CONJUGACY IN ISOMETRY GROUPS OF GENERALIZED BAIRE METRICS

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We describe the conjugacy classes of isometry groups of Baire spaces and generalized Baire spaces. In the first case each such class is uniquely characterized by a marked tree of special form and in the second by a forest of such trees. It follows in particular from the description given that the isometry groups indicated are ambivalent.

The representation of isometry groups of generalized Baire metrics as $\tilde{\ell}$-wreath products of finite symmetric groups [1] lets one study their structure in considerable detail. In the present paper we study the conjugacy classes of such groups. As an intermediate problem here it becomes necessary to describe the conjugacy classes of isometry groups of Baire metric spaces studied in [2-4].

1. Isometry Groups of Generalized Baire Metrics

Let $\Sigma=\{\mathcal{M}_\alpha\}_{\alpha\in Z}$ be a family of finite sets indexed by the integers, $\mathcal{M}=\prod\limits_{\alpha\in Z}\mathcal{M}_\alpha$, $E=(\mathcal{M},\rho)$ be a generalized Baire space over Σ in the sense of [1], $\mathrm{Is}\,E$ be its isometry group. According to [1] the group $\mathrm{Is}\,E$ is similar to the $\tilde{\ell}$-wreath product of the symmetric groups $S(\mathcal{M}_\alpha)$, $\alpha\in Z$. Hence any element of this group can be represented uniquely as a tableau

$$u=[a_\alpha({}^{(\alpha-1)}\bar{x})]_{\alpha\in Z}, \tag{1}$$

where ${}^{(\alpha-1)}\bar{x}$ is the left infinite initial segment of the collection $\bar{x}=(x_\alpha)_{\alpha\in Z}\in\mathcal{M}$ with extreme right coordinate $x_{\alpha-1}$, $a_\alpha({}^{(\alpha-1)}\bar{x})$ is a function defined on ${}^{(\alpha-1)}\mathcal{M}=\prod\limits_{i<\alpha}\mathcal{M}_i$ with values in $S(\mathcal{M}_\alpha)$. The tableau (1) acts on any element $\bar{m}=(m_\alpha)_{\alpha\in Z}$ as follows:

$$\bar{m}^u=(m_\alpha^{a_\alpha({}^{(\alpha-1)}\bar{m})})_{\alpha\in Z}. \tag{2}$$

This action corresponds to the rule of multiplication of tableaux according to which the product of the tableaux $u=[a_\alpha({}^{(\alpha-1)}\bar{x})]_{\alpha\in Z}$ and $v=[b_\alpha({}^{(\alpha-1)}\bar{x})]_{\alpha\in Z}$ will be the tableau $[a_\alpha({}^{(\alpha-1)}\bar{x})\times b_\alpha({}^{(\alpha-1)}(\bar{x}^{u_{\alpha-1}}))]_{\alpha\in Z}$. The identity element of the group $\mathrm{Is}\,E$ is the tableau e all of whose coordinates are identically equal to the substitution ε. The inverse to (1) will be the tableau $[a_\alpha^{-1}({}^{(\alpha-1)}(\bar{x}^{u^{-1}_{\alpha-1}}))]_{\alpha\in Z}$, where $u_{\alpha-1}$ is the initial tableau of u with extreme right coordinate $[u]_{\alpha-1}$. As established in [1], $\mathrm{Is}\,E$ consists of locally bounded tableaux of the form (1), i.e., those such that for any $\bar{x}\in\mathcal{M}$ there exists an $\alpha\in Z$, for which ${}^{(\alpha)}\bar{x}={}^{(\alpha)}(\bar{x}^u)$. We call a locally bounded tableau $u\in\mathrm{Is}\,E$ semibounded if there exists an $\alpha\in Z$ for it such that for all $\bar{x}\in\mathcal{M}$ one has ${}^{(\alpha)}(\bar{x}^u)={}^{(\alpha)}\bar{x}$. All semibounded tableaux form a normal subgroup $\overline{\mathrm{Is}}\,E$ of $\mathrm{Is}\,E$. The metric on E induces a metric on $\overline{\mathrm{Is}}\,E$ giving this group the structure of a generalized Baire space. $\overline{\mathrm{Is}}\,E$ is the union of an increasing sequence of normal subgroups I_α, $\alpha\in Z$ consisting of all tableaux of depth $\geqslant\alpha$, i.e., those whose coordinates with indices $\leqslant\alpha$ are equal to ε.

<u>LEMMA 1.</u> For any $\alpha\in Z$ the group I_α is isomorphic to a Cartesian power of isometry groups of a Baire space over the family of sets $\{\mathcal{M}_\beta\}_{\beta>\alpha}$.

<u>Proof.</u> According to [3] isometries of a Baire space over $\mathcal{M}^{(\alpha)}=\prod\limits_{\beta>\alpha}\mathcal{M}_\beta$ can be represented by right infinite collections $[a_1^{(\alpha)}, a_2^{(\alpha)}(x_1),\ldots]$, where $a_1^{(\alpha)}\in S(\mathcal{M}_{\alpha+1})$, $a_i^{(\alpha)}(\bar{x}_{i-1})$ is a function

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defined on the set $\mathcal{M}_{\alpha+1}\times\ldots\times\mathcal{M}_{\alpha+i-1}$ with values in the group $S(\mathcal{M}_{\alpha+i})$. For an arbitrary tableau $u=[a_\gamma(^{(\gamma-1)}\bar{x})]_{\gamma\in Z}$ from I_α and fixed $\bar{t}\in{}^{(\alpha)}\mathcal{M}$ we denote by $u(\bar{t})$ the transformation of the set $\mathcal{M}^{(\alpha)}$ defined by the tableau $[a_{\gamma+1}(\bar{t}), a_{\gamma+2}(\bar{t}, x_{\gamma+1}), \ldots]$. It is contained in the isometry group of a Baire space over $\mathcal{M}^{(\alpha)}$. Thus there is defined a map $u\to(u(\bar{t})/\bar{t}\in{}^{(\alpha)}\mathcal{M})$ of the group I_α into a Cartesian power of a group of isometries of a Baire space over $\mathcal{M}^{(\alpha)}$. One can verify directly that it is an isomorphism and the lemma is proved.

2. Conjugacy Classes of Isometry Groups of Baire Spaces

Let T be a Baire space over the family $\{\mathcal{M}_\alpha\}_{\alpha\in N}$, $|\mathcal{M}_\alpha|=n_\alpha$, $\mathrm{Is}\,T$ be its isometry group, $T^{(k)}=\mathcal{M}_1\times\ldots\times\mathcal{M}_k$, $k\in N$. Each isometry $u\in\mathrm{Is}\,T$ induces, for any $k\in N$, a transformation on $T^{(k)}$ which we denote by u_k. If $u=[a_1, a_2(x_1), \ldots]$, where $a_1\in S(\mathcal{M}_1)$, $a_i(\bar{x}_{i-1})\in S(\mathcal{M}_i)^{\mathcal{M}_1\times\ldots\times\mathcal{M}_{i-1}}$ corresponds to an initial segment of length k of this tableau. Let φ_{k-1} be the projection (with respect to the last coordinate) of $T^{(k)}$ to $T^{(k-1)}$. For a cycle $\pi=(t_k^{(1)},\ldots,t_k^{(l)})$ over $T^{(k)}$ we denote by $\varphi_{k-1}(\pi)$ the cycle on $T^{(k-1)}$ obtained by projection of sequences $t_k^{(i)}$, $1\leqslant i\leqslant l$ and elimination of repeated occurrences of elements. The cycle type analog for transformations from $\mathrm{Is}\,T$ is a tree of orbits defined as follows.

The vertices of the tree of orbits $\mathcal{D}(u)$ of the isometry u are situated at levels numbered by the natural numbers. To a root vertex corresponds level zero; it has label 1. For any $k\in N$ at the k-th level are situated vertices corresponding to components of the decomposition of the substitution u_k of the set $T^{(k)}$ into a product of independent cycles. Here each vertex is assigned a label which is the length of the corresponding cycle. The order of disposition of the vertices of a given level is inessential. The vertex of level k corresponding to the cycle π_k is joined with the vertex of level $k-1$ corresponding to the cycle π_{k-1} if and only if $\varphi_{h-1}(\pi_k)=\pi_{k-1}$, $k\geqslant 1$.

The tree $\mathcal{D}(u)$ for any $u\in\mathrm{Is}\,T$ satisfies the following conditions for all $k\in N$:

a) the sum of the labels of all vertices of level k is equal to $n_1\ldots n_k$;

b) if a vertex of level $k+1$ with label s is joined with a vertex of level k having label t, then $t|s$;

c) if a vertex of level k with label s is joined with ℓ vertices of level $k+1$ having labels $t_1,\ldots,t_\ell$ and only with them, then $\sum\limits_{i=1}^{\ell} t_i = sn_k$.

By an isomorphism of labeled trees $\mathcal{D}_1$ and $\mathcal{D}_2$ we shall mean an isomorphism of them under which a root vertex of $\mathcal{D}_1$ corresponds to a root vertex of $\mathcal{D}_2$ and for all $k\in N$ vertices of level k of $\mathcal{D}_1$ correspond to vertices of level k of $\mathcal{D}_2$ with the same labels. Let Γ_T be the set of all pairwise nonisomorphic labeled trees satisfying a)-c). Clearly the cardinality of Γ_T is equal to c. If $\mathcal{D}=\mathcal{D}(u)$, $u\in\mathrm{Is}\,T$ then its subtree $\overline{\mathcal{D}}_k$ obtained by throwing out vertices of $\mathcal{D}$ of all levels starting with the $(k+1)$-st and the edges issuing from them can be considered naturally as the tree of orbits of the transformation u_k. For any $k\in N$ there is defined an imbedding $\varphi_k:\overline{\mathcal{D}}_k\to\overline{D}_{k+1}$, where $\varphi_{k+1}|_{\overline{\mathcal{D}}_k}=\varphi_k$. Thus there is defined a direct spectrum $\langle\overline{\mathcal{D}}_k,\varphi_k\rangle_{k\in N}$, whose limit graph $\varinjlim\langle\overline{\mathcal{D}}_k,\varphi_k\rangle$ is isomorphic with $\mathcal{D}$. One can define such a limit representation for any graphs from Γ_T.

LEMMA 2. Trees $\mathcal{D}$ and $\mathcal{D}'$ from Γ_T are isomorphic if and only if there exists a system of isomorphisms $\delta_k:\overline{\mathcal{D}}_k\to\overline{\mathcal{D}}'_k$ such that for any $k\in N$ the diagram

$$\begin{array}{ccc} \overline{\mathcal{D}}_k & \xrightarrow{\delta_k} & \overline{\mathcal{D}}'_k \\ {\scriptstyle\varphi_k}\downarrow & & \downarrow{\scriptstyle\varphi'_k} \\ \overline{\mathcal{D}}_{k+1} & \xrightarrow{\delta_{k+1}} & \overline{\mathcal{D}}'_{k+1} \end{array} \tag{3}$$

is commutative.

By the symbol $\mathrm{pr}\,\bar{t}_k$ we denote the projection of $\bar{t}_k\in T^{(k)}$ with respect to the last coordinate and for any permutation $\pi_k=\begin{pmatrix}\bar{t}\\ \bar{t}'\end{pmatrix}\in S(T^{(k)})$ we set $\mathrm{pr}\ \pi_k=\begin{pmatrix}\mathrm{pr}\ \bar{t}\\ \mathrm{pr}\cdot\bar{t}'\end{pmatrix}$, $k\in N$. The two-rowed tableau $\mathrm{pr}\,\pi_k$ is a multipermutation since elements in its rows are repeated. We call the permutation π_k compatible if in $\mathrm{pr}\,\pi_k$ equal elements of the first row are only in identical columns.

LEMMA 3. The permutation π_k is compatible if and only if $\pi_k \in S(T^{(k-1)}) \sim S(\mathcal{M}_k)$.

Proof. Any multipermutation μ equal elements of whose first row are in identical columns determines a permutation $r(\mu)$ uniquely which is obtained after crossing out identical columns except for one in μ. If $\pi_k \in S(T^{(k)})$ is compatible, then $\pi_k' = r(\mathrm{pr}\,\pi_k) \in S(T^{(k-1)})$ is defined where for any $(m_1, \dots, m_k) \in T^{(k)}$ we have $((m_1, \dots, m_{k-1})^{\pi_k'}, m_k')$. Thus with a compatible permutation π_k there is associated a pair $[\pi_k', a(\bar{x}_{k-1})]$, where $a(\bar{x}_{k-1})$ is a function from $T^{(k-1)}$ to $S(\mathcal{M}_k)$ which on an arbitrary collection $(m_1, \dots, m_{k-1})$ assumes the value which carries m_k to m_k'. Thus, each compatible permutation from $S(T^{(k)})$ is contained in $S(T^{(k-1)})\ S(\mathcal{M}_k)$. The compatibility of permutations from this subgroup is obvious. The lemma is proved.

For $\pi_k \in S(T^{(k)})$ we set $\pi_k^{(1)} = r(\mathrm{pr}\,\pi_k)$, $\pi_k^{(l)} = r(\mathrm{pr}\,\pi_k^{(l-1)})$, $1 < l \leqslant k$. We call the permutation π_k strongly compatible if for all l, $1 \leqslant l \leqslant k$ the permutation $\pi_k^{(l)}$ is compatible.

LEMMA 4. The permutation $\pi_k \in S(T^{(k)})$ is contained in the subgroup $G_k = \mathop{\S}\limits_{i=1}^{k} S(\mathcal{M}_i)$ if and only if it is strongly compatible.

According to what was said above one can refine the cycle type of strongly compatible permutations by associating with them the (finite) tree of orbits. Here it is easy to see that to permutations which are conjugate in G_k there correspond isomorphic trees of orbits.

Let $\langle \pi_i \rangle_{i \in N}$ be a family of permutations such that $\pi_i \in S(T^{(i)})$. We call an isometry $w \in \mathrm{Is}\,T$ a limit for the family π_i if for any $i \in N$ one has $w_i = \pi_i$.

LEMMA 5. A limit permutation for the family $\langle \pi_i \rangle_{i \in N}$ exists if and only if for any $i \in N, \pi_i$ is compatible while $r(\mathrm{pr}\,\pi_{i+1}) = \pi_i$.

Proof. The necessity of the condition is obvious. We verify its sufficiency. Since for any $i \in N$, π_i is compatible, by Lemma 3 it is contained in $S(T^{(i-1)}) S(\mathcal{M}_i)$, i.e., is defined by a tableau of the form $[\pi_{i-1}', a_i(\bar{x}_{i-1})]$, $\pi_i \in S(T^{(i-1)})$, $a_i(\bar{x}_{i-1}) \in S(\mathcal{M}_i)^{\mathcal{M}^{(i)}}$. Here since $r(\mathrm{pr}\,\pi_i)$ is the same as the projection of this tableau with respect to the last coordinate, $\pi_{i-1}' = \pi_{i-1}$ for any $i \in N$. Hence $\pi_i \in G_i$ also and the family $\langle \pi_i \rangle_{i \in N}$ is a thread from the inverse spectrum $\langle G_i, \lambda_i \rangle_{i \in N}$, where $\lambda_i \colon G_{i+1} \to G_i$ is the homomorphism of projection with respect to the last coordinate. Consequently, it uniquely determines an element w of the limit group of this spectrum which is $\mathrm{Is}\,T$. Since for w for all $i \in N$ we have $w_i = \pi_i$ the lemma is proved.

THEOREM 1. Let $u, v \in \mathrm{Is}\,T$. The tableaux u and v are conjugate in this group if and only if their trees of orbits $\mathcal{D}$ and $\mathcal{D}'$ are isomorphic.

Proof. Necessity. Let u and v be conjugate elements. Then for any $k \in N$ the elements u_k and v_k are conjugate in G_k, i.e., their trees of orbits $\bar{\mathcal{D}}_k$ and $\bar{\mathcal{D}}_k'$ are isomorphic. Thus, for any $k \in N$ one fixes an isomorphism $\delta_k \colon \bar{\mathcal{D}}_k \to \bar{\mathcal{D}}_k'$. Moreover, $\mathcal{D} \simeq \varinjlim \langle \bar{\mathcal{D}}_k, \varphi_k \rangle$, $\mathcal{D}' \simeq \varinjlim \langle \bar{\mathcal{D}}_k', \varphi_k' \rangle$, i.e., there are defined imbeddings $\varphi_k \colon \bar{\mathcal{D}}_k \to \bar{\mathcal{D}}_{k+1}$, $\varphi_k' \colon \mathcal{D}_k' \to \bar{\mathcal{D}}_{k+1}'$. But since for arbitrary $k \in N$ the diagram (3) in this situation is commutative, it follows from Lemma 2 that $\mathcal{D} \simeq \mathcal{D}'$.

Sufficiency. Let the graphs $\mathcal{D}$ and $\mathcal{D}'$ be isomorphic. Then the corresponding systems $\langle \bar{\mathcal{D}}_k, \varphi_k \rangle$ and $\langle \bar{\mathcal{D}}_k', \varphi_k' \rangle$ satisfy the hypotheses of Lemma 2. Consequently, for any $k \in N$ the trees $\bar{\mathcal{D}}_k$ and $\bar{\mathcal{D}}_k'$ are isomorphic. Hence the strongly compatible permutations u_k and v_k are conjugate, i.e., their cycle types are the same. Let $u_k = \prod\limits_{l=1}^{s} u_k^{(l)}$, $v_k = \prod\limits_{l=1}^{s} v_k^{(l)}$ be the decompositions of u_k and v_k into products of independent cycles where the cycles are indexed so that if $u_k^{(\ell)}$ corresponds to the vertex t of the tree $\bar{\mathcal{D}}_k$ then $v_k^{(\ell)}$ corresponds to the vertex $\delta_k(t)$ of the tree $\bar{D}_k'$. Setting $u_k^{(l)} = (a_1^{(l)}, \dots, a_{r_l}^{(l)})$, $v_k^{(l)} = (b_1^{(l)}, \dots, b_{r_l}^{(l)})$, $1 \leqslant l \leqslant s$ we define the permutation

$$w_k = \begin{pmatrix} a_1^{(1)} & \dots & a_{r_1}^{(1)} & \dots & a_1^{(s)} & \dots & a_{r_s}^{(s)} \\ b_1^{(1)} & \dots & b_{r_1}^{(1)} & \dots & b_1^{(s)} & \dots & b_{r_s}^{(s)} \end{pmatrix} \in S(T^{(l)}),$$

which joins u_k and v_k. Crossing out the last coordinate in the sequences $a_i^{(j)}$ and $b_i^{(j)}$, $j = 1, \dots, s$; $i = 1, \dots, r_j$ from the transformation w_k we construct the multipermutation

$$\bar{w}_{k-1} = \begin{pmatrix} \mathrm{pr}\, a_1^{(1)} & \dots & \mathrm{pr}\, a_{r_1}^{(1)} & \dots & \mathrm{pr}\, a_1^{(s)} & \dots & \mathrm{pr}\, a_{r_s}^{(s)} \\ \mathrm{pr}\, b_1^{(1)} & \dots & \mathrm{pr}\, b_{r_1}^{(1)} & \dots & \mathrm{pr}\, b_1^{(s)} & \dots & \mathrm{pr}\, b_{r_s}^{(s)} \end{pmatrix}.$$

We show that in it identical elements of the first row are in equal columns. Let $\mathrm{pr}\, a_m^{(i)} = \mathrm{pr}\, a_n^{(j)}$, $1 \leqslant i, j \leqslant s$, $1 \leqslant m \leqslant r_i$, $1 \leqslant n \leqslant r_j$, $m \geqslant n$. Two cases are possible.

1) $i = j$, i.e., belong to one cycle $u_k^{(i)}$. We consider the permutations u_k^{m-n}, v_k^{m-n}. Since the trees of orbits of u_k and v_k are isomorphic, the trees of orbits of u_k^{m-n}, and v_k^{m-n} will also be isomorphic. For the permutation u_k^{m-n} one has $(a_n^{(j)})^{u_k^{m-n}} = a_m^{(i)}$. It follows from this and the equality of the projections of $a_m^{(i)}$ and $a_n^{(j)}$ that in the multipermutation $\bar{u}_{k-1}$ all elements obtained from the cycle $u_k^{(i)}$ are equal to one another. Hence in the permutation $r(\bar{u}_{k-1}^{m-n}) = u_{k-1}^{m-n}$ a fixed point corresponds to this cycle. Consequently, v_{k-1} has a fixed point corresponding to the cycle of v_k which contains $b_m^{(i)}$ and $b_n^{(j)}$. Hence $\mathrm{pr}\, b_m^{(i)} = \mathrm{pr}\, b_n^{(j)}$.

2) $i \neq j$, i.e., $a_m^{(i)}$, $a_n^{(j)}$ are contained in different cycles $u_k^{(i)}$, $u_k^{(j)}$. It follows from the compatibility of u_k that the cycle $(\mathrm{pr}\, a_1^{(i)}, \dots, \mathrm{pr}\, a_{r_i}^{(i)})$ is the same as the cycle $(\mathrm{pr}\, a_1^{(j)}, \dots, \mathrm{pr}\, a_{r_j}^{(j)})$. Since the trees of orbits of u_k and v_k are isomorphic and the vertices of $\bar{\mathcal{D}}_h$ corresponding to the cycles $u_k^{(i)}$, $u_k^{(j)}$ under this isomorphism correspond to vertices of $\bar{\mathcal{D}}_k'$ which correspond to the cycles $v_k^{(i)}$, $v_k^{(j)}$ it follows from the equality of cycles indicated that $(\mathrm{pr}\, b_1^{(i)}, \dots, \mathrm{pr}\, b_{r_i}^{(i)}) = (\mathrm{pr}\, b_1^{(j)}, \dots, \mathrm{pr}\, b_{r_j}^{(j)})$. Hence if $\mathrm{pr}\, a_m^{(i)} = \mathrm{pr}\, a_n^{(j)}$ then $\mathrm{pr}\, b_m^{(i)} = \mathrm{pr}\, b_n^{(j)}$.

Thus, for any $k \in N$ the permutation w_k is compatible while $r(\bar{w}_{k-1}) = w_{k-1}$. Thus, the hypotheses of Lemma 5 hold for the system of permutations w_k, $k \in N$. Consequently, there exists a limit isometry $w' \in \mathrm{Is}\, T$ such that $(w')_k = w_k$, $k \in N$. Since for it $uw' = w'v$, the tableaux u and v are conjugate in Is T and the theorem is proved.

THEOREM 2. For any Baire space T there exists a one-to-one correspondence between the conjugacy classes of the group Is T and the trees of the set Γ_T.

Proof. By Theorem 1 to each conjugacy class of Is T corresponds a tree from Γ_T and different conjugacy classes correspond to nonisomorphic trees. Hence it suffices to verify that the map defined in this way is surjective, i.e., any tree from Γ_T can be the tree of orbits of an isometry of the space T. Let $\mathcal{D} \in \Gamma_T$ be a tree. We explicitly write down the process of constructing a tableau $u = [a_1, a_2(x_1), \dots] \in \mathrm{Is}\, T$ such that $\mathcal{D}(u) = \mathcal{D}$. As a_1 we take a permutation from $S(\mathcal{M}_1)$ whose cycle type is the same as the collection of labels of vertices of the first level of the tree $\mathcal{D}$. Let us assume that the coordinates $a_1, \dots, a_{h-1}(\bar{x}_{h-2})$ of the tableau u are defined. We fix a vertex $\bar{t}$ (k − 1)-st level of the tree $\mathcal{D}$ and consider all vertices of level k incident with it. Let $\pi = (\bar{a}_1, \dots, \bar{a}_l)$ be the cycle from the decomposition of u_{k-1} corresponding to t. Ascribing to each of the sequences a_i all coordinates, elements of $\mathcal{M}_h$, we get $l \cdot |\mathcal{M}_h|$ different points. Since $\mathcal{D}$ satisfies conditions a)-b) the sum of the labels of vertices of $\mathcal{D}$ incident to $\bar{t}$ is also equal to $l \cdot |\mathcal{M}_h|$. Hence the $l \cdot |\mathcal{M}_h|$ points indicated can be divided into parts whose cardinalities are equal to the labels of vertices incident to t and a cycle formed from each part. Making the analogous constructions for all vertices of level k − 1 we get a decomposition of a permutation u_k whose first k − 1 coordinates are the functions already defined. In this way its k-th coordinate is also defined. According to Lemma 5, from this we get that there exists a tableau $u \in \mathrm{Is}\, T$ whose initial segment is u_h, $k \in N$ and D(u) = D. The theorem is proved.

3. Conjugacy Classes of the Group Is E

Let $u \in \mathrm{Is}\, E$ be an isometry, $t \in \mathcal{M}$, and k_t be the largest index such that u_{k_t} does not change $^{(k_t)}t$ or the symbol ∞, if no such index exists. Since u is a locally bounded tableau, k_t is defined for each $t \in \mathcal{M}$. By the spectrum of the tableau u we mean the family $\mathrm{sp}\, u = \langle k_t / t \in \mathcal{M} \rangle$ and we set $\mathcal{M}_u = \{t \in \mathcal{M} / k_t \neq \infty\}$. The disc $B(\xi^k, t)$ in the space E (ξ, $0 < \xi < 1$ being the fixed number figuring in the definition of the metric ρ [1]) is defined by the equality $B(\xi^k, t)$ $\{^{(k)} t * x / x \in \mathcal{M}^{(k+1)}\}$ where * is the operation of concatenation. We call the discs $B(\xi^{k_t}, t)$, $k_t \in \mathrm{sp}\, u$, $t \in \mathcal{M}_u$ significant for the permutation u. Two significant discs either do not intersect or coincide while for any such disc B it follows from $x \in B$ that $x^u \in B$. Hence the restriction u_B of the isometry u to each of its significant discs $B(\xi^{k_t}, t)$, $t \in \mathcal{M}_u$ is defined and coincides with the transformation which is induced by the tableau $u({}^{(k_t)}t)$. Let $\mathfrak{B}_u$ be the set of different significant discs of the isometry u.

LEMMA 6. For an arbitrary $u \in \mathrm{Is}\,E$ one has a decomposition

$$u = \bigoplus_{B \in \mathfrak{B}_u} u_B, \tag{4}$$

where $\oplus$ is the symbol for the direct sum of permutations in the sense of [5].

We call (4) the spectral decomposition of the isometry u. Any of the discs $B \in \mathfrak{B}_u$ is isometric to a Baire space over a suitable family and u_B defines an isometry of this space onto itself. Consequently, according to Paragraph 2 there corresponds to it a tree of orbits $\mathscr{D}(u_B)$. Thus the isometry u is associated with the marked graph $\bigcup_{B \in \mathfrak{B}_u} \mathscr{D}(u_B)$ which is a forest which we denote by F(u). We shall call it the forest of orbits of the transformation u. All trees of the forest F(u) have identically directed crowns which are indexed by integers and the cardinality of the set of vertices of each level is equal to c. To each vertex of $\mathscr{D}(u_B)$ uniquely corresponds the collection ${}^{(k_t)}t$, where $k_t \in \mathrm{sp}\,u$, $B = B(\xi^{k_t}, t)$ $t \in \mathscr{M}_u$ which we call its coordinate collection. Coordinate collections are close if they have a common initial segment.

We call two forests F_1 and F_2 of the form indicated equivalent if there exists a bijective map φ of the set of root vertices of trees of F_1, and of the same set for F_2, which satisfies the following conditions:

1) the images of the root vertices of level k of trees from F_1 under the map φ will be root vertices of trees of level k from F_2;
2) if ${}^{(k)}t$ is the coordinate collection of a vertex of a tree $\mathscr{D}$ from F_1 and ${}^{(k)}t'$ is the coordinate collection of a vertex of the tree $\mathscr{D}'$ from F_2 ${}^{(k)}t' = \varphi({}^{(k)}t)$ then the collections ${}^{(k)}t$, ${}^{(k)}t'$ are close and $\mathscr{D}$ is isomorphic to $\mathscr{D}'$.

THEOREM 3. Let E be an arbitrary generalized Baire space. Isometries u and v of this space are conjugate in the group Is E if and only if their forests of orbits F(u) and F(v) are equivalent.

Proof. Necessity. Let u and v be conjugate isometries and $u = \bigoplus_{B \in \mathfrak{B}_u} u_B$, $v = \bigoplus_{B \in \mathfrak{B}_v} v_B$ be their spectral decompositions. For $w \in \mathrm{Is}\,E$, $B \in \mathfrak{B}_w$ we denote by $\bar{w}_B$ the transformation of $\mathscr{M}$ which acts on B like w and trivially on $\mathscr{M} \setminus B$. It follows from the conjugacy of u and v that there exists a bijection $\delta: \mathfrak{B}_u \to \mathfrak{B}_v$ such that if $\delta(B) = B'$, $B = B(\xi^{k_t}, t) \in \mathfrak{B}_u$, $B' = B'(\xi^{k_{t'}}, t') \in \mathfrak{B}_v$ then $k_t = k_{t'}$ and $\bar{u}_B$ is conjugate with $\bar{v}_{B'}$ in Is E. From the bijection δ we construct a map φ defining an equivalence of forests of orbits F(u) and F(v) as follows. Each disc $B = B(\xi^{k_t}, t)$ uniquely determines a collection ${}^{(k_t)}t$ which will be the coordinate collection for root vertices of trees of orbits of the transformation $u|_B$ the restriction of u to B. Let $\mathscr{M}^u = \{{}^{(k_t)}t / B(\xi^{k_t}, t) \in \mathfrak{B}_u\}$, $\mathscr{M}^v = \{{}^{(k_{t'})}t' / B'(\xi^{k_{t'}}, t') \in \mathfrak{B}_v\}$ be the sets of coordinate collections of vertices of trees from F(u) and F(v). For any $t \in \mathscr{M}$ let $\varphi({}^{(k)}t) = {}^{(k)}t'$, where $\delta(B(\xi^k, t)) = B'(\xi^k, t')$. Since δ is a bijection, φ will be a bijective map. Since $\bar{u}_B$ and $\bar{v}_{B'}$ are conjugate by an element of Is E, ${}^{(k)}t$ and ${}^{(k)}t'$ are close and the cyclic groups $\langle \bar{u}_B \rangle$ and $\langle \bar{v}_{B'} \rangle$ are similar. Hence the cyclic groups generated by the transformations $\bar{u}_B|_B = u_B$ and $\bar{v}_{B'}|_{B'} = v_{B'}$ acting on the discs B and B' which are isometric to the same Baire space will also be similar. Consequently, the trees of orbits of the transformations u_B and $v_{B'}$ are isomorphic for arbitrary $B \in \mathfrak{B}_u$. Thus, the map φ constructed satisfies conditions 1 and 2 from the definition of equivalence of forests, i.e., F(u) and F(v) are equivalent.

Sufficiency. Let u and v be isometries such that F(u) is equivalent to F(v). Then there exists a bijection $\varphi: \mathscr{M}^u \to \mathscr{M}^v$ for which it follows from $\varphi({}^{(k_t)}t) = {}^{(k_{t'})}t'$ that $k_t = k_{t'}$ and for the discs $B(\xi^{k_t}, t) \in \mathfrak{B}_u$, $B'(\xi^{k_{t'}}, t') \in \mathfrak{B}_v$ the trees $\mathscr{D}(u_B)$ and $\mathscr{D}(v_{B'})$ are isomorphic. We define an isometry $\tilde{u}_B$ of the disc B' by setting, for any $z = {}^{(k_t)}t' * z^{(k_t+1)} \in B'$:

$$z^{\tilde{u}_B} = {}^{(k_t)}t' * (({}^{(k_t)}t * z^{(k_t+1)})^{u_B})^{(k_t+1)}.$$

Since the cyclic groups $\langle u_B\rangle$ and $\langle\tilde u_B\rangle$ are similar, the trees $\mathcal{D}(u_B)$ and $\mathcal{D}(\tilde u_B)$ are isomorphic. Consequently, $\mathcal{D}(\tilde u_B)$ is isomorphic to $\mathcal{D}(v_{B'})$. Since $\tilde u_B$ and $v_{B'}$ are isometries of the Baire space B', from Theorem 1 we get that $\tilde u_B$ and $v_{B'}$ are conjugate in Is B', i.e., there exists a $\tilde w_B \in \mathrm{Is}\,B'$ such that $\tilde u_B \tilde w_B = \tilde w_B \tilde v_{B'}$. From the isometry $\tilde w_B$ we define a transformation w_B by setting, for arbitrary $z = {}^{(k_t)}t{*}z^{(k_t+1)} \in B$

$$z^{w_B} = {}^{(k_t)}t{*}(({}^{(k_t)}t'{*}z^{(k_t+1)})^{\tilde w_B})^{(k_t+1)}.$$

The family $\langle w_B/B\in\mathfrak{B}_u\rangle$ uniquely determines a permutation w acting on elements of B like w_B for all $B\in\mathfrak{B}_u$ and leaving elements from $\mathcal{M}\setminus\left(\bigcup\limits_{B\in\mathfrak{B}_u} B\right)$ unchanged. This transformation is contained in Is E and it satisfies uw = wv. Consequently, the isometries u and v are conjugate in Is E and the theorem is proved.

The forest of orbits of an arbitrary isometry of E satisfies the following conditions:

a) for any $k\in Z$ there exists a Baire space $T^{(k)}$ such that all trees of level k of this forest are contained in $\Gamma_{T(k)}$;

b) for all $k\in Z$ the space $T^{(k+1)}$ is obtained from $T^{(k)}$ by projection of all collections with respect to the first coordinate.

We denote by $F\Gamma_E$ the set of pairwise nonisomorphic forests satisfying a) and b).

THEOREM 4. For any generalized Baire space E there exists a one-to-one correspondence between conjugacy classes of the group Is E and forests of the set $F\Gamma_E$.

The proof is analogous to the proof of Theorem 2 and we omit it.

We recall that a group is called ambivalent if each of its elements is conjugate with its inverse.

THEOREM 5. 1) For any Baire space T the group Is T is ambivalent.

2) For any generalized Baire space E the groups Is E and $\bar{\mathrm{I}}$s E are ambivalent.

Proof. 1) It suffices to see that for each $u\in \mathrm{Is}\,T$ the tree $\mathcal{D}(u)$ is isomorphic to the tree $\mathcal{D}(u^{-1})$. But this is actually so because for any $k\in N$ the permutations u_k and u_k^{-1} have identical cycle types.

2) We show that the forest, F(u) and $F(u^{-1})$ are equivalent. If $u = \bigoplus\limits_{B\in\mathfrak{B}_u} u_B$ is the spectral decomposition for $u\in \mathrm{Is}\,E$ then the spectral decomposition for u^{-1} can be written in the form $u^{-1} = \bigoplus\limits_{B\in\mathfrak{B}_u} u_B^{-1}$. Since u_B, u_B^{-1} are isometries of the Baire space B, $\mathcal{D}(u_B)$ is isomorphic to $\mathcal{D}(u_B^{-1})$ by the first part of the theorem. From this we immediately get that F(u) is equivalent to $F(u^{-1})$. If $u\in \bar{\mathrm{I}}\mathrm{s}\,E$, i.e., u has finite depth $k\in Z$ then defining the element w for u, u^{-1} from their spectral decompositions as in the proof of Theorem 3, we get that its depths is at least k, i.e., $w\in \bar{\mathrm{I}}\mathrm{s}\,E$. Consequently, for any $u\in \bar{\mathrm{I}}\mathrm{s}\,E$ the elements u, u^{-1} are conjugate in $\bar{\mathrm{I}}$s E and this group is ambivalent. The theorem is proved.

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