J. Biomechanics, 1975, Vol. 8, pp. 135–142. Pergamon Press. Printed in Great Britain

TENSION FIELD THEORY AND THE STRESS IN STRETCHED SKIN*

D. A. DANIELSON and S. NATARAJAN

Department of Applied Mathematics and Computer Science, University of Virginia, Charlottesville, Virginia 22901, U.S.A.

Abstract—Tension field theory is generalized to membranes undergoing arbitrarily large deformations. The theory is used to model operations on the human skin performed by plastic surgeons. It is shown that the tension rays are geodesics of the surface over which the skin slides. The theory is applied to a plane sheet having an elliptical hole which is closed into a straight line coinciding with its major axis, to obtain a formula for the maximum stress on a closed skin defect. The theory is also applied to a model for the Z-plasty operation, and the theoretical equations are compared with published experimental results.

1. INTRODUCTION

In a previous paper (Danielson, 1973) we have recorded the equations of the nonlinear theory of membranes which apply to all problems involving the deformation of skin. The extreme complexity of these equations has until now prevented us from solving very many practical problems. Human skin is highly nonlinear and anisotropic; the stress–strain relations vary from individual to individual, and even from one region to another of a single person's body. The skin cannot support negative stresses, and hence buckles easily, so that the deformed skin may not lie in a plane.

However, in some operations performed on the skin, one of the principal stress components in the final deformed skin is positive and the other one is negative or negligible. For such problems, the large deflection membrane equations can be replaced by a relatively simple "tension field theory". In the theory we study the plane state of stress in which the only non-zero stress component is the positive stress along a "tension ray". The tension rays can be easily seen by grasping a pinch of skin on the back of the hand; note how the force is carried by straight rays of stretched skin. (In other problems, the tension rays may exist but not be so easily seen.) The only experimental law which is needed is the uniaxial stress–strain relation along each of the tension rays. We think that this could be determined *in vivo* with the aid of a simple instrument.

Tension field theory has been developed by Wagner (1929), Reissner (1938), Kondo (1938, 1955),† Iai (1943), Stein and Hedgepeth (1961), and Mansfield (1969, 1970). Each of these authors has contributed new ideas, so the theory may now be said to be well understood. However, all of the previous authors have dealt with the case in which the stress along a tension ray is linearly related to the displacement gradient. Since the skin is a very soft substance, the linearity assumption does not hold, and we will have to generalize tension field theory so that it applies to membranes undergoing arbitrarily large deformations.

* *Received* 25 *March* 1974.

† We are indebted to Prof. E. Reissner for informing us of the work of Kondo.

We expect that the theory developed in this paper will apply only to those operations in which the deformations of the skin are quite large, such as the excision of large scars or the transposition of large flaps. (The reader is referred to the book by Borges (1973), for a description of operations on the skin currently performed by plastic surgeons.) This is because we will assume that the tension rays extend all the way to the outer boundaries of the skin, and we will completely ignore the initial stresses present naturally in the skin.

In the first part of this paper we study the model problem of an elliptical hole in an isotropic elastic membrane. In Sec. 2 we assume that the hole is only slightly deformed, in order to obtain some analytical information about the possible extent of the tension ray field. In this case the membrane is governed by the linear theory of plane elasticity. Then in Sec. 3 we pull the hole in further until it is completely closed. We develop tension field theory, and calculate the maximum stress on the closed defect. In Sec. 4 we show how the analysis developed in Sec. 3 can be generalized to problems of any geometrical and material properties. Finally, in Sec. 5 we compare the theory with the available experimental results for the Z-plasty operation.

2. ELLIPTICAL HOLE, SMALL DEFORMATION—LINEAR, PLANE THEORY

We begin our analysis by studying the model problem of an elliptical hole in an infinite, flat, isotropic elastic membrane. We let a denote the semi-major axis of the ellipse, let b denote the semi-minor axis, and establish a rectangular coordinate system (x,y) with origin at the center of the hole (refer to Fig. 1). We assume that each point on the boundary of the hole is pulled towards the x-axis, in a direction parallel to the y-axis, forming another ellipse with semi-major axis a and semi-minor axis $b - \epsilon$. If the deformations are small enough ($\epsilon \ll b$), we would expect the problem to be governed by the equations of the plane stress or plane strain theory of elasticity.

The solution to the equations of the plane theory of elasticity for the case of an elliptical hole subjected to arbitrary displacements is contained in the book by

Muskhelishvili (1953). In our problem, the displacement of a point on the top half of the hole is:

$$u_x = 0, u_y = -\epsilon\sqrt{1 - \frac{x^2}{a^2}},$$

$$\text{for } y = b\sqrt{1 - \frac{x^2}{a^2}} \text{ and } |x| \le a. \quad (1)$$

Here u_x is the displacement in the x-direction, and u_y is the displacement in the y-direction. We also require that the stresses die out to zero at infinity. With these boundary conditions, the *principal* stresses σ_1 and σ_2 in the sheet turn out to be

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{\mu\epsilon}{R\kappa|\xi^2 - m|^2}\Bigg[2m - \xi^2 - \bar{\xi}^2 \pm \Bigg| [2\xi^3\left(\bar{\xi} + \frac{m}{\bar{\xi}}\right) + m(1 - \kappa m) - \xi^2(3 - 2\kappa m^2) - \xi^4(\kappa + m)]/(\xi^2 - m)\Bigg|\Bigg], \quad (2)$$

the plus sign being taken for σ_1 and the minus sign for σ_2. The complex variable ξ in these formulas is related to the original coordinates (x,y) by the transformation equations

$$x + iy = R\left(\xi + \frac{m}{\xi}\right), \quad (3)$$

and $\bar{\xi}$ denotes the complex conjugate of ξ. The variables R and m are defined by

$$R = \frac{a + b}{2}, m = \frac{a - b}{a + b}. \quad (4)$$

The elastic constants μ and κ are related to Poisson's ratio ν and Young's modulus E by the equations

$$\mu = \frac{E}{2(1 + \nu)}$$

$$\kappa = \begin{cases} 3 - 4\nu, \text{ for plane strain.} \\ \\ \dfrac{3 - \nu}{1 + \nu}, \text{ for plane stress.} \end{cases} \quad (5)$$

In the lower part of Fig. 1 we have shown for the case $a/b = 2$, $\nu = 0$, that in each of several regions in the deformed membrane the signs of σ_1 and σ_2 remain unchanged.* The regions shown are symmetrical about the x and y-axes. We note in passing the following formula for the angle β, between the x-axis and the asymptote to the curve along which $\sigma_1 = 0$:

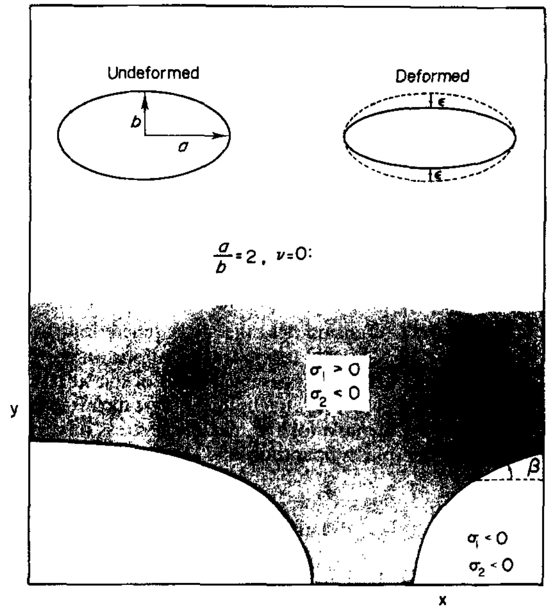

Fig. 1. Elliptical hole, small deformation—signs of principal stresses.

$$\tan\beta = \sqrt{\frac{4\sqrt{4 + 2(\kappa + m)^2} - [8 + (\kappa + m)^2]}{(\kappa + m)(\kappa + m + 4)}} \quad (6)$$

For the case shown in Fig. 1, $\beta \approx 13°$.

Note from Fig. 1 that in the shaded region one of the principal stresses is positive and the other one is negative. The regions to the far left and far right of the ellipse do not carry a load. As we pull the ellipse together, we would intuitively not expect these features to change drastically. Consequently, in the next section when we deal with the completely closed ellipse, we will assume that the tension is carried by rays which lie in a region similar to the shaded region of Fig. 1.

3. ELLIPTICAL HOLE, LARGE DEFORMATION—TENSION FIELD THEORY

In this section we suppose that the elliptical hole has been pulled in much further than in Sec. 2, so that there is no hole in the final deformed membrane. We suppose that the two edges of the hole are sutured together tightly along the x-axis to carry the force exerted by the membrane. Whereas in the previous section the membrane was taken to be infinite in extent, in tension field theory we have to prescribe a finite outer boundary. For simplicity we will assume that the membrane is fixed along outer boundaries which are straight, parallel to the x-axis, and a distance D away from the x-axis. We assume that D is big enough so that the conclusions about the extent of the tension field arrived at in Sec. 2 are valid, and small enough so that the tension rays extend to the outer boundary ($2b < D < 6b$, say). Guided by our work in Sec. 2, we would anticipate the

* In the case $\nu > 0$ there are small regions above and below the ellipse in which both of the principal stresses are positive. For simplicity, we will in this paper deal only with the case in which these regions do not exist. If necessary, the effects of these regions can later be included within the framework of the present analysis.

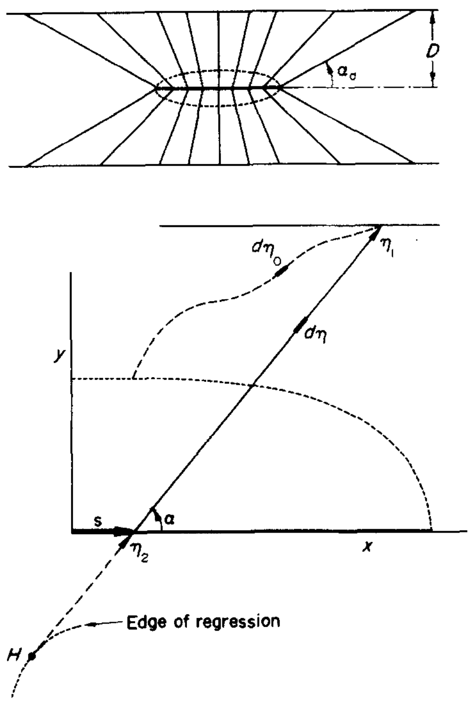

Fig. 2. Elliptical hole, large deformation—tension field lines.

tension ray field to have the extent sketched in the top of Fig. 2.

Following Reissner (1938) and Kondo (1938), we use as our coordinates (ζ,α), the system associated with the principal directions of stress. This coordinate system is orthogonal on the final, deformed membrane, and a line element dz on the deformed membrane is given by

$$dz^2 = h_1^2\,d\zeta^2 + h_2^2\,d\alpha^2. \tag{7}$$

We suppose that one non-zero (positive) stress component σ is in the direction of ζ. The tangential equilibrium equations for membranes in the absence of surface loads are exactly

$$\frac{\partial(h_2\sigma)}{\partial\zeta} = 0,\ \sigma\frac{\partial h_1}{\partial\alpha} = 0. \tag{8}$$

Solving the second of these equations, we see that the scale factor h_1 is a function of ζ only, and hence by redefining our ζ coordinate we can effectively choose $h_1 = 1$. It follows that (ζ,α) form a geodesic coordinate system, and the tension rays are geodesics (straight lines). In such a coordinate system, h_2 must obey the equation (Struik, 1961)

$$\frac{\partial^2 h_2}{\partial\zeta^2} = 0. \tag{9}$$

Solving for h_2,

$$h_2 = f_1(\alpha) + \zeta f_2(\alpha). \tag{10}$$

Now adjacent tension rays intersect at a point H on a curve called the edge of regression. We let $f_2(\alpha)$ be arc length along the edge of regression, and choose $f_2(\alpha) = 1$. Then α is the angle between a tension ray and the x-axis (refer to the bottom part of Fig. 2), and

$$\eta = h_2 = f_1(\alpha) + \zeta \tag{11}$$

represents simply the distance measured from H along a tension ray. From now on in this paper we will use the non-orthogonal system (η,α) as our basic coordinates. The solution to the first equilibrium equation in (8) is thus

$$\sigma = \frac{f(\alpha)}{\eta}, \tag{12}$$

where $f(\alpha)$ is as yet an arbitrary function of α only.

For the strain measure along a tension ray we use the Eulerian strain γ defined by

$$\gamma = \frac{d\eta - d\eta_0}{d\eta}, \tag{13}$$

where $d\eta_0$ is the length of the fiber on the undeformed membrane which, when deformed, becomes a fiber of length $d\eta$ on a tension ray. We note that we may integrate (13) to get:

$$\int_{\eta_2}^{\eta_1} \gamma\,d\eta = \Delta. \tag{14}$$

Here $\Delta(\alpha)$ is the change in length of the line of particles which becomes a tension ray on the deformed membrane, $\eta_1(\alpha)$ is the value of η at the outer boundary and $\eta_2(\alpha)$ is the value of η at the inner boundary. In writing (14), we have assumed for simplicity that the edge of regression lies below the x-axis, so that $\eta_1 > \eta_2 > 0$. In our problem, the distances η_1 and η_2 are given by:

$$\begin{aligned}\eta_1 &= D\csc\alpha - s'\sin\alpha\\ \eta_2 &= -s'\sin\alpha.\end{aligned} \tag{15}$$

Here $s(\alpha)$ is the distance from the origin to the point of intersection of a tension ray with the x-axis, and primes denote differentiation with respect to α.

The most important quantity in determining how a wound will heal is probably the maximum stress acting upon it. The maximum stress σ_{max} will occur somewhere on the x-axis, where η has the smallest value η_2. We also note in passing that the magnitude F_y of the y-component of the force pulling on the upper half of the hole is given by:

$$F_y = t\int_{-a}^{a} \sigma(\eta_2)\sin\alpha\,ds. \tag{16}$$

Here t is the thickness of the membrane, that we assume for simplicity to be a constant.

A recent survey of the many papers in which the stress–strain law of skin has been investigated is contained in the thesis of North (1972). We mention here as typical examples the work of Kenedi *et al.* (1965), and Ridge and Wright (1966). For the purposes of illustration, we adopt in this section the following stress–strain relation, which yields particularly simple analytical formulas and yet still exhibits the type of behavior typical to skin and other soft tissues:

$$\sigma = C\gamma^2. \tag{17}$$

Here C is an elastic constant having the dimensions of force per unit area. If we now solve (17) for γ, use (12), and perform the integral in (14), we can determine the function $f(\alpha)$:

$$f = \frac{C\Delta^2}{4(\sqrt{\eta_1} - \sqrt{\eta_2})^2}. \tag{18}$$

It follows from (12) that the maximum value of σ is given by:

$$\sigma_{\max} = \frac{C}{4}\max_{\alpha}\left[\frac{\Delta}{\sqrt{\eta_1\eta_2} - \eta_2}\right]^2. \tag{19}$$

We can determine an upperbound to the function $\Delta(\alpha)$.* Using the fact that the length of the line of particles which becomes a tension ray must be greater than or equal to the length of a straight line, we obtain

$$\Delta \le D\csc\alpha - \sqrt{D^2\csc^2\alpha - 2Dy + y^2}, \tag{20}$$

where for an elliptical hole

$$y = b\sqrt{1 - \frac{s^2}{a^2}}. \tag{21}$$

By symmetry, at the origin $\alpha = \pi/2$ and $\Delta(\pi/2) = b$. Thus, if the maximum value of σ occurs at $\alpha = \pi/2$, we will know the value for $\Delta(\pi/2)$ in equation (19) precisely.

To find the distribution $s(\alpha)$ of the tension rays, we follow Iai (1943) and Mansfield (1969) and maximize the total strain energy of the strained membrane. For a problem in which the tractions vanish on the part of the boundary where they are prescribed, it can be shown that the strain energy is just the negative of the complementary energy. It follows from the nonlinear principle of stationary complementary energy that the strain energy must be stationary at the equilibrium point (see Koiter (1973) for a clear discussion of this principle in the theory of elasticity). If the strain energy per unit deformed area is denoted by $\Phi(\gamma)$, then

$$\sigma = \frac{d\Phi}{d\gamma} \tag{22}$$

and the total strain energy U is given by:

$$U = \int_{\alpha_a}^{\pi/2}\int_{\eta_2}^{\eta_1} \Phi\eta\, d\eta\, d\alpha. \tag{23}$$

Here α_a is the value of α at $s = a$. For our assumed stress–strain relation (17),

$$\Phi = \frac{1}{3}C\gamma^3 = \frac{\sigma^{3/2}}{3\sqrt{C}}. \tag{24}$$

Plugging (24) into (23), using (12) and (18), and performing the η-integration, we obtain

$$U = \int_{\alpha_a}^{\pi/2} S\, d\alpha, \tag{25}$$

where

$$S = \frac{C\Delta^3}{12(\sqrt{\eta_1} - \sqrt{\eta_2})^2}. \tag{26}$$

The exact distribution for $s(\alpha)$ is obtained by maximizing the functional U. By variational calculus, a differential equation for $s(\alpha)$ may be obtained, which is identical in form to equation (3.10) in Mansfield's (1969) paper. However, since we do not know the function $\Delta(\alpha)$ precisely, we have to settle for an approximate solution for $s(\alpha)$. We assume that s is a quadratic function of α, satisfying the conditions $s(\pi/2) = 0$ and $s(\alpha_a) = a$, so that:

$$s = a\left[\frac{\frac{\pi}{2} - \alpha}{\frac{\pi}{2} - \alpha_a} + k\left(\frac{\pi}{2} - \alpha\right)(\alpha - \alpha_a)\right]. \tag{27}$$

In order to determine the constant k, we plug (27) into (25) and numerically integrate to find the value of k which maximizes U. (When evaluating (26), we replace the inequality sign in (20) by an equality sign.)

Following these procedures, we find that the maximum value of the stress σ always occurs at $\alpha = \pi/2$, if α_a is constrained to lie within certain bounds. Hence we can obtain the following formula for $\sigma_{\max}$ by setting

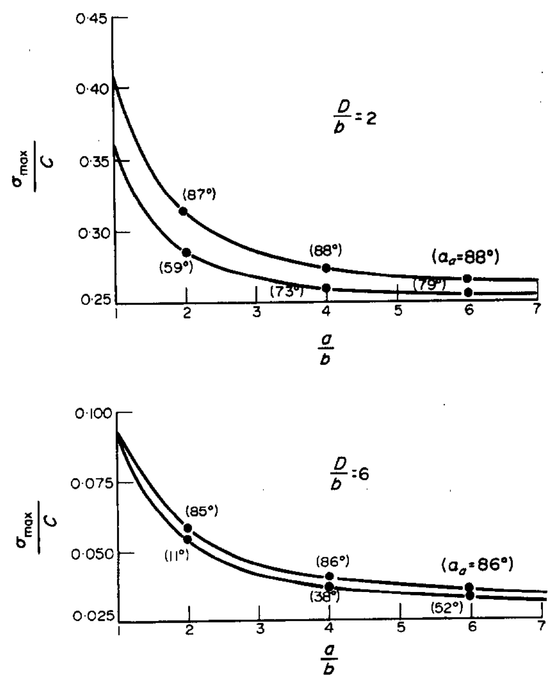

Fig. 3. Elliptical hole, large deformation—maximum stress.

* We have been unable to figure out a simple way to determine the precise value for $\Delta(\alpha)$.

$\alpha = \pi/2$ in the formula (19):

$$\sigma_{\max} = \frac{Cb^2}{4a^2\left[\dfrac{1}{\dfrac{\pi}{2}-\alpha_a} + k\left(\dfrac{\pi}{2}-\alpha_a\right)\right]^2 \left[\sqrt{1 + \dfrac{D}{a\left[\dfrac{1}{\dfrac{\pi}{2}-\alpha_a} + k\left(\dfrac{\pi}{2}-\alpha_a\right)\right]}} - 1\right]^2}. \tag{28}$$

It can be shown that for fixed α_a, as a and D increase, $\sigma_{\max}$ decreases. As b increases, $\sigma_{\max}$ increases. In Fig. 3 we have plotted $\sigma_{\max}$ for various values of a/b, D/b and α_a. To obtain the lower curve in each graph, the minimum possible value of α_a was taken. To obtain the upper curve in each graph, the maximum possible value of α_a was taken. The values of $\sigma_{\max}$ vary only slightly as α_a ranges between these two bounds. (If α_a is not constrained to lie between these two bounds, there is a stress singularity along the x-axis.)

In the limit as $a/b \to \infty$, the tension rays become parallel, and the stress along a tension ray becomes a constant. Thus the formula (28) asymptotes to the value $\sigma_{\max} \to Cb^2/D^2$.

Before leaving this problem, we note that 'dog-ears' may form in the completely buckled region to the left and right of the ellipse. If the dog-ears are too large, the surgeon may have to excise them. Unfortunately, the present theory can only say that the dog-ears lie within wedge-shaped regions to the left and right of the ellipse. The determination of the actual extent and height of the dog-ears would appear to be an interesting problem for future research.

4. OTHER PROBLEMS—MODIFICATIONS OF THE THEORY

The theory developed in the preceding sections may be modified as necessary to provide the best model for a given operation. For any problem, the basic steps performed in the preceding two sections may be followed. First, the solution to the linear equations of elasticity should be obtained, if possible, in order to find the regions in which tension field theory should apply. We have also found it helpful to make crude rubber models in order to determine the extent and direction of the tension rays. Once some rough idea of the correct tension field has been obtained, then the procedure developed in Sec. 3 may be followed, possibly with modifications. We have listed below some of the modifications necessary for specific problems.

A. *Other shapes of holes and outer boundaries*

For holes of other than elliptical shape, the formula (21) must be changed. For example, for a lenticular-shaped hole,

$$y = b\left(1 - \frac{s^2}{a^2}\right). \tag{29}$$

If the outer boundaries are not lines parallel to the x-axis, then the constant D becomes a function of α.

B. *Curved surfaces*

If the skin is stretched over a curved surface, as when a joint is flexed, we must impose a kinematical constraint condition on the membrane requiring that the deformed membrane lie on top of the curved surface. We have discussed this condition in an earlier paper (Danielson, 1973). The principal coordinates (ζ,α) on the deformed membrane still are geodesic coordinates, and the tension rays are still geodesics on the curved surface. The equation (12) in Sec. 3 must be replaced by

$$\sigma = \frac{f(\alpha)}{h_2(\zeta,\alpha)}. \tag{30}$$

The scale factor h_2 is the solution to the differential equation

$$\frac{\partial^2 h_2}{\partial \zeta^2} + K h_2 = 0. \tag{31}$$

Here $K(\zeta,\alpha)$ is the Gaussian curvature of the curved surface. Some of the other formulas in Sec. 3 may also need to be changed to correspond to the curved geometry.

It is interesting to note that if the membrane of Sec. 3 is wrapped around a circular cylinder of radius d, and the elliptical hole in the membrane has its major axis parallel to the axis of the cylinder, then the expression for $\sigma_{\max}$ is identical to (28), with $D = \pi d$.

C. *Other stress–strain relations*

Experiments will show that it is necessary for some problems to use stress–strain laws more complex than (17). In general, we could have

$$\sigma = \sum(\gamma,\alpha,\eta), \tag{32}$$

where $\sum$ is a function of (γ,α,η). We could thereby include the effects of nonlinearity, anisotropy and nonhomogeneity.

5. Z-PLASTY—THEORY AND EXPERIMENTS

In this section we compare our theory with the available experimental results. The excellent experimental work of Furnas and Fischer (1971) on the Z-plasty is apparently the only work in which the force required to close a variety of wounds has been measured *in vivo*. An incision in the shape of a Z was made by Furnas and Fischer in the skin on one side of an anaesthetized

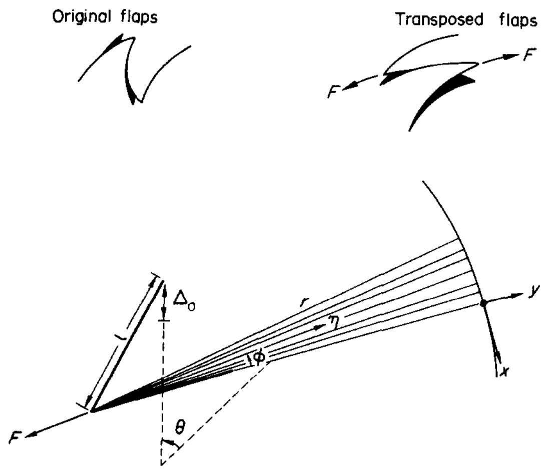

Fig. 4. Z-plasty operation—geometry.

dog. The skin was then cut away from the subcutaneous tissue under the Z, forming two flaps. A fish hook line was attached to the tip of each flap. Paired opposing forces of magnitude F were applied to the lines in order to transpose the flaps (refer to the upper half of Fig. 4). Then, the flaps were sutured, and a variety of observations about the new geometry was made. Several Z-plasties of differing geometry were performed on other dogs.

We will use our theory to calculate the force F required to transpose the flaps. We will not deal with the more difficult problem of explaining the observed geometrical changes. Nor will we investigate the problem of how the tension rays redistribute themselves after the flaps are sutured in place.

Let us first consider the case of a symmetrical Z-plasty having equal limbs of length l and angle θ between each limb (refer to Fig. 4). The geometry of the transposed flaps has been discussed by many authors, such as Limberg (1929) and McGregor (1957). The transposed flaps form a reversed Z, with limbs of equal length l and angle θ. The central limb of the transposed Z-plasty lies along the line joining the outer points of the original Z-plasty, and vice versa.

Ordinarily, in the absence of any experimental results, it would be necessary to study the linear, plane theory of elasticity to determine the regions where tension field theory should apply. However, from the excellent experimental observations of Furnas and Fischer, we can directly see that the force F is carried by a narrow band of tension rays emanating from the tip of each transposed flap. We suppose that the outer boundary of the membrane is fixed at a distance r from the tip of a transposed flap. We take (η,ϕ) as our coordinates on the transposed flap, where η is the distance measured from the tip of a flap and ϕ is the angle a tension ray makes with the central limb of a flap. We also establish a Cartesian coordinate system with origin on the outer boundary, as shown in Fig. 4. We assume that F is large enough so that we can ignore the effect of the natural tension in the skin. In analogy with equation (16) of Sec. 3, the magnitude F_y of the y-component of the force pulling on the outer boundary will be

$$F_y = tr\int_0^{\phi_l} \sigma(\eta_1)\cos\phi \,\mathrm{d}\phi, \tag{33}$$

where ϕ_l is the angle from the central limb to the outer tension ray ($\phi_l < \theta$). The stress σ is still given by (12). (We ignore the stress singularity at the tip of the flap.) In our present problem,

$$\eta_1 = r, \quad \eta_2 = 0. \tag{34}$$

If we again assume the stress–strain law (17), the function f is still given by (18). Plugging these results into (33), we find that

$$F_y = \frac{Ct}{4r}\int_0^{\phi_l} \Delta^2\cos\phi\,\mathrm{d}\phi. \tag{35}$$

Similarly, the expression for the magnitude F_x of the force in the x-direction is

$$F_x = \frac{Ct}{4r}\int_0^{\phi_l} \Delta^2\sin\phi\,\mathrm{d}\phi. \tag{36}$$

The magnitude F of the total force is then

$$F = \sqrt{F_x^2 + F_y^2}. \tag{37}$$

We have to make a kinematical assumption in order to evaluate the quantity $\Delta(\phi)$. The simplest is to assume that

$$\Delta = \Delta_0 g_1(\phi), \tag{38}$$

where $g_1(\phi)$ is a function of ϕ having the properties $g_1(0) = 1$, $g_1(\phi_l) = 0$, and where $\Delta_0 = \Delta(0)$. It then follows from (35) to (38) that

$$F \propto \frac{\Delta_0^2}{r} g_2(\phi_l), \tag{39}$$

where $g_2(\phi_l)$ is a function solely of ϕ_l. Now if we make the additional kinematical assumption that ϕ_l is independent of l, θ and r, then we see that

$$F \propto \frac{\Delta_0^2}{r}. \tag{40}$$

The constant Δ_0 is simply one-half of the elongation achieved by the Z-plasty along the central limb of the original incision, as shown in Fig. 4. From straightforward geometry we obtain:

$$\Delta_0 = \frac{l}{2}(\sqrt{5 - 4\cos\theta} - 1). \tag{41}$$

Furnas and Fischer observed that the deformation of the skin around the largest Z-plasties extended all the way to the outer edges of the dogs, whereas the deformation around the smaller Z-plasties did not extend so far. As a first approximation, we will assume that the tension rays extend out the same distance from

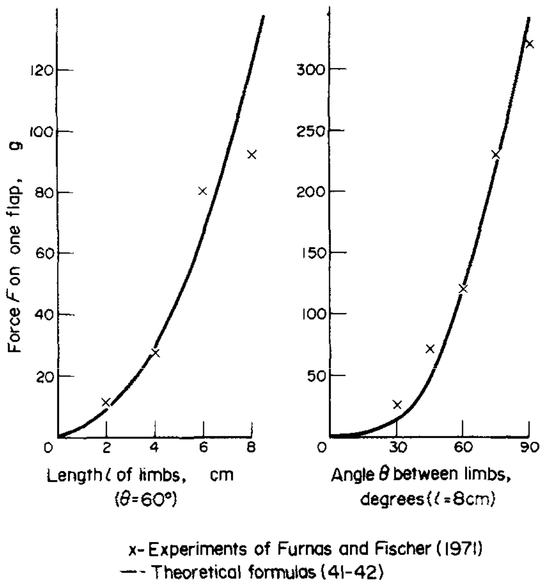

Fig. 5. Z-plasty operation—theoretical and experimental results.

the tip of each flap, so that r is a constant. It follows from (40) that

$$F = 14\Delta_0^2, \qquad (42)$$

where we have determined the multiplicative constant by requiring that $F = 120$ g for $\theta = 60°$ and $l = 8$ cm. In Fig. 5 the theoretical formulas (41–42) are plotted together with the experimental results of Furnas and Fischer. The theory agrees remarkably well with the experimental results, particularly in view of the fact that the applied load varied by a factor of 10. Although we have chosen the multiplicative constant so that (42) will agree with the experiments in which θ was varied, the same constant provides agreement with the experiments in which l was varied.

This theory can easily be generalized to the case of Z-plasties having limbs of unequal lengths, and differing angles between adjacent limbs. In fact, it can also be applied to the Z-plasty operation where a web is turned into a cleft, described in detail by Furnas (1965) and Furnas and Fischer (1971). For all of these problems, the force F is still given by the formula (42), but we must use more complicated formulas than (41) to calculate Δ_0. The evaluation of Δ_0 is a matter of tedious, but straightforward, geometry. We present below the results for the general Z-plasty on a web, shown in Fig. 6. The details of the derivation of these formulas and a comparison of them with the results of Furnas and Fischer are contained in the thesis of Natarajan (1974):

$$\Delta_0 = l\,\frac{\cos\beta + \cos\delta - \cos\alpha - 1}{1 + \cos\alpha}, \qquad (43)$$

where the angle $BDC = \delta$ is given by

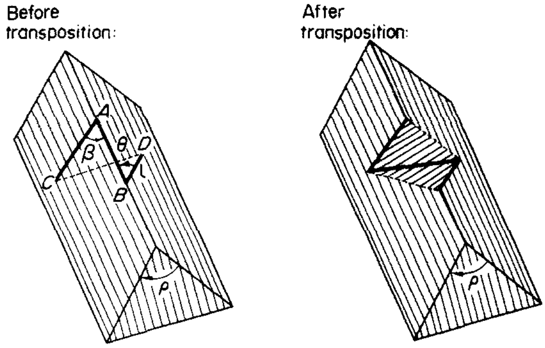

Fig. 6. Z-plasty on a web—geometry.

Here θ and β are the angles between the limbs, ρ is the peak angle between the planes, α is the angle between the skew lines BA and CD. In the case $\rho = \pi$ and $\beta = \theta$ (and therefore $\alpha = \theta + \delta$), the above formulas reduce to our previous result (41).

6. CONCLUSION

We have treated theoretically the limiting case of an extremely thin membrane with fully developed tension field lines. Future experiments will undoubtably show that the tension field lines occupy only a limited portion of the region around an operation on the skin, due to the restraining effects of the foundation over which the skin slides and to the stresses present naturally in the skin. It will be necessary to develop an "incomplete" tension field theory. Hopefully, some of the techniques developed for aircraft structures can be carried over to skin. We refer the prospective investigator to the book by Kuhn (1956) and the references therein.

Acknowledgements—This research was financially supported by NSF Grant GP-38197, and by Public Health Service Research Career Development Award 1 K04 GM70733-01 from the Institute of General Medical Sciences. We gratefully acknowledge the assistance of Dr. E. H. Mansfield for helpful criticism of a preliminary draft of this paper.

REFERENCES

Borges, A. F. (1973) *Elective Incisions and Scar Revision.* Little, Brown & Co., Boston.

Danielson, D. A. (1973) Human skin as an elastic membrane. *J. Biomechanics* **6,** 539–546.

Furnas, D. W. (1965) The tetrahedral Z-plasty. *Plastic and Reconstructive Surgery* **35,** 291–302.

Furnas, D. W. and Fischer, G. W. (1971) The Z-plasty: biomechanics and mathematics. *Br. J. Plastic Surg.* **24,** 144–160.

Iai, T. (1943) Method of solution by means of the theorem of maximum strain energy of the problems of elasticity. *J. Soc. Aeron. Sci. Nippon* **10,** 276.

$$\tan\delta = \frac{\sqrt{2(1-\cos\theta) - [\cos\beta(1-\cos\theta) + \sin\beta\sin\theta\cos\rho]^2}}{1 - \cos\beta(1-\cos\theta) - \sin\beta\sin\theta\cos\rho}. \qquad (44)$$

Kenedi, R. M., Gibson, T. and Daley, C. H. (1965) Bioengineering studies of the human skin—II, *Proc. Symp. on Biomech. and Related Bio-Eng. Topics* (Edited by R. M. Kenedi), pp. 147–158. Pergamon Press, Oxford.
Koiter, W. T. (1973) On the principle of stationary complementary energy in the nonlinear theory of elasticity. *SIAM J. appl. Math.* **25**, 424–434.
Kondo, K. (1938) The general solution of the flat tension field. *J. Soc. Aeron. Sci. Nippon* **5**, 887–901.
Kondo, K., Iai, T., Moriguti, S. and Murasaki, T. (1955) *Tension-field Theory. Memoirs of the Unifying Study of the Basic Problems in Engineering Sciences by Means of Geometry I* (Edited by Kondo, K.), pp. 417–441. Gakujutsu Bunken Fukyu-Kai, Tokyo.
Kuhn, P. (1956) *Stresses in Aircraft and Shell Structures.* McGraw-Hill, New York.
Limberg, A. A. (1929) Skin plasty by transposition of two interchanging triangular flaps. *J. Sovremenoy Chir.* **4**, 22.
Mansfield, E. H. (1969) Tension field theory. *Proc. 12th Int. Cong. Appl. Mech.* (Edited by M. Hetenyi and W. G. Vincenti), pp. 305–320. Springer, New York.
Mansfield, E. H. (1970) Load transfer via a wrinkled membrane. *Proc. Roy. Soc. Lond.* (A)**316**, 269–289.
McGregor, I. A. (1957) The theoretical basis of the Z-plasty. *Br. J. Plastic Surg.* **9**, 256–259.
Muskhelishvili, N. I. (1953) *Some Basic Problems of the Mathematical Theory of Elasticity.* Noordhoff, Groningen, Holland.
Natarajan, S. (1974) Theoretical analysis of the stresses involved in wound repair and Z-plasty surgery, Thesis, Dept. of Appl. Math. and Computer Sci., University of Virginia, Charlottesville, Virginia.
North, J. F. (1972) Impact characteristics of human skin and subcutaneous tissue, Thesis, Bioengineering Unit, University of Strathclyde, Glasgow, Scotland.
Reissner, E. (1938) On tension field theory. *Proc. 5th Int. Congr. Appl. Mech.*, 88–92.
Ridge, M. D. and Wright, V. (1966) The directional effects of skin. *J. invest. Dermatol.* **46**, 341–346.
Stein, M. and Hedgepeth, J. M. (1961) Analysis of partly wrinkled membranes. NASA TN D-813.
Struik, D. J. (1961) *Differential Geometry*, p. 138. Addison-Wesley, Reading, Mass.
Wagner, H. (1929) Flat sheet metal girders with a very thin metal web. *Z. Flugtechn. Motorluftschiffahrt* **20**, 200–314; Translation into English, NACA TM 604–606.

NOMENCLATURE

a, b	Semi-major, semi-minor axes of ellipse (Fig. 1)
C	Elastic constant in (17)
d	Radius of cylinder
D	Distance from x-axis to outer boundary (Fig. 2)
E	Young's modulus
f	Function given by (18)
f_1, f_2	Functions in (10)
F	Total force acting on Z-plasty (Fig. 4)
F_x, F_y	Components of force in x and y directions
g_1, g_2	functions in (38–39)
h_1, h_2	Scale factors in (7)
k	Constant in (27)
K	Gaussian curvature of curved surface
l	Length of limb of Z-plasty (Fig. 4)
m	Ratio defined in (4)
r	Distance from tip of Z-plasty to outer boundary (Fig. 4)
R	Length defined in (4)
s	Distance along x-axis between origin and tension ray (Fig. 2)
S	Function given by (26)
t	Thickness of membrane
u_x, u_y	Components of displacement vector in x and y directions
U	Total strain energy of membrane
x, y	Rectangular coordinates
α	Angle tension ray makes with x-axis (Fig. 2)
α_a	Value of α at $x = a$ (Fig. 2)
β	Angle given by (6) and shown in Fig. 1
γ	Strain measure defined in (13)
δ	angle given by (44)
Δ	Change in length of line of particles which becomes a tension ray
Δ_0	Distance shown in Fig. 4
ϵ	Displacement shown in Fig. 1
ζ	One of principal coordinates (equation 7)
η	Distance measured along a tension ray
η_0	Distance measured on undeformed membrane along line of particles which becomes a tension ray
η_1, η_2	Values of η at outer and inner boundaries (Fig. 2)
θ	Angle between limbs of Z-plasty
κ, μ	Elastic constants given by (5)
ν	Poisson's ratio
ξ	Complex variable given by (3)
ρ	Peak angle between the planes (Fig. 6)
σ	Stress along a tension ray
σ_{max}	Maximum value of stress, given by (28)
σ_1, σ_2	Principal stresses given by (2)
$\sum$	General function in (32)
ϕ	Angle between tension ray and central limb of Z-plasty (Fig. 4)
ϕ_l	Angle between outer tension ray and central limb of Z-plasty
Φ	Strain energy, per unit deformed area.