Applied Mathematical Sciences, Vol. 10, 2016, no. 44, 2189 - 2204
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ams.2016.64158

Analytical Mechanics of Membrane Shells: a Review

Lev N. Rabinskiy

Moscow Aviation Institute (National Research University)
125993, Volokolamskoie shosse 4, A-80, GSP-3, Moscow, Russia

Nadezhda P. Shoumova

Moscow Aviation Institute (National Research University)
125993, Volokolamskoie shosse 4, A-80, GSP-3, Moscow, Russia

Sergey I. Zhavoronok

Institute of Applied Mechanics of Russian Academy of Sciences
125040, Leningradsky prospekt 7, Moscow, Russia

Abstract

In this work, we review main mathematical models of membrane shells as well as the main problem statements of structural membranes theory. The basic assumptions of membrane theories are briefly discussed. Some mathematical problems close to the membrane theory such as the wrinkling of thin shells are considered. An emphasis is made on the analytical models formulated in terms of two-dimensional continua.

Keywords: Shells, Structural Membranes, Statics, Stability, Wrinkling

1 Introduction

Membrane shells are widely used nowadays as various elements of temporary buildings, gas holders [1], aerostatic flying vehicles [2], aerodynamic brakes for re-entry systems [3], light-weight spacecraft structures [4] and in many others engineering applications [5, 6]. Structural membranes can be highly pressurized to conserve the initial geometry [7] or low-pressurized, equipped by the stiffener system [8] protecting the structure from buckling.

In general, a thin shell has to be considered as a membrane shell if:

- a shell has only a *tensile* strength and no *compression* and *bending* strength;
- a shell does not have its own form; the shell shape is determined by the active external force system.

The earliest results concerning the membrane shells theory were obtained by S. Finsterwalder [9, 10]. In the earliest publications [11, 12] as well as in the latest ones [13] membrane shell's states were divided into *biaxial* and *uniaxial* depending on the values of the *principal stresses*. The biaxial membrane state allows one to determine the configuration; contrarily in uniaxial state the shell shape depends entirely on the acting forces. These forces can be subdivided to the statical ones, including the supporting a form of shell, and the payloads of a structure; the final form depends on the combination of the first and second force fields. The loads are often induced by external liquids [13, 15, 16, 17] or gaseous media [18, 19, 20, 21, 22, 24, 25, 26, 27, 28].

1.1 Main problems of a structural membrane theory

The following problems statements can be formulated [9, 11, 12]:

1. The current configuration and the load system are given, and the initial configuration is to be found.
2. The initial configuration, the load system, and the boundary conditions are given, and the current state needs to be found.
3. The current configuration, the loads, and the stresses are given, as well as the load perturbation; the new current configuration must be found.

The first problem can be formulated as a linear boundary value problem for the system of partial differential equations [12]. A solution of this kind of problem allows one to design a membrane on the groundwork of the known loaded state. This problem is defined only for a *biaxial* membrane [12]. The problem of the second kind is essentially nonlinear because of the unknown acting configuration which is needed to formulate the governing equations system. It must be noted that the biaxial state can transform to the uniaxial one under small load variations; the type of the stress state of the shell is initially unknown. Problems of dynamic interaction of membranes with external flows can be reduced to the third class. In case of small load disturbances with respect to the stationary equilibrium state (e. g. the statically pressurized membrane) the initial-boundary value problem of third class can be linearized.

1.2 Main assumptions of membrane shell theory

The "two-dimensional" modeling of deforming membranes is based on a set of assumptions. The full range of theories can be subdivided conditionally to the following subclasses depending on these assumptions:

- inextensible membranes [11, 13, 29];
- extensible [12, 30], highly extensible [31, 32] and hyperelastic [33, 34] membranes;
- a class of networks [35, 36, 37, 38, 39, 40] has to be marked out.

An improved theory of initially pressurized membrane shells made from an *inextensible* material was formulated by A. A. Alekseev [11]. The following basic assumption system was proposed [11, 13]:

1. shell's model consists in a surface S;
2. shell's material is inextensible and heteroresistive:

   $$\varepsilon_\alpha < 0 \Leftrightarrow \sigma_\alpha = 0; \quad \varepsilon_\alpha = 0 \Leftrightarrow \sigma_\alpha \in \mathrm{R}_+,$$

   here ε_α and σ_α are principal strains and stresses, $\alpha = 1, 2$;
3. a reference configuration corresponds to maximum lengths of all line elements on S *and* to maximum volume enveloped by a shell;
4. a reference configuration can be determined accurate to small bending of shell surface;
5. a current configuration corresponds to equilibrium state depending on the acting force system.

For an *extensible* membrane the assumption system must be updated. The compressive stress cannot appear as well as for inextensible membranes, but the assumption Nr. 2 is replaced by the following ones:

- the material is heteroresistive: $\sigma_\alpha = 0 \Leftrightarrow \varepsilon_\alpha < 0$; $\varepsilon_\alpha > 0 \Leftrightarrow \sigma_\alpha = \sigma_\alpha(\varepsilon)$, where the arbitrary constraint equation is introduced,

and the third assumption must be omitted. A wide range of extensible membrane models exists; some of them are discussed below.

2 Tension field theory

2.1 Basics of tension field theory

One of the main problem occurred in the mathematical formulation of the general membrane theory consists in the simultaneous existence of biaxial, uniaxial, and zero-axial stress regions with initially unknown boundaries. For a membrane this means the localized buckling. Thin shells are able to carry only small compressive stresses before buckling; "... if such shells are pulled in one direction and compressed in the other... a large number of high-aspect-ratio buckles forms with creases oriented in the tension direction" ([41], p. 266). While the critical stress goes down in the "membrane limit" of shell model, i. e. if the bending rigidity vanishes, "... the number of buckles becomes very large, thus forming a typical "tension field" " ([41], p. 266) or uniaxial stress state with forming of wrinkles. The appropriate model so-called "Tension Filed Theory" (TFT) "... is a theory of continuously distributed wrinkles in membranes" ([42], p. 35). Thus, "... the underlying idea of TFT is to replace the complex wrinkled surface... with a smoothed out *pseudo surface*... which could be imagined as a continuum distribution of wrinkles... and constitutes a regular surface in space" ([41], p. 266). In other words, the sequences of infinitesimal more-and-more wrinkled states of a membrane appear in the uniaxial area so that "... the limiting state of a fiber might be described as an infinitesimal zig-zag around a mean path" ([43], p. 94). The real curvature in the wrinkled area cannot be determined if the bending stiffness vanishes [35, 39, 42, 44, 45, 46, 47], and TFT only "... deals with the deformation of the *mean membrane surface* and does not attempt to describe the field of displacements normal to this mean surface" ([43], p. 93).

The earliest TFT [35] has been developed by H. Wagner for shear-loaded thin metal webs. E. Reissner formulated his TFT [44] considering the wrinkling regions existing if the compressive stresses appear. The further TFT improvement was proposed by E. H. Mansfield [46] an D. J. Steigmann [47].

2.2 Wrinkling criteria

The membrane theory requires the determination of the biaxial, uniaxial, and slacked zones; the wrinkling becomes a "... structural answer in order to avoid compression stresses" ([48], p. 2552) in the membrane. Three main types of criteria were proposed:

- Principal Stress Criterion [49, 50, 51, 52]:

$$\begin{aligned} \sigma_2 > 0 \quad &\Rightarrow \quad \text{biaxial (taut) state} \\ \sigma_1 > 0,\ \sigma_2 < 0 \quad &\Rightarrow \quad \text{uniaxial (wrinkled) state} \\ \sigma_1 < 0 \quad &\Rightarrow \quad \text{zero} - \text{axial (slacked) state} \end{aligned}$$

– the constitutive equations are expressed through stresses; this criterion yields "... a misjudgement of the real membrane status" ([48], p. 2553);

- Principal Strain Criterion [50, 51, 52, 53]:

$$\begin{aligned} \varepsilon_1 \geq 0,\ \varepsilon_2 \geq \nu\varepsilon_1 &\ \Rightarrow\ \text{biaxial (taut) state} \\ \varepsilon_1 \geq 0,\ \varepsilon_2 \leq -\nu\varepsilon_1 &\ \Rightarrow\ \text{uniaxial (wrinkled) state} \\ \varepsilon_1 \leq 0 &\ \Rightarrow\ \text{zero} - \text{axial (slacked) state} \end{aligned}$$

– this criterion properly estimates the membrane status, but it can be applied only to *isotropic* membranes;

- Mixed Criterion (D. G. Roddeman, [54, 55, 56, 57]):

$$\begin{aligned} \sigma_2 > 0 &\ \Rightarrow\ \text{biaxial (taut) state} \\ \varepsilon_1 > 0,\ \sigma_2 \leq 0 &\ \Rightarrow\ \text{uniaxial (wrinkled) state} \\ \varepsilon_1 \leq 0 &\ \Rightarrow\ \text{zero} - \text{axial (slacked) state} \end{aligned}$$

– this criterion is useable for anisotropic materials.

The wrinkling criteria have to be a basis to construct the constitutive equations. For instance, Wagner's theory [35] uses a principal strain $\varepsilon_1 > 0$ as a constitutive variable, and the principal stresses are defined by the Hookean law. The Reissner theory [44] is based on the "artificial", or "temporary" anisotropy of the membrane material where the axis of anisotropy coincide with the principal stresses directions. This way allows one to account for the wrinkling but is treated by some authors as a "not physical model" [48]. Several improved approaches were proposed to model the stress state of wrinkled structural membranes; the most useful of them are briefly discussed below.

2.3 Strain tensor modification

A *modified strain tensor* concept [48, 54, 55, 58, 59, 60] was proposed assuming that the wrinkling state does not affect the principal stress directions $\mathbf{s}_{1,2}$ [59]. To describe a new fictitious non-wrinkled configuration a modified deformation gradient is derived on the basis of a "wrinkling parameter" $\beta = (l' - l)/l$ [48] where l corresponds to the real length of a wrinkled element, and l' to the length of the fictitious non-wrinkled one in the principal strain direction $\mathbf{e}_2$ orthogonal to the wrinkles direction. The modified deformation gradient $\mathbf{F}'$ and the Green-Lagrange strain $\mathbf{E}'$ can be introduced as follows:

$$\mathbf{F}' = (\mathbf{I} - \beta \mathbf{e}_1 \otimes \mathbf{e}_1) \cdot \mathbf{F}; \quad 2\mathbf{E}' = \mathbf{F}'^T \cdot \mathbf{F}' - \mathbf{I},$$

here $\mathbf{I}$ is the second-rank unit tensor. The stress tensor construction can be based on the modified Green-Lagrange tensor $\mathbf{E}'$. L. Tänzer and al. used the representation referred to the initial membrane configuration [60] contrarily to the formulation [55] referred to the actual configuration. An improved algorithm of the search for the wrinkling direction was proposed in [57].

The improved formulation on the basis of the mixed wrinkling criterion and allowing one to model anisotropic membranes was proposed in [48]. The modified strain is represented as the sum of the elastic $\mathbf{E}^e$ and wrinkling $\mathbf{E}^w$ terms, therefore the fictitious wrinkling strain can be defined as follows:

$$\mathbf{E}' = \mathbf{E}^e + \mathbf{E}^w \quad \Rightarrow \quad \mathbf{E}^w = \mathbf{E}' - S_1 \mathbf{C}^{-1} : (\mathbf{s}_1 \otimes \mathbf{s}_1),$$

where $\mathbf{C}$ is the fourth-rank elasticity tensor and S_1 is the maximum principal stress of Piola-Kirchhoff and $\mathbf{s}_1$ is its direction. The transformation into the principal strains eigenvector space and accounting of the second principal stress allows one to improve the convergence of the numerical simulation [48].

2.4 Membrane's material modification

This approach does not change any kinematical variable but modifies *constitutive tensors*. A method so-called "stiffness/compliance modification" uses the variable Poisson ratio to modify the stiffness matrix $\mathbf{D}_k$ in the wrinkling direction [53, 61, 62]. The 3×3 "effective elasticity matrix" is defined as

$$\mathbf{D}_1 = E/(1-\lambda^2)\mathbf{D}',\ D'_{11} = D'_{22} = 1,\ D'_{12} = D'_{21} = \lambda,\ D'_{33} = (1-\lambda)/2$$

[53], where λ is the "effective Poisson's ratio" that is expressed as $\lambda = -\varepsilon_2/\varepsilon_1$ for an uniaxial stress state [61]. An improved algorithm allowing one to eliminate the singularity due to $\lambda = 1$ was proposed in [53] where the new effective matrix $\mathbf{D}_2$ depends on the angle of rotation of principal strains $\varepsilon_1, \varepsilon_2$:

$$\mathbf{D}_k = \begin{cases} \mathbf{0}, & \varepsilon_1 \le 0; \\ \mathbf{D}_2, & 0 < \varepsilon_1,\ \nu\varepsilon_1 < -\varepsilon_2; \\ \mathbf{D}_1,\ \lambda = \nu & \text{otherwise.} \end{cases}$$

Another way consists in the use of a penalty parameter; "...whenever compression occurs...the corresponding components of the constitutive tensor in the direction of the compressive stress are penalized to weaken the compressive stiffness" ([63], p. 773). This approach was developed to use together with the finite element method and allows one to use a coarse mesh in wrinkled regions; to avoid the singularity of the stiffness matrix the small but nonzero stiffness in the direction of the minimum principal stress is assigned [64].

An efficient computational strategy can be based on the analogy between wrinkling and plastic strain in a material [63, 65]:

- the summary strain is represented as $\mathbf{E}' = \mathbf{E}^e + \mathbf{E}^p$ as for elastic-plastic materials, but $\mathbf{E}^p \equiv \mathbf{E}^w$ here;

- vanishing of the compressive stiffness in the wrinkling direction is similar to the perfect plasticity with no hardening;
- the plastic flow direction coincides with the wrinkling direction $\mathbf{w}$.

Thus, the second-rank tensor $\mathbf{C}^{ew}$ of the effective stiffness can be expressed through the constitutive one $\mathbf{C}$ as follows ([63], p. 786):

$$\mathbf{C}^{ew} = \mathbf{C} - \left(1 - P^2\right) (\mathbf{w} \cdot \mathbf{C} \cdot \mathbf{w})^{-1} [(\mathbf{C} \cdot \mathbf{w}) \otimes (\mathbf{w} \cdot \mathbf{C})]$$

where P is the penalty factor allowing a small compressive stiffness to stabilize the computation.

2.5 Relaxed energy approach

The improved TFT of Reissner [44] extended to the nonlinear deformed state was developed by Alan C. Pipkin who introduced a new minimum energy principle "...in such a way that it deals with carriers of crinkles directly" ([43], p. 94) where the *crinkles* are infinitesimally wrinkled states of a membrane. A quasiconvexification approach was used to construct the equivalent strain-energy function for a membrane in presence of wrinkled zones [45, 47]:

$$W_R\left(\lambda_1, \lambda_2\right) = \begin{cases} W\left(\lambda_1, v(\lambda_2)\right), & \lambda_1 > 1,\ \lambda_2 < v(\lambda_1); \\ W\left(v(\lambda_1), \lambda_2\right), & \lambda_2 > 1,\ \lambda_1 < v(\lambda_2); \\ 0, & \lambda_1 \le 0,\ \lambda_2 \le 0, \end{cases}$$

where W is the strain energy expressed through the principal stretches λ_1, λ_2, and $v(\lambda_\gamma)$ is the "natural width" of a material strip in its simple tension along the γ-th axis [45, 47]. It was shown that the appropriate choice of the "relaxed energy density" [45] allows one to obtain the main assumptions directly from the energy function, for instance, no compressive stress can appear, e t.c.

It has to be noted that most of the known papers deal with the plane membranes contrarily to several earlier works ([11, 13] and others).

3 Improved modeling of membrane wrinkling

3.1 Analytical modeling of wrinkling

The modeling of the local wrinkling is one of main problems of the modern membrane theory; A. Libai has noted (1992) that "...more effort should be put into "incomplete tension fields" which describes the transition from shell to wrinkled membrane ([41], p. 267). The improved statements as well as the solution methods of the wrinkling problem will be considered below.

In general, this problem was posed for networks [35, 36, 37, 40, 42, 43], membranes [44, 46, 66], and also for shells with small but nonvanishing bending stiffness [67, 68]. Two types of wrinkling patterns were found [35, 69], the observed near the critical load so-called "near threshold" pattern with deformations being "...small perturbations of the initial flat state" [70] and the "far-from-threshold" pattern of a thin-walled structure "...carrying a...load well in excess of the initial buckling value" ([53], p. 631). The first one can be described as the well-known postbuckling by the appropriate equations, whereas the second one "...requires a nonstandard perturbation theory around the singular membrane limit" ([70], p. 18227), i. e. for the stress vanishing in the compression direction. The tension field theory [44] and its further developments such as the "relaxed energy" approach [42, 45] mentioned above "...correctly characterize the stress distribution and the corresponding extent of wrinkled region, but cannot identify...the wavelength...of wrinkles, their amplitude" ([70], p. 18227). Thus, the improved tension field theory is able to find the *conditions of infinitesimal wrinkling*, but do not deals with the *finite wrinkling* (similarly to the linearized buckling theory that predicts the critical loads but predicts not the postbuckled state).

In [70, 71] the Föppl–von Kármán shell theory equations were used as a groundwork to describe the "far-from-threshold" wrinkling regime of a thin sheet. The asymptotics was constructed using two dimensionless parameters, a confinement α and a bendablity $\varepsilon \sim D$ where D is the traditional bending stiffness, for highly bendable sheets ($D \to 0$ and $\varepsilon \to 0$). It has been shown that "...the bendability parameter radically affects the wrinkle pattern" ([71], p. 9717, and [70]). An interesting analogy with the fluid mechanics was found, where "...the Reynolds number defines the viscous (low Re) and inviscid (high Re) limits of the Navier-Stokes equations" ([71], p. 9718). In the shell wrinkling problem, the bendability parameter ε defines two limits of Föppl–von Kármán equations and the mentioned wrinkling regimes. In the low-bendability regime (corresponding to a shell with finite bending stiffness D) a proper description of the wrinkling is allowed on the basis of a traditional buckling model, so that "...the sheet accomodates compression even in the buckled state" ([71], p. 9717) and the state $\sigma_2 < 0$ remains possible. Quite the contrary in the high-bendability regime "...only tiny levels of of compression which vanishes at $\varepsilon \to 0$ can be supported" ([71], p. 9717), i. e. *the shell becomes a membrane with no compression strength*. Thus, the wrinkling of a shell with finite stretching modulus and zero bending modulus can be *qualitatively* predicted by an "...perturbation theory around the singular membrane limit...wrinkles do not cost any bending energy and compression can be completely relaxed" ([71], p. 9717-9718) using the "tension field theories" [44, 69] or "relaxed energy" [42, 45]. Contrarily to this one, an asymptotic approach used in [70, 71] and introduced earlier for the uniaxial tension problem in [67, 68] allows one to obtain not only the extent of wrinkled region from the leading order of the expansion but also the wavelength and amplitude of wrinkles from the subleading order ([70], p. 18227). The wrinkling of curved sheets on deforming substrate was considered in [72] where the wavelength λ estimate given in [68] has been constructed on the basis of the variable effective stiffness; also the problem of crystalline sheet confinement on spherically shaped substrates was solved in [73] using the approach [70, 71].

3.2 Numerical and applied wrinkling models

The other side of the wrinkling problem in membrane theory consists in the numerical simulation of the wrinkled state of a structural membrane. Three types of wrinkling modeling can be noted; "...the most detailed but...most difficult approach is a three dimensional analysis" ([48], p. 2551).

The "two-dimensional" methods and wrinkling models useful for the numerical simulation of real structures were developed on the groundwork of the general tension field theory [35, 44] and two different approaches [74], the "deformation tensor modification" and the "stiffness/compliance modification"; the appropriate literature survey can be found in [75] and in [76].

These approaches allow one to find the wrinkling area and direction but do not allow one to obtain neither wrinkling wavelengths nor wrinkling amplitudes; "...to enable the *computational* modeling of detailed wrinkling deformations both membrane and bending stiffness must be considered in the analytical model based one geometrically nonlinear analysis with large displacements and rotations" ([74], p. 1517) as it was realized, for instance, in [77]; in [78] the finite element approach was replaced by the distributed transfer function method. Several authors attempted to use the buckling approach applied to the nonlinear highly flexible shell model [79]. The further investigations have shown that the direct use of the post-buckling computation scheme yields inevitable convergence problems [74] due to the specificity of wrinkling mechanics described analytically [70] – [73]. A FSI numerical approach was proposed in [27]; other methods are also useful [28].

4 Conclusions

The brief review of main results obtained in the structural membranes modeling shows that the direct numerical simulation providing *quantitative* results is not yet well developed due to high computer power required and convergence problems, while the analytical methods of *qualitative* analysis seem to be underestimated. The further improvement of the tension field theory can be achieved on the groundwork of the approaches of analytical dynamics of unilaterally constrained systems [80] extended to two-dimensional continua.

Acknowledgements. Authors are supported by the Russian Foundation for Basic Researches, grants No.16-01-00751-a and No. 14-01-00480-a.

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Received: May 8, 2016; Published: July 6, 2016