λ, μ are the Lamé moduli. Accordingly

$$\mathbf{C}[\mathbf{A}] = \lambda(\operatorname{tr}\mathbf{A})\mathbf{I} + 2\mu(\operatorname{Sym}\mathbf{A}), \tag{9}$$

for any tensor $\mathbf{A}$, where $\mathbf{I}$, with orthogonal components δ_{ij}, is the three-dimensional identity. The well-known necessary and sufficient conditions for strong ellipticity are

$$\mu > 0 \quad \text{and} \quad \lambda + 2\mu > 0, \tag{10}$$

while positive-definiteness as a function of the (infinitesimal) strain is equivalent to the stronger conditions

$$\mu > 0 \quad \text{and} \quad \kappa > 0, \tag{11}$$

where

$$\kappa = \lambda + \frac{2}{3}\mu, \tag{12}$$

is the bulk modulus.

Our approach is based on a development of the virtual-work equation in powers of thickness. Thus, let $\mathbf{u}(\mathbf{x};\epsilon)$ be a one-parameter family of displacements and let $\mathbf{u}(\mathbf{x}) = \mathbf{u}(\mathbf{x};0)$ be an equilibrium displacement. Then, the relevant expression is

$$\int_{\partial R} \mathbf{t}\cdot\dot{\mathbf{u}}\,\mathrm{d}a = \left(\int_R U\,\mathrm{d}v\right)^{\cdot}, \tag{13}$$

where $\dot{\mathbf{u}}(\mathbf{x})$ – the virtual displacement – is the derivative of $\mathbf{u}(\mathbf{x};\epsilon)$ with respect to the parameter ϵ, evaluated at $\epsilon = 0$, and

$$\mathbf{t} = \mathbf{PN}, \tag{14}$$

is the assigned traction on the boundary ∂R with exterior unit normal $\mathbf{N}$. It is well-known [17] that if the strain energy is a positive-definite function of the (infinitesimal) strain, then in standard mixed position/traction problems an equilibrium displacement field $\mathbf{u}(\mathbf{x})$ satisfies $\Delta\mathscr{E} \geqslant 0$, where

$$\Delta\mathscr{E} = \mathscr{E}[\mathbf{u}+\Delta\mathbf{u}] - \mathscr{E}[\mathbf{u}], \tag{15}$$

and

$$\mathscr{E} = \int_R U\,\mathrm{d}v - \int_{\partial R_2} \mathbf{t}\cdot\mathbf{u}\,\mathrm{d}a, \tag{16}$$

is the potential energy of the body. Here, $\partial R_2 = \partial R \setminus \partial R_1$ is a part of the boundary where traction is assigned, while ∂R_1 is the part where displacement is prescribed and, accordingly, where $\Delta\mathbf{u}$ vanishes.

The expression (6) for the strain energy function may be used to evaluate $\mathscr{E}$ on the displacement ansatz

$$\mathbf{u} = \sum_{n=0}^{N} \varsigma^n \boldsymbol{\delta}_n, \tag{17}$$

where ς is a through-thickness coordinate, N is a fixed positive integer, and the $\boldsymbol{\delta}_n$ depend on in-plane or surface coordinates. For example, in Michell's exact solution for a plate subjected to uniform lateral pressure (see [16, Art. 307] and [18]), the displacement field includes terms up to order ς^4. However, rather than model a given ansatz, we instead consider a simpler alternative based on a specified order of truncation of the energy in powers of thickness.

2. Plate theory

2.1. Thickness expansion of the energy

We write the virtual-work form (13) of the three-dimensional equation of equilibrium in terms of iterated integrals and expand the thickness-wise integrals therein for small values of the thickness h of the thin plate. For simplicity we take h to be uniform. The associated strong forms are then derived from the fundamental lemma of the calculus of variations. The resulting two-dimensional system depends on the relationship of the base surface Ω to the three-dimensional body. The classical example in which the base surface coincides with the midsurface is emphasized for the sake of illustration.

The reference placement of the plate is described by

$$\mathbf{x} = \mathbf{r} + \varsigma\mathbf{n}, \tag{18}$$

where $\mathbf{r} \in \Omega$, $\mathbf{n}$ is the fixed orientation of the plate and $\varsigma \in [-h/2, h/2]$. The origin of the position $\mathbf{r}$ is assumed to lie on Ω. Let

$$\hat{\mathbf{u}}(\mathbf{r},\varsigma) = \tilde{\mathbf{u}}(\mathbf{r}+\varsigma\mathbf{n}) \quad \text{and} \quad \widehat{\mathbf{H}}(\mathbf{r},\varsigma) = \widetilde{\mathbf{H}}(\mathbf{r}+\varsigma\mathbf{n}). \tag{19}$$

Then, using $\mathrm{d}\tilde{\mathbf{u}} = \widetilde{\mathbf{H}}\,\mathrm{d}\mathbf{x}$ we obtain

$$(\widetilde{\mathbf{H}}\mathbf{1})\mathrm{d}\mathbf{r} + \widetilde{\mathbf{H}}\mathbf{n}\,\mathrm{d}\varsigma = \mathrm{d}\hat{\mathbf{u}} = (\nabla\hat{\mathbf{u}})\mathrm{d}\mathbf{r} + \hat{\mathbf{u}}'\,\mathrm{d}\varsigma, \tag{20}$$

where, here and henceforth, $\nabla(\cdot)$ is the (two-dimensional) gradient with respect to $\mathbf{r}$ at fixed ς and the notation $(\cdot)'$ is used to denote $\partial(\cdot)/\partial\varsigma$ at fixed $\mathbf{r}$. Further

$$\mathbf{1} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n}, \tag{21}$$

where $\mathbf{I}$ is the identity for three-space, is the (two-dimensional) identity on the translation (vector) space Ω' of Ω. This may be used to derive an orthogonal decomposition of tensors into tangential and normal parts [19]. Thus, if $\mathbf{A}$ is a linear map from three-space to itself, then

$$\mathbf{A} = \mathbf{A}\mathbf{I} = \mathbf{A}\mathbf{1} + \mathbf{A}\mathbf{n} \otimes \mathbf{n}, \tag{22}$$

and if $\mathbf{B}$ is such a map, then

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{A}\mathbf{1} \cdot \mathbf{B}\mathbf{1} + \mathbf{A}\mathbf{n} \cdot \mathbf{B}\mathbf{n}. \tag{23}$$

The cross terms vanish because $\mathbf{1}\mathbf{n} = \mathbf{0}$. Accordingly, from (20) we derive

$$\widetilde{\mathbf{H}}\mathbf{1} = \nabla\hat{\mathbf{u}}, \quad \widetilde{\mathbf{H}}\mathbf{n} = \hat{\mathbf{u}}', \tag{24}$$

and

$$\widehat{\mathbf{H}} = \nabla\hat{\mathbf{u}} + \hat{\mathbf{u}}' \otimes \mathbf{n}. \tag{25}$$

The total strain energy $\mathscr{S}$ in a given deformation is

$$\mathscr{S} = \int_R U(\widetilde{\mathbf{H}}(\mathbf{x}))\mathrm{d}v = \int_\Omega \int_{-h/2}^{h/2} U(\widehat{\mathbf{H}}(\mathbf{r},\varsigma))\mathrm{d}\varsigma\,\mathrm{d}a. \tag{26}$$

We assume the conditions of Fubini's theorem to hold [20]. Thus the integrand is assumed to possess such continuity as is needed to justify interchanging the order of integration. This ensures that the order does not affect the model to be derived. We write the through-thickness integral in the form

$$W = \int_{-h/2}^{h/2} G(\varsigma)\mathrm{d}\varsigma, \tag{27}$$

where $G(\cdot) = U(\widehat{\mathbf{H}}(\mathbf{r},\cdot))$ and W is the strain energy density of Ω. If $\widetilde{\mathbf{H}}(\mathbf{x})$ is sufficiently smooth, then by Leibniz' rule and Taylor's theorem

$$W = hG_0 + \frac{1}{24}h^3 G_0'' + \ldots, \tag{28}$$

where the subscript $(\cdot)_0$ identifies function values at $\varsigma = 0$ and where, by the chain rule

$$G_0 = U(\widehat{\mathbf{H}}_0), \quad G_0' = \mathbf{P}_0 \cdot \widehat{\mathbf{H}}_0', \quad G_0'' = \mathbf{P}_0' \cdot \widehat{\mathbf{H}}_0' + \mathbf{P}_0 \cdot \widehat{\mathbf{H}}_0''. \tag{29}$$

For uniform materials without initial stress

$$U(\widehat{\mathbf{H}}_0) = \frac{1}{2}\mathbf{P}_0 \cdot \widehat{\mathbf{H}}_0, \quad \mathbf{P}_0 = \mathbf{C}[\widehat{\mathbf{H}}_0] \quad \text{and} \quad \mathbf{P}_0' = \mathbf{C}[\widehat{\mathbf{H}}_0'], \tag{30}$$

where from (25)

$$\widehat{\mathbf{H}}_0 = \nabla\mathbf{u} + \mathbf{a} \otimes \mathbf{n}, \quad \widehat{\mathbf{H}}_0' = \nabla\mathbf{a} + \mathbf{b} \otimes \mathbf{n} \quad \text{and} \quad \widehat{\mathbf{H}}_0'' = \nabla\mathbf{b} + \mathbf{c} \otimes \mathbf{n}, \tag{31}$$

and

$$\mathbf{u} = \hat{\mathbf{u}}_0, \quad \mathbf{a} = \hat{\mathbf{u}}_0', \quad \mathbf{b} = \hat{\mathbf{u}}_0'' \quad \text{and} \quad \mathbf{c} = \hat{\mathbf{u}}_0''', \tag{32}$$

are mutually independent functions of $\mathbf{r}$. These are the coefficient vectors in the thickness-wise expansion (cf. (17))

$$\hat{\mathbf{u}} = \mathbf{u} + \varsigma\mathbf{a} + \frac{1}{2}\varsigma^2\mathbf{b} + \frac{1}{6}\varsigma^3\mathbf{c} + \ldots. \tag{33}$$

The order h^3 expansion of the strain energy is thus given by

$$\mathscr{S} = S + o(h^3), \tag{34}$$

where

$$S = \int_{\Omega} W \, \mathrm{d}a, \tag{35}$$

and

$$W(\nabla \mathbf{u}, \nabla \mathbf{a}, \nabla \mathbf{b}, \mathbf{a}, \mathbf{b}, \mathbf{c}) = \frac{1}{2} h \mathbf{P}_0 \cdot \widehat{\mathbf{H}}_0 + \frac{1}{24} h^3 (\mathbf{P}_0' \cdot \widehat{\mathbf{H}}_0' + \mathbf{P}_0 \cdot \widehat{\mathbf{H}}_0''), \tag{36}$$

is the function obtained by substituting (29)–(31) into (28). We emphasize that this is not the complete expression for the energy associated with the order ς^3 ansatz. The latter generates additional higher-order terms in h. Here, we are concerned with the problem of constructing an accurate order h^3 model rather than a complete model for a given ansatz.

To proceed, let C^* be the line orthogonal to Ω and intersecting ∂R at a point with position $\mathbf{r}$. Let $\partial R_C = \partial\Omega \times C$, where C is the collection of such lines, be the cylindrical generating surface of the plate-like region R obtained by translating the points of $\partial\Omega$ along their associated lines C^*. Let s measure arclength on the curve $\partial\Omega$ with unit tangent $\boldsymbol{\tau}$ and rightward unit normal $\boldsymbol{\nu} = \boldsymbol{\tau} \times \mathbf{n}$.

For illustrative purposes we assume the lateral surfaces of the plate to be free of traction and thus that traction can be non-zero only on ∂R_C. The virtual-work of the boundary tractions may thus be expanded to obtain

$$\int_{\partial R} \mathbf{t} \cdot (\tilde{\mathbf{u}})^{\cdot} \, \mathrm{d}a = \int_{\partial\Omega} (\mathbf{p}_u \cdot \dot{\mathbf{u}} + \mathbf{p}_a \cdot \dot{\mathbf{a}} + \mathbf{p}_b \cdot \dot{\mathbf{b}}) \mathrm{d}s + o(h^3), \tag{37}$$

where

$$\mathbf{p}_u = h\mathbf{t}_0 + \frac{1}{24} h^3 \mathbf{t}_0'', \quad \mathbf{p}_a = \frac{1}{12} h^3 \mathbf{t}_0' \quad \text{and} \quad \mathbf{p}_b = \frac{1}{24} h^3 \mathbf{t}_0. \tag{38}$$

We observe that the potential energy $\mathscr{E}$, and the energy difference $\Delta\mathscr{E}$ relative to an equilibrium state, admit the expansions

$$\mathscr{E} = E + o(h^3) \quad \text{and} \quad \Delta\mathscr{E} = \Delta E + o(h^3), \tag{39}$$

where

$$E = h\mathscr{E}_1 + h^3 \mathscr{E}_3 \quad \text{and} \quad \Delta E = h\Delta\mathscr{E}_1 + h^3 \Delta\mathscr{E}_3, \tag{40}$$

in which $\mathscr{E}_n$ and $\Delta\mathscr{E}_n$ are independent of h. The condition $\Delta\mathscr{E} \geqslant 0$, when divided by h, yields $\Delta\mathscr{E}_1 + o(h)/h \geqslant 0$. Passing to the limit, we obtain $\Delta\mathscr{E}_1 \geqslant 0$ and conclude that equilibria minimize the membrane potential energy if the plate is arbitrarily thin. If attention is restricted to displacement gradients that are skew at the midsurface, so that the midsurface is unstrained, and if the edge traction $\mathbf{t}_0$ vanishes, then $\mathscr{E}_1$ and $\Delta\mathscr{E}_1$ vanish identically and the same argument yields $\Delta\mathscr{E}_3 \geqslant 0$. In this case admissible deformations of the plate correspond to pure bending and equilibria minimize the associated potential energy. These ideas underlie the approach to membrane and bending theory via Gamma convergence. However, in the case of combined bending and stretching of a finite-thickness plate in which terms of order h and h^3 are retained, the inequality $\Delta\mathscr{E} \geqslant 0$ satisfied by equilibria does not imply that $\Delta E \geqslant 0$. This is the reason why the method of Gamma convergence, which is concerned exclusively with the derivation of the limiting minimization problem, has not succeeded in generating a single model for combined bending and stretching, except in the fortuitous circumstance when the two effects can be decoupled [21]. Accordingly, we do not require that E be minimized at an equilibrium state. Instead, the Euler equations for E, together with associated boundary conditions, are regarded as an approximate model for equilibria. We elaborate on this in the next subsection and return to the issue of minimizers in Section 2.4.

2.2. Euler equations and edge conditions

The equilibrium equations for the thin plate are identified with the Euler equations for the order h^3 truncation of the strain energy. Given that this truncation does not account for all terms associated with the order ς^3 ansatz, it is not surprising that the Euler equations obtained do not agree precisely with those of the exact theory, specialized to the same ansatz. Our purpose, here and in Section 2.3, is to identify adjustments needed to bring the order h^3 model into compliance with the exact theory, at least insofar as the differential equations holding in the interior of the plate are concerned.

For the order h^3 truncation given by (36), the Euler equations are

$$\operatorname{div}(W_{\nabla \mathbf{u}}) = \mathbf{0}, \quad \operatorname{div}(W_{\nabla \mathbf{a}}) = W_{\mathbf{a}}, \quad \operatorname{div}(W_{\nabla \mathbf{b}}) = W_{\mathbf{b}} \quad \text{and} \quad W_{\mathbf{c}} = \mathbf{0}, \tag{41}$$

where $\operatorname{div}(\cdot)$ is the two-dimensional divergence operation on Ω (Section 1). To obtain the various terms in these equations we compare the variation of (36) to

$$\dot{W} = W_{\nabla \mathbf{u}} \cdot \nabla \dot{\mathbf{u}} + W_{\nabla \mathbf{a}} \cdot \nabla \dot{\mathbf{a}} + W_{\nabla \mathbf{b}} \cdot \nabla \dot{\mathbf{b}} + W_{\mathbf{a}} \cdot \dot{\mathbf{a}} + W_{\mathbf{b}} \cdot \dot{\mathbf{b}} + W_{\mathbf{c}} \cdot \dot{\mathbf{c}}. \tag{42}$$

Using the major symmetry of the tensor of elastic moduli, we find that

$$W_{\nabla \mathbf{u}} = h\mathbf{P}_0\mathbf{1} + \frac{1}{24}h^3\mathbf{P}_0''\mathbf{1}, \quad W_{\nabla \mathbf{a}} = \frac{1}{12}h^3\mathbf{P}_0'\mathbf{1}, \quad W_{\nabla \mathbf{b}} = \frac{1}{24}h^3\mathbf{P}_0\mathbf{1}, \tag{43}$$

and

$$W_{\mathbf{a}} = h\mathbf{P}_0\mathbf{n} + \frac{1}{24}h^3\mathbf{P}_0''\mathbf{n}, \quad W_{\mathbf{b}} = \frac{1}{12}h^3\mathbf{P}_0'\mathbf{n}, \quad W_{\mathbf{c}} = \frac{1}{24}h^3\mathbf{P}_0\mathbf{n}. \tag{44}$$

The last of the Euler Eq. (41) implies that the midsurface is in a state of plane stress; i.e. that

$$\mathbf{P}_0\mathbf{n} = \mathbf{0}. \tag{45}$$

With this satisfied, the second and third of Eq. (41) reduce, respectively, to

$$2\operatorname{div}(\mathbf{P}_0'\mathbf{1}) = \mathbf{P}_0''\mathbf{n} \quad \text{and} \quad \operatorname{div}(\mathbf{P}_0\mathbf{1}) = 2\mathbf{P}_0'\mathbf{n}. \tag{46}$$

It is noteworthy that these results do not involve the small parameter h and thus purport to hold uniformly over the plate. They may be reduced to algebraic equations to be solved for the vector fields $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. To see this we use $(30)_2$ to recast (45) in the form

$$\mathbf{A}_n\mathbf{a} = -(\mathbf{C}[\nabla \mathbf{u}])\mathbf{n}, \tag{47}$$

where $\mathbf{A}_n$ is the acoustic tensor defined by

$$\mathbf{A}_n\mathbf{v} = (\mathbf{C}[\mathbf{v} \otimes \mathbf{n}])\mathbf{n}, \tag{48}$$

for any vector $\mathbf{v}$. This is positive-definite by virtue of the strong-ellipticity inequality (7). Then, (47) yields $\mathbf{a}$ uniquely in terms of $\nabla \mathbf{u}$. Next, we combine $(30)_3$ with $(46)_2$ to derive

$$2\mathbf{A}_n\mathbf{b} = \operatorname{div}(\mathbf{P}_0\mathbf{1}) - 2(\mathbf{C}[\nabla \mathbf{a}])\mathbf{n}. \tag{49}$$

In the same way, $(46)_1$ may be recast as

$$\mathbf{A}_n\mathbf{c} = 2\operatorname{div}(\mathbf{P}_0'\mathbf{1}) - (\mathbf{C}[\nabla \mathbf{b}])\mathbf{n}. \tag{50}$$

Substitution of the solutions to (47), (49) and (50) into $(41)_1$ delivers

$$\operatorname{div}\left(h\mathbf{P}_0\mathbf{1} + \frac{1}{24}h^3\mathbf{P}_0''\mathbf{1}\right) = \mathbf{0}, \tag{51}$$

where

$$\mathbf{P}_0\mathbf{1} = (\mathbf{C}[\nabla \mathbf{u} + \mathbf{a} \otimes \mathbf{n}])\mathbf{1} \quad \text{and} \quad \mathbf{P}_0''\mathbf{1} = (\mathbf{C}[\nabla \mathbf{b} + \mathbf{c} \otimes \mathbf{n}])\mathbf{1}. \tag{52}$$

We note that (47) and (49), respectively, determine $\mathbf{a}$ as a function of the gradient of $\mathbf{u}$ and $\mathbf{b}$ as a function of the first and second gradients of $\mathbf{u}$. Then, (50) yields $\mathbf{c}$ in terms of the first, second and third gradients of $\mathbf{u}$. Accordingly, (51) reduces to a fourth-order equation for the midsurface displacement.

Regarding edge conditions, we consider the simplest case in which $\partial\Omega$ consists of the union of disjoint arcs $\partial\Omega_1$ and $\partial\Omega_2$ where essential and natural boundary conditions, respectively, are specified. These must, of course, be consistent with the existence of a solution to the equations holding in the interior. This requirement imposes compatibility conditions of a non-standard kind. For example, suppose three-dimensional position is assigned on $\partial R_{C_1} = \partial\Omega_1 \times C$. Then the variation $(\tilde{\mathbf{u}})^{\cdot}$ vanishes on ∂R_{C_1}, and it follows from (33) that $\dot{\mathbf{u}}, \dot{\mathbf{a}}$ and $\dot{\mathbf{b}}$ vanish on $\partial\Omega_1$. We refer to this as a clamped edge. Evidently $\mathbf{u}$, $\mathbf{a}$ and $\mathbf{b}$ are assigned on $\partial\Omega_1$. However, the latter two fields cannot be assigned arbitrarily. The assigned values must agree with the continuous extensions to $\partial\Omega_1$ of the functions $\mathbf{a}$ and $\mathbf{b}$ delivered by the interior Eqs. (47) and (50). This effectively means that $\mathbf{u}$ and its normal derivative $\mathbf{u}_\nu$ are assigned. For, the displacement gradient may be decomposed in the form [6]

$$\nabla\mathbf{u} = \mathbf{u}_s \otimes \boldsymbol{\tau} + \mathbf{u}_\nu \otimes \boldsymbol{\nu}, \tag{53}$$

where $\boldsymbol{\tau} = \mathbf{r}_s$ and $\boldsymbol{\nu} = \mathbf{r}_\nu$, in which the tangential derivative $\mathbf{u}_s$ is obtained by differentiating $\mathbf{u}$ with respect to arclength on $\partial\Omega_1$. The continuous extension to $\partial\Omega_1$ of the field $\mathbf{a}$ derived from (47) is thus controlled by the boundary values of $\mathbf{u}$ and $\mathbf{u}_\nu$. In the same way we stipulate that the values of $\mathbf{b}$ on $\partial\Omega_1$ agree with the continuous extension of the solution to (50). We then consider $\mathbf{a}$ and $\mathbf{b}$ to be assigned accordingly and thus require their variations to vanish on $\partial\Omega_1$. In general these edge values are fully specified only after the problem has been solved, and so some parts of the data on ∂R_{C_1} must effectively adjust a posteriori to the problem at hand, rather than being imposed a priori. The alternative to this state of affairs is to use three-dimensional theory in a region adjoining the boundary and then to match its predictions to those of the foregoing interior equations [22,23]. Indeed, this idea is the focus of some current efforts in the development of finite-element methods for plates and shells. Here, however, we intend that the two-dimensional model apply to the entire body, and this requires a compromise with respect to boundary data.

If traction is assigned on $\partial R_{C_2} = \partial\Omega_2 \times C \subset \partial R_2$, then, in principle, the variations $\dot{\mathbf{u}}, \dot{\mathbf{a}}$ and $\dot{\mathbf{b}}$ are arbitrary on $\partial\Omega_2$. This in turn implies that natural boundary conditions involving $\mathbf{p}_u, \mathbf{p}_a$ and $\mathbf{p}_b$ are satisfied (cf. (37)). However, the differential equation holding in the interior is of the fourth-order and the specification of three vector conditions on an edge will in general overspecify the problem. To correct for this we take the view that the boundary value of $\mathbf{b}$ is fixed a posteriori by the continuous extension to $\partial\Omega_2$ of the solution to (49) delivered by the equilibrium deformation. Accordingly, its variation vanishes and no third boundary condition emerges. Accordingly, the order h^3 expansion of the virtual-work equation, valid for all types of boundary data considered, becomes

$$\int_\Omega \dot{W}\,\mathrm{d}a = \int_{\partial\Omega} (\mathbf{p}_u \cdot \dot{\mathbf{u}} + \mathbf{p}_a \cdot \dot{\mathbf{a}})\mathrm{d}s. \tag{54}$$

The natural boundary conditions are

$$(W_{\nabla\mathbf{u}})\boldsymbol{\nu} = \mathbf{p}_u \quad \text{and} \quad (W_{\nabla\mathbf{a}})\boldsymbol{\nu} = \mathbf{p}_a. \tag{55}$$

These are seen, on substitution of (38) and (43), to be exact consequences of the three-dimensional traction condition $\mathbf{t} = \mathbf{P}\boldsymbol{\nu}$, obtained from it by differentiating with respect to ς and evaluating the resulting expressions at $\varsigma = 0$. The natural boundary condition associated with variations in $\mathbf{b}$, which we do not impose, is $(W_{\nabla\mathbf{b}})\boldsymbol{\nu} = \mathbf{p}_b$. Using $(38)_1$ and $(43)_3$, we see that this is simply the order h truncation of $(55)_1$. Although it is an exact consequence of the three-dimensional traction condition, its imposition would have the undesired effect of separating the scales a priori in the manner described in the Introduction. We elaborate on this point in the next section.

2.3. Constrained theory

The foregoing model involves the ς-derivatives of the stress, evaluated at the midsurface. Comparison with the exact theory is facilitated by introducing the decomposition (cf. (22))

$$\widehat{\mathbf{P}} = \widehat{\mathbf{P}}\mathbf{1} + \widehat{\mathbf{P}}\mathbf{n} \otimes \mathbf{n}, \tag{56}$$

of the three-dimensional stress $\mathbf{P}(\mathbf{x}) = \widehat{\mathbf{P}}(\mathbf{r}, \varsigma)$ into Eq. (1) of equilibrium without body force; i.e.

$$\mathrm{Div}\,\widehat{\mathbf{P}} = \mathbf{0}. \tag{57}$$

Thus

$$\mathrm{div}(\widehat{\mathbf{P}}\mathbf{1}) + \widehat{\mathbf{P}}'\mathbf{n} = \mathbf{0}. \tag{58}$$

We assume throughout that the stress is related to the displacement field by the constitutive Eq. (2) in which the (constant) moduli satisfy the strong-ellipticity condition. It is known that in the presence of strong ellipticity, and for standard mixed boundary value problems, displacement fields satisfying (57) (or (58)) are of class C^∞ on the interior of the body [24]. Accordingly, it is legitimate to differentiate (58) with respect to ς as many times as needed to derive a system that characterizes a given ansatz completely. For example, evaluating (58) and its first and second derivatives on the midsurface, we obtain

$$\mathrm{div}(\mathbf{P}_0\mathbf{1}) + \mathbf{P}'_0\mathbf{n} = \mathbf{0}, \quad \mathrm{div}(\mathbf{P}'_0\mathbf{1}) + \mathbf{P}''_0\mathbf{n} = \mathbf{0} \quad \text{and} \quad \mathrm{div}(\mathbf{P}''_0\mathbf{1}) + \mathbf{P}'''_0\mathbf{n} = \mathbf{0}. \tag{59}$$

The first and second of these are consistent with (46) provided that

$$\mathbf{P}'_0\mathbf{n} = \mathbf{0} \quad \text{and} \quad \mathbf{P}''_0\mathbf{n} = \mathbf{0}. \tag{60}$$

Then, (51) is seen to be a consequence of the exact theory provided that

$$\mathbf{P}'''_0\mathbf{n} = \mathbf{0}. \tag{61}$$

Together with (45), the differential equations of the order h^3 model are compatible with those of the exact theory if and only if the midsurface values of the through-thickness derivatives $\widehat{\mathbf{P}}^{(n)}\mathbf{n}$ vanish for $n = 0, 1, 2, 3$. The tractions

$$\mathbf{t}_{|\pm h/2} = \mathbf{P}_0\mathbf{n} \pm \frac{h}{2}\mathbf{P}'_0\mathbf{n} + \frac{h^2}{8}\mathbf{P}''_0\mathbf{n} \pm \frac{h^3}{48}\mathbf{P}'''_0\mathbf{n} + O(h^4), \tag{62}$$

at the lateral surfaces of the plate, are thus of order h^4 at most. This in turn is consistent with our expression for the potential energy of an edge-loaded plate. In classical treatments of plate theory [25], and their modern extensions [26], it is customary to impose conditions like (45), (60) or (61) a priori, to ensure that the lateral surfaces are free of traction to the desired order in thickness. Here, we observe that consistency with the exact differential equations holding in the interior is achieved simultaneously.

Eqs. (45), (60) and (61) are necessary conditions for the plate to be in a state of plane stress throughout its thickness. That plane stress is approximately satisfied has been shown rigorously in the work of John [27] on interior estimates for shell equations that apply away from edges. It was subsequently adopted as an a priori assumption by Koiter in his work on shell theory [14,15]. In turn, the latter work has been justified through comparison with the asymptotic behavior of solutions to the three-dimensional equations by Ciarlet and co-workers [12]. However, the imposition of (45), (60) and (61) a posteriori in the equilibrium equations of the order h^3 model implies that the midsurface displacement field is subject to (51) and to the remaining restrictions implied by Eq. (46); namely

$$\mathrm{div}(\mathbf{P}_0\mathbf{1}) = \mathbf{0} \quad \text{and} \quad \mathrm{div}(\mathbf{P}'_0\mathbf{1}) = \mathbf{0}. \tag{63}$$

We have seen that these are necessary conditions for the exact theory of plane stress (see also [16, Arts. 90, 301, 302] and [28]). However, they are too restrictive to be of use in mixed boundary value problems for combined stretching and bending. This is because natural boundary conditions associated with (63) entail specification of the midsurface value of the exact traction distribution together with that of its through-thickness derivative. A principal aim of plate theory is to circumvent this requirement. Instead, its purpose is to facilitate analysis when only limited statistical information about the traction distribution is available; namely, the resultant forces and moments per unit length of an edge. The analyst must expect that this gain will be offset by a loss in the form of greater complexity of the differential equations. This is amply illustrated by the fourth-order Eq. (51) and attendant boundary condition $(55)_1$, which are necessary, but not sufficient, for the simpler exact equations in the presence of the restrictions (45), (60) and (61).

Thus, the imposition of (45), (60) and (61) has the effect of separating the scales, forcing the order h and h^3 equations to be satisfied separately and requiring exact traction data. This is precisely the undesired feature of the approaches based on asymptotic expansions or Gamma convergence mentioned in Section 1. To circumvent it, we incorporate the plane stress condition directly into the expression (36) for the strain energy function and regard the associated Euler equations and edge conditions as furnishing the appropriate model. Thus, our approach generalizes that of Koiter to account for all terms emerging at order h^3. In particular, we do not make further assumptions regarding the relative orders of magnitude of bending and strain gradient terms.

Our view is that such information should emerge from solutions to the model equations and associated boundary conditions, via asymptotic or other methods of analysis.

We note that the expression (36) for the order h^3 strain energy does not involve $\mathbf{P}_0''\mathbf{n}$ or $\mathbf{P}_0'''\mathbf{n}$. Accordingly, we limit ourselves to the constraints

$$\mathbf{P}_0\mathbf{n} = \mathbf{0} \quad \text{and} \quad \mathbf{P}_0'\mathbf{n} = \mathbf{0}. \tag{64}$$

These in turn accommodate the existence of larger lateral forces. Indeed, in the classical theory of plate-bending, the net lateral distributed force scales precisely as h^3. This can be seen from Eqs. (12.4) and (12.5) of [25]. The treatment of the constraints is simplified considerably by the strong ellipticity of the three-dimensional energy. This facilitates their explicit solution in terms of the gradients of the midsurface displacement, obviating the need for Lagrange multipliers. Moreover, the constraints are compatible with data at the lateral surfaces in the three-dimensional parent model; these effectively become part of the domain in the surrogate two-dimensional theory. This contrasts with alternative approaches in which constraints of the Kirchhoff–Love type are imposed a priori at the purely kinematic level, without regard to constitutive equations or boundary data, and without regard to a length scale (here, the thickness h).

The resulting energy per unit area, which we again denote by W, is

$$W = \frac{1}{2}h\mathbf{P}_0\mathbf{1}\cdot\nabla\mathbf{u} + \frac{1}{24}h^3(\mathbf{P}_0'\mathbf{1}\cdot\nabla\bar{\mathbf{a}} + \mathbf{P}_0\mathbf{1}\cdot\nabla\bar{\mathbf{b}}), \tag{65}$$

where $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ are the unique solutions to $(64)_{1,2}$, respectively; i.e.

$$\mathbf{A}_n\bar{\mathbf{a}} = -(\mathbf{C}[\nabla\mathbf{u}])\mathbf{n} \quad \text{and} \quad \mathbf{A}_n\bar{\mathbf{b}} = -(\mathbf{C}[\nabla\bar{\mathbf{a}}])\mathbf{n}. \tag{66}$$

Substituting the solutions into (65), we derive an expression for the energy that depends on the first, second and third gradients of the midsurface displacement field $\mathbf{u}(\mathbf{r})$. The third-order gradient yields a contribution to the Euler equations that involves the third-order spatial derivatives of its coefficient, $\mathbf{P}_0\mathbf{1}$, in the expression for the energy. This is a linear function of the first-order gradient of the displacement field. Accordingly, the Euler equations again reduce to a fourth-order system of partial differential equations. For reasons discussed in Section 2.5, we work with an equivalent expression for the strain energy obtained by integrating (65) over Ω and applying the Green–Stokes theorem to the final term. This furnishes the order h^3 strain energy in the form

$$S = \int_\Omega F(\nabla\mathbf{u}, \nabla^2\mathbf{u})\,\mathrm{d}a + \frac{1}{24}h^3\int_{\partial\Omega}\bar{\mathbf{b}}\cdot\mathbf{P}_0\mathbf{1}\boldsymbol{\nu}\,\mathrm{d}s, \tag{67}$$

where $\nabla^2\mathbf{u}$ is the second gradient (the gradient of the gradient) of $\mathbf{u}(\mathbf{r})$, and

$$F(\nabla\mathbf{u}, \nabla^2\mathbf{u}) = \frac{1}{2}h\mathbf{P}_0\mathbf{1}\cdot\nabla\mathbf{u} - \frac{1}{24}h^3\bar{\mathbf{b}}\cdot\mathrm{div}(\mathbf{P}_0\mathbf{1}) + \frac{1}{24}h^3\mathbf{P}_0'\mathbf{1}\cdot\nabla\bar{\mathbf{a}}. \tag{68}$$

2.4. The constrained theory for isotropic materials

We illustrate the foregoing for isotropic materials. This is facilitated by the formulas (cf. (9))

$$(\mathbf{C}[\mathbf{A}])\mathbf{1} = \lambda(\mathrm{tr}\mathbf{A})\mathbf{1} + 2\mu(\mathrm{Sym}\,\mathbf{A})\mathbf{1} \quad \text{and} \quad (\mathbf{C}[\mathbf{A}])\mathbf{n} = \lambda(\mathrm{tr}\mathbf{A})\mathbf{n} + 2\mu(\mathrm{Sym}\,\mathbf{A})\mathbf{n}. \tag{69}$$

The acoustic tensor $\mathbf{A}_n$ is

$$\mathbf{A}_n = (\lambda + 2\mu)\mathbf{n}\otimes\mathbf{n} + \mu\mathbf{1}, \tag{70}$$

in which the Lamé moduli λ, μ satisfy (10).

It proves convenient to decompose the midsurface displacement in the form

$$\mathbf{u} = \mathbf{v} + w\mathbf{n}, \tag{71}$$

where $\mathbf{v} = \mathbf{1}\mathbf{u}$ is the displacement in the plane of the plate and $w = \mathbf{u}\cdot\mathbf{n}$ is the transverse displacement.

Thus

$$\nabla\mathbf{u} = \nabla\mathbf{v} + \mathbf{n}\otimes\nabla w. \tag{72}$$

We also have

$$(\mathbf{C}[\nabla\mathbf{u}])\mathbf{n} = \lambda\theta\mathbf{n} + \mu\nabla w, \tag{73}$$

where

$$\theta = \operatorname{div}\mathbf{v}, \tag{74}$$

is the two-dimensional dilation. Eq. $(66)_1$ then delivers

$$\bar{\mathbf{a}} = \boldsymbol{\alpha} + a\mathbf{n}, \tag{75}$$

where

$$\boldsymbol{\alpha} = \mathbf{1a} = -\nabla w \quad \text{and} \quad a = \mathbf{a}\cdot\mathbf{n} = -\left(\frac{\lambda}{\lambda+2\mu}\right)\theta. \tag{76}$$

To obtain $\bar{\mathbf{b}}$ we observe, from $(66)_2$, that the relation holding between $\bar{\mathbf{b}}$ and $\nabla\bar{\mathbf{a}}$ is identical to that existing between $\bar{\mathbf{a}}$ and $\nabla\mathbf{u}$. Thus

$$\bar{\mathbf{b}} = \boldsymbol{\beta} + b\mathbf{n}, \tag{77}$$

where

$$\boldsymbol{\beta} = \mathbf{1b} = -\nabla a \quad \text{and} \quad b = \mathbf{b}\cdot\mathbf{n} = -\left(\frac{\lambda}{\lambda+2\mu}\right)\operatorname{div}\boldsymbol{\alpha}. \tag{78}$$

We then combine these results with (65), in which

$$\mathbf{P}_0\mathbf{1} = \lambda(\operatorname{div}\mathbf{v} + a)\mathbf{1} + 2\mu\operatorname{Sym}(\nabla\mathbf{v}) \quad \text{and} \quad \mathbf{P}_0'\mathbf{1} = \lambda(\operatorname{div}\boldsymbol{\alpha} + b)\mathbf{1} + 2\mu\operatorname{Sym}(\nabla\boldsymbol{\alpha}), \tag{79}$$

to derive

$$W = W_s + W_b, \tag{80}$$

where

$$W_s = h\left[\frac{\lambda\mu}{\lambda+2\mu}\theta^2 + \mu|\operatorname{Sym}\nabla\mathbf{v}|^2\right] + \frac{1}{12}h^3\left\{\frac{\lambda^2\mu}{(\lambda+2\mu)^2}\theta(\Delta\theta) + \frac{\lambda\mu}{\lambda+2\mu}\nabla^2\theta\cdot(\operatorname{Sym}\nabla\mathbf{v})\right\}, \tag{81}$$

and

$$W_b = \frac{1}{12}h^3\left[\frac{\lambda\mu}{\lambda+2\mu}(\Delta w)^2 + \mu|\nabla^2 w|^2\right], \tag{82}$$

are the stretching and bending energies, respectively.

Using the two-dimensional Cayley–Hamilton formula

$$\mathbf{M}^2 = (\operatorname{tr}\mathbf{M})\mathbf{M} - (\det\mathbf{M})\mathbf{1}, \tag{83}$$

we reduce these to

$$W_s = \frac{1}{2}h\frac{E}{1-v^2}[\theta^2 - 2(1-v)\det(\operatorname{Sym}\nabla\mathbf{v})] + \frac{1}{24}h^3\frac{Ev}{1-v^2}\nabla^2\theta\cdot(\operatorname{Sym}\nabla\mathbf{v}) + \frac{1}{12}h^3\frac{\lambda^2\mu}{(\lambda+2\mu)^2}\theta(\Delta\theta), \tag{84}$$

and

$$W_b = \frac{1}{2}D[(\Delta w)^2 - 2(1-v)\det(\nabla^2 w)], \tag{85}$$

where

$$D = \frac{1}{12}h^3\frac{4\mu(\lambda+\mu)}{\lambda+2\mu} = \frac{1}{12}h^3\frac{E}{1-v^2}, \tag{86}$$

is the flexural rigidity of the plate, and

$$\kappa = \frac{E}{3(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}, \tag{87}$$

in which E is Young's modulus and ν is Poisson's ratio.

In the exact theory of plane stress for uniform isotropic materials, the equations of equilibrium without body force yield the restriction that θ is a plane harmonic function [16,28]. Accordingly, the final term in (84) vanishes. With this adjustment our expression for W coincides precisely with the order h^3 expansion of the equilibrium strain energy given in Love [16, Arts. 90 and 301]). It is interesting that W_b is also the order h^3 energy associated with bending under generalized plane stress (see [16, Art. 304]).

If we were to require that equilibria $\mathbf{u}(\mathbf{r})$ minimize the order h^3 truncation of the potential energy, then the operative Legendre–Hadamard inequality would imply that $f(t)$ is convex at $t=0$ [29], where $f(t) = W(\nabla\mathbf{u}, \nabla^2\mathbf{u}, \mathbf{u}_{,\alpha\beta\gamma} + t\mathbf{a}b_\alpha b_\beta b_\gamma)$ and W is the strain energy density expressed in terms of the displacement gradients. Here $\mathbf{a}$ is an arbitrary three-vector and b_α are arbitrary scalars. From the foregoing formulas for W it is easy to see that the dependence on $\nabla^3\mathbf{u}$ is linear and thus that $f(t)$ is linear. Accordingly, the Legendre–Hadamard inequality is satisfied identically, as an equality. However, it is known that in the exact linear theory for plane stress without bending, the thickness expansion of the strain energy per unit area terminates at order h^5 [16, p. 469]. The coefficient of h^5 is a homogeneous, quadratic function of the third-order derivatives of the in-plane midsurface displacement field. Because of this term, the equilibrium energy is readily seen to satisfy the Legendre–Hadamard inequality, provided – as is typically assumed – that the three-dimensional strain energy is a positive-definite (hence convex) function of the (infinitesimal) strain. This state of affairs leads us to conclude that it is not appropriate to require a given truncation of the energy (here, order h^3) to satisfy the Legendre–Hadamard condition. For, if it does not, then the operative version of the inequality may well be satisfied by the neglected terms. In other words, as argued in Section 2.1, truncated expansions of the energy in which different orders of h are retained should not be associated with minimization problems in their own right. Accordingly, it is to be expected that such expansions yield meaningful stationary principles but that they do not in general furnish minimum principles. In classical plate theory [25] and its non-linear extensions [14], this issue is side-stepped without comment simply by suppressing the higher-order in-plane displacement gradients. In [26] it is addressed by regularizing the order h^3 expansion of the energy density through the addition of ad hoc strain gradient terms. Here, we forego such regularizations as we are interested in establishing an order h^3 model having a direct connection to three-dimensional elasticity.

For isotropic materials, (68) reduces to

$$F = h\left[\frac{\lambda\mu}{\lambda+2\mu}\theta^2 + \mu|\mathrm{Sym}\,\nabla\mathbf{v}|^2\right] - \frac{1}{8}h^3\frac{\lambda\mu\kappa}{(\lambda+2\mu)^2}|\nabla\theta|^2 - \frac{1}{24}h^3\frac{\lambda\mu}{\lambda+2\mu}\Delta\mathbf{v}\cdot\nabla\theta + W_b, \tag{88}$$

where W_b is given by (84) or (85). The use of (68) in place of (65) thus affects the stretching term only. The relevant Legendre–Hadamard inequality, which is violated, is the requirement that $g(t) = F(\nabla\mathbf{u}, \mathbf{u}_{,\alpha\beta} + t\mathbf{a}b_\alpha b_\beta)$ be convex at $t=0$ [29].

2.5. Euler equations and edge conditions in the constrained theory

To obtain edge conditions, suppose three-dimensional position is assigned on $\partial R_{C_1} = \partial\Omega_1 \times C$. The continuous extension to $\partial\Omega_1$ of the field $\bar{\mathbf{a}}$ derived from $(66)_1$ is controlled by the boundary values of $\mathbf{u}$ and $\mathbf{u}_\nu$ (cf. (53)). In place of (37), we stipulate that the values of $\mathbf{b}$ on $\partial\Omega_1$ agree with the continuous extension of the solution $\bar{\mathbf{b}}$ to $(66)_2$. We then consider $\mathbf{a}$ and $\mathbf{b}$ to be assigned accordingly and require their variations to vanish on $\partial\Omega_1$. If traction is assigned on $\partial R_{C_2} = \partial\Omega_2 \times C$, then, in principle, the variations $\dot{\mathbf{u}}, \dot{\mathbf{a}}$ and $\dot{\mathbf{b}}$ are arbitrary on $\partial\Omega_2$. However, as $\bar{\mathbf{a}}$ is determined by the edge values of $\mathbf{u}_s$ and $\mathbf{u}_\nu$, the variation of $\bar{\mathbf{a}}$ is controlled by those of $\mathbf{u}$ and $\mathbf{u}_\nu$. This in turn implies that natural boundary conditions involving $\mathbf{p}_u$ and $\mathbf{p}_a$, to be derived later, must be satisfied (see (100) and (118)). By the same reasoning, the value on $\partial\Omega_2$ of $\bar{\mathbf{b}}$ is controlled by $\mathbf{u}$ and its normal derivatives through the second-order. These are not assigned, and so a natural boundary condition emerges that involves $\mathbf{p}_b$. However, as in the model without constraints, the differential equation holding in the interior is of the fourth-order and the specification of three vector conditions on an edge will in general overspecify the problem. To correct for this we take the view that the boundary value of $\mathbf{b}$ is fixed a posteriori by the continuous extension to $\partial\Omega_2$ of the function $\bar{\mathbf{b}}$ delivered by the equilibrium deformation. Accordingly, its variation vanishes and no third boundary condition emerges. Further, if $\partial\Omega_2$ is dead-loaded, then the variation of the boundary term in (67) reduces to the integral over $\partial\Omega_1$ of the variation $\frac{1}{24}h^3\bar{\mathbf{b}}\cdot\dot{\mathbf{P}}_0\mathbf{1}\boldsymbol{\nu}$, plus an integral over $\partial\Omega_2$ that cancels its counterpart in (37). This follows by imposing the exact condition $\mathbf{t}_0 = \mathbf{P}_0\mathbf{1}\boldsymbol{\nu}$, together with $(38)_3$. Now, Eqs. (53) and (66) imply that the values of $\dot{\mathbf{P}}_0\mathbf{1}\boldsymbol{\nu}$ on $\partial\Omega_1$ are linearly related to the (vanishing) values thereon of $\dot{\mathbf{u}}_s$ and $\dot{\mathbf{u}}_\nu$. Accordingly, the order h^3 expansion of the virtual-work equation becomes

$$\int_\Omega \dot{F}\,\mathrm{d}a = \int_{\partial\Omega_2}(\mathbf{p}_u\cdot\dot{\mathbf{u}} + \mathbf{p}_a\cdot\dot{\mathbf{a}})\mathrm{d}s. \tag{89}$$

The development of the Euler equations and boundary conditions is straightforward but involved. This is facilitated by the use of Cartesian tensor notation. Thus

$$\dot{F} = (\partial F/\partial u_{i,\alpha})\dot{u}_{i,\alpha} + (\partial F/\partial u_{i,\alpha\beta})\dot{u}_{i,\alpha\beta}, \tag{90}$$

where $u_i = \mathbf{u}\cdot\mathbf{e}_i$, $\{\mathbf{e}_i\}$ is a fixed orthonormal basis, and subscripts preceded by commas are used to denote partial derivatives with respect to the plane Cartesian coordinates $r_\alpha = \mathbf{r}\cdot\mathbf{e}_\alpha$ on Ω. Latin indices range over {1, 2, 3} and Greek over {1, 2}. Thus, $u_\alpha = v_\alpha$ and $u_3 = w$. We recast this as

$$\dot{F} = \dot{u}_i[(\partial F/\partial u_{i,\alpha\beta})_{,\alpha\beta} - (\partial F/\partial u_{i,\alpha})_{,\alpha}] + \varphi_{\alpha,\alpha}, \tag{91}$$

where

$$\varphi_\alpha = \dot{u}_i[\partial F/\partial u_{i,\alpha} - (\partial F/\partial u_{i,\alpha\beta})_{,\beta}] + (\partial F/\partial u_{i,\alpha\beta})\dot{u}_{i,\beta}. \tag{92}$$

The contribution of the divergence $\varphi_{\alpha,\alpha}$ to (89) is equivalent to that of the flux $\varphi_\alpha\nu_\alpha$, integrated over the edge. There, we use (53) to obtain

$$\varphi_\alpha\nu_\alpha = T_{i\alpha}\nu_\alpha\dot{u}_i + M_{i\alpha\beta}\nu_\alpha\nu_\beta\dot{u}_{i\nu} + M_{i\alpha\beta}\nu_\alpha\tau_\beta\dot{u}_{is}, \tag{93}$$

where

$$T_{i\alpha} = N_{i\alpha} - M_{i\beta\alpha,\beta}, \quad \text{with } N_{i\alpha} = \partial F/\partial u_{i,\alpha} \quad \text{and} \quad M_{i\alpha\beta} = \partial F/\partial u_{i,\alpha\beta}. \tag{94}$$

The third term in (93), involving the arclength derivative of the virtual displacement, may be re-written as

$$M_{i\alpha\beta}\nu_\alpha\tau_\beta\dot{u}_{is} = (M_{i\alpha\beta}\nu_\alpha\tau_\beta\dot{u}_i)_s - (M_{i\alpha\beta}\nu_\alpha\tau_\beta)_s\dot{u}_i, \tag{95}$$

in which the net contribution to the boundary work of the first term on the right vanishes if $\partial\Omega$ is smooth. This term generates corner forces if the edge is only piecewise smooth, with a finite number of jumps in the tangent $\boldsymbol{\tau}$ to the edge. Corner forces do not arise in the unconstrained theory of Section 2.2. The handling of piecewise smooth edges is entirely straightforward but omitted here for the sake of brevity. For smooth edges, the effective part of the flux $\varphi_\alpha\nu_\alpha$ may thus be reduced to

$$[T_{i\alpha}\nu_\alpha - (M_{i\alpha\beta}\nu_\alpha\tau_\beta)_s]\dot{u}_i + M_{i\alpha\beta}\nu_\alpha\nu_\beta\dot{u}_{i\nu}. \tag{96}$$

In the same way, we use (53) in (89) and compute the variation of $\bar{\mathbf{a}}$ in terms of the variations of the normal and tangential displacement gradients $\mathbf{u}_\nu$ and $\mathbf{u}_s$. For *smooth* edges, the net boundary work in (89) may then be written as

$$\int_{\partial\Omega}(\mathbf{p}_u\cdot\dot{\mathbf{u}} + \mathbf{p}_a\cdot\dot{\mathbf{a}})\mathrm{d}s = \int_{\partial\Omega}(\mathbf{f}\cdot\dot{\mathbf{u}} + \mathbf{m}\cdot\dot{\mathbf{u}}_\nu)\mathrm{d}s, \tag{97}$$

for certain functions $\mathbf{f}$ and $\mathbf{m}$, to be discussed. Collecting these expressions, we reduce (89) to

$$\int_{\partial\Omega}\varphi_\alpha\nu_\alpha\,\mathrm{d}s - \int_\Omega \dot{u}_i T_{i\alpha,\alpha}\,\mathrm{d}a = \int_{\partial\Omega}(f_i\dot{u}_i + m_i\dot{u}_{i\nu})\mathrm{d}s, \tag{98}$$

from which the Euler equation

$$T_{i\alpha,\alpha} = 0 \text{ in } \Omega, \tag{99}$$

follows immediately. On a clamped edge, denoted by $\partial\Omega_1$, the midsurface displacement $\mathbf{u}$ and its normal derivative $\mathbf{u}_\nu$ are assigned. Eq. (98) reduces to an identity and no further conditions apply. On a part of the boundary where traction is assigned, the variations of $\mathbf{u}$ and $\mathbf{u}_\nu$ are arbitrary. The balance (98) then furnishes

$$T_{i\alpha}\nu_\alpha - (M_{i\alpha\beta}\nu_\alpha\tau_\beta)_s = f_i \quad \text{and} \quad M_{i\alpha\beta}\nu_\alpha\nu_\beta = m_i \text{ on } \partial\Omega_2. \tag{100}$$

To render the various equations explicit for isotropic materials, we compare the variation

$$\dot{F} = N_{i\alpha}\dot{u}_{i,\alpha} + M_{i\alpha\beta}\dot{u}_{i,\alpha\beta}, \tag{101}$$

to the similar expression for $\dot{F}$ obtained directly from (88). The comparison yields the required formulas for $N_{i\alpha}$ and $M_{i\alpha\beta}$, where $M_{i\alpha\beta}$ is symmetric in the second pair of subscripts (cf. $(94)_3$). Accordingly, the procedure used to generate $M_{i\alpha\beta}$ from a specific strain energy function must take this symmetry into account.

In view of the structure of (88), it is natural to separate the contributions of $\mathbf{v}$ and w. For example, the variation holding fixed all variables other than $w_{,\alpha}$ furnishes $(\partial F/\partial w_{,\alpha})\dot{w}_{,\alpha} = 0$, implying that $\partial F/\partial w_{,\alpha}$ vanishes; and hence, from $(94)_2$, that

$$N_{3\alpha} = 0. \tag{102}$$

In the same way, we find

$$(\partial F/\partial w_{,\alpha\beta})\dot{w}_{,\alpha\beta} = D[(\Delta w)\delta_{\alpha\beta} - (1-\nu)k_{\alpha\beta}]\dot{w}_{,\alpha\beta}, \tag{103}$$

where

$$\mathbf{k} = (\Delta w)\mathbf{1} - \nabla^2 w, \tag{104}$$

is the cofactor of $\nabla^2 w$. The term in square brackets possesses the required symmetry, and so $(94)_3$ furnishes

$$M_{3\alpha\beta} = D[(\Delta w)\delta_{\alpha\beta} - (1-\nu)k_{\alpha\beta}]. \tag{105}$$

Next, we use (88) to derive

$$(\partial F/\partial v_{\alpha,\beta})\dot{v}_{\alpha,\beta} = 2h\left(\frac{\lambda\mu}{\lambda+2\mu}v_{\lambda,\lambda}\delta_{\alpha\beta} + \mu\epsilon_{\alpha\beta}\right)\dot{v}_{\alpha,\beta}, \tag{106}$$

where $2\epsilon_{\alpha\beta} = v_{\alpha,\beta} + v_{\beta,\alpha}$. It follows from the symmetry of the parenthetical term on the right-hand side that the skew part of $\partial F/\partial v_{\alpha,\beta}$ vanishes, and hence from $(94)_2$ that

$$N_{\alpha\beta} = h\left[\frac{2\lambda\mu}{\lambda+2\mu}v_{\lambda,\lambda}\delta_{\alpha\beta} + \mu(v_{\alpha,\beta}+v_{\beta,\alpha})\right]. \tag{107}$$

Finally, (88) yields

$$(\partial F/\partial v_{\lambda,\alpha\beta})\dot{v}_{\lambda,\alpha\beta} = -\frac{1}{4}h^3\frac{\lambda\mu\kappa}{(\lambda+2\mu)^2}\theta_{,\beta}\dot{\theta}_{,\beta} - \frac{1}{24}h^3\frac{\lambda\mu}{\lambda+2\mu}(\dot{v}_{\lambda,\beta\beta}v_{\alpha,\alpha\lambda} + v_{\lambda,\beta\beta}\dot{v}_{\alpha,\alpha\lambda}). \tag{108}$$

We write the right-hand side as a linear function of $\dot{v}_{\lambda,\alpha\beta}$, retaining coefficients that possess the requisite symmetry with respect to α and β. The skew coefficients, while non-zero, are irrelevant. Comparison of the result to $(94)_3$ then furnishes

$$M_{\lambda\alpha\beta} = -\frac{1}{8}h^3\frac{\lambda\mu\kappa}{(\lambda+2\mu)^2}(\theta_{,\alpha}\delta_{\beta\lambda} + \theta_{,\beta}\delta_{\alpha\lambda}) - \frac{1}{24}h^3\frac{\lambda\mu}{\lambda+2\mu}\left[\delta_{\alpha\beta}\theta_{,\lambda} + \frac{1}{2}(\Delta v_\alpha\delta_{\beta\lambda} + \Delta v_\beta\delta_{\alpha\lambda})\right]. \tag{109}$$

We are now in a position to record the Euler Eq. (99). For $i = 3$ the conventional plate-bending equation

$$D\Delta(\Delta w) = 0, \tag{110}$$

follows from (99), (94) and (105), and from the vanishing of the divergence of $\mathbf{k}$, which is an immediate consequence of (104) (see also [25]). This equation has been justified by the method of Gamma convergence in the case of bending without stretching [30]. The decoupling of transverse and in-plane displacements occurring in linear plate theory (when the midsurface is a plane of symmetry of the material properties) allows the method

Gamma convergence to be applied, with the same result, in the presence of non-trivial in-plane displacement fields [21].

The three-dimensional stress in the presence of pure bending ($\mathbf{v}$ vanishing identically) is predicted to be

$$\mathbf{P}(\mathbf{x}) = \varsigma\mathbf{P}_0' + O(\varsigma^2), \tag{111}$$

where $\mathbf{P}_0'$ satisfies $\mathbf{P}_0'\mathbf{n} = \mathbf{0}$. Accordingly, the traction at the lateral surfaces vanishes with an error of order h^2.

For $i = 1, 2$ the reduction of (99), which is somewhat more involved, furnishes

$$\frac{1}{3}h^3\frac{\lambda\mu(\lambda+\mu)}{(\lambda+2\mu)^2}\nabla(\Delta\theta) + h\left(\frac{3\kappa\mu}{\lambda+2\mu}\nabla\theta + \mu\Delta\mathbf{v}\right) = \mathbf{0}. \tag{112}$$

Forming the divergence, we obtain

$$\frac{4\lambda\mu(\lambda+\mu)}{\lambda+2\mu}\Delta\left[\frac{1}{12}h^2\left(\frac{\lambda}{\lambda+2\mu}\right)\Delta\theta + \theta\right] = 0, \tag{113}$$

which belongs to a class of equations whose general solution is known (see [31,32]). Eq. (112) may then be treated as a decoupled system of Poisson equations for the components of $\mathbf{v}$ in which the 'forcing' term is a gradient. The integrability condition for this system, namely

$$\Delta\omega = 0, \tag{114}$$

where $2\omega = v_{1,2} - v_{2,1}$ is the in-plane material rotation, arises in the theories of plane stress and generalized plane stress [28]. The same theories require that θ also be harmonic. This furnishes a solution to (113), but deviations, associated with a singular perturbation, are also possible. It is to be expected that deviations from harmonicity will be localized in boundary layers adjoining the edges, necessary to accommodate general force or moment data.

To derive explicit boundary conditions, Eqs. (75) and (76) are used to obtain the variation

$$(\bar{\mathbf{a}})^{\cdot} = -\nabla\dot{w} - \frac{\lambda}{\lambda+2\mu}\dot{\theta}\mathbf{n}, \tag{115}$$

where

$$\nabla\dot{w} = \boldsymbol{\tau}\dot{w}_s + \boldsymbol{\nu}\dot{w}_\nu \quad \text{and} \quad \dot{\theta} = \boldsymbol{\tau}\cdot\dot{\mathbf{v}}_s + \boldsymbol{\nu}\cdot\dot{\mathbf{v}}_\nu. \tag{116}$$

Forming the inner product with $\mathbf{p}_a$ and using a decomposition similar to (95) for the arclength derivatives, we find, for smooth edges, that the net contribution of $\mathbf{p}_a$ to the boundary work is given by the integral over $\partial\Omega$ of

$$\left\{(\boldsymbol{\tau}\cdot\mathbf{p}_a)_s\mathbf{n} + \frac{\lambda}{\lambda+2\mu}[(\mathbf{n}\cdot\mathbf{p}_a)\boldsymbol{\tau}]_s\right\}\cdot\dot{\mathbf{u}} - \left[(\boldsymbol{\nu}\cdot\mathbf{p}_a)\mathbf{n} + \frac{\lambda}{\lambda+2\mu}(\mathbf{n}\cdot\mathbf{p}_a)\boldsymbol{\nu}\right]\cdot\dot{\mathbf{u}}_\nu. \tag{117}$$

Thus

$$\mathbf{f} = \mathbf{p}_u + (\boldsymbol{\tau}\cdot\mathbf{p}_a)_s\mathbf{n} + \frac{\lambda}{\lambda+2\mu}[(\mathbf{n}\cdot\mathbf{p}_a)\boldsymbol{\tau}]_s \quad \text{and} \quad \mathbf{m} = -(\boldsymbol{\nu}\cdot\mathbf{p}_a)\mathbf{n} - \frac{\lambda}{\lambda+2\mu}(\mathbf{n}\cdot\mathbf{p}_a)\boldsymbol{\nu}. \tag{118}$$

For $i = 3$ the edge conditions $(100)_{1,2}$ reduce to

$$D[\boldsymbol{\nu}\cdot\nabla(\Delta w) + (1-\nu)(w_{,\alpha\beta}\nu_\alpha\tau_\beta)_s] = -\mathbf{n}\cdot\mathbf{f} \quad \text{and} \quad D[\nu\Delta w + (1-\nu)w_{,\alpha\beta}\nu_\alpha\nu_\beta] = \mathbf{n}\cdot\mathbf{m}, \tag{119}$$

respectively. The remaining boundary conditions involve the derivatives of $\mathbf{v}$. Their explicit forms, which are too lengthy to be recorded here, are derived similarly.

Given its place in the expression for the boundary work, we interpret $\mathbf{f}$ as the force per unit length of $\partial\Omega$. If the exact traction distribution on a cylindrical generating surface is independent of the through-thickness coordinate ς, then the prescription of the force is equivalent to that of the midsurface value of the three-dimensional traction; and thus, by (14), to the prescription of $\mathbf{P}_0\mathbf{1}\boldsymbol{\nu}$, which lies in the plane of the plate. This furnishes data for the leading-order (membrane) term in (113), and so we expect the higher-order terms to be negligibly small in regions adjoining the boundary provided that the edge traction is parallel to the plate (see [16, Art. 301]).

It is well-known that classical plate theory is compatible with the existence of pure bending moments at an edge. This corresponds to the conditions $m_\alpha = 0$. The present model replaces these with the less restrictive requirement that m_α be proportional to the components of the edge normal v_α.

3. Shell theory

The derivation of shell theory proceeds along similar lines. Accordingly, we omit much of the detail and concentrate on the adjustments to the foregoing development needed to accommodate the differential geometry of a shell.

3.1. Kinematics in shell space

We seek equations of equilibrium for the shell involving as independent variables the coordinates θ^α that parametrize a curved base surface Ω. To this end we use the standard normal-coordinate parametrization of three-dimensional space in the vicinity of the base surface [6–8]. Thus

$$\mathbf{x}(\theta^\alpha, \varsigma) = \mathbf{r}(\theta^\alpha) + \varsigma\mathbf{n}(\theta^\alpha), \tag{120}$$

where $\mathbf{r}(\theta^\alpha)$ is the parametrization of Ω with unit normal field $\mathbf{n}(\theta^\alpha)$ and ς is the coordinate in the direction perpendicular to Ω, the latter corresponding to $\varsigma = 0$. This generalizes (18), to which it reduces if the base surface is flat. The lateral surfaces of the thin three-dimensional body are assumed to correspond to constant values of ς. These are separated by the distance h, the thickness of the shell. For simplicity we again assume the thickness to be uniform. In all cases h is assumed to be small against any other length scale at hand, such as a spanwise dimension or a local radius of curvature. As in the discussion of plates, we develop the model for the classical example in which the base surface coincides with the midsurface.

The orientation of Ω is induced by the assumed right-handedness of the coordinate system $(\theta^\alpha, \varsigma)$; thus, $\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{n} > 0$, where $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} \equiv \partial\mathbf{r}/\partial\theta^\alpha$ span the tangent plane $T_{\Omega(p)}$ to Ω at the point p with coordinates θ^α. The curvature $\boldsymbol{\kappa}$ of the base surface is the symmetric linear map from $T_{\Omega(p)}$ to itself defined by the Weingarten equation

$$\mathrm{d}\mathbf{n} = -\boldsymbol{\kappa}\,\mathrm{d}\mathbf{r}, \tag{121}$$

where $\mathrm{d}\mathbf{r} = \mathbf{a}_\alpha\,\mathrm{d}\theta^\alpha$ and $\mathrm{d}\mathbf{n} = \mathbf{n}_{,\alpha}\,\mathrm{d}\theta^\alpha$. Accordingly

$$\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{r} + \varsigma\,\mathrm{d}\mathbf{n} + \mathbf{n}\,\mathrm{d}\varsigma = \mathbf{G}(\mathrm{d}\mathbf{r} + \mathbf{n}\,\mathrm{d}\varsigma), \tag{122}$$

where

$$\mathbf{G} = \boldsymbol{\mu} + \mathbf{n}\otimes\mathbf{n}, \quad \boldsymbol{\mu} = \mathbf{1} - \varsigma\boldsymbol{\kappa}, \tag{123}$$

and

$$\mathbf{1} = \mathbf{I} - \mathbf{n}\otimes\mathbf{n}, \tag{124}$$

is the (two-dimensional) identity on $T_{\Omega(p)}$. The latter may be used to derive an orthogonal decomposition of tensors into tangential and normal parts analogous to (22).

The differential volume induced by the coordinates is

$$\mathrm{d}v = [\mathrm{d}\mathbf{x}_1, \mathrm{d}\mathbf{x}_2, \mathrm{d}\mathbf{x}_3] = (\det\mathbf{G})[\mathrm{d}\mathbf{r}_1, \mathrm{d}\mathbf{r}_2, \mathbf{n}\,\mathrm{d}\varsigma], \tag{125}$$

where $\mathrm{d}\mathbf{x}_\alpha = \mathbf{G}\,\mathrm{d}\mathbf{r}_\alpha$ and $\mathrm{d}\mathbf{x}_3 = \mathbf{G}\mathbf{n}\,\mathrm{d}\varsigma$, with $\mathrm{d}\mathbf{r}_1 = \mathbf{a}_1\,\mathrm{d}\theta^1$ and $\mathrm{d}\mathbf{r}_2 = \mathbf{a}_2\,\mathrm{d}\theta^2$, and $[\cdot, \cdot, \cdot]$ is the scalar triple product. Thus

$$\mathrm{d}v = \mu\,\mathrm{d}\varsigma\,\mathrm{d}a, \tag{126}$$

where $\mathrm{d}a = \mathbf{n}\cdot\mathrm{d}\mathbf{r}_1 \times \mathrm{d}\mathbf{r}_2$ is the differential area on Ω, and

$$\mu = 1 - 2H\varsigma + \varsigma^2 K, \tag{127}$$

is the (two-dimensional) determinant of $\boldsymbol{\mu}$ in which

$$H = \frac{1}{2}\operatorname{tr}\boldsymbol{\kappa} \quad \text{and} \quad K = \det\boldsymbol{\kappa}, \tag{128}$$

are the mean and Gaussian curvatures of Ω, respectively. This may be written

$$\mu = (1 - \varsigma\kappa_1)(1 - \varsigma\kappa_2), \tag{129}$$

where κ_α – the *principal curvatures* – are the eigenvalues of $\boldsymbol{\kappa}$. The transformation from $(\theta^\alpha, \varsigma)$ to $\mathbf{x}$ is one-to-one and orientation preserving if and only if $\mu > 0$. This property obtains in the region of space containing the base surface in which

$$|\varsigma| < \min\{r_1, r_2\}, \tag{130}$$

where $r_\alpha = |\kappa_\alpha|^{-1}$ are the principal radii of curvature [6–8].

Let C^* be the line orthogonal to Ω and intersecting R at a point with surface coordinates θ^α. Let $\partial R_C = \partial\Omega \times C$, where C is the collection of such curves, be the generating surface of the thin shell-like region R obtained by translating the points of $\partial\Omega$ along their associated lines C^*. Let s measure arclength on the curve $\partial\Omega$ with unit tangent $\boldsymbol{\tau}$ and rightward unit normal $\boldsymbol{\nu} = \boldsymbol{\tau} \times \mathbf{n}$. The oriented differential surface area induced by the (s, ς)-parametrization of ∂R_C is $\mathbf{N}\,da = d\mathbf{x}_\tau \times d\mathbf{x}_n$, where $d\mathbf{x}_n = \mathbf{n}\,d\varsigma$ and $d\mathbf{x}_\tau = \boldsymbol{\mu}(d\mathbf{r})$, with $d\mathbf{r} = \boldsymbol{\tau}\,ds$. These are obtained from (122) by setting $ds = 0$ and $d\varsigma = 0$, respectively. Thus

$$\mathbf{N}\,da = \mathbf{G}\boldsymbol{\tau}\,ds \times \mathbf{G}\mathbf{n}\,d\varsigma = \mathbf{G}^*\boldsymbol{\nu}\,ds\,d\varsigma, \tag{131}$$

where $\mathbf{G}^*$ is the cofactor of $\mathbf{G}$. From $(123)_1$ it follows easily that $\mathbf{G}^* = \mu(\boldsymbol{\mu}^{-1} + \mathbf{n} \otimes \mathbf{n})$ and

$$\mathbf{N}\,da = \boldsymbol{\varphi}\boldsymbol{\nu}\,d\varsigma\,ds; \quad da = |\boldsymbol{\varphi}\boldsymbol{\nu}|\,d\varsigma\,ds, \quad \text{where } \boldsymbol{\varphi} = \boldsymbol{\mu}^*, \tag{132}$$

and $\boldsymbol{\mu}^* = \mu\boldsymbol{\mu}^{-1}$ is the (two-dimensional) cofactor of $\boldsymbol{\mu}$. The Cayley–Hamilton formula

$$\boldsymbol{\kappa}^2 = 2H\boldsymbol{\kappa} - K\mathbf{1}, \tag{133}$$

may be used to confirm [7] that

$$\boldsymbol{\varphi} = \mathbf{1} + \varsigma(\boldsymbol{\kappa} - 2H\mathbf{1}). \tag{134}$$

The representation

$$\boldsymbol{\kappa} = \kappa_\nu \boldsymbol{\nu} \otimes \boldsymbol{\nu} + \kappa_\tau \boldsymbol{\tau} \otimes \boldsymbol{\tau} + \tau(\boldsymbol{\nu} \otimes \boldsymbol{\tau} + \boldsymbol{\tau} \otimes \boldsymbol{\nu}), \tag{135}$$

where κ_ν, κ_τ and τ are the normal curvatures and twist of Ω on the $(\boldsymbol{\nu}, \boldsymbol{\tau})$-axes, then yields $2H = \kappa_\nu + \kappa_\tau$ and

$$\boldsymbol{\varphi}\boldsymbol{\nu} = (1 - \varsigma\kappa_\tau)\boldsymbol{\nu} + (\varsigma\tau)\boldsymbol{\tau}. \tag{136}$$

In Section 3.2 we encounter the integral

$$\int_R U\,dv = \int_\Omega \int_C G\,d\varsigma\,da; \quad G = \mu U. \tag{137}$$

We again require that the integrand G fulfill the conditions of Fubini's theorem. Relevant to our development are the values on the base surface of the ς-derivatives of G through the second order. The derivatives are denoted by primes and their values on the base surface are indicated by appending the subscript $(_0)$. Thus

$$G_0 = U_0, \quad G_0' = U_0' - 2HU_0 \quad \text{and} \quad G_0'' = U_0'' - 4HU_0' + 2KU_0. \tag{138}$$

Integrals of the type

$$\int_{\partial R_C} J\,da = \int_{\partial\Omega} \int_C I\,d\varsigma\,ds; \quad I = |\boldsymbol{\varphi}\boldsymbol{\nu}|J, \tag{139}$$

also play a role, where again the conditions of Fubini's theorem are assumed to hold. We will need the values on $\partial\Omega$ of the ς-derivatives of I through the second-order. To this end we form

$$|\boldsymbol{\varphi}\boldsymbol{\nu}|^2 = (1 - \varsigma\kappa_\tau)^2 + \varsigma^2\tau^2, \tag{140}$$

differentiate with respect to ς, and evaluate the results at $\varsigma = 0$ to obtain

$$|\boldsymbol{\varphi}\boldsymbol{v}|_0 = 1, \quad |\boldsymbol{\varphi}\boldsymbol{v}|_0' = -\kappa_\tau, \quad |\boldsymbol{\varphi}\boldsymbol{v}|_0'' = \tau^2, \tag{141}$$

and

$$I_0 = J_0, \quad I_0' = J_0' - \kappa_\tau J_0, \quad I_0'' = J_0'' - 2\kappa_\tau J_0' + \tau^2 J_0. \tag{142}$$

The model requires expressions for the three-dimensional displacement gradient and its through-thickness derivatives. If $\tilde{\mathbf{u}}(\mathbf{x})$ is the displacement with gradient $\widetilde{\mathbf{H}}(\mathbf{x}) = D\tilde{\mathbf{u}}$, then $d\tilde{\mathbf{u}} = \widetilde{\mathbf{H}}\,d\mathbf{x}$. We define

$$\hat{\mathbf{u}}(\theta^\alpha, \varsigma) = \tilde{\mathbf{u}}(\mathbf{r}(\theta^\alpha) + \varsigma\mathbf{n}(\theta^\alpha)) \quad \text{and} \quad \widehat{\mathbf{H}}(\theta^\alpha, \varsigma) = \widetilde{\mathbf{H}}(\mathbf{r}(\theta^\alpha) + \varsigma\mathbf{n}(\theta^\alpha)). \tag{143}$$

Then

$$\widehat{\mathbf{H}}(\boldsymbol{\mu}\,d\mathbf{r} + \mathbf{n}\,d\varsigma) = d\hat{\mathbf{u}} = \hat{\mathbf{u}}_{,\alpha}\,d\theta^\alpha + \hat{\mathbf{u}}'\,d\varsigma = (\nabla\hat{\mathbf{u}})d\mathbf{r} + \hat{\mathbf{u}}'\,d\varsigma, \tag{144}$$

where $\hat{\mathbf{u}}_{,\alpha} = \partial\hat{\mathbf{u}}/\partial\theta^\alpha, \hat{\mathbf{u}}' = \partial\hat{\mathbf{u}}/\partial\varsigma$ and

$$\nabla\hat{\mathbf{u}} = \hat{\mathbf{u}}_{,\alpha} \otimes \mathbf{a}^\alpha, \tag{145}$$

is the surface displacement gradient. Here $\{\mathbf{a}^\alpha\}$ is dual to $\{\mathbf{a}_\alpha\}$ on $T_{\Omega(p)}$ and use has been made of $d\theta^\alpha = \mathbf{a}^\alpha \cdot d\mathbf{r}$. Thus

$$\widehat{\mathbf{H}}\mathbf{1}\boldsymbol{\mu} = \nabla\hat{\mathbf{u}} \quad \text{and} \quad \widehat{\mathbf{H}}\mathbf{n} = \hat{\mathbf{u}}', \tag{146}$$

where the decomposition $\widehat{\mathbf{H}} = \widehat{\mathbf{H}}\mathbf{1} + \widehat{\mathbf{H}}\mathbf{n} \otimes \mathbf{n}$ has been used. It follows that

$$\widehat{\mathbf{H}} = (\nabla\hat{\mathbf{u}})\boldsymbol{\mu}^{-1} + \hat{\mathbf{u}}' \otimes \mathbf{n}, \tag{147}$$

yielding

$$\begin{aligned} &\widehat{\mathbf{H}}' = (\nabla\hat{\mathbf{u}}')\boldsymbol{\mu}^{-1} + (\nabla\hat{\mathbf{u}})(\boldsymbol{\mu}^{-1})' + \hat{\mathbf{u}}'' \otimes \mathbf{n} \quad \text{and} \\ &\widehat{\mathbf{H}}'' = (\nabla\hat{\mathbf{u}}'')\boldsymbol{\mu}^{-1} + 2(\nabla\hat{\mathbf{u}}')(\boldsymbol{\mu}^{-1})' + (\nabla\hat{\mathbf{u}})(\boldsymbol{\mu}^{-1})'' + \hat{\mathbf{u}}''' \otimes \mathbf{n}. \end{aligned} \tag{148}$$

To find the associated base surface values we restrict ς in accordance with (130) and differentiate $\boldsymbol{\mu}\boldsymbol{\mu}^{-1} = \mathbf{1}$, which is independent of ς, to obtain

$$(\boldsymbol{\mu}^{-1})' = -\boldsymbol{\mu}^{-1}\boldsymbol{\mu}'\boldsymbol{\mu}^{-1} \quad \text{and} \quad (\boldsymbol{\mu}^{-1})'' = \boldsymbol{\mu}^{-1}(2\boldsymbol{\mu}'\boldsymbol{\mu}^{-1}\boldsymbol{\mu}' - \boldsymbol{\mu}'')\boldsymbol{\mu}^{-1}, \tag{149}$$

where

$$\boldsymbol{\mu}' = -\boldsymbol{\kappa} \quad \text{and} \quad \boldsymbol{\mu}'' = \mathbf{0}. \tag{150}$$

Accordingly

$$\boldsymbol{\mu}_0^{-1} = \mathbf{1}, \ (\boldsymbol{\mu}^{-1})_0' = \boldsymbol{\kappa} \quad \text{and} \quad (\boldsymbol{\mu}^{-1})_0'' = 2\boldsymbol{\kappa}^2. \tag{151}$$

Finally

$$\widehat{\mathbf{H}}_0 = \nabla\mathbf{u} + \mathbf{a} \otimes \mathbf{n}, \ \widehat{\mathbf{H}}_0' = \nabla\mathbf{a} + (\nabla\mathbf{u})\boldsymbol{\kappa} + \mathbf{b} \otimes \mathbf{n}, \tag{152}$$

and

$$\widehat{\mathbf{H}}_0'' = \nabla\mathbf{b} + 2(\nabla\mathbf{a})\boldsymbol{\kappa} + 2(\nabla\mathbf{u})\boldsymbol{\kappa}^2 + \mathbf{c} \otimes \mathbf{n}, \tag{153}$$

where

$$\mathbf{u} = \hat{\mathbf{u}}_0, \quad \mathbf{a} = \hat{\mathbf{u}}_0', \ \mathbf{b} = \hat{\mathbf{u}}_0'' \quad \text{and} \quad \mathbf{c} = \hat{\mathbf{u}}_0''', \tag{154}$$

are mutually *independent* functions of θ^α. These are the coefficient vectors in the thickness-wise expansion

$$\hat{\mathbf{u}} = \mathbf{u} + \varsigma\mathbf{a} + \frac{1}{2}\varsigma^2\mathbf{b} + \frac{1}{6}\varsigma^3\mathbf{c} + \ldots \tag{155}$$

of the three-dimensional displacement.

3.2. Descent from three-dimensional elasticity

Our approach, as in the development of plate theory, is to write the virtual-work form (13) of the three-dimensional equation of equilibrium in terms of iterated integrals and to expand the integrals over C therein for small values of the thickness h of the thin body. The associated strong forms are then derived from the fundamental lemma. We again illustrate for the classical example in which the base surface coincides with the midsurface.

We have $C = [-h/2, h/2]$ and (cf. (137))

$$W = \int_C G\,\mathrm{d}\varsigma = \int_{-h/2}^{h/2} G\,\mathrm{d}\varsigma, \tag{156}$$

where W is the strain energy per unit area of Ω. For fixed values of the surface coordinates θ^α this is a function of h, which we assume to be smooth enough to admit of the expansion

$$W = hG_0 + \frac{1}{24}h^3 G_0'' = h\left(1 + \frac{1}{12}h^2 K\right)U_0 + \frac{1}{24}h^3\left(U_0'' - 4HU_0'\right), \tag{157}$$

modulo terms smaller than order h^3, obtained by combining Leibniz' rule with Taylor's theorem and (138). Here, as in (29), we have

$$U_0 = U(\widehat{\mathbf{H}}_0), \quad U_0' = \mathbf{P}_0 \cdot \widehat{\mathbf{H}}_0' \quad \text{and} \quad U_0'' = \mathbf{P}_0' \cdot \widehat{\mathbf{H}}_0' + \mathbf{P}_0 \cdot \widehat{\mathbf{H}}_0'', \tag{158}$$

where for uniform materials without initial stress

$$U(\widehat{\mathbf{H}}_0) = \frac{1}{2}\mathbf{P}_0 \cdot \widehat{\mathbf{H}}_0, \;\; \mathbf{P}_0 = \mathbf{C}[\widehat{\mathbf{H}}_0] \quad \text{and} \quad \mathbf{P}_0' = \mathbf{C}[\widehat{\mathbf{H}}_0']. \tag{159}$$

However, in place of (31) we use (152) and (153), where the gradients of the various vector fields are defined by a formula like (145). The energy per unit area is thus specified as a function of $\nabla\mathbf{u}$, $\nabla\mathbf{a}$, $\nabla\mathbf{b}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, as in the case of plates. The Euler equation for $\mathbf{c}$ again yields (45), where the orientation $\mathbf{n}$ is now that of the curved base surface. The remaining Euler equations, together with the relevant edge conditions, are straightforward generalizations of those for plates (see Section 2.2) and therefore omitted.

Instead, as in Section 2.3, we impose the constraints

$$\mathbf{P}_0\mathbf{n} = \mathbf{0} \quad \text{and} \quad \mathbf{P}_0'\mathbf{n} = \mathbf{0}, \tag{160}$$

obtaining

$$W = \left[\left(\frac{1}{2}h - \frac{1}{24}h^3 K\right)\mathbf{P}_0\mathbf{1} + \frac{1}{24}h^3\mathbf{P}_0'\mathbf{1}\boldsymbol{\kappa}\right] \cdot \nabla\mathbf{u} + \frac{1}{24}h^3\left[2\mathbf{P}_0\mathbf{1}(\boldsymbol{\kappa} - 2H\mathbf{1}) + \mathbf{P}_0'\mathbf{1}\right] \cdot \nabla\mathbf{a} + \frac{1}{24}h^3\mathbf{P}_0\mathbf{1} \cdot \nabla\mathbf{b}, \tag{161}$$

where the symmetry of $\boldsymbol{\kappa}$ has been used to permute factors in some of the inner products. To render this expression explicit for a specific elastic material we use (160) to obtain $\mathbf{a} = \bar{\mathbf{a}}$ and $\mathbf{b} = \bar{\mathbf{b}}$, where

$$\mathbf{A}_n\bar{\mathbf{a}} = -(\mathbf{C}[\nabla\mathbf{u}])\mathbf{n}, \quad \mathbf{A}_n\bar{\mathbf{b}} = -(\mathbf{C}[\nabla\bar{\mathbf{a}} + (\nabla\mathbf{u})\boldsymbol{\kappa}])\mathbf{n}, \tag{162}$$

and $\mathbf{A}_n$ is again the acoustic tensor defined by (48) in which $\mathbf{n}$ is the orientation of the base surface. We then compute

$$\mathbf{P}_0\mathbf{1} = (\mathbf{C}[\nabla\mathbf{u} + \bar{\mathbf{a}} \otimes \mathbf{n}])\mathbf{1} \quad \text{and} \quad \mathbf{P}_0'\mathbf{1} = (\mathbf{C}[\nabla\bar{\mathbf{a}} + (\nabla\mathbf{u})\boldsymbol{\kappa} + \bar{\mathbf{b}} \otimes \mathbf{n}])\mathbf{1}, \tag{163}$$

where $\mathbf{1}$ is defined by (124), and substitute the results into the expression for W.

We assume the lateral surfaces of the shell to be free of traction and thus that traction can be non-zero only on ∂R_C. This means that boundary traction makes no contribution to the integral over Ω of the thickness-wise expansion of the weak form of the equations. Its contribution is thus confined to an integral over $\partial\Omega$. To obtain it, we combine (139) and (142) to derive the order h^3 expansion

$$\int_C I\,d\varsigma = \int_{-h/2}^{h/2} I\,d\varsigma = hI_0 + \frac{1}{24}h^3 I_0'' = hJ_0 + \frac{1}{24}h^3\left(J_0'' - 2\kappa_\tau J_0' + \tau^2 J_0\right). \tag{164}$$

Using this with $J = \mathbf{t}\cdot(\tilde{\mathbf{u}})^{\cdot}$ gives the order h^3 formula

$$\int_{\partial R} \mathbf{t}\cdot(\tilde{\mathbf{u}})^{\cdot}\,da = \int_{\partial\Omega} (\mathbf{p}_u\cdot\dot{\mathbf{u}} + \mathbf{p}_a\cdot\dot{\mathbf{a}} + \mathbf{p}_b\cdot\dot{\mathbf{b}})ds, \tag{165}$$

where

$$\mathbf{p}_u = h\left(1 + \frac{1}{24}h^2\tau^2\right)\mathbf{t}_0 + \frac{1}{24}h^3\left(\mathbf{t}_0'' - 2\kappa_\tau\mathbf{t}_0'\right), \quad \mathbf{p}_a = \frac{1}{12}h^3\left(\mathbf{t}_0' - \kappa_\tau\mathbf{t}_0\right), \quad \mathbf{p}_b = \frac{1}{24}h^3\mathbf{t}_0. \tag{166}$$

The discussion of boundary conditions in plate theory applies equally here. Thus we invoke $(166)_3$, apply Stokes' theorem to the final term in (161), and transform the order h^3 expansion of the virtual-work Eq. (13) to

$$\int_\Omega \dot{F}\,da = \int_{\partial\Omega} (\mathbf{p}_u\cdot\dot{\mathbf{u}} + \mathbf{p}_a\cdot\dot{\mathbf{a}})ds, \tag{167}$$

where

$$\begin{aligned} F = {} & \left[\left(\frac{1}{2}h - \frac{1}{24}h^3K\right)\mathbf{P}_0\mathbf{1} + \frac{1}{24}h^3\mathbf{P}_0'\mathbf{1}\boldsymbol{\kappa}\right]\cdot\nabla\mathbf{u} + \frac{1}{24}h^3\left[2\mathbf{P}_0\mathbf{1}(\boldsymbol{\kappa} - 2H\mathbf{1}) + \mathbf{P}_0'\mathbf{1}\right]\cdot\nabla\mathbf{a} \\ & - \frac{1}{24}h^3\mathbf{b}\cdot\operatorname{div}(\mathbf{P}_0\mathbf{1}), \end{aligned} \tag{168}$$

in which $\mathbf{a} = \bar{\mathbf{a}}$ and $\mathbf{b} = \bar{\mathbf{b}}$ are the (unique) solutions to $(162)_1$ and $(162)_2$, respectively. The latter yield $\mathbf{a}$ and $\mathbf{b}$ as functions of the first and second gradients of $\mathbf{u}$ which, in turn, may be written as functions of the *partial* derivatives $\mathbf{u}_{,\alpha}$ and $\mathbf{u}_{,\alpha\beta}$ of the midsurface displacement field. Accordingly, the energy is determined by these derivatives. Here, Stokes' theorem has been used in the form

$$\int_\Omega \psi^\alpha_{;\alpha}\,dv = \int_{\partial\Omega} \psi^\alpha \nu_\alpha\,ds, \tag{169}$$

with $\psi^\alpha = \mathbf{P}_0\mathbf{1}\mathbf{a}^\alpha\cdot\mathbf{b}$ and $\operatorname{div}(\mathbf{P}_0\mathbf{1}) = (\mathbf{P}_0\mathbf{1}\mathbf{a}^\alpha)_{;\alpha}$, where the subscript preceded by a semi-colon is the covariant derivative. The notation is defined by (179) below. Further, we assume for the sake of notational simplicity that Ω can be covered by a single coordinate chart. Our results remain valid in the general case in which Ω is covered by the union of such charts.

3.3. Euler equations and edge conditions

The dependence of F on $\mathbf{u}_{,\alpha}$ and $\mathbf{u}_{,\alpha\beta}$ implies that its variation may be written in the form

$$\dot{F} = \mathbf{L}^\alpha\cdot\dot{\mathbf{u}}_{,\alpha} + \mathbf{M}^{\alpha\beta}\cdot\dot{\mathbf{u}}_{,\alpha\beta}, \tag{170}$$

where, as in Section 2.5, the coefficients $\mathbf{L}^\alpha$ and $\mathbf{M}^{\alpha\beta}$ are determined by direct comparison with the similar decomposition obtained by using (168), specialized to a particular material. Further, the vectors $\mathbf{M}^{\alpha\beta}$ are symmetric in the superscripts. Here we use the *mixed* notation advocated by Libai and Simmonds [8]. For our present purposes, this decomposition has the slightly inconvenient feature that the individual terms in it are not scalar invariants on Ω, although their sum is, of course, invariant. To facilitate the manipulations to follow, we use

$$\dot{\mathbf{u}}_{;\alpha\beta} = \dot{\mathbf{u}}_{,\alpha\beta} - \dot{\mathbf{u}}_{,\lambda}\Gamma^\lambda_{\alpha\beta}, \tag{171}$$

where $\Gamma^\lambda_{\alpha\beta}$ are the Christoffel symbols induced by the coordinates on Ω and the subscripted semi-colon stands for covariant differentiation. It follows that

$$\dot{F} = \mathbf{N}^\alpha\cdot\dot{\mathbf{u}}_{,\alpha} + \mathbf{M}^{\alpha\beta}\cdot\dot{\mathbf{u}}_{;\alpha\beta}, \tag{172}$$

where

$$\mathbf{N}^{\alpha} = \mathbf{L}^{\alpha} + \mathbf{M}^{\lambda\mu}\Gamma^{\alpha}_{\mu\lambda}. \tag{173}$$

Unlike $\mathbf{L}^{\alpha}$, this transforms as a contravariant surface vector under transformations of the surface coordinates. It follows from (172) that $\mathbf{M}^{\alpha\beta}$ transforms as a contravariant surface tensor. This decomposition furnishes the generalization of (101) to shells.

The presence of the Christoffel symbols in (173) reminds us of the divergence

$$\mu^{\beta\alpha}_{;\beta} = \mu^{\beta\alpha}_{,\beta} + \mu^{\beta\alpha}\Gamma^{\lambda}_{\lambda\beta} + \mu^{\beta\lambda}\Gamma^{\alpha}_{\lambda\beta}, \tag{174}$$

of a contravariant surface tensor $\mu^{\beta\alpha}$. Let $\mu^{\beta\alpha}_{i}; i = 1, 2, 3$ be three such tensors and let $\mathbf{M}^{\beta\alpha} = \mu^{\beta\alpha}_{i}\mathbf{e}_{i}$, where $\{\mathbf{e}_{i}\}$ is a fixed orthonormal basis. Multiplying each of the divergences by the corresponding element of the basis and adding, we obtain

$$\mathbf{M}^{\beta\alpha}_{;\beta} = \mathbf{M}^{\beta\alpha}_{,\beta} + \mathbf{M}^{\beta\alpha}\Gamma^{\lambda}_{\lambda\beta} + \mathbf{M}^{\beta\lambda}\Gamma^{\alpha}_{\lambda\beta}. \tag{175}$$

The extension to shells of definition $(94)_1$, namely

$$\mathbf{T}^{\alpha} = \mathbf{N}^{\alpha} - \mathbf{M}^{\beta\alpha}_{;\beta}, \tag{176}$$

combined with the convention $\mathbf{u}_{;\alpha} = \mathbf{u}_{,\alpha}$ and the standard product rule for covariant derivatives, allows us to reduce (172) to the form

$$\dot{F} = \varphi^{\alpha}_{;\alpha} - \dot{\mathbf{u}} \cdot \mathbf{T}^{\alpha}_{;\alpha}, \tag{177}$$

where

$$\varphi^{\alpha} = \mathbf{T}^{\alpha} \cdot \dot{\mathbf{u}} + \mathbf{M}^{\alpha\beta} \cdot \dot{\mathbf{u}}_{,\beta}, \tag{178}$$

and

$$\mathbf{T}^{\alpha}_{;\alpha} = \mathbf{T}^{\alpha}_{,\alpha} + \mathbf{T}^{\beta}\Gamma^{\alpha}_{\alpha\beta} = a^{-1/2}(a^{1/2}\mathbf{T}^{\alpha})_{,\alpha}, \tag{179}$$

is the surface divergence of $\mathbf{T}^{\alpha}$. Here $a = \det(a_{\alpha\beta})$, where $a_{\alpha\beta} = \mathbf{a}_{\alpha} \cdot \mathbf{a}_{\beta}$ is the metric induced by the coordinates on Ω, and we have used the well-known fact [6] that $\Gamma^{\alpha}_{\alpha\beta} = a^{-1/2}(a^{1/2})_{,\beta}$.

Stokes' theorem yields

$$\int_{\Omega} \dot{F}\,\mathrm{d}a = \int_{\partial\Omega} \varphi^{\alpha}\nu_{\alpha}\,\mathrm{d}s - \int_{\Omega} \dot{\mathbf{u}} \cdot \mathbf{T}^{\alpha}_{;\alpha}\,\mathrm{d}a, \tag{180}$$

which, with (167), furnishes the Euler equation

$$\mathbf{T}^{\alpha}_{;\alpha} = \mathbf{0} \text{ in } \Omega. \tag{181}$$

The rather complicated component forms of this equation follow by decomposing $\mathbf{N}^{\alpha}$ and $\mathbf{M}^{\alpha\beta}$ into tangential and normal parts. Thus

$$\mathbf{N}^{\alpha} = N^{\beta\alpha}\mathbf{a}_{\beta} + N^{\alpha}\mathbf{n}, \ \mathbf{M}^{\alpha\beta} = M^{\lambda\alpha\beta}\mathbf{a}_{\lambda} + M^{\alpha\beta}\mathbf{n}. \tag{182}$$

Using the Gauss and Weingarten equations [6]

$$\mathbf{a}_{\alpha;\beta} = \kappa_{\alpha\beta}\mathbf{n} \quad \text{and} \quad \mathbf{n}_{,\alpha} = -\kappa^{\beta}_{\alpha}\mathbf{a}_{\beta}, \tag{183}$$

the latter being equivalent to (121), we reduce (181) to

$$N^{\mu\alpha}_{\alpha} - N^{\alpha}\kappa^{\mu}_{\alpha} - (M^{\beta\alpha}_{\beta} + M^{\lambda\beta\alpha}\kappa_{\lambda\beta})\kappa^{\mu}_{\alpha} - (M^{\mu\beta\alpha}_{\beta} - M^{\beta\alpha}\kappa^{\mu}_{\beta})_{\alpha} = 0, \tag{184}$$

and

$$N^{\alpha}_{\alpha} + N^{\beta\alpha}\kappa_{\beta\alpha} - (M^{\beta\alpha}_{\beta} + M^{\lambda\beta\alpha}\kappa_{\lambda\beta})_{\alpha} - (M^{\mu\beta\alpha}_{\beta} - M^{\beta\alpha}\kappa^{\mu}_{\beta})\kappa_{\mu\alpha} = 0, \tag{185}$$

where

$$M_{\beta}^{\mu\beta\alpha} = \mathbf{a}^{\mu} \cdot \mathbf{M}_{\beta}^{\beta\alpha}. \tag{186}$$

As for the boundary term, we use (178) with a decomposition of the displacement gradient into normal and tangential derivatives. Thus

$$\mathbf{u}_{,\alpha} = \tau_{\alpha}\mathbf{u}_{s} + v_{\alpha}\mathbf{u}_{v}, \tag{187}$$

where τ_{α} and v_{α}, respectively, are the covariant components of the unit tangent $\boldsymbol{\tau}$ and unit normal $\boldsymbol{v}$ to the edge. Next, we use integration by parts for the terms involving the arclength derivative. For smooth edges, the effective contribution to the boundary term in (180) is given by the integral over $\partial\Omega$ of

$$[\mathbf{T}^{\alpha}v_{\alpha} - (\mathbf{M}^{\alpha\beta}v_{\alpha}\tau_{\beta})_{s}] \cdot \dot{\mathbf{u}} + \mathbf{M}^{\alpha\beta}v_{\alpha}v_{\beta} \cdot \dot{\mathbf{u}}_{v}. \tag{188}$$

As in the case of plates, for smooth edges the effective contribution to the boundary work reduces to the integral over $\partial\Omega$ of $\mathbf{f} \cdot \dot{\mathbf{u}} + \mathbf{m} \cdot \dot{\mathbf{u}}_{v}$, where $\mathbf{f}$ and $\mathbf{m}$ are related to the exact traction distribution in a manner determined by material constitution. With (181) satisfied, the virtual-work statement thus reduces to

$$\int_{\partial\Omega} \{[\mathbf{T}^{\alpha}v_{\alpha} - (\mathbf{M}^{\alpha\beta}v_{\alpha}\tau_{\beta})_{s}] \cdot \dot{\mathbf{u}} + \mathbf{M}^{\alpha\beta}v_{\alpha}v_{\beta} \cdot \dot{\mathbf{u}}_{v}\}\,\mathrm{d}s = \int_{\partial\Omega} (\mathbf{f} \cdot \dot{\mathbf{u}} + \mathbf{m} \cdot \dot{\mathbf{u}}_{v})\mathrm{d}s. \tag{189}$$

Essential boundary conditions entail the specification of $\mathbf{u}$ and $\mathbf{u}_{v}$. As for plates, this is tantamount to assignment of the midsurface displacement and the field $\mathbf{a}$, corresponding to a clamped edge. The dual natural boundary conditions are

$$\mathbf{f} = \mathbf{T}^{\alpha}v_{\alpha} - (\mathbf{M}^{\alpha\beta}v_{\alpha}\tau_{\beta})_{s} \quad \text{and} \quad \mathbf{m} = \mathbf{M}^{\alpha\beta}v_{\alpha}v_{\beta}. \tag{190}$$

For specific materials, the Euler equations are rendered explicit in terms of the midsurface displacement field $\mathbf{u}$ by following the procedure outlined for plates and writing the variation of the energy (168) in the form (172), paying due attention to the symmetry requirement. Likewise, $\mathbf{f}$ and $\mathbf{m}$ are made explicit in terms of the three-dimensional traction distribution by following the procedure already described for plates. This laborious exercise is left to the interested reader.

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