Journal of Algebra 452 (2016) 401–426

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Automorphisms of restricted parabolic trees and Sylow p-subgroups of the finitary symmetric group

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A R T I C L E I N F O

Article history:
Received 29 October 2014
Available online 4 February 2016
Communicated by Martin Liebeck

MSC:
20B27
20E08
20B22
20B35
20F65
20B07

Keywords:
Finitary symmetric group
Restricted parabolic trees
Automorphism groups of forests
Sylow p-subgroups

A B S T R A C T

In the paper we introduce the notion of a k-adic restricted parabolic tree D_k and investigate the group $Aut\ D_k$ of automorphisms of this tree. In particular, we characterize the Sylow p-subgroups in the subgroup $Aut_f\ D_p$ of finitary automorphisms of a p-adic restricted parabolic tree. Then we use the characterization for the classification of Sylow p-subgroups in the finitary symmetric group $FS_{\mathbb{N}}$.

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http://dx.doi.org/10.1016/j.jalgebra.2015.12.019

1. Introduction

In the past few decades the groups acting on trees have found many applications, especially in group theory, harmonic analysis, geometry, dynamics and representation theory. For instance, in group theory the groups acting on trees provide constructions of groups with specific properties. These constructions usually employ the group of automorphisms of an infinite homogeneous rooted tree.

Let (T, v_0) denote the rooted tree with root v_0. The set $V(T)$ of all vertices of T is partitioned into subsets of vertices lying at the same distance to the root v_0. The set L_i of vertices on distance i to the root is called the *i-th level* of the tree. The tree is called *spherically homogeneous*, if for every i there exists a number k_i such that for every vertex $v \in L_i$ the number of elements of L_{i+1} which are adjacent to v is equal to k_i. In this case the vector $\overline{k}_n = (k_1, k_2, \dots, k_{n-1})$ is called the *spherical index* of T, and the tree is denoted by $(T_{\overline{k}_n}, v_0)$. If $\overline{k}_n = (k, k, \dots, k)$, then $(T_{\overline{k}_n}, v_0)$ is called k-adic and denoted by $(T_{k,n}, v_0)$.

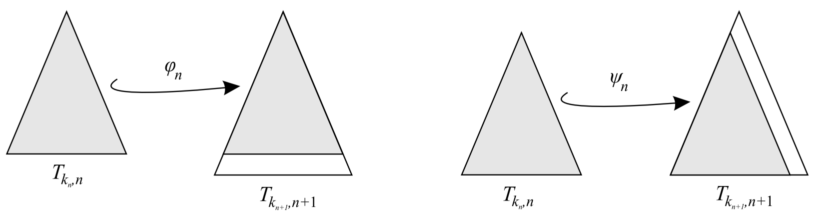
The infinite homogeneous rooted tree $(T_{\overline{k}}, v_0)$ with root v_0 and spherical index $\overline{k} = (k_1, k_2, \dots)$ is the direct limit

$$(T_{\overline{k}}, v_0) \cong \varinjlim_{n} \left((T_{\overline{k}_n}, v_0), \varphi_n\right),$$

of finite homogeneous rooted trees $(T_{\overline{k}_n}, v_0)$ with root v_0, spherical index $\overline{k}_n = (k_1, k_2, \dots, k_{n-1})$ and embeddings $\varphi_n : (T_{\overline{k}_n}, v_0) \hookrightarrow (T_{\overline{k}_{n+1}}, v_0)$ shown in Fig. 1a.

The group $Aut\,(T_k, v_0)$ of automorphisms of an infinite k-adic ($k \geq 2$) rooted tree (T_k, v_0) is an object of particular interest and has been widely investigated. For instance, it contains subgroups which are just infinite groups, groups of intermediate growth or Burnside type groups. A lot of interesting results have been obtained in this direction by L. Bartholdi, R. Grigorchuk, S. Sidki, V. Nekrashevych and others (see e.g. [2,18]). Moreover, certain groups acting on infinite rooted trees initialized the studies of self-similar group actions on spaces [3,15]. Another interesting result concerns the distribution of orders of random elements and their Hausdorff dimension of automorphism groups of the infinite homogeneous rooted tree [1].

The group $Aut\,(T_k, v_0)$ is profinite and hence the Sylow theorems are valid: for every prime p there exists a Sylow p-subgroup of $Aut\,(T_k, v_0)$ (in a topological sense) and



(a) Embedding φ_n (b) Embedding ψ_n

Fig. 1. Embeddings of homogeneous rooted trees.

every two Sylow p-subgroups are conjugated in $Aut\,(T_k, v_0)$. In particular, all Sylow p-subgroups of $Aut\,(T_p, v_0)$ are isomorphic to $P_\infty = \underset{i=1}{\overset{\infty}{\wr}} C_p^{(i)}$.

In the case of locally finite groups, the Sylow theory is much more complicated and the classification of all Sylow p-subgroups is hardly possible [13]. For example, the results in [14] show that it is impossible to classify all Sylow p-subgroups in the Hall's universal locally finite group. In our paper we address this problem in the group of automorphisms of a certain infinite tree, whose construction seems also quite natural. To begin, we propose another embedding of finite homogeneous rooted trees $\psi_n : (T_{\overline{k}_n}, v_n) \hookrightarrow (T_{\overline{k}_{n+1}}, v_{n+1})$ shown in Fig. 1b. Then we construct the direct limit

$$D_{\overline{k}} = \varinjlim_{n} \left((T_{\overline{k}_n}, v_n), \psi_n\right)$$

which is an infinite tree, different from $(T_{\overline{k}}, v_0)$. In contrast to $(T_{\overline{k}}, v_0)$, the tree $D_{\overline{k}}$ has no root, but does have leaves. Moreover, the horospheres of $D_{\overline{k}}$, which correspond to levels of $(T_{\overline{k}}, v_0)$, are infinite. It seems that this tree has not been investigated so far, even though it has interesting properties, significantly different than the properties of $(T_{\overline{k}}, v_0)$. It is also a particular example of a parabolic tree, while the concept of parabolic trees is fundamental in the investigations of tree lattices [5].

Our main results involve the characterization of the group $Aut\,D_{\overline{k}}$ of automorphisms of the tree $D_{\overline{k}}$ and its applications. We give a thorough description of the group and list some of its properties. In particular, we prove the following

Theorem 1 *(Properties of $Aut\ D_{\overline{k}}$).*

1. *The group $Aut\ D_{\overline{k}}$ is a product of its subgroups $Aut_f\ D_{\overline{k}}$ of finitary automorphisms and the stabilizer $Stab(v_0)$ of the vertex v_0:*

$$Aut\ D_{\overline{k}} = Aut_f\ D_{\overline{k}} \cdot Stab(v_0).$$

2. *The subgroup $Aut_f\ D_{\overline{k}}$ is locally finite and not residually finite.*
3. *The subgroup $Stab(v_0)$ is residually finite and the group $Aut\ D_{\overline{k}}$ is locally residually finite.*
4. *The group $Aut(T_k, w)$ isomorphically embeds into $Stab(v_0) \leq Aut\,D_{k+1}$ and the group $Stab(v_0) \leq Aut\ D_k$ isomorphically embeds into $Aut\ (T_k, w)$.*

Then we investigate in detail the subgroup $Aut_f\ D_k$ of finitary automorphisms of the k-adic restricted parabolic tree D_k. In particular, in the case $k = p$ is a prime, we provide a complete characterization of the Sylow p-subgroups in the group $Aut_f\ D_p$.

Theorem 2 *(Sylow p-subgroups of $Aut_f\ D_p$).*

1. *Every Sylow p-subgroup of $Aut_f\ D_p$ acts transitively on every horosphere H_i, $i = 0, 1, \ldots,$ of the tree D_p.*
2. *Every two Sylow p-subgroups of $Aut_f\ D_p$ are locally conjugated in $Aut_f\ D_p$ and conjugated in $Aut\ D_p$.*
3. *Every finitary totally imprimitive uniserial p-group embeds in every Sylow p-subgroup of $Aut_f\ D_p$.*

The notion of a totally imprimitive uniserial p-group was introduced by P. Neumann in [16] and refers to a p-group acting on a set A, such that for every index $i \in \mathbb{N}$ there exists a unique system of imprimitivity on A with blocks of size p^i.

The action of an arbitrary automorphism of D_p induces a permutation on every horosphere of the tree and, moreover, it is determined by a permutation of vertices of H_0. As the horospheres of D_p are countable, each automorphism of D_p may be identified with a permutation on the set $\mathbb{N}$, while a finitary automorphism corresponds to a finitary permutation. Thus each Sylow p-subgroup of $Aut_f\ D_p$ may be identified with a p-subgroup of $FS_{\mathbb{N}}$. We use this observation for the characterization of transitive Sylow p-subgroups of the finitary symmetric group $FS_{\mathbb{N}}$.

Theorem 3 *(Transitive Sylow p-subgroups of $FS_{\mathbb{N}}$).*

1. *By identification of all vertices of the horosphere H_0 with the elements from the set $\mathbb{N}$, every Sylow p-subgroup of $Aut_f\ D_p$ is a transitive Sylow p-subgroup of $FS_{\mathbb{N}}$.*
2. *Every two transitive Sylow p-subgroups of $FS_{\mathbb{N}}$ are conjugated in $S_{\mathbb{N}}$.*

Theorem 3 shows that a weak Sylow condition on maximal p-subgroups holds in the class of transitive subgroups of $FS_{\mathbb{N}}$. Recall that in a finite symmetric group S_n the Sylow p-subgroups are either all transitive (in case n being a prime power) or all intransitive (otherwise). Thus in the finitary symmetric group we find both of these kinds of Sylow p-subgroups. The Sylow p-subgroups of $FS_{\mathbb{N}}$ were characterized in the sixties by Ivanuta [11] in combinatorial terms.

Here we propose a geometric approach to characterization of all Sylow p-subgroups of $FS_{\mathbb{N}}$. Namely, we characterize all the subgroups as groups of automorphisms of a specially constructed forest $\mathcal{F}_{\mathcal{A}}$.

Theorem 4 *(Intransitive Sylow p-subgroups of $FS_{\mathbb{N}}$).*

1. *For every p-forest $\mathcal{F}_{\mathcal{A}}$ and the choice Ψ of the respective Sylow p-subgroups of groups of automorphisms of particular trees, the group $P(\mathcal{F}_{\mathcal{A}}, \Psi)$ is a Sylow p-subgroup of $FS_{\mathbb{N}}$.*

2. *For every Sylow p-subgroup Q of the finitary symmetric group there exist a forest $\mathcal{F}$ and a choice Ψ of respective Sylow p-subgroups in the groups of automorphisms of the constituent trees, such that $Q = P(\mathcal{F}, \Psi)$.*
3. *Let $A = P(\mathcal{F}_A, \Psi_A)$ and $B = P(\mathcal{F}_B, \Psi_B)$ be Sylow p-subgroups of $FS_{\mathbb{N}}$, defined respectively by the forests $\mathcal{F}_A$ and $\mathcal{F}_B$ and the choices Ψ_A and Ψ_B of the Sylow p-subgroups. Then A and B are isomorphic if and only if the forests $\mathcal{F}_A$ and $\mathcal{F}_B$ are isomorphic.*
   In particular, A and B are isomorphic if and only if A and B are conjugated in $S_{\mathbb{N}}$.

The paper is organized as follows. In Section 2 we introduce the notion of a restricted parabolic tree and describe a particular construction of such tree. Then we discuss the properties of these trees, especially indicating the differences between them and infinite rooted trees.

Section 3 contains the characterization of the group of automorphisms of the restricted parabolic tree. We introduce a canonical representation for an automorphism of D_k and distinguish two subgroups in $Aut\, D_k$: the group $Aut_f\, D_k$ of finitary automorphisms and the stabilizer $Stab(v_0)$ of the trunk. Then we list our observations on the properties of $Aut_f\, D_k$ and prove Theorem 1.

In Section 4 we investigate Sylow p-subgroups of $Aut_f\, D_p$ and we prove Theorem 2. Transitive and intransitive Sylow p-subgroups of $FS_{\mathbb{N}}$ are discussed in Section 5 along with our proofs of Theorems 3 and 4.

2. Restricted parabolic trees

Let $\overline{T}$ be an arbitrary infinite locally finite tree. A one-way infinite path r_v starting at vertex $v \in V(\overline{T})$ is called a *ray*. We introduce the equivalence relation $\backsim$ on the set $\mathcal{R}(\overline{T})$ of all rays in $\overline{T}$ as follows. For every pair $r_{v_1}, r_{v_2} \in \mathcal{R}(\overline{T})$ of rays we put

$$r_{v_1} \backsim r_{v_2} \iff r_{v_1} \cap r_{v_2} \text{ is an infinite path.}$$

Each of the equivalence classes $E_v = [r_v]_\backsim$ is called an *end* of tree $\overline{T}$. A tree with one specified end is called *ended tree*. In our work we investigate ended trees with exactly one end, however in general an ended tree can have more than one end.

Definition 1. (See [5].) An infinite locally finite tree with a unique end E is called a *parabolic* tree with end E.

A parabolic tree with end E will be denoted by (D, E). It is clear that every vertex v of a tree (D, E) determines a unique ray r_v, i.e. all but one one-way paths starting at v are finite and end up at leaves. Each vertex $w \neq v$ in r_v is called and *ancestor of* v while v is called *descendant of* w. If w and v are on distance 1 from each other, then v is called *direct descendant* of w.

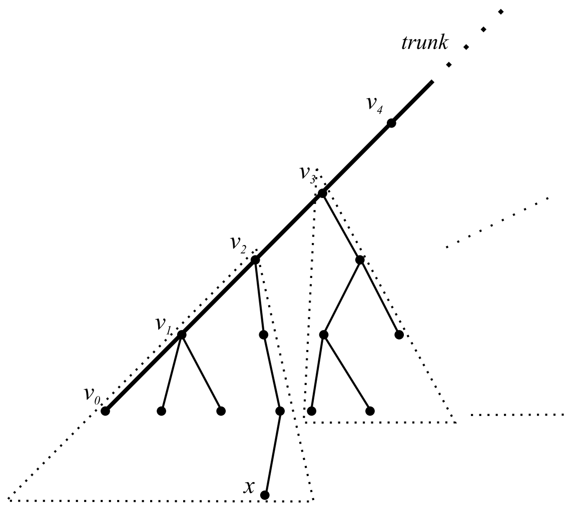


Fig. 2. Parabolic tree with basic rooted tree $T_b(v_2)$ and hanging tree $T(v_3)$.

We say that (D_1, E_1) and (D_2, E_2) are isomorphic as parabolic trees, if there exists a graph isomorphism $\varphi : D_1 \longrightarrow D_2$ such that the $\varphi(E_1) = E_2$.

Let us fix a leaf v_0 of the parabolic tree D and call the ray $r_{v_0} = (v_0, v_1, \ldots)$ a *trunk*. Then we consider D as a graph containing a single trunk and denote it by (D, E_{v_0}). Each vertex v_i of the trunk splits tree D into two subtrees. The rooted subtree $T_b(v_i)$ with root v_i is called the *basic rooted tree* and is the finite rooted subtree containing all vertices descendant to v_i in D. The other part is a parabolic tree with trunk starting at v_i. It is uniquely determined by an infinite sequence $T(v_n)$, $n = i+1, i+2, \ldots$, of so-called *hanging trees* each of which is the rooted subtree with root v_n for some $n = i+1, i+2, \ldots$, and contains all vertices descendant to v_n in D, except those lying on the trunk (see Fig. 2). Thus every parabolic tree may be represented as a sequence of finite rooted trees: $D = (T_b(v_i), T(v_{i+1}), \ldots)$. We call this sequence an *i-th basic tree representation* of tree D.

For $r \in \mathbb{Z}$ we denote

$$H_r(D) = \{w \in V(D) \mid d(v_0, v_i) - d(v_i, w) = r\},$$

where v_i is the vertex on the trunk which is the root of the hanging subtree $T(v_i)$ containing vertex w. The set $H_r(D)$ is called the r-th *horosphere* of the parabolic tree (D, v_0). Obviously, the trunk starts always in horosphere H_0 and there may also exist nonempty horospheres of negative indices. For example, the vertex x in Fig. 2 is contained in horosphere -1. We call a parabolic tree (D, v_0) *horospherically homogeneous*, if for every $r \in \mathbb{Z}$ there exists a natural number k_r such that for every $v \in H_r(D)$ we have

$$|\{w \in H_{r-1}(D) \mid w \text{ is descendant to } v\}| = k_r.$$

We note that if (D, v_0) is a horospherically homogeneous tree, then $H_r = \emptyset$ for $r < 0$ and thus the tree is defined by a left-infinite sequence $\overline{k} = (\ldots, k_1, k_0)$ which is called the *horospherical index* of tree (D, v_0). Further, if $\overline{k} = (\ldots, k, k)$, then (D, v_0) is called *k-adic parabolic tree*. In the following we focus our considerations on parabolic trees with no negative horospheres.

Definition 2. A parabolic tree (D, v_0) is called *restricted parabolic* if all leaves of D are contained in $H_0(D)$.

Our discussion shows that a horospherically homogeneous tree is always restricted parabolic. Now we introduce a specific construction of a restricted parabolic tree.

2.1. A word construction of restricted parabolic tree

Let $X = (\ldots, X_1, X_0)$ be a left-infinite sequence of nonempty pairwise disjoint finite sets, called alphabets, such that $|X_i| = k_i$ for $i \in \mathbb{N}$. Additionally, we assume that each of the alphabets X_i contains a special element $x_i^0 \in X_i$. We define the left-infinite word u as $u = \ldots x_{r+1}x_r$, where $x_i \in X_i$ for $i \geq r$. The sequence $\varepsilon = \ldots x_1^0 x_0^0$ is called a zero word. Let W_l, $l \geq 0$, be the set of all left-infinite almost zero words, i.e. words of the type $(\ldots x_{l+1}x_l)$, where $x_i \in X_i$ for all $i \geq l$ and for which there exists $k \geq l$ such that $x_j = x_j^0$ for all $j \geq k$. Note that every word w in W_l ends with a letter from X_l. We put $\overline{k} = (\ldots, k_1, k_0)$ and define the graph $D_{\overline{k}}$ as follows:

1. $V(D_{\overline{k}}) = \bigcup_{l=0}^{\infty} W_l$ is the set of vertices;
2. Two vertices $u, v \in V(D_{\overline{k}})$ are connected with an edge if and only if $u \in W_l$ and $v \in W_{l-1}$ and $v = ux$ for certain $x \in X_{l-1}$ (here ux denotes the concatenation of word u with letter x).

We choose the trunk of $D_{\overline{k}}$ to be the ray that begins in the zero word $\varepsilon \in W_0$. Then $D_{\overline{k}}$ is a horospherically homogeneous restricted parabolic tree with the horospherical index $\overline{k}$ and is called the word parabolic tree. We note also that $W_r = H_r(D_{\overline{k}})$.

If additionally $k_i = k$ for all $i \in \mathbb{N}$ then we obtain a *k-adic restricted parabolic tree*, which we denote by D_k. A part of the 3-adic restricted parabolic tree D_3 is shown in Fig. 3.

Since every ray in a parabolic tree is uniquely determined by a vertex, in which it starts, we may identify a given ray $r_v \in \mathcal{R}(D_{\overline{k}})$ with the vertex v itself. For every vertex $v \in H_n(D_{\overline{k}})$ the unique rooted subtree $(T_{\overline{k}|_n}, v)$ with root v containing all descendants of v in $D_{\overline{k}}$ is called the *branch* $U(v)$ of $D_{\overline{k}}$. Here $\overline{k}|_n$ denotes the subsequence of n last terms in the sequence $\overline{k}$. The branch $U(\ldots x_2^0 x_1^1)$ is depicted in Fig. 3. If v belongs to the trunk and we additionally remove from the branch $U(v)$ all vertices contained in the trunk (except the vertex v) together with adjacent subtrees, then we obtain the deleted branch

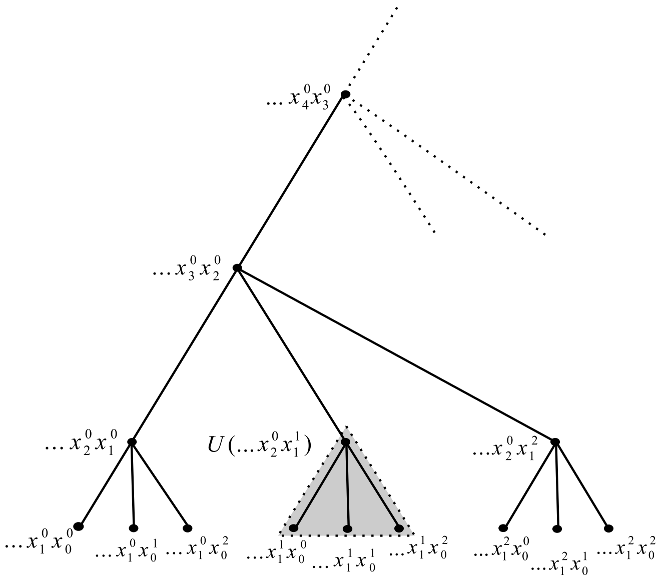


Fig. 3. 3-adic restricted parabolic tree D_3 with selected branch.

$U_d(v)$ which coincides with the hanging tree $T(v)$. It is clear that $U_d(v) = (T_{\overline{k^*},n}, v)$, where $\overline{k^*} = (k_0, k_1, \dots, k_{n-2}, k_{n-1} - 1)$.

2.2. Properties of restricted parabolic trees

1. Every horospherically homogeneous restricted parabolic tree of the horospherical index $\overline{k}$ is isomorphic to the word parabolic tree $D_{\overline{k}}$.

2. The horospherically homogeneous restricted parabolic tree can be obtained as a direct limit of a system of finite level-homogeneous rooted trees with special embeddings. One can embed the spherically homogeneous rooted tree $(T_{\overline{k}_n}, v_n)$ of spherical index $\overline{k}_n = (k_0, k_1, \dots, k_{n-1})$ as a subtree with root contained at level 1 of the homogeneous rooted tree $(T_{\overline{k}_{n+1}}, v_{n+1})$ of spherical index $\overline{k}_{n+1} = (k_0, k_1, \dots, k_n)$ in a manner shown in Fig. 1b.

Then we obtain the following

Lemma 5. *The horospherically homogeneous restricted parabolic tree $D_{\overline{k}}$ of the horospherical index $\overline{k} = (\dots, k_2, k_1, k_0)$ is isomorphic to the direct limit of the spherically homogeneous rooted trees $T_{\overline{k}_n}$ of spherical indices $\overline{k}_n = (k_0, k_1, \dots, k_{n-1})$, i.e.*

$$D_{\overline{k}} = \varinjlim_{n} \left((T_{\overline{k}_n}, v_n), \psi_n \right).$$

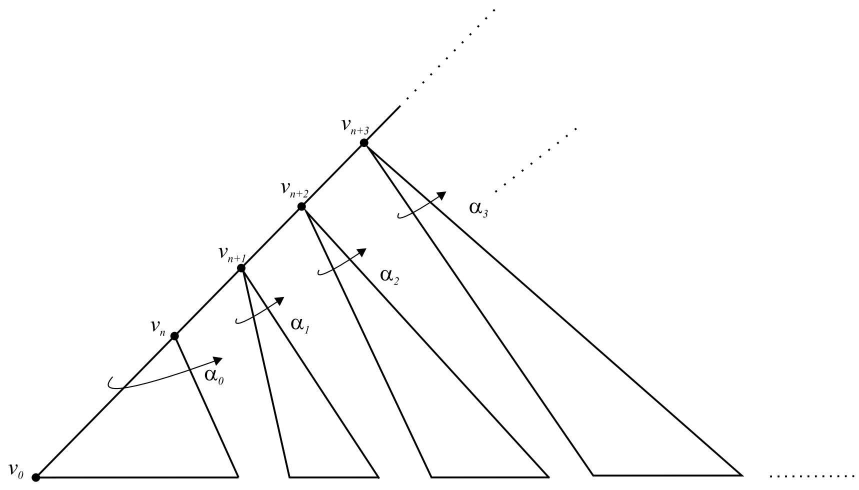


Fig. 4. The action of an automorphism u on subtrees of $D_{\overline{k}}$.

3. The k-adic restricted parabolic tree D_k is isomorphic to its restricted parabolic subtree, obtained by cutting off a finite number of the first horospheres together with the incident edges. Hence D_k has a self-similar structure.

3. Automorphisms of restricted parabolic trees

Since every automorphism u of $D_{\overline{k}}$ fixes some ray r pointwise ([19], Corollary 4.2), u can be described in terms of its action on particular branches of $D_{\overline{k}}$. Namely, let v be the vertex contained in the trunk of $D_{\overline{k}}$ and belonging to the fixed ray r. Then obviously $u(v) = v$. Assume $v = v_n \in H_n(D_{\overline{k}})$. Then u is determined by the action α_n of u on the basic rooted tree $T_b(v_n)$ and the actions α_i, $i > n$, of u on the hanging trees $T(v_i)$, where $v_i \in H_i(D_{\overline{k}})$ are vertices contained in the trunk (see Fig. 4).

Hence we represent every automorphism $u \in Aut\ D_{\overline{k}}$ as an infinite sequence

$$u = [\dots, \alpha_{n+2}, \alpha_{n+1}, \alpha_n], \tag{1}$$

where $\alpha_n \in Aut\ (T_{\overline{k}_n}, v_n)$ for $\overline{k}_n = (k_0, k_1, \dots, k_{n-1})$ and $\alpha_i \in Aut\ (T_{\overline{k}_i^*}, v_i)$ for $\overline{k}_i^* = (k_0, k_1, \dots, k_{i-1} - 1)$, $i > n$. The representation (1) is relevant to the n-th basic tree representation of tree $D_{\overline{k}}$.

If $u = [\dots, \alpha_{n+2}, \alpha_{n+1}, \alpha_n]$ and $w = [\dots, \beta_{m+2}, \beta_{m+1}, \beta_m]$ are two automorphisms of $D_{\overline{k}}$, then:

$$\begin{aligned} u^{-1} &= [\dots, \alpha_{n+2}^{-1}, \alpha_{n+1}^{-1}, \alpha_n^{-1}], \\ u \cdot w &= [\dots, \alpha_{t+2}\beta_{t+2}, \alpha_{t+1}\beta_{t+1}, \alpha_t\beta_t], \qquad t = \max\{n, m\}. \end{aligned}$$

If we additionally require α_n to be the action on the branch $U(v_n)$, where v_n is the vertex of minimal horosphere, contained both in the fixed ray and the trunk, then the

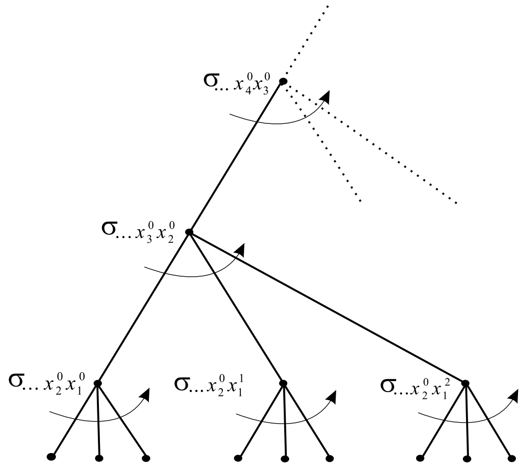


Fig. 5. The automorphism u of D_3.

representation (1) is unique. In this case the action is denoted by $\dot{\alpha_n}$ and the unique representation of $u \in Aut\ D_{\overline{k}}$, given by

$$u = [\dots, \alpha_{n+2}, \alpha_{n+1}, \dot{\alpha_n}]$$

is called the *canonical representation of* u.

Another useful way of representing a tree automorphism is by giving its *portrait*. Using the word construction of restricted parabolic tree described in section 2.1, we adapt the definition of the portrait introduced in [4].

Let $u \in Aut\ D_{\overline{k}}$ be an automorphism of the restricted parabolic tree $D_{\overline{k}}$, and let $\underline{x} = (\dots x_{l+1}, x_l)$ be a left-infinite word corresponding to a vertex in the horosphere $H_l(D_{\overline{k}})$. For every $x_{l-1} \in X_{l-1}$ we have:

$$u((\dots x_{l+1}, x_l, x_{l-1})) = u(\dots x_{l+1}, x_l)y,$$

where $y \in X_{l-1}$. Thus u induces a permutation $\sigma_{\underline{x}} : x_{l-1} \mapsto y$ of the alphabet X_{l-1}. We call permutation $\sigma_{\underline{x}} \in Sym(X_{l-1})$ the vertex permutation. For the given automorphism u by $\Sigma(u)$ we denote the set of all vertex permutations induced by u in all vertices of the tree $D_{\overline{k}}$.

Now, we represent the automorphism $u \in Aut\ D_{\overline{k}}$ by decorating every vertex of the tree $D_{\overline{K}}$ with the respective vertex permutation from $\Sigma(u)$. The resulting labelled tree $D_{\overline{k},\Sigma(u)}$ uniquely determines u (see Fig. 5) and is called the *portrait* of u.

Definition 3. An automorphism $u \in Aut\ D_{\overline{k}}$ is called a *finitary automorphism* of $D_{\overline{k}}$ if u permutes only a finite subset of $V(D_{\overline{k}})$.

Observe that the definition is equivalent to the condition that u permutes only a finite subset of the horosphere $H_0(D_{\overline{k}})$. It is clear that the set of all finitary automorphisms of

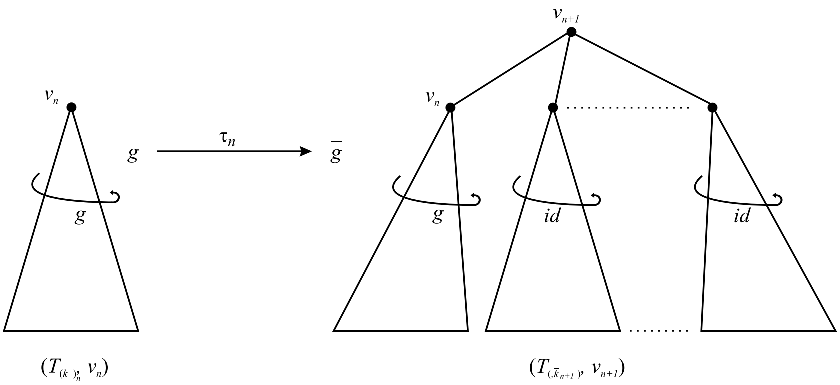


Fig. 6. The embedding τ_n.

$D_{\overline{k}}$ forms a subgroup of $Aut\, D_{\overline{k}}$. We denote this subgroup by $Aut_f\, D_{\overline{k}}$. An automorphism u of $D_{\overline{k}}$ is finitary if and only if it has a representation $u = [\ldots, id, id, \alpha_n]$ for some $n \in \mathbb{N}$. Thus every finitary automorphism $u \in Aut_f\; D_{\overline{k}}$ is associated with an automorphism α_n of a basic rooted tree $(T_{\overline{k}_n}, v_n)$ and conversely, for every automorphism α of the rooted tree $(T_{\overline{k}_n}, v_n)$ we may construct a respective finitary automorphism of $D_{\overline{k}}$. From these observations it follows that $Aut_f\; D_{\overline{k}}$ can be constructed as a direct limit of $Aut\; T_{\overline{k}_n}$. Namely, let

$$\tau_n : Aut\; (T_{\overline{k}_n}, v_n) \longrightarrow\; Aut\; (T_{\overline{k}_{n+1}}, v_{n+1})$$

be an embedding defined as follows. For every $g \in\; Aut\; (T_{\overline{k}_n}, v_n)$ we put $\tau_n(g)$ to be the automorphism $\bar{g} \in\; Aut\; (T_{\overline{k}_{n+1}}, v_{n+1})$ whose action on $(T_{\overline{k}_{n+1}}, v_{n+1})$ is given by the action of g on a subtree of height n in the tree $(T_{\overline{k}_{n+1}}, v_{n+1})$ rooted at level 1 while all vertices outside this subtree are fixed points of $\bar{g}$ (see Fig. 6).

The elements of the direct limit of $\left(Aut\; (T_{\overline{k}_n}, v_n), \tau_n\right)$ are exactly those automorphisms of the restricted parabolic tree $D_{\overline{k}}$, which permute only a finite subset of $V(D_{\overline{k}})$. Hence we obtain:

Lemma 6.

$$Aut_f\; D_{\overline{k}} \cong \varinjlim_{n} \left(Aut\; (T_{\overline{k}_n}, v_n), \tau_n\right). \tag{2}$$

Proof. For a fixed n, let $\varphi \in Aut\; (T_{\overline{k}_n}, v_n)$. Since φ acts trivially on $V(D_{\overline{k}}) \setminus V(T_{\overline{k}_n}, v_n)$ then the extension of φ in $Aut\; D_{\overline{k}}$ is a finitary automorphism of $D_{\overline{k}}$. Conversely, every finitary automorphism ψ of $D_{\overline{k}}$ permutes only a finite subset X of vertices of $D_{\overline{k}}$ and hence there exists a number n, such that $X \subseteq V(T_{\overline{k}_n}, v_n)$. Thus, the restriction of ψ on $(T_{\overline{k}_n}, v_n)$ is an element from $Aut\; (T_{\overline{k}_n}, v_n)$. $\square$

Before we state our next proposition we define the following embedding

$$\chi_n : \mathop{\wr}_{i=1}^{n} S_{k_i} \longrightarrow \mathop{\wr}_{i=1}^{n+1} S_{k_i}$$

of wreath products of symmetric groups S_{k_i} acting on the sets $\{1, 2, \dots, k_i\}$. For every $h = [g_1, g_2(x_1), \dots, g_n(x_1, \dots, x_{n-1})] \in \mathop{\wr}_{i=1}^{n} S_{k_i}$ the image of h under χ_n is defined as

$$\chi_n(h) = [id, g_2'(x_1), g_3'(x_1, x_2), \dots, g_n'(x_1, x_2, \dots, x_{n-1}), g_{n+1}'(x_1, x_2, \dots, x_{n-1}, x_n)],$$

where

$$\begin{aligned} g_2'(x_1) &= \begin{cases} g_1, & \text{if } x_1 = 1, \\ id, & \text{if } x_1 \neq 1. \end{cases} \\ g_i'(x_1, x_2, \dots, x_{i-1}) &= \begin{cases} g_{i-1}(x_2, x_3, \dots, x_{i-1}), & \text{if } x_1 = 1, \\ id, & \text{if } x_1 \neq 1. \end{cases} \end{aligned}$$

Simple calculations show that for every $n \in \mathbb{N}$ the mapping χ_n is a monomorphism.

Proposition 7. *The group $Aut_f\ D_{\overline{k}}$ has the following properties:*

1. *The action of $Aut_f\ D_{\overline{k}}$ on every horosphere of $D_{\overline{k}}$ is totally imprimitive.*
2. *$Aut_f\ D_{\overline{k}} \cong \varinjlim_{n} \left(\mathop{\wr}_{i=1}^{n} S_{k_i}, \chi_n \right)$, where S_{k_i} denotes the symmetric group on a set of k_i elements and χ_n is the respective embedding.*

Proof. *(1).* Observe first that the systems of imprimitivity in $Aut_f\ D_{\overline{k}}$ are determined by the structure of the tree. Namely, every system of imprimitivity is a partition of the horosphere H_0, which is determined by the subtrees rooted at the vertices of a chosen horosphere. In particular, the system of imprimitivity $\mathcal{S}_0$, determined by H_0 is the trivial partition into singletons. The system $\mathcal{S}_n$ is determined by H_n as a partition of H_0 into tuples of $k_0 \cdot k_1 \cdot \ldots \cdot k_{n-1}$ vertices, each tuple lying in the unique common subtree rooted at the horosphere H_n. Now it is clear that $(\mathcal{S}_n)_{n\in\mathbb{N}}$ is an infinite series of systems of imprimitivity of $Aut_f\ D_{\overline{k}}$, such that $\mathcal{S}_{n-1}$ is a subpartition of $\mathcal{S}_n$. Thus the action of $Aut_f\ D_{\overline{k}}$ on the horosphere H_0 is totally imprimitive.

Now, if we cut off the first n horospheres of $D_{\overline{k}}$ together with the adjacent edges, we obtain the restricted parabolic tree $D_{\overline{k}_n}$. It is clear that $Aut_f\ D_{\overline{k}_n}$ coincides with the restriction of $Aut_f D_{\overline{k}}$ onto the tree $D_{\overline{k}_n}$. Moreover, the horosphere $\dot{H}_0$ of $D_{\overline{k}_n}$ corresponds to the horosphere H_n in $D_{\overline{k}}$ and, by the above arguments, the action of $Aut_f\ D_{\overline{k}_n}$ on $\dot{H}_0$ is totally imprimitive. Hence the action of $Aut_f\ D_{\overline{k}}$ on H_n is totally imprimitive, too.

(2). For the proof of the second statement we discuss in detail the embeddings τ_n used in (2). For convenience we use the following labelling of the vertices of a spherically homogeneous rooted tree $(T_{\overline{k}_n}, v_n)$. We assign to the root v_n the label 1. Then, let all vertices of the first level L_1 be labeled with elements from $\{1, 2, \dots, k_1\}$. Now, if v is a

vertex from L_i labeled with $e(v) = [x_1, x_2, \ldots, x_i] \in \prod_{j=1}^{i} \{1, 2, \ldots, k_j\}$, then all its direct descendants from L_{i+1} are labeled with vectors $[x_1, x_2, \ldots, x_i, m] \in \prod_{j=1}^{i+1} \{1, 2, \ldots, k_j\}$ for pairwise different elements $m \in \{1, 2, \ldots, k_{i+1}\}$. Hence, the set of all leaves $L_n(T)$ of the tree $(T_{\overline{k}_n}, v_n)$ can be considered as the set $\prod_{j=1}^{n} \{1, 2, \ldots, k_j\}$. Observe that for the embedding $\tau_n : (T_{\overline{k}_n}, v_n) \hookrightarrow (T_{\overline{k}_{n+1}}, v_{n+1})$ the image of a leaf $[x_1, x_2, \ldots, x_n] \in L_n(T_{\overline{k}_n}, v_n)$ is the leaf $[1, x_1, x_2, \ldots, x_n] \in L_{n+1}(T_{\overline{k}_{n+1}}, v_{n+1})$.

The action of any automorphism $\alpha \in Aut(T_{\overline{k}_n}, v_n)$ on $(T_{\overline{k}_n}, v_n)$ is uniquely determined by the way α permutes the leaves of the tree, i.e. by the permutation of the set $L_n(T) = \prod_{j=1}^{n} \{1, 2, \ldots, k_j\}$. In particular, α may be defined by

$$\alpha([x_1, x_2, \ldots, x_n]) = [\alpha_1(x_1, x_2, \ldots, x_n), \alpha_2(x_1, x_2, \ldots, x_n), \ldots, \alpha_n(x_1, x_2, \ldots, x_n)],$$

where $\alpha_i : \prod_{j=1}^{n} \{1, 2, \ldots, k_j\} \longrightarrow \{1, 2, \ldots, k_i\}$ for $i = 1, 2, \ldots, n$. Then the embedding τ_n of groups of automorphisms of rooted trees can be defined as follows:

$$\alpha \longmapsto \tau_n(\alpha),$$

where

$$\tau_n(\alpha)([x_1, \ldots, x_{n+1}]) = \begin{cases} [1, \alpha_1(x_2, \ldots, x_{n+1}), \ldots, \alpha_n(x_2, \ldots, x_{n+1})], & \text{if } x_1 = 1, \\ [x_1, x_2, \ldots, x_n, x_{n+1}], & \text{if } x_1 \neq 1. \end{cases} \tag{3}$$

It is known that the group of automorphisms of a spherically homogeneous rooted tree $(T_{\overline{k}_n}, v_n)$ is isomorphic to the wreath product of the symmetric groups S_{k_i}, $i = 1, \ldots, n$. Let us denote the isomorphism by:

$$\iota_n : Aut\,(T_{\overline{k}_n}, v_n) \longleftrightarrow \wr_{i=1}^{n} S_{k_i}.$$

Since groups $Aut\,(T_{\overline{k}_n}, v_n)$ and $\wr_{i=1}^{n} S_{k_i}$ are isomorphic, they act on the space $\prod_{j=1}^{n} \{1, 2, \ldots, k_j\}$ in the same manner and, in particular, for every $\alpha \in Aut\,(T_{\overline{k}_n}, v_n)$ we have:

$$\alpha([x_1, x_2, \ldots, x_n]) = \iota(\alpha)[x_1, x_2, \ldots, x_n].$$

Direct calculations show that the following diagram is commutative:

$$\begin{array}{ccc} Aut\,(T_{\overline{k}_n}, v_n) & \xrightarrow{\tau_n} & Aut\,(T_{\overline{k}_{n+1}}, v_{n+1}) \\ \updownarrow \iota_n & & \updownarrow \iota_{n+1} \\ \wr_{i=1}^{n} S_{k_i} & \xrightarrow{\chi_n} & \wr_{i=1}^{n+1} S_{k_i} \end{array} . \tag{4}$$

Diagram (4) determines the one-to-one correspondence of two direct systems

$$\big(Aut(T_{\overline{k}_n}, v_n), \tau_n\big) \rightleftarrows \big(\wr_{i=1}^{n} S_{k_i}, \chi_n\big),$$

and the second statement of our proposition follows. $\square$

We will denote the direct limit $\varinjlim_{n}(\wr_{i=1}^{n} S_{k_i}, \chi_n)$ by $(D)\wr_{i=1}^{\infty} S_{k_i}$. This is a particular construction of iterated wreath product, for other constructions refer to [6,8,10].

In the particular case of k-adic restricted parabolic trees D_k we note the following observation.

Remark 1. The action of $Aut_f\, D_k$ on every horosphere of D_k is transitive. The kernel of the action of $Aut_f\, D_k$ on H_i coincides with the stabilizer $Stab_{Aut_f\, D_k} H_i$. Every two permutation groups $(Aut_f\, D_k/Stab_{Aut_f\, D_k} H_i, H_i)$ and $(Aut_f\, D_k/Stab_{Aut_f\, D_k} H_j, H_j)$, $i, j \in \mathbb{N}$, are isomorphic.

Further considerations on $Aut D_{\overline{k}}$ will focus on another special type of automorphisms, which arise dually to finitary automorphisms.

Definition 4. An automorphism $s \in Aut\ D_{\overline{k}}$ is called a *hanging trees automorphism*, if s fixes the trunk of $D_{\overline{k}}$, i.e. it fixes all hanging trees of the parabolic tree $D_{\overline{k}}$.

All hanging trees automorphisms form the stabilizer $Stab(v_0)$ of the trunk. The elements of $Stab(v_0)$ are exactly the automorphisms g, which have the representation $g = [\ldots, \alpha_2, \alpha_1, id]$. In particular, in the canonical representation of g we assume that α_i for $i > 0$ is the automorphism of the hanging tree $(T_{\overline{k}_i^*}, v_i)$ rooted at vertex v_i of $D_{\overline{k}}$. Thus we have

Lemma 8. *Let $\ldots, v_1, v_0$ be vertices of the trunk of the parabolic tree $D_{\overline{k}}$. Then*

$$Stab(v_0) \cong \prod_{i=1}^{\infty} Aut\,(T_{\overline{k}_i^*}, v_i), \quad \textit{where } \overline{k}_i^* = (k_0, k_1, \ldots, k_{i-2}, k_{i-1} - 1),$$

i.e. the stabilizer of the trunk is the unrestricted direct product of certain automorphism groups of rooted trees.

Proof. Clear. $\square$

3.1. *Proof of Theorem 1*

(1). An arbitrary automorphism $g \in Aut\, D_{\overline{k}}$ of the representation

$$g = [\ldots, \alpha_{n+2}, \alpha_{n+1}, \alpha_n]$$

clearly consists of the finitary part induced by α_n, acting on a tree of height n and the hanging trees part induced by α_i, $i > n$. Therefore we have the following factorization:

$$g = [\ldots, id_{n+2}, id_{n+1}, \alpha_n] \circ [\ldots, \alpha_{n+2}, \alpha_{n+1}, id_n],$$

where id_m denotes the trivial automorphism acting on a rooted tree of height m.

(2). First, observe that $Aut_f\, D_{\overline{k}}$ is locally finite, as any finite set of finitary automorphisms generates a subgroup, which is isomorphic to a certain group of automorphisms of a finite rooted tree. Then, as a direct limit of transitive groups, $Aut_f\, D_{\overline{k}}$ acts transitively on the horosphere $H_0(D_{\overline{k}})$. Hence, by the result of P. Neumann ([16], Th. 2), the group $Aut_f\, D_{\overline{k}}$ is not residually finite.

(3). The first statement follows directly by Lemma 8. We prove only the second statement.

Let $H = \langle h_1, h_2, \ldots, h_n \rangle$, $h_i \in Aut\, D_{\overline{k}}$. Assume that $h_1, \ldots, h_k \in Aut_f\, D_{\overline{k}}$ and $h_{k+1}, \ldots, h_n \in Aut\, D_{\overline{k}} \setminus Aut_f\, D_{\overline{k}}$. Let N be the minimal height of the basic tree in $D_{\overline{k}}$, such that all finitary automorphisms $h_1, \ldots, h_k$ act trivially outside $T_{\overline{k}|_N}$. Moreover, let

$$h_j = [\ldots, \alpha_{s_j+1}, \dot{\alpha_{s_j}}], \quad j = k+1, \ldots, n$$

be the canonical representation of h_j. Take

$$M = \max\{N, s_j \mid j = k+1, \ldots, n\}.$$

Then every automorphism $f \in H$ has representation of the form:

$$f = [\ldots, f_{M+1}, f_M]$$

and hence

$$H \cong \overline{H} \leq Aut\, T_{\overline{k^*}|_M} \times \prod_{i=1}^{\infty} Aut\, T_{\overline{k}|_{M+i}} = Stab(v_M).$$

Now, take $g_1, g_2 \in H$, $g_1 = [\ldots, \beta_{M+1}, \beta_M]$, $g_2 = [\ldots, \gamma_{M+1}, \gamma_M]$, such that $g_1 \neq g_2$. Then there exists $i_0 \in \{M, M+1, \ldots\}$ such that $\beta_{i_0} \neq \gamma_{i_0}$. Let π_{i_0} be the projection of H into the group of automorphisms of finite rooted tree:

$$\pi_{i_0} : H \longrightarrow \widehat{H} \leq Aut\, T_{\overline{k}|_{i_0}},$$

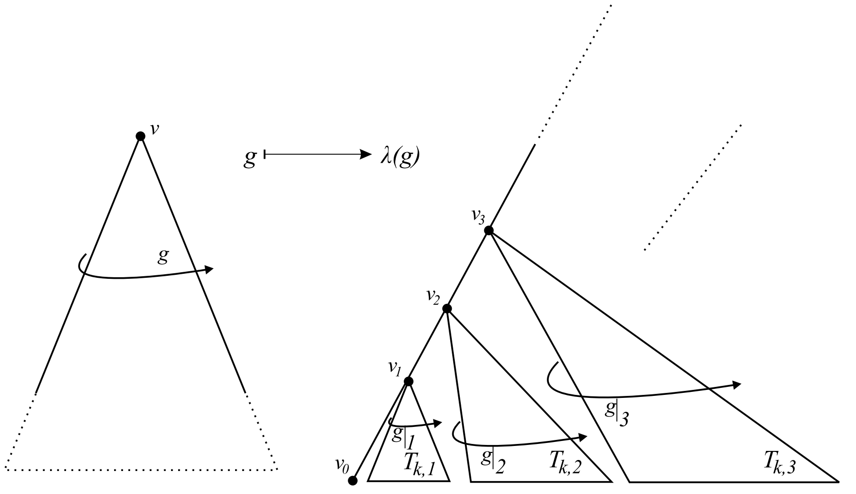


Fig. 7. The embedding of $Aut\,(T_k, w)$ into $Stab(v_0)$.

such that if $h \in H$ then $\widehat{h} = \pi_{i_0}(h)$ is the automorphism of a rooted tree of height i_0, which agrees on $T_{\overline{k}|_{i_0}}$ with the action of h. Indeed π_{i_0} is a projection, since $h \in Stab(v_M)$. Moreover it is clear that

$$\widehat{g_1} \neq \widehat{g_2}$$

and $\widehat{H}$ is finite. Thus H residually finite.

(4). Let us construct an embedding $\lambda : Aut\,(T_k, w) \longrightarrow Stab(v_0)$, as presented in Fig. 7.

Namely, let g be an automorphism of the k-adic infinite rooted tree (T_k, v). By $g|_i$ we denote the restriction of the automorphism g to the first i levels of (T_k, v). It is clear that $g|_i \in Aut\,(T_{k,i}, v)$. Now let D_k^* denote a restricted parabolic tree in which every vertex v_i contained in the trunk is the root of a hanging k-adic rooted tree $(T_{k,i}, v_i)$. We define $\lambda(g)$, the automorphism of D_k^*, in the following way: for every vertex v_i from the trunk $(i \in \mathbb{N})$ the automorphism $\lambda(g)$ acts on the hanging tree $(T_{k,i}, v_i)$ according to $g|_i$. It is clear that λ is a monomorphism.

Now it is enough to observe that every automorphism of D_k^* can be naturally extended to the automorphism of the tree D_{k+1} by fixing all vertices of D_{k+1} which are not contained in D_k^*. It is clear that $\lambda(g)$ stabilizes the trunk. Hence $Aut(T_k, v) \hookrightarrow Stab(v_0) \leq Aut\, D_{k+1}$.

Now, consider another embedding $L : Stab(v_0) \longrightarrow Aut\,(T_k, w_0)$, which maps an automorphism $g = [\ldots, g_3, g_2, g_1] \in Stab(v_0) \leq Aut\, D_k$ to a respective automorphism $L(g)$ of an infinite k-adic rooted tree (T_k, w_0) in the way presented in Fig. 8. The action of $L(g)$ on (T_k, w_0) is determined by the action of g on the hanging subtrees of D_k in the following way. In (T_k, w_0) we choose a ray starting at the root w_0 and label all the vertices along this ray consequently with $w_0, w_1, \ldots$. Then we define the automorphism

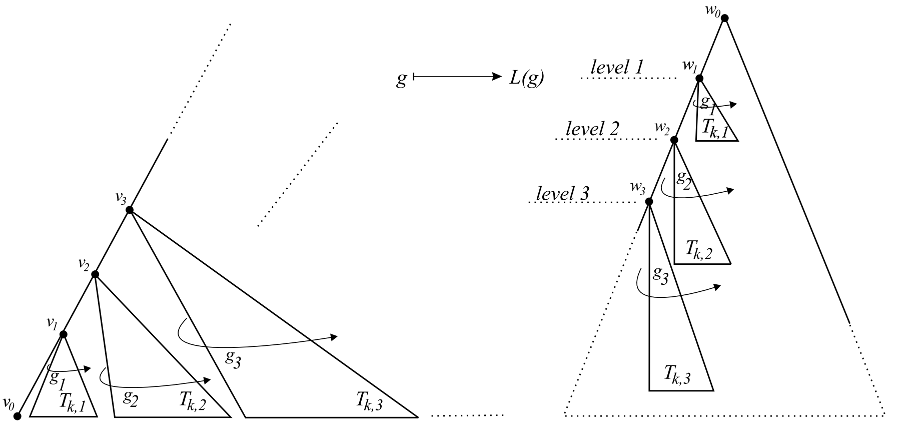


Fig. 8. The embedding of $Stab(v_0)$ into $Aut\,T_{k,w}$.

$L(g)$ to act on the first i levels of the (infinite) rooted subtrees with roots w_1, w_2, etc., according to the action of g on the respective hanging subtrees $(T_{k^*,i}, v_i)$ in D_k. By definition, outside the subtrees (T_{k^*}, w_i), $i = 0, 1, 2, \ldots$, the automorphism $L(g)$ acts trivially. Clearly, L is a group monomorphism and the statement follows. $\square$

Remark 2. It is clear that $Aut_f\, D_{\overline{k}} \cap Stab(v_0)$ is isomorphic to the restricted direct product $\prod^{\times} Aut(T_{\overline{k}_i^*}, v_i)$, i.e. $Aut\, D_{\overline{k}}$ is factorized by the subgroups $Aut\, D_{\overline{k}}$ and $Stab(v_0)$, but it is not the general product of these subgroups.

Remark 3. In the case of $k = p$, p – prime, the quasi-cyclic group C_{p^∞} naturally embeds into $Aut_f\, D_p$. To see this it is enough to observe that $Aut_f\, D_p$ contains automorphisms f_i, $i \in \mathbb{N}$, which define nontrivial vertex permutations only in the first i vertices lying on the trunk. All these nontrivial vertex permutations are equal to a chosen p-cycle σ. The portrait of f_3 is shown in Fig. 9.

Now, obviously $\langle f_i \rangle \cong C_{p^i}$. Moreover, the embeddings $\langle f_i \rangle \hookrightarrow \langle f_{i+1} \rangle$ are correspondent to the embeddings $C_{p^i} \hookrightarrow C_{p^{i+1}}$ for the limit construction of C_{p^∞}. Thus $C_{p^\infty} \subseteq Aut_f\, D_p$.

4. Sylow p-subgroups of $Aut_f\, D_p$

4.1. Local systems and Sylow p-subgroups

Let us fix a prime p. We first note few observations on the local structure of $Aut_f\, D_p$. By v_i we denote the vertex of the trunk of D_p lying on the i-th horosphere. In the previous section we have shown that $Aut_f\, D_p$ contains an increasing series Σ of finite subgroups:

$$G_1 \subset G_2 \subset \ldots, \qquad \text{where} \quad G_i \cong Aut(T_{k,i}, v_i),\ i = 1, 2, \ldots,$$

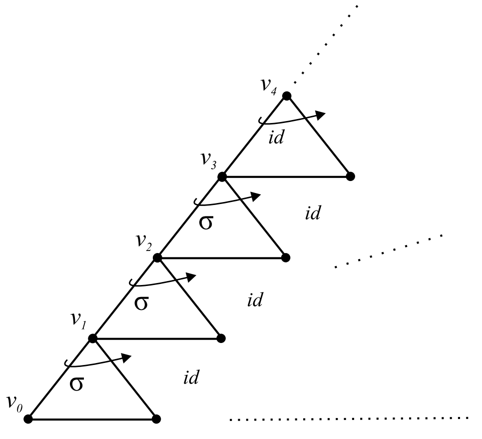


Fig. 9. The portrait of f_3.

such that $\bigcup_{i=1}^{\infty} G_i = Aut_f\ D_p$. Here $Aut(T_{k,i}, v_i)$ is considered as a subgroup of $Aut_f\ D_p$, which acts nontrivially on the basic rooted subtree $(T_{k,i}, v_i)$ of D_p. It is clear that if $j > i$, then G_i is subnormal in G_j, as G_i is a direct factor of the stabilizer of the $(j-i)$-th level in G_j. Moreover, every finitely generated subgroup of $Aut_f\ D_p$ is finite and hence it is contained in G_i for some i. Thus along with the group $Aut_f\ D_p$ we have a local system of subnormal subgroups. We recall here a useful result of Rae [17]:

Lemma 9. *Let Σ be a local system of subgroups of G, such that the inclusion of two subgroups implies that one is subnormal in the other. Then for every Sylow p-subgroup P of G and every subgroup $S \in \Sigma$ the intersection $S \cap P$ is a Sylow p-subgroup in S.*

Since $\Sigma = \{G_i \mid G_i \cong Aut(T_{k,i}, v_i), i = 1, 2, \ldots\}$ is a local system in $Aut_f\ D_p$, by Lemma 9 we have the following:

Remark 4. Every Sylow p-subgroup P of $Aut_f\, D_p$ defines an increasing series Q_i of Sylow p-subgroups in $Aut(T_{k,i}, v_i)$, such that $Q_i = P \cap Aut(T_{k,i}, v_i)$.

4.2. Construction

Now we construct a Sylow p-subgroup of $Aut_f\, D_p$ using the characterization of $Aut_f\, D_p$ as a direct limit of wreath products of symmetric groups, which we gave in Proposition 7.

Given tree T we recall that a p-automorphism of T is an automorphism of order p. This directly implies that for a p-automorphism, all vertex permutations are necessarily p-cycles, all of which are powers of a certain p-cycle α. It is well known (see [1] or [9] for example) that the set $P_n(\alpha)$ of all p-automorphisms of a finite p-adic rooted tree $(T_{p,n}, v_0)$ (defined for a certain p-cycle α) constitutes a Sylow p-subgroup of $Aut\ (T_{p,n}, v_0)$ and

$$P_n(\alpha) \cong \wr_{i=1}^{n} C_p^{(i)} = \mathcal{P}_n, \tag{5}$$

i.e. $\iota_n(P_n(\alpha)) = \mathcal{P}_n$, where ι_n is the group isomorphism of $\wr_{i=1}^{n} S_p^{(i)}$ and $Aut(T_{p,i}, v)$. Clearly, $P_n(\alpha)$ is not a unique Sylow p-subgroup of $Aut(T_{p,n}, v_0)$ as two different p-cycles α and β may define two different conjugate subgroups $P_n(\alpha) \neq P_n(\beta)$. In the following we refer to P_n as to any Sylow p-subgroup of $Aut\,(T_{p,n}, v_0)$.

Now, we use this characterization for the construction of a Sylow p-subgroup in $Aut_f\, D_p$. Simple calculations show that for every $n \in \mathbb{N}$ the embedding χ_n maps $\mathcal{P}_n$ into $\mathcal{P}_{n+1}$. Thus $(\mathcal{P}_n, \chi_n)$ is a direct system of wreath powers of cyclic groups, and by Proposition 7 the direct limit of this direct system is isomorphic to the respective D-wreath product of cyclic groups:

$$\mathcal{DP}_\infty = \varinjlim_{n}(\mathcal{P}_n, \chi_n) \cong (D) \wr_{i=1}^{\infty} C_p^{(i)}. \tag{6}$$

Due to the isomorphism ι_n of P_n and $\mathcal{P}_n$ and the correspondence of the embeddings χ_n and τ_n discussed earlier we have the following system:

$$\begin{array}{ccccccccccc}
P_0 & \xrightarrow{\tau_0} & P_1 & \xrightarrow{\tau_1} & \dots & \xrightarrow{\tau_{n-1}} & P_n & \xrightarrow{\tau_n} & P_{n+1} & \xrightarrow{\tau_{n+1}} & \dots \\
\updownarrow \iota_0 & & \updownarrow \iota_1 & & \vdots & & \updownarrow \iota_n & & \updownarrow \iota_{n+1} & & \vdots \\
\mathcal{P}_0 & \xrightarrow{\chi_0} & \mathcal{P}_1 & \xrightarrow{\chi_1} & \dots & \xrightarrow{\chi_{n-1}} & \mathcal{P}_n & \xrightarrow{\chi_n} & \mathcal{P}_{n+1} & \xrightarrow{\chi_{n+1}} & \dots
\end{array}$$

thus we may define $DP_\infty = \varinjlim_{n}(P_n, \tau_n)$, where $DP_\infty \cong \mathcal{DP}_\infty$. The obtained Sylow p-subgroup DP_∞ is not unique as it depends on the particular choice of the chain of Sylow p-subgroups in $Aut\,(T_{p,n}, v_0)$, $n = 1, 2, \dots$.

4.3. Proof of Theorem 2

(1). Let P be a Sylow p-subgroup of $Aut_f\, D_p$. By Remark 4 we have $P = \bigcup_{i=1}^{\infty} Q_i$, where Q_i is a Sylow p-subgroup of $G_i = Aut(T_{p,i}, v_i)$. Now, let x and y be vertices from a horosphere H_i, $i \geq 0$. Then x and y can be considered as leaves of the basic rooted tree $(T_{p,m}, v_m)$ for some sufficiently large $m \in \mathbb{N}$. Since Q_m acts transitively on every level of $(T_{p,m}, v_m)$, there exists a p-automorphism $\alpha \in Q_m \subset P$ such that $\alpha(x) = y$. Thus, P acts transitively on H_i.

(2). Let P and P' be two Sylow p-subgroups of $Aut_f\, D_p$ and let $\{Q_i\}$ and $\{Q_i'\}$ be the respective series of Sylow p-subgroups of $Aut(T_{k,i}, v_i)$, described in Remark 4. First we show that for the i-th horosphere the elements of P define vertex permutations, which are powers of a p-cycle α_i. Let $u \in P$ be a finitary automorphism and let $x_1, x_2, \dots, x_t$ be all vertices of the i-th horosphere of D_p on which u act nontrivially. Then there exists a sufficiently large index m, such that all vertices $x_1, \dots, x_t$ belong to the basic tree $(T_{p,m}, v_m)$ and u acts trivially outside that tree. Since the restriction of P onto

$(T_{p,m}, v_m)$ is a Sylow p-subgroup Q_m, which is isomorphic to the m-iterated wreath power of a cyclic group C_p of order p, then the vertex permutations of u at vertex x_i, $i = 1, 2, \dots, t$, is a power of a p-cycle $\alpha_i \in C_p$.

Now, suppose subgroups P and Q act on the horospheres of D_p according to p-cycles $\alpha_1, \alpha_2, \dots, \alpha_i, \dots$ and $\beta_1, \beta_2, \dots, \beta_i, \dots$ respectively. Then for every index i there exists a permutation γ_i such that $\beta_i = \gamma_i^{-1} \alpha_i \gamma_i$.

Let $f \in Aut\, D_p$ be the p-automorphism, such that for every $i \in \mathbb{N}$ the vertex permutations on the i-th horosphere are powers of γ_i, and let f_i be the restriction of f on the basic tree $(T_{p,i}, v_i)$ rooted on the i-th horosphere. Then $Q_i = f_i^{-1} Q_i' f_i$ and $P = f^{-1} P' f$.

(3). Let G be a totally imprimitive p-group which is p-uniserial on the countable set A and assume the action to be finitary. For every $i \in \mathbb{N}$ group G has a unique system of imprimitivity with blocks of size p^i, which is a subsystem of a system with blocks of size p^{i-1}. Hence we have a collection of partitions of A, such that each partition contains sets with the same cardinality p^i for some $i \in \mathbb{N}$. These partitions naturally correspond to a restricted parabolic tree D_p. Namely, we start with the p-partition and put each domain of imprimitivity from this partition as the set of leaves of a p-adic rooted tree of height 1. Then we group these trees by packs of p into the domains from the p^2-partition and construct trees of height two. We repeat this construction inductively by increasing the heights of trees.

We note that for every two elements $d, d' \in A$ there exists a block of imprimitivity, which contains both d and d'. Indeed. Being totally imprimitive and p-uniserial, group G contains an infinite ascending sequence of finite blocks of imprimitivity (see Lemma 8.3A in [7] for reference):

$$\Delta_1 \subset \Delta_2 \subset \dots,$$

such that $|\Delta_i| = p^i$. For each i the respective system of imprimitivity consists of block Δ_i and its possible images $g(\Delta_i)$, $g \in G$. In particular, the p-partition corresponding to the system of imprimitivity of rank p is

$$\Delta_1, g_1(\Delta_1), g_2(\Delta_1), \dots \quad , g_i \in G.$$

Assume that $d \in g(\Delta_1)$ and $d' \in g'(\Delta_1)$. If $g(\Delta_1) = g'(\Delta_1)$ then $g(\Delta_1)$ is the desired block of imprimitivity. Otherwise, consider the system of imprimitivity of rank p^k, $k \in \mathbb{N}$. Obviously we have $d \in g(\Delta_k)$ and $d' \in g'(\Delta_k)$. If there exists k_0 such that $g(\Delta_{k_0}) = g'(\Delta_{k_0})$, then $g(\Delta_{k_0})$ contains both d and d' and we are done. So assume there is no such k_0, i.e. for every $k \in \mathbb{N}$ we have $g(\Delta_k) \cap g'(\Delta_k) = \emptyset$, as any two blocks of the same size are either equal or disjoint. It follows that $\Delta_k \cap g^{-1}g'(\Delta_k) = \emptyset$, i.e. Δ_k is contained in the support of $g^{-1}g'$ for any $k \in \mathbb{N}$. Hence the infinite set $\bigcup_{i=1}^{\infty} \Delta_k$ is contained in the support of $g^{-1}g'$, which contradicts the assumption on the action of G to be finitary.

As a result, with the presented construction we obtain a connected restricted parabolic tree $D_p(G)$. Clearly, as G is a finitary p-group acting on $D_p(G)$, then G is contained in a Sylow p-subgroup of $Aut_f\, D_p$.

Thus, Theorem 2 follows. □

Remark 5. As a consequence of Theorem 2, we get a classification of the conjugacy classes of Sylow p-subgroups in $Aut_f\, D_p$. Let Λ be a sequence of p-cycles λ_i. By $DP(\Lambda)$ we denote the subgroup of $Aut_f\, D_p$, which acts on the i-th horosphere according to powers of λ_i. Then $DP(\Lambda)$ is a Sylow p-subgroup in $Aut_f\, D_p$, conjugated to DP_∞ in $Aut\, D_p$. Moreover, unless $p = 2$ or $p = 3$, there exist uncountably many conjugacy classes of Sylow p-subgroups in $Aut_f\, D_p$.

5. Sylow p-subgroups of $FS_{\mathbb{N}}$

5.1. Transitive Sylow p-subgroups of $FS_{\mathbb{N}}$

We translate our previous results to problem of characterization of transitive Sylow p-subgroups in the finitary symmetric group. Consider an automorphism u of a finite p-adic rooted tree $(T_{p,n}, v_0)$. Since u preserves the distances between vertices, it is clear that u maps every leaf to another leaf. Thus u acts as a permutation on the set of all leaves of tree $(T_{p,n}, v_0)$. Moreover, the action of u on leaves uniquely determines the action of u on the rest of the tree. Hence we have a natural correspondence between the automorphisms of the p-adic rooted tree $(T_{p,n}, v_0)$ and the permutations of the set of its leaves.

Let $S_{\mathbb{N}}$ denote the symmetric group on the set of all natural numbers [7]. The *finitary symmetric group* $FS_{\mathbb{N}}$ may be represented as a union of finite symmetric groups. Namely, let $M_1 \subset M_2 \subset \ldots$ be an increasing sequence of finite sets such that $\bigcup_{i=1}^{\infty} M_i = \mathbb{N}$, and let $S(M_i)$ be the symmetric group on the set M_i, $i \in \mathbb{N}$. Then

$$FS_{\mathbb{N}} = \bigcup_{i=1}^{\infty} S(M_i).$$

Further, we identify the set $\mathbb{N}$ of all natural numbers with the vertices of the horosphere H_0, enumerated from left to right. This allows us to identify subgroups of $Aut_f\, D_p$ with subgroups of $FS_{\mathbb{N}}$.

Proof of Theorem 3. *(1.)* For the proof we choose the following representation of the finitary symmetric group:

$$FS_{\mathbb{N}} = \bigcup_{i=1}^{\infty} S(\mathbb{F}_p^i).$$

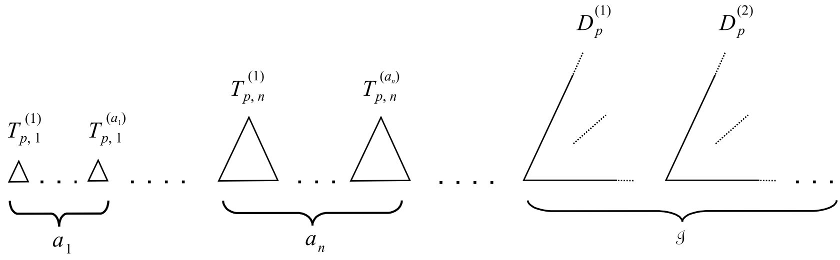


Fig. 10. The forest $\mathcal{F}_{\mathcal{A}}$ respective to the sequence $\mathcal{A}$.

Moreover, P_n is a Sylow p-subgroup of $S(\mathbb{F}_p^n) \cong S_{p^n}$ and $\varinjlim_{n}(P_n) = DP_\infty$. Hence the statement follows.

(2.) The proof of the second statement follows easily from Theorem 2. If P is a transitive Sylow p-subgroup of $FS_{\mathbb{N}}$ then it is a p-group, which is uniserial and totally imprimitive on $\mathbb{N}$. Hence, by statement (3) of Theorem 2, group P embeds into a Sylow p-subgroup DP of $Aut_f\ D_p(P)$. By the maximality of P we have $P = DP$.

Now, if we have two transitive Sylow p-subgroups P and P' in $FS_{\mathbb{N}}$ and the corresponding trees $D_p(P)$ and $D_p(P')$, then either $D_p(P) = D_p(P')$ or one can reorder the leaves (i.e. apply a permutation $\alpha \in S_{\mathbb{N}}$) of $D_p(P)$ in order to get the tree $D_p(P')$.

In the first case P is conjugated to P' in $S_{\mathbb{N}}$ by Theorem 2, in the latter case conjugation by α of P and P' reduces the problem to the first case. $\square$

5.2. Characterization of intransitive Sylow p-subgroups of $FS_{\mathbb{N}}$

In combinatorial terms the Sylow p-subgroups of the finitary symmetric group were characterized up to isomorphism by Ivanuta in [11]. Below we propose a classification of Sylow p-subgroups of $FS_{\mathbb{N}}$ based on groups of automorphisms of the p-forests of finite rooted and restricted parabolic trees.

Given a sequence

$$\mathcal{A} = (\mathcal{I}, a_0, a_1, a_2, \ldots), \tag{7}$$

where $\mathcal{I} \in \mathbb{N} \cup \{\infty\}$ and $0 \le a_i < p$ for all $i \in \mathbb{N}$, we define the p-forest $\mathcal{F}_{\mathcal{A}}$ to be the union of a_n finite p-adic rooted trees isomorphic to $T_{p,n}$ for every $n > 0$, and $\mathcal{I}$ p-adic restricted parabolic trees isomorphic to D_p (Fig. 10):

$$\mathcal{F}_{\mathcal{A}} = \bigsqcup_{j=1}^{a_1} T_{p,1}^{(j)} \sqcup \ldots \sqcup \bigsqcup_{j=1}^{a_n} T_{p,n}^{(j)} \sqcup \ldots \sqcup \bigsqcup_{j=1}^{\mathcal{I}} D_p^{(j)}.$$

The set of all leaves of a p-forest $\mathcal{F}$ is naturally partitioned into subsets of leaves of particular constituent trees, each being a subset of order of a power of p or infinite. Thus every p-forest determines a partition of the set of all leaves, which we call a p-partition

$\pi_{\mathcal{F}}$. Conversely, an arbitrary p-partition π of a countably infinite set determines a unique p-forest $\mathcal{F}_\pi$.

By $\oplus$ we denote the sum of permutation groups, i.e. if (G, X) and (H, Y) are two permutation groups such that $X \cap Y = \emptyset$, then

$$(G, X) \oplus (H, Y) = (G \times H, X \cup Y).$$

In a natural way, the operation $\oplus$ may be extended to an arbitrary family of permutation groups acting on disjoint sets.

Now, given a p-forest $\mathcal{F}_{\mathcal{A}}$, for every rooted tree of height n in $\mathcal{F}_{\mathcal{A}}$ we choose a Sylow p-subgroup of $Aut\,(T_{p,n}, v)$ isomorphic to P_n, $n \in \mathbb{N}$, and for every restricted parabolic tree we choose a Sylow p-subgroup of $Aut_f\, D_p$ isomorphic to DP_∞. This way we obtain a group $P(\mathcal{F}_{\mathcal{A}}, \Psi)$ acting on the forest $\mathcal{F}_{\mathcal{A}}$:

$$P(\mathcal{F}_{\mathcal{A}}, \Psi) \cong \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{a_i} P_i \oplus \bigoplus_{j=1}^{\mathcal{I}} DP_\infty.$$

We emphasize here that the above definition involves a choice of particular Sylow p-subgroups of the respective group. Hence the group resulting from our construction depends both on the p-forest $\mathcal{A}$ and the choice Ψ of particular Sylow p-subgroups.

Using the above construction we give a geometric analogue of results in [11].

Proof of Theorem 4. *(1).* Given a p-forest $\mathcal{F}_{\mathcal{A}}$ and the respective sequence $\mathcal{A} = (\mathcal{I}, a_0, a_1, a_2, \ldots)$ we choose the particular Sylow p-subgroups as follows.

Let $D_{p,n}^{(j)}$ denote the $U(v_n)$ branch of $D_p^{(j)}$ (v_n is the n-th vertex of the trunk of $D_p^{(j)}$). Further, let $N_{\omega,n}^{(j)} \subset N_\omega^{(j)}$ denote the set of leaves of $D_{p,n}^{(j)}$, and $N_{p^i}^{(j)}$ denote the set of leaves of a finite p-adic rooted tree $T_{p,i}^{(j)}$. Then let

$$\mathcal{F}_n = \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{a_i} T_{p,i}^{(j)} \sqcup \bigsqcup_{i=1}^{n} D_{p,n+i}^{(i)}.$$

Now we can embed $\mathcal{F}_{n-1}$ into $\mathcal{F}_n$ as shown at Fig. 11. The set of leaves of $\mathcal{F}_n$ is the union

$$M_n = \bigsqcup_{i=1}^{n} \bigsqcup_{j=1}^{a_i} N_{p^i}^{(j)} \sqcup \bigsqcup_{i=1}^{n} N_{\omega,n+i}^{(i)}.$$

Given the set M_n we denote by $S(M_n)$ the group of all permutations on $\mathbb{N}$ that fix $\mathbb{N} \backslash M_n$ point-wise. It is clear, that $S(M_n)$ is isomorphic to the finite symmetric group on M_n.

Then if P_i is a particular Sylow p-subgroup of a particular finite rooted tree from $\mathcal{F}_n$ then

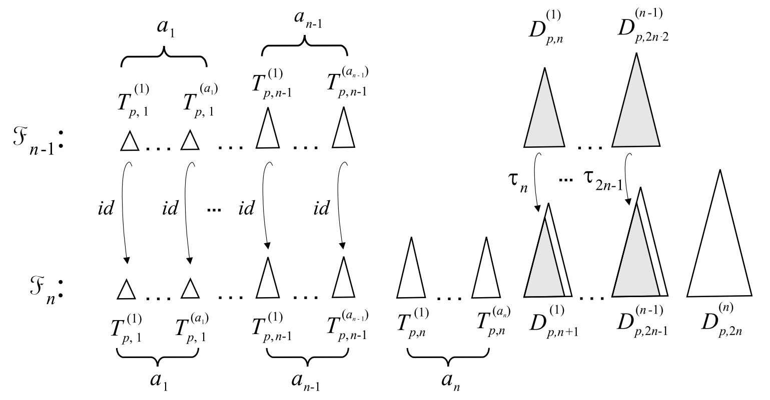


Fig. 11. Embedding $\mathcal{F}_{n-1}$ into $\mathcal{F}_n$.

$$Q_n = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{a_i} P_i \oplus \bigoplus_{i=1}^{n} P_{n+i},$$

is a Sylow p-subgroup of $S(M_n)$. This fact becomes evident if we just calculate the orders of Q_n and $S(M_n)$.

Hence, we have the following diagram

$$\begin{array}{ccccccc} Q_1 & < \ldots < & Q_n & < & Q_{n+1} & < \ldots & P(\mathcal{F}_\mathcal{A}, \Psi) = \bigcup\limits_{n=1}^{\infty} Q_n \\ \wedge & & \wedge & & \wedge & & \\ S(M_1) & < \ldots < & S(M_n) & < & S(M_{n+1}) & < \ldots & FS_\mathbb{N} = \bigcup\limits_{n=1}^{\infty} S(M_n) \end{array}$$

where Q_n is a Sylow p-subgroup of $S(M_n)$ for all $n \in \mathbb{N}$. Consequently, $P(\mathcal{F}_\mathcal{A}, \Psi)$ is a Sylow p-subgroup of $FS_\mathbb{N}$.

(2). Let Q be a Sylow p-subgroup of $FS_\mathbb{N}$. Any orbit of transitivity of Q is either infinite or finite with p^n elements, $n \in \mathbb{N}$. Let Q have a_0 orbits of length 1; a_1 orbits of length p; a_2 orbits of length p^2; $\ldots$; and $\mathcal{I}$ orbits of infinite cardinality, $\mathcal{I} \in \mathbb{N} \cup \{\infty\}$. Observe that $a_i < p$ for all $i \in \mathbb{N}$. If $a_k \geq p$ for some $k \in \mathbb{N}$ then there exists an element of $FS_\mathbb{N} \setminus Q$, permuting p orbits of length p^k and then Q would not be a maximal p-subgroup of $FS_\mathbb{N}$. Hence, $\mathcal{A} = (\mathcal{I}, a_0, a_1, a_2, \ldots)$ is a sequence for which we may construct the respective p-forest $\mathcal{F}_\mathcal{A}$ and a p-partition $\pi_\mathcal{F}$. Moreover, Q induces on its orbits (and hence on the constituent trees) the respective Sylow p-subgroups (P_n on every tree of height p^n and DP_∞ on D_p). Thus, the statement follows.

(3). Assume that $\mathcal{F}_\mathcal{A}$ and $\mathcal{F}_\mathcal{B}$ are isomorphic, i.e. $\mathcal{A} = \mathcal{B}$, and let $\{\theta_i\}_{i=1}^{\infty}$, $\{\vartheta_i\}_{i=1}^{\infty}$ be the respective orbits of transitivity. We put

$$\alpha = \begin{pmatrix} \theta_1 & \theta_2 & \ldots & \theta_i & \ldots \\ \vartheta_1 & \vartheta_2 & \ldots & \vartheta_i & \ldots \end{pmatrix}$$

then $\alpha^{-1} A \alpha = B$.

Conversely, let

$$A = P(\mathcal{A}, \Psi_1) = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{a_i} P_i \oplus \bigoplus_{j=1}^{\mathcal{I}} DP_\infty \cong \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{b_i} P_i \oplus \bigoplus_{j=1}^{\mathcal{J}} DP_\infty = P(\mathcal{B}, \Psi_2) = B,$$

where $\mathcal{A} = (\mathcal{I}_\infty, a_0, a_1, a_2, \ldots)$, $\mathcal{B} = (\mathcal{J}, b_0, b_1, b_2, \ldots)$ are the sequences for which the respective p-forests $\mathcal{F}_\mathcal{A}$, $\mathcal{F}_\mathcal{B}$ are defined (as shown in Fig. 10). Let π_A and π_B denote the p-partitions determined by the p-forests $\mathcal{F}_\mathcal{A}$ and $\mathcal{F}_\mathcal{B}$, respectively.

The center $Z(A)$ of the group A is the direct sum of $Z(P_i)$, $i \in \mathbb{N}$ (DP_∞ is the group with trivial center). Additionally, P_i' (the derived subgroup of P_i) contains $Z(P_i)$ for all $i \geq 2$ and P_1 is the abelian (cyclic) group [12]. Thus, $P_1^{a_1} = Z(A) \setminus A'$. Similarly, $P_1^{b_1} = Z(B) \setminus B'$. Since $A \simeq B$ we obtain $P_1^{a_1} \simeq P_1^{b_1}$ and $a_1 = b_1$.

Now we cut off the lower level of the forests $\mathcal{F}_\mathcal{A}$ and $\mathcal{F}_\mathcal{B}$. In other words, we delete all leaves of $\mathcal{F}_\mathcal{A}$ and $\mathcal{F}_\mathcal{B}$ together with all adjacent edges. Then one can repeat the reasoning shown above for groups induced by A and B on these truncated forests and obtain $a_2 = b_2$, etc. Thus, we obtain that $a_i = b_i$ for all $i \in \mathbb{N}$ and, hence, $\bigoplus_{j=1}^{\mathcal{I}} DP_\infty \simeq \bigoplus_{j=1}^{\mathcal{J}} DP_\infty$. Since DP_∞ cannot be represented as a direct product of non-trivial subgroups, then $\mathcal{I}$ and $\mathcal{J}$ have the same cardinality.

It is also clear that A and B are isomorphic if and only if A and B are conjugate in $S_\mathbb{N}$. $\square$

Acknowledgment

The authors are very grateful to the anonymous referee for valuable comments that helped to improve the manuscript and correct some mistakes.

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