Problem 7: You want to perform 1-kNN-classification based on

i) L_1-norm

ii) L_2-norm

Prove or disprove: The L_2-distance $d_2(\boldsymbol{x},\boldsymbol{y}) = \left(\sum_{i=1}^{d}(x_i - y_i)^2\right)^{\frac{1}{2}}$ between two points $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^d$ is always smaller or equal than the L_1-distance $d_1(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{d}|x_i - y_i|$.

$$a_i := x_i - y_i$$

$$d_2(\vec{x},\vec{y}) = \left(\sum_{i=1}^{d} a_i\right)^{1/2} = \left[a_1^2 + a_2^2 + \dots + a_d^2\right]^{1/2}$$

$$d_1(\vec{x},\vec{y}) = \sum_{i=1}^{d} |a_i| = |a_1| + |a_2| + \dots + |a_d|$$

to prove: $\left[a_1^2 + a_2^2 + \dots + a_d\right]^{1/2} \le |a_1| + |a_2| + \dots + |a_d| \quad \Big|\,(\ )^2$

$$a_1^2 + a_2^2 + \dots + a_d^2 \le \left[|a_1| + |a_2| + \dots + |a_d|\right]^2$$

$$a_1^2 + a_2^2 + \dots + a_d^2 \le |a_1|^2 + |a_2|^2 + \dots + |a_d|^2 + C \qquad (I)$$

$$C = \sum_{i=1}^{d} \sum_{j=1, j\neq i}^{d} |a_i| \cdot |a_j| \ge 0$$

Since C is greater or equal to zero, (I) is valid, which proves that the L2-distnace between two points is always smaller than or equal to the L1-distance.

$\square$

Problem 8: Prove or disprove: Consider two arbitrary points $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^2$. If $\boldsymbol{x}$ is the nearest neighbor of $\boldsymbol{y}$ regarding the L_2-norm then $\boldsymbol{x}$ is the nearest neighbor of $\boldsymbol{y}$ regarding the L_1-norm.

Consider three points $\vec{y} = (0,0)^T$, $\vec{x} = (4,3)^T$ and $\vec{x}' = (0,6)$

$$d_2(\vec{x},\vec{y}) = \left[(4-0)^2 + (3-0)^2\right]^{1/2} = \sqrt{16+9} = 5$$

$$d_2(\vec{x}',\vec{y}) = \sqrt{6^2} = 6$$

=> x is closer to y regarding the L2-distance

$$d_1(\vec{x},\vec{y}) = |4| + |3| = 7$$

$$d_1(\vec{x}', \vec{y}) = |6| = 6$$

But x' is closer to y regarding the L1-distance, which disproves the statment of problem 8 via example.