

## Chapter 1

# Example problem: SUPG-stabilised solution of the 2D advection diffusion equation

In this example we discuss the SUPG-stabilised solution of the 2D advection-diffusion problem

### Two-dimensional advection-diffusion problem in a rectangular domain

Solve

$$\text{Pe} \sum_{i=1}^2 w_i(x_1, x_2) \frac{\partial u}{\partial x_i} = \sum_{i=1}^2 \frac{\partial^2 u}{\partial x_i^2} + f(x_1, x_2), \quad (1)$$

in the rectangular domain  $D = \{(x_1, x_2) \in [0, 1] \times [0, 2]\}$ , with Dirichlet boundary conditions

$$u|_{\partial D} = u_0, \quad (2)$$

where the *Peclet number*,  $\text{Pe}$  the boundary values,  $u_0$ , the source function  $f(x_1, x_2)$ , and the components of the "wind"  $w_i(x_1, x_2)$  ( $i = 1, 2$ ) are given.

We set  $f(x_1, x_2) = 0$  and assign the boundary conditions such that

$$u_0(x_1, x_2) = \tanh(1 - \alpha(x_1 \tan \Phi - x_2)), \quad (3)$$

For large values of  $\alpha$ , this boundary data approaches a step, oriented at an angle  $\Phi$  against the  $x_1$ -axis.

In the computations we will impose the "wind"

$$\mathbf{w}(x_1, x_2) = \begin{pmatrix} \sin(6x_2) \\ \cos(6x_1) \end{pmatrix}, \quad (4)$$

illustrated in this vector plot:



Figure 1.1 Plot of the wind.

The figures below show plots of the solution for  $\Phi = 45^\circ$ ,  $\alpha = 50$  and a Peclet number of  $Pe = 200$ , with and without SUPG stabilisation. The wire-mesh plot shows the solution computed on a  $10 \times 10$  mesh, the shaded surface represents the solution obtained from an unstabilised solution on a  $150 \times 150$  mesh. Note how SUPG stabilisation "suppresses the wiggles" on the relatively coarse mesh.



Figure 1.2 Plot of the SUPG-stabilised solution at different levels of mesh refinement.



Figure 1.3 Plot of the unstabilised solution at different levels of mesh refinement.

## 1.1 The driver code

Overall, the structure of the driver code is very similar to that used for the [problem without stabilisation](#).

### 1.1.1 To be written:

- Discuss SUPG theory.
- Implementation and the role of basis, shape and test functions (our equations are isoparametric)

Until we get around to completing this example, here's the driver code. Fairly self-explanatory, isn't it?

```
//LIC// =====
//LIC// This file forms part of oomph-lib, the object-oriented,
//LIC// multi-physics finite-element library, available
//LIC// at http://www.oomph-lib.org.
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//LIC// Foundation, Inc., 51 Franklin Street, Fifth Floor, Boston, MA
//LIC// 02110-1301 USA.
//LIC//
//LIC// The authors may be contacted at oomph-lib@maths.man.ac.uk.
//LIC//
//LIC//=====
//Driver for a simple 2D adv diff problem with SUPG stabilisation
//Generic routines
#include "generic.h"
// The Poisson equations
#include "advection_diffusion.h"
// The mesh
#include "meshes/rectangular_quadmesh.h"
using namespace std;
```

```

using namespace oomph;

//=====start_of_namespace=====
/// Namespace for global parameters: Unforced problem with
/// boundary values corresponding to a steep tanh step profile
/// oriented at 45 degrees across the domain.
//=====
namespace GlobalPhysicalParameters
{
    /// Peclet number
    double Peclet=200.0;

    /// Parameter for steepness of step in boundary values
    double Alpha=50.0;

    /// Parameter for angle of step in boundary values: 45 degrees
    double TanPhi=1.0;

    /// Some "solution" for assignment of boundary values
    void get_boundary_values(const Vector<double>& x, Vector<double>& u)
    {
        u[0]=tanh(1.0-Alpha*(TanPhi*x[0]-x[1]));
    }

    /// Zero source function
    void source_function(const Vector<double>& x_vect, double& source)
    {
        source=0.0;
    }

    /// Wind
    void wind_function(const Vector<double>& x, Vector<double>& wind)
    {
        wind[0]=sin(6.0*x[1]);
        wind[1]=cos(6.0*x[0]);
    }
} // end of namespace

////////////////////////////////////
////////////////////////////////////
////////////////////////////////////

//===== start_of_problem_class=====
/// 2D AdvectionDiffusion problem on rectangular domain, discretised
/// with refineable 2D QAdvectionDiffusion elements. The specific type
/// of element is specified via the template parameter.
//=====
template<class ELEMENT>
class SUPGAdvectionDiffusionProblem : public Problem
{
public:

    /// Constructor: Pass pointer to source and wind functions, and
    /// flag to indicate if stabilisation is to be used.
    SUPGAdvectionDiffusionProblem(
        AdvectionDiffusionEquations<2>::AdvectionDiffusionSourceFctPt source_fct_pt,
        AdvectionDiffusionEquations<2>::AdvectionDiffusionWindFctPt wind_fct_pt,
        const bool& use_stabilisation);

    /// Destructor. Empty
    ~SUPGAdvectionDiffusionProblem() {}

    /// Update the problem specs before solve: Reset boundary conditions
    /// to the values from the tanh solution and compute stabilisation
    /// parameter.
    void actions_before_newton_solve();

    /// Update the problem after solve (empty)
    void actions_after_newton_solve() {}

    /// Doc the solution.
    void doc_solution();

    /// Overloaded version of the problem's access function to
    /// the mesh. Recasts the pointer to the base Mesh object to
    /// the actual mesh type.
    RectangularQuadMesh<ELEMENT>* mesh_pt()
    {
        return dynamic_cast<RectangularQuadMesh<ELEMENT>*>(
            Problem::mesh_pt());
    }
private:

    /// DocInfo object
    DocInfo Doc_info;

```

```

/// Pointer to source function
AdvectionDiffusionEquations<2>::AdvectionDiffusionSourceFctPt Source_fct_pt;

/// Pointer to wind function
AdvectionDiffusionEquations<2>::AdvectionDiffusionWindFctPt Wind_fct_pt;

/// Flag to indicate if stabilisation is to be used
bool Use_stabilisation;
}; // end of problem class
//=====start_of_constructor=====
/// Constructor for AdvectionDiffusion problem: Pass pointer to
/// source function and wind functions and flag to indicate
/// if stabilisation is to be used.
//=====
template<class ELEMENT>
SUPGAdvectionDiffusionProblem<ELEMENT>::SUPGAdvectionDiffusionProblem(
    AdvectionDiffusionEquations<2>::AdvectionDiffusionSourceFctPt source_fct_pt,
    AdvectionDiffusionEquations<2>::AdvectionDiffusionWindFctPt wind_fct_pt,
    const bool& use_stabilisation)
: Source_fct_pt(source_fct_pt), Wind_fct_pt(wind_fct_pt),
  Use_stabilisation(use_stabilisation)
{
    // Set output directory
    if (use_stabilisation)
    {
        Doc_info.set_directory("RESLT_stabilised");
    }
    else
    {
        Doc_info.set_directory("RESLT_unstabilised");
    }
    // Setup mesh
    // # of elements in x-direction
    unsigned n_x=40;
    // # of elements in y-direction
    unsigned n_y=40;
    // Domain length in x-direction
    double l_x=1.0;
    // Domain length in y-direction
    double l_y=2.0;
    // Build and assign mesh
    Problem::mesh_pt() =
        new RectangularQuadMesh<ELEMENT>(n_x,n_y,l_x,l_y);

    // Set the boundary conditions for this problem: All nodes are
    // free by default -- only need to pin the ones that have Dirichlet
    // conditions here
    unsigned num_bound = mesh_pt()->nboundary();
    for(unsigned ibound=0;ibound<num_bound;ibound++)
    {
        unsigned num_nod= mesh_pt()->nboundary_node(ibound);
        for (unsigned inod=0;inod<num_nod;inod++)
        {
            mesh_pt()->boundary_node_pt(ibound,inod)->pin(0);
        }
    } // end loop over boundaries

    // Complete the build of all elements so they are fully functional
    // Loop over the elements to set up element-specific
    // things that cannot be handled by the (argument-free!) ELEMENT
    // constructor: Pass pointer to source function
    unsigned n_element = mesh_pt()->nelement();
    for(unsigned i=0;i<n_element;i++)
    {
        // Upcast from GeneralisedElement to the present element
        ELEMENT *el_pt = dynamic_cast<ELEMENT*>(mesh_pt()->element_pt(i));
        //Set the source function pointer
        el_pt->source_fct_pt() = Source_fct_pt;
        //Set the wind function pointer
        el_pt->wind_fct_pt() = Wind_fct_pt;
        // Set the Peclet number
        el_pt->pe_pt() = &GlobalPhysicalParameters::Peclet;
    }
    // Setup equation numbering scheme
    cout <<"Number of equations: " << assign_eqn_numbers() << std::endl;
} // end of constructor
//=====start_of_actions_before_newton_solve=====
/// Update the problem specs before solve: (Re-)set boundary conditions
//=====
template<class ELEMENT>
void SUPGAdvectionDiffusionProblem<ELEMENT>::actions_before_newton_solve()
{
    // How many boundaries are there?
    unsigned num_bound = mesh_pt()->nboundary();

    //Loop over the boundaries

```

```

for(unsigned ibound=0;ibound<num_bound;ibound++)
{
    // How many nodes are there on this boundary?
    unsigned num_nod=mesh_pt()->nboundary_node(ibound);
    // Loop over the nodes on boundary
    for (unsigned inod=0;inod<num_nod;inod++)
    {
        // Get pointer to node
        Node* nod_pt=mesh_pt()->boundary_node_pt(ibound,inod);
        // Extract nodal coordinates from node:
        Vector<double> x(2);
        x[0]=nod_pt->x(0);
        x[1]=nod_pt->x(1);
        // Get boundary value
        Vector<double> u(1);
        GlobalPhysicalParameters::get_boundary_values(x,u);
        // Assign the value to the one (and only) nodal value at this node
        nod_pt->set_value(0,u[0]);
    }
}
// Now loop over all elements and set the stabilisation parameter
unsigned n_element = mesh_pt()->nelement();
for(unsigned i=0;i<n_element;i++)
{
    // Upcast from GeneralisedElement to the present element
    ELEMENT *el_pt = dynamic_cast<ELEMENT*>(mesh_pt()->element_pt(i));

    // Use stabilisation?
    if (Use_stabilisation)
    {
        //Compute stabilisation parameter
        el_pt->compute_stabilisation_parameter();
    }
    else
    {
        //Compute stabilisation parameter
        el_pt->switch_off_stabilisation();
    }
}
} // end of actions before solve
//=====start_of_doc=====
/// Doc the solution
//=====
template<class ELEMENT>
void SUPGAdvectionDiffusionProblem<ELEMENT>::doc_solution()
{
    ofstream some_file;
    char filename[100];
    // Number of plot points: npts x npts
    unsigned npts=5;
    // Output solution
    //-----
    sprintf(filename,"%s/soln%i.dat",Doc_info.directory().c_str(),
        Doc_info.number());
    some_file.open(filename);
    mesh_pt()->output(some_file,npts);
    some_file.close();
} // end of doc
//===== start_of_main=====
/// Driver code for 2D AdvectionDiffusion problem
//=====
int main()
{
    //Set up the problem with stabilisation
    {
        bool use_stabilisation=true;

        // Create the problem with 2D nine-node elements from the
        // QAdvectionDiffusionElement family. Pass pointer to
        // source and wind function.
        SUPGAdvectionDiffusionProblem<QSUPGAdvectionDiffusionElement<2,3> >
            problem(&GlobalPhysicalParameters::source_function,
                &GlobalPhysicalParameters::wind_function,
                use_stabilisation);

        // Solve the problem
        problem.newton_solve();

        //Output the solution
        problem.doc_solution();
    }

    //Set up the problem without stabilisation
    {

```

```
bool use_stabilisation=false;

// Create the problem with 2D nine-node elements from the
// QAdvectionDiffusionElement family. Pass pointer to
// source and wind function.
SUPGAdvectionDiffusionProblem<QSUPGAdvectionDiffusionElement<2,3> >
problem(&GlobalPhysicalParameters::source_function,
        &GlobalPhysicalParameters::wind_function,
        use_stabilisation);

// Solve the problem
problem.newton_solve();

//Output the solution
problem.doc_solution();

}

} // end of main
```

---

## 1.2 Source files for this tutorial

- The source files for this tutorial are located in the directory:

`demo_drivers/advection_diffusion/two_d_adv_diff_SUPG/`

- The driver code is:

`demo_drivers/advection_diffusion/two_d_adv_diff_SUPG/two_d_adv_diff_↵  
SUPG.cc`

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## 1.3 PDF file

A [pdf version](#) of this document is available.