

Chapter 1

Example problem: The spatially-adaptive solution of the azimuthally Fourier-decomposed 3D Helmholtz equation

In this document we discuss the spatially-adaptive finite-element-based solution of the 3D Helmholtz equation in cylindrical polar coordinates, using a Fourier-decomposition of the solution in the azimuthal direction.

The driver code is very similar to the one discussed in [another tutorial](#) – the main purpose of the current tutorial is to demonstrate the use of spatial adaptivity on unstructured meshes.

1.1 A specific example

We will solve the azimuthally Fourier-decomposed Helmholtz equation

$$\nabla^2 u_N(r, z) + \left(k^2 - \frac{N^2}{r^2}\right) u_N(r, z) = 0, \quad (1)$$

where N is the azimuthal wavenumber, in the finite domain $1 < \sqrt{r^2 + z^2} < 3$. We impose the Sommerfeld radiation condition at the outer boundary of the computational domain at $\sqrt{r^2 + z^2} = 3$, using a Dirichlet-to-Neumann mapping, and apply flux boundary condition on the surface of the unit-sphere (where $\sqrt{r^2 + z^2} = 1$) such that the exact solution is given by

$$u_N(r, z) = u_N^{[exact]}(r, z) = \sum_{l=N}^{N_{\text{terms}}} h_l^{(1)}(k\sqrt{r^2 + z^2}) P_l^N\left(\frac{z}{\sqrt{r^2 + z^2}}\right).$$

This solution corresponds to the superposition of several outgoing waves that emerge from the unit sphere.

The two plots below show a comparison between the exact and computed solutions for $N_{\text{terms}} = 6$, a Fourier wavenumber of $N = 1$, and a (squared) Helmholtz wavenumber of $k^2 = 10$.



Figure 1.1 Plot of the exact (green) and computed (red) real parts of the solution of the Fourier-decomposed Helmholtz equation for $N=1$ and a wavenumber of $k^2 = 10$.



Figure 1.2 Plot of the exact (green) and computed (red) imaginary parts of the solution of the Fourier-decomposed Helmholtz equation for $N=1$ and a wavenumber of $k^2 = 10$.

1.2 The numerical solution

The driver code for this problem is very similar to the one discussed in [another tutorial](#).
Running `sdiff` on the driver codes

```
demo_drivers/fourier_decomposed_helmholtz/sphere_scattering/sphere_↵
scattering.cc
```

and

demo_drivers/fourier_decomposed_helmholtz/sphere_scattering/unstructured_↵
sphere_scattering.cc

shows the main differences required to discretise the computational domain with an adaptive, unstructured mesh:

- The provision of the functions `actions_before/after_adapt()` to detach/re-attach the `Face↵` `Elements` that are used to enforce the Neumann boundary conditions before and after every spatial adaptation, and to pass the physical parameters to the newly created bulk elements.
- The generation of an unstructured mesh whose curvilinear boundaries are represented by `GeomObjects` – this ensures that the domain boundaries become increasingly well resolved under mesh refinement.

That's all!

1.3 Code listing

Here's a listing of the complete driver code:

```
//LIC// =====
//LIC// This file forms part of oomph-lib, the object-oriented,
//LIC// multi-physics finite-element library, available
//LIC// at http://www.oomph-lib.org.
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//LIC// 02110-1301 USA.
//LIC//
//LIC// The authors may be contacted at oomph-lib@maths.man.ac.uk.
//LIC//
//LIC//=====
//Driver for Fourier-decomposed Helmholtz problem
#include <complex>
#include <cmath>
//Generic routines
#include "generic.h"
// The Helmholtz equations
#include "fourier_decomposed_helmholtz.h"

// The mesh
#include "meshes/triangle_mesh.h"
// Get the Bessel functions
#include "oomph_crbond_bessel.h"
using namespace oomph;
using namespace std;

////////////////////////////////////
////////////////////////////////////
////////////////////////////////////

//==== start_of_namespace_planar_wave=====
/// Namespace to test representation of planar wave in spherical
/// polars
//=====
namespace PlanarWave
{

    /// Number of terms in series
    unsigned N_terms=100;

    /// Wave number
    double K=3.0*MathematicalConstants::Pi;

    /// Imaginary unit
    std::complex<double> I(0.0,1.0);
```

```

/// Exact solution as a Vector of size 2, containing real and imag parts
void get_exact_u(const Vector<double>& x, Vector<double>& u)
{
    // Switch to spherical coordinates
    double R=sqrt(x[0]*x[0]+x[1]*x[1]);

    double theta;
    theta=atan2(x[0],x[1]);

    // Argument for Bessel/Hankel functions
    double kr = K*R;

    // Need half-order Bessel functions
    double bessel_offset=0.5;
    // Evaluate Bessel/Hankel functions
    Vector<double> jv(N_terms);
    Vector<double> yv(N_terms);
    Vector<double> djv(N_terms);
    Vector<double> dyv(N_terms);
    double order_max_in=double(N_terms-1)+bessel_offset;
    double order_max_out=0;

    // This function returns vectors containing
    // J_k(x), Y_k(x) and their derivatives
    // up to k=order_max, with k increasing in
    // integer increments starting with smallest
    // positive value. So, e.g. for order_max=3.5
    // jv[0] contains J_{1/2}(x),
    // jv[1] contains J_{3/2}(x),
    // jv[2] contains J_{5/2}(x),
    // jv[3] contains J_{7/2}(x).
    CRBond_Bessel::bessjyv(order_max_in,
                           kr,
                           order_max_out,
                           &jv[0],&yv[0],
                           &djv[0],&dyv[0]);

    // Assemble exact solution (actually no need to add terms
    // below i=N_fourier as Legendre polynomial would be zero anyway)
    complex<double> u_ex(0.0,0.0);
    for(unsigned i=0;i<N_terms;i++)
    {
        //Associated Legendre functions
        double p=Legendre_functions_helper::plgndr2(i,0,cos(theta));

        // Set exact solution
        u_ex+=(2.0*i+1.0)*pow(I,i)*
            sqrt(MathematicalConstants::Pi/(2.0*kr))*jv[i]*p;
    }

    // Get the real & imaginary part of the result
    u[0]=u_ex.real();
    u[1]=u_ex.imag();
}

//end of get_exact_u

/// Plot
void plot()
{
    unsigned nr=20;
    unsigned nz=100;
    unsigned nt=40;
    ofstream some_file("planar_wave.dat");
    for (unsigned i_t=0;i_t<nt;i_t++)
    {
        double t=2.0*MathematicalConstants::Pi*double(i_t)/double(nt-1);
        some_file << "ZONE I=" << nz << ", J=" << nr << std::endl;

        Vector<double> x(2);
        Vector<double> u(2);
        for (unsigned i=0;i<nr;i++)
        {
            x[0]=0.001+double(i)/double(nr-1);
            for (unsigned j=0;j<nz;j++)
            {
                x[1]=double(j)/double(nz-1);
                get_exact_u(x,u);
                complex<double> uu=complex<double>(u[0],u[1])*exp(-I*t);
                some_file << x[0] << " " << x[1] << " "
                    << uu.real() << " " << uu.imag() << "\n";
            }
        }
    }
}

```

```

////////////////////////////////////
////////////////////////////////////
////////////////////////////////////

//==== start_of_namespace=====
// Namespace for the Fourier decomposed Helmholtz problem parameters
//=====
namespace ProblemParameters
{
    /// \short Square of the wavenumber
    double K_squared=10.0;

    /// Fourier wave number
    int N_fourier=3;

    /// Number of terms in computation of DtN boundary condition
    unsigned Nterms_for_DtN=6;

    /// Number of terms in the exact solution
    unsigned N_terms=6;

    /// Coefficients in the exact solution
    Vector<double> Coeff(N_terms,1.0);

    /// Imaginary unit
    std::complex<double> I(0.0,1.0);

    /// Exact solution as a Vector of size 2, containing real and imag parts
    void get_exact_u(const Vector<double>& x, Vector<double>& u)
    {
        // Switch to spherical coordinates
        double R=sqrt(x[0]*x[0]+x[1]*x[1]);

        double theta;
        theta=atan2(x[0],x[1]);

        // Argument for Bessel/Hankel functions
        double kr = sqrt(K_squared)*R;

        // Need half-order Bessel functions
        double bess_offset=0.5;
        // Evaluate Bessel/Hankel functions
        Vector<double> jv(N_terms);
        Vector<double> yv(N_terms);
        Vector<double> djv(N_terms);
        Vector<double> dyv(N_terms);
        double order_max_in=double(N_terms-1)+bess_offset;
        double order_max_out=0;

        // This function returns vectors containing
        // J_k(x), Y_k(x) and their derivatives
        // up to k=order_max, with k increasing in
        // integer increments starting with smallest
        // positive value. So, e.g. for order_max=3.5
        // jv[0] contains J_{1/2}(x),
        // jv[1] contains J_{3/2}(x),
        // jv[2] contains J_{5/2}(x),
        // jv[3] contains J_{7/2}(x).
        CRBond_Bessel::bessjyv(order_max_in,
                               kr,
                               order_max_out,
                               &jv[0],&yv[0],
                               &djv[0],&dyv[0]);

        // Assemble exact solution (actually no need to add terms
        // below i=N_fourier as Legendre polynomial would be zero anyway)
        complex<double> u_ex(0.0,0.0);
        for(unsigned i=N_fourier;i<N_terms;i++)
        {
            //Associated_legendre_functions
            double p=Legendre_functions_helper::plgndr2(i,N_fourier,
                                                         cos(theta));

            // Set exact solution
            u_ex+=Coeff[i]*sqrt(MathematicalConstants::Pi/(2.0*kr))*(jv[i]+I*yv[i])*p;
        }

        // Get the real & imaginary part of the result
        u[0]=u_ex.real();
        u[1]=u_ex.imag();
    }
} //end of get_exact_u

/// \short Get -du/dr (spherical r) for exact solution. Equal to prescribed
/// flux on inner boundary.
void exact_minus_dudr(const Vector<double>& x, std::complex<double>& flux)

```

```

{
    // Initialise flux
    flux=std::complex<double>(0.0,0.0);

    // Switch to spherical coordinates
    double R=sqrt(x[0]*x[0]+x[1]*x[1]);

    double theta;
    theta=atan2(x[0],x[1]);

    // Argument for Bessel/Hankel functions
    double kr=sqrt(K_squared)*R;
    // Helmholtz wavenumber
    double k=sqrt(K_squared);
    // Need half-order Bessel functions
    double bessel_offset=0.5;
    // Evaluate Bessel/Hankel functions
    Vector<double> jv(N_terms);
    Vector<double> yv(N_terms);
    Vector<double> djv(N_terms);
    Vector<double> dyv(N_terms);
    double order_max_in=double(N_terms-1)+bessel_offset;
    double order_max_out=0;

    // This function returns vectors containing
    // J_k(x), Y_k(x) and their derivatives
    // up to k=order_max, with k increasing in
    // integer increments starting with smallest
    // positive value. So, e.g. for order_max=3.5
    // jv[0] contains J_{1/2}(x),
    // jv[1] contains J_{3/2}(x),
    // jv[2] contains J_{5/2}(x),
    // jv[3] contains J_{7/2}(x).
    CRBond_Bessel::bessjyv(order_max_in,
                           kr,
                           order_max_out,
                           &jv[0],&yv[0],
                           &djv[0],&dyv[0]);

    // Assemble exact solution (actually no need to add terms
    // below i=N_fourier as Legendre polynomial would be zero anyway)
    complex<double> u_ex(0.0,0.0);
    for(unsigned i=N_fourier;i<N_terms;i++)
    {
        //Associated Legendre functions
        double p=Legendre_functions_helper::plgndr2(i,N_fourier,
                                                    cos(theta));

        // Set flux of exact solution
        flux+=Coeff[i]*sqrt(MathematicalConstants::Pi/(2.0*kr))*p*
            ( k*(djv[i]+I*dyv[i]) - (0.5*(jv[i]+I*yv[i])/R) );
    }

    // end of exact_normal_derivative

} // end of namespace

////////////////////////////////////
////////////////////////////////////
////////////////////////////////////

//===== start_of_problem_class=====
/// Problem class
//=====
template<class ELEMENT>
class FourierDecomposedHelmholtzProblem : public Problem
{
public:

    /// Constructor
    FourierDecomposedHelmholtzProblem();

    /// Destructor (empty)
    ~FourierDecomposedHelmholtzProblem() {}

    /// Update the problem specs before solve (empty)
    void actions_before_newton_solve() {}

    /// Update the problem after solve (empty)
    void actions_after_newton_solve() {}

    /// \short Doc the solution. DocInfo object stores flags/labels for where the
    /// output gets written to
    void doc_solution(DocInfo& doc_info);

```

```

/// Recompute gamma integral before checking Newton residuals
void actions_before_newton_convergence_check()
{
    if (!CommandLineArgs::command_line_flag_has_been_set("--square_domain"))
    {
        Helmholtz_outer_boundary_mesh_pt->setup_gamma();
    }
}

/// Actions before adapt: Wipe the mesh of prescribed flux elements
void actions_before_adapt();

/// Actions after adapt: Rebuild the mesh of prescribed flux elements
void actions_after_adapt();

/// Check gamma computation
void check_gamma(DocInfo& doc_info);

private:

/// \short Create BC elements on outer boundary
void create_outer_bc_elements();

/// Create flux elements on inner boundary
void create_flux_elements_on_inner_boundary();

/// \short Delete boundary face elements and wipe the surface mesh
void delete_face_elements( Mesh* const & boundary_mesh_pt)
{
    // Loop over the surface elements
    unsigned n_element = boundary_mesh_pt->nelement();
    for(unsigned e=0;e<n_element;e++)
    {
        // Kill surface element
        delete boundary_mesh_pt->element_pt(e);
    }

    // Wipe the mesh
    boundary_mesh_pt->flush_element_and_node_storage();
}

#ifdef ADAPTIVE

/// Pointer to the "bulk" mesh
RefineableTriangleMesh<ELEMENT>* Bulk_mesh_pt;
#else

/// Pointer to the "bulk" mesh
TriangleMesh<ELEMENT>* Bulk_mesh_pt;
#endif

/// \short Pointer to mesh containing the DtN boundary
/// condition elements
FourierDecomposedHelmholtzDtNMesh<ELEMENT>* Helmholtz_outer_boundary_mesh_pt;

/// on the inner boundary
Mesh* Helmholtz_inner_boundary_mesh_pt;

/// Trace file
ofstream Trace_file;
}; // end of problem class
//=====start_of_actions_before_adapt=====
/// Actions before adapt: Wipe the mesh of face elements
//=====
template<class ELEMENT>
void FourierDecomposedHelmholtzProblem<ELEMENT>::actions_before_adapt()
{
    // Kill the flux elements and wipe the boundary meshes
    if (!CommandLineArgs::command_line_flag_has_been_set("--square_domain"))
    {
        delete_face_elements(Helmholtz_outer_boundary_mesh_pt);
    }
    delete_face_elements(Helmholtz_inner_boundary_mesh_pt);
    // Rebuild the Problem's global mesh from its various sub-meshes
    rebuild_global_mesh();
} // end of actions_before_adapt
//=====start_of_actions_after_adapt=====
/// Actions after adapt: Rebuild the face element meshes
//=====
template<class ELEMENT>
void FourierDecomposedHelmholtzProblem<ELEMENT>::actions_after_adapt()
{
    // Complete the build of all elements so they are fully functional

    // Loop over the Helmholtz bulk elements to set up element-specific
    // things that cannot be handled by constructor: Pass pointer to

```

```
// wave number squared
unsigned n_element = Bulk_mesh_pt->nelement();
for(unsigned e=0;e<n_element;e++)
{
    // Upcast from GeneralisedElement to Helmholtz bulk element
    ELEMENT *el_pt = dynamic_cast<ELEMENT*>(Bulk_mesh_pt->element_pt(e));

    //Set the k_squared pointer
    el_pt->k_squared_pt() = &ProblemParameters::K_squared;
    // Set pointer to Fourier wave number
    el_pt->fourier_wavenumber_pt()=&ProblemParameters::N_fourier;
}
// Create prescribed-flux elements and BC elements
// from all elements that are adjacent to the boundaries and add them to
// Helmholtz_boundary_meshe
create_flux_elements_on_inner_boundary();
if (!CommandLineArgs::command_line_flag_has_been_set("--square_domain"))
{
    create_outer_bc_elements();
}
// Rebuild the Problem's global mesh from its various sub-meshe
rebuild_global_mesh();

} // end of actions_after_adapt
//=====start_of_constructor=====
// Constructor for Fourier-decomposed Helmholtz problem
//=====
template<class ELEMENT>
FourierDecomposedHelmholtzProblem<ELEMENT>::
FourierDecomposedHelmholtzProblem()
{
    // Open trace file
    Trace_file.open("RESLT/trace.dat");

    // Create circles representing inner and outer boundary
    double x_c=0.0;
    double y_c=0.0;
    double r_min=1.0;
    double r_max=3.0;
    Circle* inner_circle_pt=new Circle(x_c,y_c,r_min);
    Circle* outer_circle_pt=new Circle(x_c,y_c,r_max);

    // Edges/boundary segments making up outer boundary
    //-----
    Vector<TriangleMeshCurveSection*> outer_boundary_line_pt(4);

    // Number of segments used for representing the curvilinear boundaries
    unsigned n_segments = 20;

    // All poly boundaries are defined by two vertices
    Vector<Vector<double> > boundary_vertices(2);

    // Bottom straight boundary on symmetry line
    //-----
    boundary_vertices[0].resize(2);
    boundary_vertices[0][0]=0.0;
    boundary_vertices[0][1]=-r_min;
    boundary_vertices[1].resize(2);
    boundary_vertices[1][0]=0.0;
    boundary_vertices[1][1]=-r_max;
    unsigned boundary_id=0;
    outer_boundary_line_pt[0]=
    new TriangleMeshPolyLine(boundary_vertices,boundary_id);
    if (CommandLineArgs::command_line_flag_has_been_set("--square_domain"))
    {
        // Square outer boundary:
        //-----
        Vector<Vector<double> > boundary_vertices(4);
        boundary_vertices[0].resize(2);
        boundary_vertices[0][0]=0.0;
        boundary_vertices[0][1]=-r_max;
        boundary_vertices[1].resize(2);
        boundary_vertices[1][0]=r_max;
        boundary_vertices[1][1]=-r_max;
        boundary_vertices[2].resize(2);
        boundary_vertices[2][0]=r_max;
        boundary_vertices[2][1]=r_max;
        boundary_vertices[3].resize(2);
        boundary_vertices[3][0]=0.0;
        boundary_vertices[3][1]=r_max;
        boundary_id=1;
        outer_boundary_line_pt[1]=
        new TriangleMeshPolyLine(boundary_vertices,boundary_id);
    }
    else
    {
        // Outer circular boundary:

```



```

//-----
// The intrinsic coordinates for the beginning and end of the curve
double s_start = -0.5*MathematicalConstants::Pi;
double s_end   = 0.5*MathematicalConstants::Pi;

boundary_id = 1;
outer_boundary_line_pt[1]=
    new TriangleMeshCurviline(outer_circle_pt,
                               s_start,
                               s_end,
                               n_segments,
                               boundary_id);
}
// Top straight boundary on symmetry line
//-----
boundary_vertices[0][0]=0.0;
boundary_vertices[0][1]=r_max;
boundary_vertices[1][0]=0.0;
boundary_vertices[1][1]=r_min;
boundary_id=2;
outer_boundary_line_pt[2]=
    new TriangleMeshPolyLine(boundary_vertices,boundary_id);

// Inner circular boundary:
//-----

// The intrinsic coordinates for the beginning and end of the curve
double s_start = 0.5*MathematicalConstants::Pi;
double s_end   = -0.5*MathematicalConstants::Pi;

boundary_id = 3;
outer_boundary_line_pt[3]=
    new TriangleMeshCurviline(inner_circle_pt,
                               s_start,
                               s_end,
                               n_segments,
                               boundary_id);

// Create closed curve that defines outer boundary
//-----
TriangleMeshClosedCurve *outer_boundary_pt =
    new TriangleMeshClosedCurve(outer_boundary_line_pt);

// Use the TriangleMeshParameters object for helping on the manage of the
// TriangleMesh parameters. The only parameter that needs to take is the
// outer boundary.
TriangleMeshParameters triangle_mesh_parameters(outer_boundary_pt);

// Specify maximum element area
double element_area = 0.1;
triangle_mesh_parameters.element_area() = element_area;

#ifdef ADAPTIVE

// Build "bulk" mesh
Bulk_mesh_pt=new RefineableTriangleMesh<ELEMENT>(triangle_mesh_parameters);
// Create/set error estimator
Bulk_mesh_pt->spatial_error_estimator_pt()=new Z2ErrorEstimator;

// Choose error tolerances to force some uniform refinement
Bulk_mesh_pt->min_permitted_error()=0.00004;
Bulk_mesh_pt->max_permitted_error()=0.0001;
#else
// Pass the TriangleMeshParameters object to the TriangleMesh one
Bulk_mesh_pt= new TriangleMesh<ELEMENT>(triangle_mesh_parameters);
#endif
// Check what we've built so far...
Bulk_mesh_pt->output("mesh.dat");
Bulk_mesh_pt->output_boundaries("boundaries.dat");

if (!CommandLineArgs::command_line_flag_has_been_set("--square_domain"))
{
    // Create mesh for DtN elements on outer boundary
    Helmholtz_outer_boundary_mesh_pt=
        new FourierDecomposedHelmholtzDtNMesh<ELEMENT>(
            r_max,ProblemParameters::Nterms_for_DtN);

    // Populate it with elements
    create_outer_bc_elements();
}
// Create flux elements on inner boundary
Helmholtz_inner_boundary_mesh_pt=new Mesh;
create_flux_elements_on_inner_boundary();

// Add the several sub meshes to the problem
add_sub_mesh(Bulk_mesh_pt);

```

```

add_sub_mesh(Helmholtz_inner_boundary_mesh_pt);
if (!CommandLineArgs::command_line_flag_has_been_set("--square_domain"))
{
    add_sub_mesh(Helmholtz_outer_boundary_mesh_pt);
}
// Build the Problem's global mesh from its various sub-meshes
build_global_mesh();
// Complete the build of all elements so they are fully functional
unsigned n_element = Bulk_mesh_pt->nelement();
for(unsigned i=0;i<n_element;i++)
{
    // Upcast from GeneralisedElement to the present element
    ELEMENT *el_pt = dynamic_cast<ELEMENT*>(Bulk_mesh_pt->element_pt(i));

    //Set the k_squared pointer
    el_pt->k_squared_pt()=&ProblemParameters::K_squared;

    // Set pointer to Fourier wave number
    el_pt->fourier_wavenumber_pt()=&ProblemParameters::N_fourier;
}

// Setup equation numbering scheme
cout <<"Number of equations: " << assign_eqn_numbers() << std::endl;
} // end of constructor
//=====start_of_check_gamma=====
// Check gamma computation: \f$ \gamma = -du/dn \f$
//=====
template<class ELEMENT>
void FourierDecomposedHelmholtzProblem<ELEMENT>::check_gamma(DocInfo& doc_info)
{
    // Compute gamma stuff
    Helmholtz_outer_boundary_mesh_pt->setup_gamma();

    ofstream some_file;
    char filename[100];
    sprintf(filename,"%s/gamma_test%i.dat",doc_info.directory().c_str(),
            doc_info.number());
    some_file.open(filename);

    //first loop over elements e
    unsigned nel=Helmholtz_outer_boundary_mesh_pt->nelement();
    for (unsigned e=0;e<nel;e++)
    {
        // Get a pointer to element
        FourierDecomposedHelmholtzDtNBoundaryElement<ELEMENT>* el_pt=
            dynamic_cast<FourierDecomposedHelmholtzDtNBoundaryElement<ELEMENT>*>
            (Helmholtz_outer_boundary_mesh_pt->element_pt(e));

        //Set the value of n_intpt
        const unsigned n_intpt =el_pt->integral_pt()->nweight();

        // Get gamma at all gauss points in element
        Vector<std::complex<double>> gamma(
            Helmholtz_outer_boundary_mesh_pt->gamma_at_gauss_point(el_pt));

        //Loop over the integration points
        for(unsigned ipt=0;ipt<n_intpt;ipt++)
        {
            //Allocate and initialise coordiante
            Vector<double> x(el_pt->dim()+1,0.0);

            //Set the Vector to hold local coordinates
            unsigned n=el_pt->dim();
            Vector<double> s(n,0.0);
            for(unsigned i=0;i<n;i++)
            {
                s[i]=el_pt->integral_pt()->knot(ipt,i);
            }

            //Get the coordinates of the integration point
            el_pt->interpolated_x(s,x);

            complex<double> flux;
            ProblemParameters::exact_minus_dudr(x,flux);
            some_file << atan2(x[0],x[1]) << " "
                << gamma[ipt].real() << " "
                << gamma[ipt].imag() << " "
                << flux.real() << " "
                << flux.imag() << " "
                << std::endl;

        } // end of loop over integration points
    } // end of loop over elements
    some_file.close();
}

```

```

//end of output_gamma
//=====start_of_doc=====
// Doc the solution: doc_info contains labels/output directory etc.
//=====
template<class ELEMENT>
void FourierDecomposedHelmholtzProblem<ELEMENT>::doc_solution(DocInfo& doc_info)
{
    ofstream some_file;
    char filename[100];
    // Number of plot points: npts x npts
    unsigned npts=5;

    // Output solution
    //-----
    sprintf(filename,"%s/soln%i.dat",doc_info.directory().c_str(),
            doc_info.number());
    some_file.open(filename);
    Bulk_mesh_pt->output(some_file,npts);
    some_file.close();
    // Output exact solution
    //-----
    sprintf(filename,"%s/exact_soln%i.dat",doc_info.directory().c_str(),
            doc_info.number());
    some_file.open(filename);
    Bulk_mesh_pt->output_fct(some_file,npts,ProblemParameters::get_exact_u);
    some_file.close();

    // Doc error and return of the square of the L2 error
    //-----
    double error,norm;
    sprintf(filename,"%s/error%i.dat",doc_info.directory().c_str(),
            doc_info.number());
    some_file.open(filename);
    Bulk_mesh_pt->compute_error(some_file,ProblemParameters::get_exact_u,
                             error,norm);
    some_file.close();

    // Doc L2 error and norm of solution
    cout << "\nNorm of error : " << sqrt(error) << std::endl;
    cout << "Norm of solution: " << sqrt(norm) << std::endl << std::endl;

    // Write norm of solution to trace file
    Bulk_mesh_pt->compute_norm(norm);
    Trace_file << norm << std::endl;
    if (!CommandLineArgs::command_line_flag_has_been_set("--square_domain"))
    {
        // Check gamma computation
        check_gamma(doc_info);
    }
} // end of doc
//=====start_of_create_outer_bc_elements=====
// Create BC elements on outer boundary
//=====
template<class ELEMENT>
void FourierDecomposedHelmholtzProblem<ELEMENT>::create_outer_bc_elements()
{
    // Outer boundary is boundary 1:
    unsigned b=1;
    // Loop over the bulk elements adjacent to boundary b
    unsigned n_element = Bulk_mesh_pt->nboundary_element(b);
    for(unsigned e=0;e<n_element;e++)
    {
        // Get pointer to the bulk element that is adjacent to boundary b
        ELEMENT* bulk_elem_pt = dynamic_cast<ELEMENT*>(
            Bulk_mesh_pt->boundary_element_pt(b,e));

        //Find the index of the face of element e along boundary b
        int face_index = Bulk_mesh_pt->face_index_at_boundary(b,e);

        // Build the corresponding DtN element
        FourierDecomposedHelmholtzDtNBoundaryElement<ELEMENT>* flux_element_pt = new
            FourierDecomposedHelmholtzDtNBoundaryElement<ELEMENT>(bulk_elem_pt,
                                                                    face_index);

        //Add the flux boundary element to the helmholtz_outer_boundary_mesh
        Helmholtz_outer_boundary_mesh_pt->add_element_pt(flux_element_pt);
        // Set pointer to the mesh that contains all the boundary condition
        // elements on this boundary
        flux_element_pt->
            set_outer_boundary_mesh_pt(Helmholtz_outer_boundary_mesh_pt);
    }
} // end of create_outer_bc_elements
//=====start_of_create_flux_elements=====
// Create flux elements on inner boundary
//=====

```

```

template<class ELEMENT>
void FourierDecomposedHelmholtzProblem<ELEMENT>::
create_flux_elements_on_inner_boundary()
{
    // Apply flux bc on inner boundary (boundary 3)
    unsigned b=3;
    // Loop over the bulk elements adjacent to boundary b
    unsigned n_element = Bulk_mesh_pt->nboundary_element(b);
    for(unsigned e=0;e<n_element;e++)
    {
        // Get pointer to the bulk element that is adjacent to boundary b
        ELEMENT* bulk_elem_pt = dynamic_cast<ELEMENT*>(
            Bulk_mesh_pt->boundary_element_pt(b,e));

        //Find the index of the face of element e along boundary b
        int face_index = Bulk_mesh_pt->face_index_at_boundary(b,e);

        // Build the corresponding prescribed incoming-flux element
        FourierDecomposedHelmholtzFluxElement<ELEMENT>* flux_element_pt = new
            FourierDecomposedHelmholtzFluxElement<ELEMENT>(bulk_elem_pt,face_index);

        //Add the prescribed incoming-flux element to the surface mesh
        Helmholtz_inner_boundary_mesh_pt->add_element_pt(flux_element_pt);

        // Set the pointer to the prescribed flux function
        flux_element_pt->flux_fct_pt() = &ProblemParameters::exact_minus_dudr;
    } //end of loop over bulk elements adjacent to boundary b

} // end of create flux elements on inner boundary
//===== start_of_main=====
/// Driver code for Fourier decomposed Helmholtz problem
//=====
int main(int argc, char **argv)
{
    // Store command line arguments
    CommandLineArgs::setup(argc,argv);
    // Define possible command line arguments and parse the ones that
    // were actually specified

    // Square domain without DtN
    CommandLineArgs::specify_command_line_flag("--square_domain");
    // Parse command line
    CommandLineArgs::parse_and_assign();

    // Doc what has actually been specified on the command line
    CommandLineArgs::doc_specified_flags();
    // Check if the claimed representation of a planar wave in
    // the tutorial is correct -- of course it is!
    //PlanarWave::plot();
    // Test Bessel/Hankel functions
    //-----
    {
        // Number of Bessel functions to be computed
        unsigned n=3;

        // Offset of Bessel function order (less than 1!)
        double bessell_offset=0.5;

        ofstream bessely_file("bessely.dat");
        ofstream bessely_deriv_file("dbessely.dat");

        ofstream besselj_file("besselJ.dat");
        ofstream besselj_deriv_file("dbesselJ.dat");

        // Evaluate Bessel/Hankel functions
        Vector<double> jv(n+1);
        Vector<double> yv(n+1);
        Vector<double> djv(n+1);
        Vector<double> dyv(n+1);
        double x_min=0.5;
        double x_max=5.0;
        unsigned nplot=100;
        for (unsigned i=0;i<nplot;i++)
        {
            double x=x_min+(x_max-x_min)*double(i)/double(nplot-1);
            double order_max_in=double(n)+bessell_offset;
            double order_max_out=0;

            // This function returns vectors containing
            // J_k(x), Y_k(x) and their derivatives
            // up to k=order_max, with k increasing in
            // integer increments starting with smallest
            // positive value. So, e.g. for order_max=3.5
            // jv[0] contains J_{1/2}(x),
            // jv[1] contains J_{3/2}(x),
            // jv[2] contains J_{5/2}(x),
            // jv[3] contains J_{7/2}(x).

```

```

CRBond_Bessel::bessjyv(order_max_in,x,
                      order_max_out,
                      &jv[0],&yv[0],
                      &djv[0],&dyv[0]);

bessely_file << x << " ";
for (unsigned j=0;j<=n;j++)
{
    bessely_file << yv[j] << " ";
}
bessely_file << std::endl;

besselj_file << x << " ";
for (unsigned j=0;j<=n;j++)
{
    besselj_file << jv[j] << " ";
}
besselj_file << std::endl;

bessely_deriv_file << x << " ";
for (unsigned j=0;j<=n;j++)
{
    bessely_deriv_file << dyv[j] << " ";
}
bessely_deriv_file << std::endl;

besselj_deriv_file << x << " ";
for (unsigned j=0;j<=n;j++)
{
    besselj_deriv_file << djv[j] << " ";
}
besselj_deriv_file << std::endl;

}
bessely_file.close();
besselj_file.close();
bessely_deriv_file.close();
besselj_deriv_file.close();
}

// Test Legrendre Polynomials
//-----
{
    // Number of lower indices
    unsigned n=3;

    ofstream some_file("legendre3.dat");
    unsigned nplot=100;
    for (unsigned i=0;i<nplot;i++)
    {
        double x=double(i)/double(nplot-1);
        some_file << x << " ";
        for (unsigned j=0;j<=n;j++)
        {
            some_file << Legendre_functions_helper::plgndr2(n,j,x) << " ";
        }
        some_file << std::endl;
    }
    some_file.close();
}

#ifdef ADAPTIVE
// Create the problem with 2D six-node elements from the
// TFourierDecomposedHelmholtzElement family.
FourierDecomposedHelmholtzProblem<ProjectableFourierDecomposedHelmholtzElement<
    TFourierDecomposedHelmholtzElement<3> > > problem;
#else

// Create the problem with 2D six-node elements from the
// TFourierDecomposedHelmholtzElement family.
FourierDecomposedHelmholtzProblem<TFourierDecomposedHelmholtzElement<3> >
    problem;

#endif
// Create label for output
DocInfo doc_info;

// Set output directory
doc_info.set_directory("RESLT");

// Solve for a few Fourier wavenumbers
for (ProblemParameters::N_fourier=0;ProblemParameters::N_fourier<4;
     ProblemParameters::N_fourier++)
{
    // Step number
    doc_info.number()=ProblemParameters::N_fourier;
}

```

```

#ifdef ADAPTIVE
// Max. number of adaptations
unsigned max_adapt=1;

// Solve the problem with Newton's method, allowing
// up to max_adapt mesh adaptations after every solve.
problem.newton_solve(max_adapt);
#else
// Solve the problem
problem.newton_solve();
#endif

//Output the solution
problem.doc_solution(doc_info);
}

} //end of main

```

1.4 Source files for this tutorial

- The source files for this tutorial are located in the directory:

`demo_drivers/fourier_decomposed_helmholtz/sphere_scattering/`

- The driver code is:

`demo_drivers/fourier_decomposed_helmholtz/sphere_↵
scattering/unstructured_sphere_scattering.cc`

1.5 PDF file

A [pdf version](#) of this document is available.