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Well-posedness and long time behavior of a perturbed Cahn–Hilliard system with regular potentials

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Abstract. The aim of this paper is to study the well-posedness and long time behavior, in terms of finite-dimensional attractors, of a perturbed Cahn–Hilliard equation. This equation differs from the usual Cahn–Hilliard by the presence of the term $\varepsilon(-\Delta u + f(u))$. In particular, we prove the existence of a robust family of exponential attractors as ε goes to zero.

Keywords: Cahn–Hilliard equation, regular potential, global attractor, exponential attractor

1. Introduction

We consider the following boundary value problem in a smooth and bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$:

$$\begin{cases} \partial_t u + \Delta^2 u - \Delta f(u) - \varepsilon\Delta u + \varepsilon f(u) = 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where f is the derivative of a nonconvex potential and the unknown u is the relative concentration of one phase. We assume throughout this paper that $n \leqslant 3$.

When $\varepsilon = 0$, we recover the well-known Cahn–Hilliard equation (see [11,14,16]) and when $\varepsilon > 0$, Eq. (1.1) may be viewed as a combination of the well-known Cahn–Hilliard equation and the Allen–Cahn equation (see [7–10]).

We recall that the Cahn–Hilliard equation describes the behavior of two-phase systems, in particular in spinodal decomposition, i.e. in the case of rapid separation of phases when the material is cooled down

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sufficiently and, when $\varepsilon > 0$, the equation describes a simplified model of adsorption to and desorption from the surface.

Such a model has been studied in [5,6] and [8], for a smooth nonlinearity in [6,8] and for a singular nonlinearity in [5] where questions such as existence and uniqueness of solutions, existence of the global attractor and an exponential attractor have been addressed.

We rewrite problem (1.1) as a system of second order equations:

$$\begin{cases} \partial_t u = \Delta w - \varepsilon w & \text{in } \Omega, \\ w = -\Delta u + f(u), \\ u = \Delta u = 0 & \text{on } \partial\Omega, \\ u|_{t=0} = u_0. \end{cases} \tag{1.2}$$

We denote by F the antiderivative of f vanishing at 0 and assume that

$$\begin{cases} f \in C^2(\mathbb{R}), \\ f(0) = 0, \\ f'(s) \geqslant -\kappa, \\ f(s)s \geqslant p b_{2p} s^{2p} - c \geqslant c\bigl(s^2 + s^{2p}\bigr) - c, \\ \bigl|f(s)\bigr| \leqslant \alpha b_{2p} s^{2p} + c(\alpha), \quad \forall \alpha > 0, \\ \bigl|f(s)\bigr| \leqslant c\bigl(|s|^{2p-1} + 1\bigr), \\ \frac{1}{2} b_{2p} s^{2p} - c \leqslant F(s) \leqslant \frac{3}{2} b_{2p} s^{2p} + c \end{cases} \tag{1.3}$$

for all $s \in \mathbb{R}$ and where κ, c, $c(\alpha)$ and b_{2p} are positive constants and $p \geqslant 2$ is an odd integer.

Typical choice for f is

$$f(s) = s^3 - s.$$

This paper is organized as follows. In Section 2, we give some useful assumptions and notation. Then, in Section 3, we derive uniform a priori estimates for approximated solutions which allow us to pass to the limit in the approximated problem to study the well-posedness, namely, the existence and uniqueness of a weak solution as stated in Theorem 3.1. Section 4 is dedicated to the proof of some additional regularity for the solution. It follows from the well-posedness result that the system generates a continuous semigroup in a suitable phase space, which allows to study the existence of the global attractor in Section 5. In Section 6, we prove that the fractal dimension of the global attractor is finite by studying the existence of exponential attractors. Finally, Section 7 is devoted to the proof of the continuity of exponential attractors for the perturbed system (1.1) and to the derivation of the corresponding estimate for the symmetric distance.

Remark 1. Neumann boundary conditions, namely, $\partial_n u = \partial_n \Delta u = 0$ on $\partial\Omega$, are also relevant in the context of the Allen–Cahn and Cahn–Hilliard equations. In that case, the limit problem, i.e., the Cahn–Hilliard equation, is a conservation law, in the sense that the spatial average of the order parameter u is a conserved quantity. This brings additional difficulties in the study of the continuity of exponential attractors and will be studied elsewhere.

2. Notation and assumptions

We introduce the following spaces:

$$H = L^2(\Omega), \qquad V = H_0^1(\Omega), \qquad W = H^2(\Omega) \cap H_0^1(\Omega).$$

We denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the usual norm and inner product in H.

Let

$$A_\varepsilon = -\Delta + \varepsilon I : D(A_\varepsilon) \subset H \to H$$

with $D(A_\varepsilon)= W$. The operator A_ε is a strictly positive self-adjoint linear operator with compact inverse A_ε^{-1}. Then, problem (1.2) can be reformulated as follows:

$$\begin{cases} A_\varepsilon^{-1}\,\partial_t u - \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u|_{t=0} = u_0. \end{cases} \tag{2.1}$$

For $0 \leqslant \varepsilon \leqslant 1$ and $u \in D(A_\varepsilon)$, we have the following:

$$\|\nabla u\| \leqslant \big\|A_\varepsilon^{1/2} u\big\| \leqslant \|u\|_{H^1(\Omega)} \quad \Longrightarrow \quad \big\|A_\varepsilon^{1/2} u\big\| \sim \|u\|_{H^1(\Omega)}, \tag{2.2}$$

$$\|\Delta u\| \leqslant \|A_\varepsilon u\| \leqslant c\|u\|_{H^2(\Omega)} \quad \Longrightarrow \quad \|A_\varepsilon u\| \sim \|u\|_{H^2(\Omega)}, \tag{2.3}$$

$$\big\|A_\varepsilon^{-1/2} u\big\| \leqslant c\big\|A_\varepsilon^{1/2} u\big\|, \tag{2.4}$$

$$\|u\|_{-1} \sim \big\|A_\varepsilon^{-1/2} u\big\|, \tag{2.5}$$

$$\|u\|_{-2} \sim \big\|A_\varepsilon^{-1} u\big\|, \tag{2.6}$$

$$\big(A_\varepsilon(-\Delta)^{-1} u, u\big) \sim \|u\|^2, \tag{2.7}$$

and

$$\big((-\Delta)A_\varepsilon^{-1} u, u\big) \sim \|u\|^2, \tag{2.8}$$

where c is independent of ε, $\|v\|_{-1}^2 = ((-\Delta)^{-1}v, v)$ and all the equivalences are independent of ε. Indeed, we have, if u is regular enough:

$$\|\nabla u\|^2 \leqslant \big\|A_\varepsilon^{1/2} u\big\|^2 = (A_\varepsilon u, u) = \|\nabla u\|^2 + \varepsilon\|u\|^2 \leqslant \|\nabla u\|^2 + \|u\|^2 = \|u\|_{H^1(\Omega)}^2,$$

$$\|\Delta u\|^2 \leqslant \|A_\varepsilon u\|^2 = (A_\varepsilon u, A_\varepsilon u) = \|\Delta u\|^2 + 2\varepsilon\|\nabla u\|^2 + \varepsilon^2\|u\|^2 \leqslant c\|u\|_{H^2(\Omega)}^2.$$

We then use Poincaré's inequality since $u = 0$ on $\partial\Omega$ and we deduce (2.2) and (2.3). Now, using the inclusions $V \subset H \subset V'$, the scalar product:

$$a(u, v) = (\nabla u, \nabla v) \quad \forall u, v \in V$$

defines a linear operator $A: D(A) = W \to H$. The operator A is the Laplace operator with Dirichlet boundary conditions and is a nonnegative self-adjoint operator; it has an orthonormal basis of eigenvectors $\{e_j\}_j$ associated to the eigenvalues $\{\lambda_j\}_j$, with

$$0 < \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_j \leqslant \cdots, \quad \lambda_j \to +\infty \text{ as } j \to +\infty.$$

The family $\{e_j\}_j$ may be assumed to be normalized in the norm of H, i.e.,

$$(e_i, e_j) = \delta_{ij},$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

We also note that the operator A_ε has the same orthonormal basis of eigenvectors $\{e_j\}_j$ associated to the eigenvalues $\{\lambda_{\varepsilon,j}\}_j$, with $\lambda_{\varepsilon,j} = \lambda_j + \varepsilon$. We have

$$\lambda_j \leqslant \lambda_{\varepsilon,j} \leqslant \lambda_j + 1 \quad \forall j \in \mathbb{N},$$

hence (2.4). We note that

$$A_\varepsilon(-\Delta)^{-1}u = (-\Delta + \varepsilon I)(-\Delta)^{-1}u = u + \varepsilon(-\Delta)^{-1}u,$$

hence

$$(-\Delta)^{-1}u = A_\varepsilon^{-1}u + \varepsilon A_\varepsilon^{-1}(-\Delta)^{-1}u.$$

Therefore,

$$\|u\|_{-1}^2 = \big((-\Delta)^{-1}u, u\big) = \big(A_\varepsilon^{-1}u + \varepsilon A_\varepsilon^{-1}(-\Delta)^{-1}u, u\big) = \big\|A_\varepsilon^{-1/2}u\big\|^2 + \varepsilon\big(A_\varepsilon^{-1}(-\Delta)^{-1}u, u\big).$$

Using the fact that A_ε^{-1} and $(-\Delta)^{-1}$ are nonnegative self-adjoint operators which commute, we obtain

$$\varepsilon\big(A_\varepsilon^{-1}(-\Delta)^{-1}u, u\big) \geqslant 0,$$

which yields

$$\big\|A_\varepsilon^{-1/2}u\big\|^2 \leqslant \|u\|_{-1}^2. \tag{2.9}$$

We also have $A_\varepsilon^{-1/2}(-\Delta)^{-1/2} = (-\Delta)^{-1/2}A_\varepsilon^{-1/2}$, hence

$$\begin{aligned} \|u\|_{-1}^2 &= \big((-\Delta)^{-1}u, u\big) = \big(A_\varepsilon^{-1}u + \varepsilon A_\varepsilon^{-1}(-\Delta)^{-1}u, u\big) = \big\|A_\varepsilon^{-1/2}u\big\|^2 + \varepsilon\big\|A_\varepsilon^{-1/2}(-\Delta)^{-1/2}u\big\|^2 \\ &\leqslant \big\|A_\varepsilon^{-1/2}u\big\|^2 + \big\|(-\Delta)^{-1/2}A_\varepsilon^{-1/2}u\big\|^2 \leqslant \big\|A_\varepsilon^{-1/2}u\big\|^2 + c'\big\|A_\varepsilon^{-1/2}u\big\|^2, \end{aligned}$$

which yields

$$\|u\|_{-1}^2 \leqslant c\big\|A_\varepsilon^{-1/2}u\big\|^2. \tag{2.10}$$

From estimates (2.9) and (2.10), we deduce (2.5).

We further have

$$\begin{aligned}
\|u\|_{-2}^2 &= \big((-\Delta)^{-1}u, (-\Delta)^{-1}u\big) = \big(A_\varepsilon^{-1}u + \varepsilon A_\varepsilon^{-1}(-\Delta)^{-1}u, A_\varepsilon^{-1}u + \varepsilon A_\varepsilon^{-1}(-\Delta)^{-1}u\big)\\
&= \big\|A_\varepsilon^{-1}u\big\|^2 + \varepsilon^2\big\|A_\varepsilon^{-1}(-\Delta)^{-1}u\big\|^2 + 2\varepsilon\big(A_\varepsilon^{-1}(-\Delta)^{-1}u, A_\varepsilon^{-1}u\big)\\
&= \big\|A_\varepsilon^{-1}u\big\|^2 + \varepsilon^2\big\|A_\varepsilon^{-1}(-\Delta)^{-1}u\big\|^2 + 2\varepsilon\big\|(-\Delta)^{-1/2}A_\varepsilon^{-1}u\big\|^2\\
&\leqslant \big\|A_\varepsilon^{-1}u\big\|^2 + \big\|(-\Delta)^{-1}A_\varepsilon^{-1}u\big\|^2 + 2\big\|(-\Delta)^{-1/2}A_\varepsilon^{-1}u\big\|^2 \leqslant c\big\|A_\varepsilon^{-1}u\big\|^2,
\end{aligned}$$

and, since $\varepsilon^2\|A_\varepsilon^{-1}(-\Delta)^{-1}u\|^2 + 2\varepsilon(A_\varepsilon^{-1}(-\Delta)^{-1}u, A_\varepsilon^{-1}u) \geqslant 0$, we find

$$\big\|A_\varepsilon^{-1}u\big\|^2 \leqslant \|u\|_{-2}^2.$$

Then, we deduce that

$$\big\|A_\varepsilon^{-1}u\big\|^2 \leqslant \|u\|_{-2}^2 \leqslant c\big\|A_\varepsilon^{-1}u\big\|^2,$$

where c is independent of ε.

Now,

$$\begin{aligned}
\big(A_\varepsilon(-\Delta)^{-1}u, u\big) &= \big((-\Delta + \varepsilon I)(-\Delta)^{-1}u, u\big) = \big(u + \varepsilon(-\Delta)^{-1}u, u\big) = \|u\|^2 + \varepsilon\|u\|_{-1}^2\\
&\leqslant \|u\|^2 + \|u\|_{-1}^2 \leqslant c\|u\|^2
\end{aligned}$$

and

$$\|u\|^2 \leqslant \big(A_\varepsilon(-\Delta)^{-1}u, u\big),$$

so that

$$\big(A_\varepsilon(-\Delta)^{-1}u, u\big) \sim \|u\|^2.$$

We also have

$$\begin{aligned}
\big((-\Delta)A_\varepsilon^{-1}u, u\big) &= \big((-\Delta)A_\varepsilon^{-1}u, A_\varepsilon(-\Delta)^{-1}(-\Delta)A_\varepsilon^{-1}u\big)\\
&\leqslant c\big\|(-\Delta)A_\varepsilon^{-1}u\big\|^2 \quad \text{(using (2.7))}\\
&\leqslant c\big\|A_\varepsilon A_\varepsilon^{-1}u\big\|^2 \quad \text{(using (2.3))}\\
&\leqslant c\|u\|^2
\end{aligned}$$

and

$$\|u\|^2 = \left\|A_\varepsilon^{1/2}(-\Delta)^{-1/2}(-\Delta)^{1/2}A_\varepsilon^{-1/2}u\right\|^2 \leqslant c\left\|(-\Delta)^{1/2}A_\varepsilon^{-1/2}u\right\|^2 \leqslant c\bigl((-\Delta)A_\varepsilon^{-1}u, u\bigr),$$

which yields

$$\bigl((-\Delta)A_\varepsilon^{-1}u, u\bigr) \sim \|u\|^2.$$

The variational formulation of problem (2.1) reads

Find $u : [0,T] \to V$ such that

$$\begin{cases} \bigl(A_\varepsilon^{-1}\,\partial_t u, q\bigr) + (\nabla u, \nabla q) + \bigl(f(u), q\bigr) = 0 \quad \forall q \in V, \\ u(0) = u_0, \end{cases} \tag{2.11}$$

$\forall T > 0$.

In what follows, unless mentioned explicitly, the same letter Q denotes monotone increasing functions and the same letter c denotes positive constants independent of ε, possibly changing at different occurrences.

3. Uniform a priori estimates. Existence and uniqueness of solutions

In order to prove the existence of solutions for problem (2.1), we use a Galerkin scheme. We first define an m-dimensional system which approximates the initial problem. The resolution of the system proves the existence of an approximated solution. We then obtain a priori estimates which allow us to justify the passage to the limit in the approximated problem and obtain a solution to the initial problem. We have the following theorem.

Theorem 3.1. *Let us take $u_0 \in H$. Then, there exists a unique solution u of problem* (2.1) *with initial datum u_0 such that*

$$u \in L^\infty\bigl([0,T]; H\bigr) \cap L^2\bigl([0,T]; W\bigr) \cap L^{2p}\bigl(0,T; L^{2p}(\Omega)\bigr).$$

Furthermore, if $u_0 \in V \cap L^{2p}(\Omega)$, then

$$u \in L^\infty\bigl([0,T]; V \cap L^{2p}(\Omega)\bigr) \cap L^2\bigl([0,T]; W\bigr) \quad \textit{and} \quad \partial_t u \in L^2\bigl([0,T]; H^{-1}(\Omega)\bigr).$$

Uniqueness of the solution. Let u and v be two solutions of (2.1) on the time interval $[0,T]$. We set $w = u - v$, then w verifies the following equation:

$$\begin{cases} A_\varepsilon^{-1}\,\partial_t w + \nabla w + l(t)w = 0, \\ w(0) = u(0) - v(0), \end{cases} \tag{3.1}$$

where $l(t) = \int_0^1 f'(su(t) + (1-s)v(t))\,\mathrm{d}s$. Multiplying (3.1) by w and integrating over Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|A_\varepsilon^{-1/2}w\right\|^2 + \|\nabla w\|^2 + \bigl(l(t)w, w\bigr) = 0.$$

Using the fact that $f'(s) \geqslant -\kappa$, we have

$$\big(l(t)w, w\big) \geqslant -\kappa\|w\|^2.$$

Using the following interpolation inequality:

$$\|w\|^2_{L^2(\Omega)} \leqslant c\|w\|_{-1}\|\nabla w\| \leqslant c\big\|A_\varepsilon^{-1/2}w\big\|\|\nabla w\|,$$

we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|A_\varepsilon^{-1/2}w\big\|^2 + \frac{1}{2}\|\nabla w\|^2 \leqslant \frac{c^2\kappa^2}{2}\big\|A_\varepsilon^{-1/2}w\big\|^2. \tag{3.2}$$

Applying Gronwall's lemma to (3.2), we find

$$\big\|A_\varepsilon^{-1/2}w(t)\big\|^2 + \int_0^t \big\|\nabla w(s)\big\|^2\,\mathrm{d}s \leqslant \big\|A_\varepsilon^{-1/2}\big(u(0) - v(0)\big)\big\|^2 \mathrm{e}^{c^2\kappa^2 t}. \tag{3.3}$$

Relation (3.3) shows the continuous dependence of the solution on the initial data in the H^{-1}-norm and, in particular, when $u(0) = v(0)$, it implies the uniqueness of the solution.

Existence of a local solution. We denote by E_m the space

$$E_m = \operatorname{span}\{e_1, e_2, \ldots, e_m\}$$

and by P_m the orthogonal projection from V onto E_m:

$$P_m h = \sum_{j=1}^m (h, e_j)e_j,$$

where $\{e_j\}_j$ are the eigenfunctions of the operator A_ε.

For any $m \in \mathbb{N}$, we look for functions of the form

$$u_m(t) = \sum_{j=1}^m u_{m_j}(t)e_j,$$

solving the approximated problem

$$\begin{cases} A_\varepsilon^{-1}\,\partial_t u_m - \Delta u_m + f(u_m) = 0, \\ u_m(0) = P_m u_0. \end{cases} \tag{3.4}$$

The variational formulation of problem (3.4) reads

Find $u_m : [0, T] \to E_m$ *such that*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(A_\varepsilon^{-1}\displaystyle\sum_{j=1}^m u_{m_j}(t)e_j, e_i\right) + \left(\displaystyle\sum_{j=1}^m u_{m_j}(t)\nabla e_j, \nabla e_i\right) \\ \qquad + \big(f(u_m), e_i\big) = 0, \quad i = 1, \ldots, m, \\ u_m(0) = P_m u_0. \end{cases} \tag{3.5}$$

We rewrite problem (3.5) as follows:

$$M_\varepsilon \frac{\mathrm{d}Y}{\mathrm{d}t} + MY + N(Y) = 0,$$

where $M_\varepsilon = ((A_\varepsilon^{-1}(e_i), e_j))_{i,j=1,\dots,m}$, $M = ((\nabla e_i, \nabla e_j))_{i,j=1,\dots,m}$, $Y = \begin{pmatrix} u_{m_1} \\ \vdots \\ u_{m_m} \end{pmatrix}$ and $N(Y) = \begin{pmatrix} (f(u_m), e_1) \\ \vdots \\ (f(u_m), e_m) \end{pmatrix}$.

We can easily check that matrix M_ε is invertible. Indeed, setting $X = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \neq 0$ and $X' = \sum_{i=1}^m x_i e_i$, we have

$$(M_\varepsilon X, X) = \sum_{j=1}^m \sum_{i=1}^m \left(A_\varepsilon^{-1} e_j, e_i\right) x_i x_j = \left(A_\varepsilon^{-1} X', X'\right) > 0,$$

due to the fact that A_ε^{-1} is a positive definite operator. Moreover, the matrix M is positive definite and $N(Y)$ depends continuously on Y. Applying Cauchy's theorem for a system of ordinary differential equations, it follows that there exists a time $t_m \in (0, T)$ and a unique solution Y for the equation $\frac{\mathrm{d}Y}{\mathrm{d}t} + M_\varepsilon^{-1} MY + M_\varepsilon^{-1} N(Y) = 0$ on the time interval $t \in [0, t_m[$. Based on the a priori estimates with respect to t that will be derived below for the solution $u_m(t)$, we deduce that any local solution of (2.11) is actually a global solution defined on the whole interval $[0, T]$.

Now, we give the a priori estimates for the solution $u_m(t)$.

A priori estimates. Multiplying (3.4) by u_m, we find

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left\| A_\varepsilon^{-1/2} u_m \right\|^2 + \|\nabla u_m\|^2 + \left(f(u_m), u_m\right) = 0. \tag{3.6}$$

By (1.3), we have

$$\left(f(u_m), u_m\right) \geqslant c\left(\|u_m\|^2 + \|u_m\|_{L^{2p}(\Omega)}^{2p}\right) - c|\Omega|$$

and estimate (3.6) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\| A_\varepsilon^{-1/2} u_m \right\|^2 + c\left(\|u_m\|_{H^1(\Omega)}^2 + \|u_m\|_{L^{2p}(\Omega)}^{2p}\right) \leqslant c. \tag{3.7}$$

Multiplying (3.4) by $A_\varepsilon u_m$ and using (1.3), we get

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u_m\|^2 + \|\Delta u_m\|^2 + \varepsilon \|\nabla u_m\|^2 &\leqslant \kappa \|\nabla u_m\|^2 + \kappa\varepsilon \|u_m\|^2 \\ &\leqslant \kappa\left((-\Delta u_m, u_m) + \|u_m\|^2\right) \\ &\leqslant \kappa \|u_m\|^2 + 1/2 \|\Delta u_m\|^2 + \kappa^2/2 \|u_m\|^2, \end{aligned}$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|^2 + \|\Delta u_m\|^2 + 2\varepsilon\|\nabla u_m\|^2 \leqslant c\|u_m\|^2. \tag{3.8}$$

Using the fact that $\|A_\varepsilon^{-1/2}u_m\| \leqslant c\|u_m\|_{H^1(\Omega)}$ and $\|u_m\| \leqslant c\|\Delta u_m\|$, estimates (3.7) and (3.8) give, for a positive constant α independent of ε,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|A_\varepsilon^{-1/2}u_m\big\|^2 + \alpha\big\|A_\varepsilon^{-1/2}u_m\big\|^2 + c_\alpha\|u_m\|^2_{H^1(\Omega)} + c\|u_m\|^{2p}_{L^{2p}(\Omega)} \leqslant c, \tag{3.9}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_m\|^2 + \alpha\|u_m\|^2 + c_\alpha\|\Delta u_m\|^2 + 2\varepsilon\|\nabla u_m\|^2 \leqslant c\|u_m\|^2, \tag{3.10}$$

where the constant $c_\alpha > 0$ depends on α and is independent of ε.

Applying Gronwall's lemma, estimate (3.9) leads to

$$\begin{aligned}&\big\|A_\varepsilon^{-1/2}u_m(t)\big\|^2 + \int_0^t \big(\big\|u_m(s)\big\|^2_{H^1(\Omega)} + \big\|u_m(s)\big\|^{2p}_{L^{2p}(\Omega)}\big)\mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}s\\ &\quad\leqslant c\big\|A_\varepsilon^{-1/2}u_m(0)\big\|^2\mathrm{e}^{-\alpha t} + c.\end{aligned} \tag{3.11}$$

Integrating (3.7) with respect to $s \in [t, t+1]$ and using estimate (3.11), we obtain

$$\begin{aligned}\int_t^{t+1} \big(\big\|u_m(s)\big\|^2_{H^1(\Omega)} + \big\|u_m(s)\big\|^{2p}_{L^{2p}(\Omega)}\big)\,\mathrm{d}s &\leqslant \big\|A_\varepsilon^{-1/2}u_m(t)\big\|^2 + c\\ &\leqslant c\big\|A_\varepsilon^{-1/2}u_m(0)\big\|^2\mathrm{e}^{-\alpha t} + c.\end{aligned} \tag{3.12}$$

Estimates (3.11) and (3.12) give

$$\big\|A_\varepsilon^{-1/2}u_m(t)\big\|^2 + \int_t^{t+1} \big(\big\|u_m(s)\big\|^2_{H^1(\Omega)} + \big\|u_m(s)\big\|^{2p}_{L^{2p}(\Omega)}\big)\,\mathrm{d}s \leqslant c\big\|A_\varepsilon^{-1/2}u_m(0)\big\|^2\mathrm{e}^{-\alpha t} + c,$$

so that

$$u_m \in L^2(0,T;V) \cap L^{2p}\big(0,T;L^{2p}(\Omega)\big) \cap L^\infty\big(0,T;H^{-1}(\Omega)\big).$$

Applying Gronwall's Lemma and using estimate (3.11), relation (3.8) leads to

$$\big\|u_m(t)\big\|^2 \leqslant \big\|u_m(0)\big\|^2\mathrm{e}^{-\alpha t} + c\int_0^t \big\|u_m(s)\big\|^2\mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}s \leqslant c\big\|u_m(0)\big\|^2\mathrm{e}^{-\alpha t} + c, \tag{3.13}$$

which implies that $u_m \in L^\infty(0,T;H)$. Now, integrating estimate (3.8) with respect to $s \in [t,t+1]$ and using (3.13), we obtain

$$\|u_m(t)\|^2 + \int_t^{t+1} \left(\|\Delta u_m(s)\|^2 + 2\varepsilon\|\nabla u_m(s)\|^2\right) \mathrm{d}s \leqslant c\|u_m(0)\|^2 \mathrm{e}^{-\alpha t} + c, \tag{3.14}$$

where the positive constant c is independent of m. Integrating (3.8) with respect to $t \in [0,T]$ and using (3.13), we find

$$\int_0^T \|\Delta u_m(t)\|^2 \,\mathrm{d}t \leqslant \|u_m(0)\|^2 + \int_0^T \|u_m(t)\|^2 \,\mathrm{d}t \leqslant \|u_m(0)\|^2 + c\|u_m(0)\|^2 + cT$$

and we deduce that $u_m \in L^2(0,T;W)$.

We introduce the energy functional $J : V \to \mathbb{R}$ defined by

$$J(v) = \frac{1}{2}\|\nabla v\|^2 + \int_\Omega F(v)\,\mathrm{d}x.$$

Multiplying (3.4) by $\partial_t u_m$, we obtain

$$\|A_\varepsilon^{-1/2}\,\partial_t u_m\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\nabla u_m\|^2 + \int_\Omega F(u_m)\,\mathrm{d}x\right) = 0, \tag{3.15}$$

which shows that the energy J decays along the trajectories. Integrating Eq. (3.15) with respect to t, we get

$$\begin{aligned} J\big(u_m(t)\big) + \int_0^t \|A_\varepsilon^{-1/2}\,\partial_t u_m(s)\|^2 \,\mathrm{d}s &\leqslant \frac{1}{2}\|\nabla u_m(0)\|^2 + \int_\Omega F\big(u_m(0)\big) \\ &\leqslant \|\nabla u_0\|^2 + \frac{3}{2}b_{2p}\int_\Omega u_m^{2p}(0)\,\mathrm{d}x + c \\ &\leqslant \|\nabla u_0\|^2 + c\|u_0\|^{2p}_{L^{2p}(\Omega)} + c. \end{aligned}$$

Thus, if $u_0 \in V \cap L^{2p}(\Omega)$, we have a solution u_m which verifies

$$u_m \in L^\infty\big(0,T;V \cap L^{2p}(\Omega)\big) \quad \text{and} \quad \partial_t u_m \in L^2\big(0,T;H^{-1}(\Omega)\big) \quad \forall T > 0.$$

Passage to the limit. We now pass to the limit as $m \to \infty$ and study the convergence of the sequence $(u_m)_m$. Let q satisfy $\frac{1}{2p} + \frac{1}{q} = 1$, which implies $q < 2$. According to the a priori estimates derived in the previous section, $\|u_m\|_{L^2(0,T;W)}$ is uniformly bounded and, consequently, $(u_m)_m$ is bounded in $L^q(0,T;L^q(\Omega))$. Moreover, thanks to (1.3) and the uniform boundedness of $(u_m)_m$ in $L^{2p}(0,T;L^{2p}(\Omega))$, $\|f(u_m)\|_{L^q(0,T;L^q(\Omega))}$ is uniformly bounded, which implies that $(A_\varepsilon^{-1}\,\partial_t u_m)_m$ is bounded in $L^q(0,T;L^q(\Omega))$. It follows that

$$\|\partial_t u_m\|_{L^q(0,T;W^{-2,q}(\Omega))} \leqslant c. \tag{3.16}$$

Starting from this point, all convergence relations will be intended to hold up to the extraction of suitable subsequences, generally not relabeled. Thus, we observe that weak and weak star compactness results applied to the sequence $(u_m)_m$ entail that there exists a function u such that the following properties hold:

$$u_m \rightharpoonup u \quad \text{weakly in } L^q(0,T;W), \tag{3.17}$$

$$\partial_t u_m \rightharpoonup \partial_t u \quad \text{weakly in } L^q\big(0,T;W^{-2,q}(\Omega)\big), \tag{3.18}$$

as $m \to \infty$. It follows from (3.17), (3.18) and the Aubin–Lions compactness theorem that

$$u_m \to u \quad \text{strongly in } L^q\big(0,T;L^q(\Omega)\big). \tag{3.19}$$

Consequently $u_m(t,x) \to u(t,x)$ a.e. $(t,x) \in [0,T] \times \Omega$.

Moreover, we have

$$\left.\begin{array}{l} u_m(t,x) \to u(t,x) \text{ a.e.} \\ f \text{ is a continuous function} \end{array}\right\} \quad \Longrightarrow \quad f\big(u_m(t,x)\big) \to f\big(u(t,x)\big) \text{ a.e.}$$

$$\left.\begin{array}{l} f\big(u_m(t,x)\big) \to f\big(u(t,x)\big) \text{ a.e.} \\ \big\|f(u_m)\big\|_{L^q(\Omega_T)} \leqslant \text{ constant} \end{array}\right\} \quad \Longrightarrow \quad f(u_m) \rightharpoonup f(u) \text{ weakly in } L^q(\Omega_T),$$

where we have used the weak dominated convergence theorem. Finally, we deduce that $A_\varepsilon^{-1}\,\partial_t u_m \rightharpoonup A_\varepsilon^{-1}\,\partial_t u$ weakly in $L^q(\Omega_T)$. Thus, passing to the limit in (3.4), we obtain

$$A_\varepsilon^{-1}\,\partial_t u - \Delta u + f(u) = 0 \quad \text{in } L^q(\Omega_T). \tag{3.20}$$

To prove that $u(0) = u_0$, we consider a test function $\psi \in C^1([0,T];L^{2p}(\Omega))$ such that $\psi(T) = 0$. Multiplying (2.1) by ψ and integrating over $\Omega \times [0,T]$, we obtain

$$\int_0^T \big\langle A_\varepsilon^{-1}\,\partial_t u, \psi\big\rangle\,\mathrm{d}t - \int_0^T \langle \Delta u, \psi\rangle\,\mathrm{d}t + \int_0^T \big\langle f(u), \psi\big\rangle\,\mathrm{d}t = 0. \tag{3.21}$$

Integrating by parts in (3.21), we have

$$-\int_0^T \big\langle A_\varepsilon^{-1}u, \partial_t\psi\big\rangle\,\mathrm{d}t - \big\langle A_\varepsilon^{-1}u(0), \psi(0)\big\rangle - \int_0^T \langle \Delta u, \psi\rangle\,\mathrm{d}t + \int_0^T \big\langle f(u), \psi\big\rangle\,\mathrm{d}t = 0. \tag{3.22}$$

Multiplying (3.4) by ψ and integrating over $\Omega \times [0,T]$, we obtain

$$\begin{aligned} &-\int_0^T \big\langle A_\varepsilon^{-1}u_m, \partial_t\psi\big\rangle\,\mathrm{d}t - \big\langle A_\varepsilon^{-1}P_m u(0), \psi(0)\big\rangle - \int_0^T \langle \Delta u_m, \psi\rangle\,\mathrm{d}t \\ &\quad + \int_0^T \big\langle f(u_m), \psi\big\rangle\,\mathrm{d}t = 0. \end{aligned} \tag{3.23}$$

Having $\psi \in L^{2p}(0,T;L^{2p}(\Omega))$ and $\frac{\partial\psi}{\partial t} \in L^2(0,T;H)$ we deduce that

$$-\int_0^T \langle \Delta u_m, \psi\rangle \,\mathrm{d}t \to -\int_0^T \langle \Delta u, \psi\rangle \,\mathrm{d}t$$

and

$$\int_0^T \langle A_\varepsilon^{-1} u_m, \partial_t\psi\rangle \,\mathrm{d}t \to \int_0^T \langle A_\varepsilon^{-1} u, \partial_t\psi\rangle \,\mathrm{d}t.$$

Then, passing to the limit in (3.23) as $m \to \infty$, we obtain

$$-\int_0^T \langle A_\varepsilon^{-1} u, \partial_t\psi\rangle \,\mathrm{d}t - \langle A_\varepsilon^{-1} u_0, \psi(0)\rangle - \int_0^T \langle \Delta u, \psi\rangle \,\mathrm{d}t + \int_0^T \langle f(u), \psi\rangle \,\mathrm{d}t = 0. \tag{3.24}$$

We deduce from (3.22) and (3.24) that $A_\varepsilon^{-1}u(0) = A_\varepsilon^{-1}u_0$, which implies $u(0) = u_0$.

4. Additional regularity

In this section, we will derive some additional regularity for the solution $u(t)$. We have the following theorem.

Theorem 4.1. *Let us take $u_0 \in W$. Then, there exists a unique solution u of problem* (2.1) *with initial datum u_0 such that*

$$u \in L^\infty([0,T];H^2(\Omega)) \cap L^2([0,T];H^4(\Omega)). \tag{4.1}$$

Furthermore, we have

$$\|u\|_{H^2(\Omega)} \leqslant Q(\|u_0\|_{H^2(\Omega)})\mathrm{e}^{-\alpha t} + c, \tag{4.2}$$

where the constants $c, \alpha > 0$ and the monotonic function Q are independent of ε.

The proof is based on a priori estimates that we formally derive below, a rigorous justification being based on the classical Galerkin method.

Lemma 4.2. *Let $u(t)$ be a solution of* (2.1) *with $u_0 \in W$. Then, there exists a time $T_0 = T_0(\|u_0\|_{H^2(\Omega)})$, $0 < T_0 < 1/2$, and a monotonic function Q such that*

$$\|\Delta u(t)\| \leqslant Q(\|u_0\|_{H^2(\Omega)}), \quad t \leqslant T_0(\|u_0\|_{H^2(\Omega)}). \tag{4.3}$$

Proof. Multiplying Eq. (2.1) by $\Delta^2 A_\varepsilon u(t)$, we obtain the following inequality:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + \|\Delta^2 u\|^2 + \varepsilon\|\nabla\Delta u\|^2 &= -\big(A_\varepsilon f(u), \Delta^2 u\big)\\ &\leqslant \frac{1}{2}\|A_\varepsilon f(u)\|^2 + \frac{1}{2}\|\Delta^2 u\|^2\\ &\leqslant \frac{1}{2}c\|f(u)\|^2_{H^2(\Omega)} + \frac{1}{2}\|\Delta^2 u\|^2,\end{aligned}\tag{4.4}$$

where c is independent of ε. Taking into account that $f \in C^2(\Omega)$ and that $H^2(\Omega) \subset C(\overline{\Omega})$, we can prove that there exists a monotonic function Q (depending on f) such that

$$\|f(u)\|^2_{H^2(\Omega)} \leqslant Q\big(\|u\|^2_{H^2(\Omega)}\big) \leqslant Q\big(\|\Delta u\|^2\big).\tag{4.5}$$

Indeed, we have for $n \leqslant 3$ (where n is the space dimension)

$$\begin{aligned}&\|f'(u)\|^2_{L^\infty(\Omega)} \leqslant Q_{f'}\big(\|u\|^2_{L^\infty(\Omega)}\big) \leqslant Q_{f'}\big(\|u\|^2_{H^2(\Omega)}\big), \quad \text{since } f' \in C(\overline{\Omega}),\\ &\|f''(u)\|^2_{L^\infty(\Omega)} \leqslant Q_{f''}\big(\|u\|^2_{L^\infty(\Omega)}\big) \leqslant Q_{f''}\big(\|u\|^2_{H^2(\Omega)}\big), \quad \text{since } f'' \in C(\overline{\Omega})\end{aligned}$$

and

$$\begin{aligned}\|f(u)\|^2_{H^2(\Omega)} &\leqslant c\|\Delta f(u)\|^2\\ &\leqslant c\big(\|f'(u)\Delta u\|^2 + \|f''(u)\nabla u\cdot\nabla u\|^2\big)\\ &\leqslant c\big(\|f'(u)\|^2_{L^\infty(\Omega)}\|\Delta u\|^2 + \|f''(u)\|^2_{L^\infty(\Omega)}\|\nabla u\|^4_{L^4(\Omega)}\big)\\ &\leqslant c\big(\|f'(u)\|^2_{L^\infty(\Omega)}\|\Delta u\|^2 + \|f''(u)\|^2_{L^\infty(\Omega)}\|\Delta u\|^4\big)\\ &\leqslant Q\big(\|u\|^2_{H^2(\Omega)}\big),\end{aligned}\tag{4.6}$$

for some monotonic increasing functions $Q_{f'}$ and $Q_{f''}$.

Thus, replacing inequality (4.5) in (4.4), the function $y(t) := \|\Delta u(t)\|^2$ satisfies the inequality

$$y'(t) \leqslant Q\big(y(t)\big).$$

Let $z(t)$ be a solution of the following equation:

$$z'(t) = Q\big(z(t)\big), \qquad z(0) = y(0) = \|\Delta u(0)\|^2.$$

Due to the comparison principle, there exists a time $T_0(\|u_0\|_{H^2(\Omega)}) \in (0, 1/2)$ such that we have

$$y(t) \leqslant z(t) \quad \forall t \leqslant T_0\big(\|u_0\|_{H^2(\Omega)}\big).\tag{4.7}$$

Then, the lemma is an immediate consequence of (4.7). $\square$

Lemma 4.3. *Let the above assumptions hold and let T_0 be the same as in Lemma* 4.2. *Then*, *the following estimate holds*:

$$t\big\|A_\varepsilon^{-1/2}\,\partial_t u(t)\big\|^2 \leqslant Q\big(\|u_0\|_{H^2(\Omega)}\big), \quad t \in (0, T_0], \tag{4.8}$$

for some monotonic function Q.

Proof. Multiplying (2.1) by $\partial_t u(t)$ and integrating over Ω, we obtain

$$\big\|A_\varepsilon^{-1/2}\,\partial_t u\big\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 \leqslant \big|\big(f(u), \partial_t u\big)\big| \leqslant c\big\|f(u)\big\|^2_{H^1(\Omega)} + 1/2\big\|A_\varepsilon^{-1/2}\,\partial_t u\big\|^2. \tag{4.9}$$

Integrating (4.9) over $[0, T_0]$ and taking into account (4.3) and the fact that $H^2(\Omega) \subset C(\overline{\Omega})$, we have

$$\begin{aligned}\int_0^{T_0}\big\|A_\varepsilon^{-1/2}\,\partial_t u(t)\big\|^2\,\mathrm{d}t &\leqslant c\int_0^{T_0}\big\|f\big(u(t)\big)\big\|^2_{H^1(\Omega)}\,\mathrm{d}t + \big\|\nabla u(0)\big\|^2 \\ &\leqslant Q\big(\|u_0\|_{H^2(\Omega)}\big) + \big\|\nabla u(0)\big\|^2 \\ &\leqslant Q\big(\|u_0\|_{H^2(\Omega)}\big),\end{aligned} \tag{4.10}$$

where we have used the fact that

$$\big\|f\big(u(t)\big)\big\|^2_{H^1(\Omega)} \leqslant Q\big(\big\|u(t)\big\|_{H^2(\Omega)}\big) \leqslant Q\big(\|u_0\|_{H^2(\Omega)}\big) \quad \forall t \in [0, T_0]$$

for some monotonic function Q. Differentiating (2.1) with respect to t and setting $\theta(t) = \partial_t u(t)$, we find

$$\begin{cases} A_\varepsilon^{-1}\,\partial_t\theta - \Delta\theta + f'\big(u(t)\big)\theta = 0, \\ \theta|_{\partial\Omega} = 0. \end{cases} \tag{4.11}$$

Multiplying (4.11) by $t\theta(t)$, integrating over Ω and using the fact that $f'(u) \geqslant -\kappa$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(t\big\|A_\varepsilon^{-1/2}\theta\big\|^2\big) + 2t\|\nabla\theta\|^2 &\leqslant 2\kappa t\|\theta\|^2 + \big\|A_\varepsilon^{-1/2}\theta\big\|^2 \\ &\leqslant ct\big\|A_\varepsilon^{-1/2}\theta\big\|\|\nabla\theta\| + \big\|A_\varepsilon^{-1/2}\theta\big\|^2 \\ &\leqslant t\|\nabla\theta\|^2 + c(t+1)\big\|A_\varepsilon^{-1/2}\theta\big\|^2.\end{aligned}$$

Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(t\big\|A_\varepsilon^{-1/2}\theta\big\|^2\big) \leqslant c(t+1)\big\|A_\varepsilon^{-1/2}\theta\big\|^2. \tag{4.12}$$

Applying Gronwall's lemma to estimate (4.12) over $(0, t)$ for $t \leqslant T_0 \leqslant 1$ and using (4.10), we deduce that

$$t\big\|A_\varepsilon^{-1/2}\theta(t)\big\|^2 \leqslant c\int_0^t\big\|A_\varepsilon^{-1/2}\theta(s)\big\|^2\,\mathrm{d}s \leqslant Q\big(\|u_0\|_{H^2(\Omega)}\big) \tag{4.13}$$

and Lemma 4.3 is proven. □

Lemma 4.4. *Let $u(t)$ be a solution of Eq.* (2.1) *and let $t \geqslant T_0$, where T_0 is the same as in Lemma* 4.2. *Then, the following estimate holds uniformly with respect to ε:*

$$\big\|A_\varepsilon^{-1/2}\,\partial_t u(t)\big\|^2 + \big\|u(t)\big\|^2_{H^2(\Omega)} + \int_t^{t+1} \big\|\partial_t u(s)\big\|^2_{H^1(\Omega)}\,\mathrm{d}s \leqslant \mathrm{e}^{K_1 t} Q\big(\|u_0\|_{H^2(\Omega)}\big) \quad \forall t \geqslant T_0,$$

where K_1 is a positive constant and Q is some monotonic function.

Proof. Multiplying (4.11) by $\theta(t)$, integrating over Ω and using the fact that $f'(u) \geqslant -\kappa$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|A_\varepsilon^{-1/2}\theta\big\|^2 + 2\|\nabla\theta\|^2 \leqslant 2\kappa\|\theta\|^2 \leqslant 2c\kappa\big\|A_\varepsilon^{-1/2}\theta\big\|\|\nabla\theta\| \leqslant \|\nabla\theta\|^2 + c^2\kappa^2\big\|A_\varepsilon^{-1/2}\theta\big\|^2. \tag{4.14}$$

Applying Gronwall's lemma to estimate (4.14) over (T_0, t), we obtain

$$\big\|A_\varepsilon^{-1/2}\theta(t)\big\|^2 + \int_t^{t+1}\big\|\nabla\theta(s)\big\|^2\,\mathrm{d}s \leqslant \mathrm{e}^{K_1 t}\big\|A_\varepsilon^{-1/2}\theta(T_0)\big\|^2, \quad t \geqslant T_0 \tag{4.15}$$

for some positive constant K_1. Using Lemma 4.3, estimate (4.15) gives

$$\big\|A_\varepsilon^{-1/2}\,\partial_t u(t)\big\|^2 + \int_t^{t+1}\big\|\partial_t u(s)\big\|^2_{H^1(\Omega)}\,\mathrm{d}s \leqslant \mathrm{e}^{K_1 t} Q\big(\|u_0\|_{H^2(\Omega)}\big), \quad t \geqslant T_0. \tag{4.16}$$

Interpreting the parabolic Eq. (2.1) as an elliptic boundary value problem,

$$\Delta u(t) - f\big(u(t)\big) = h(t) := A_\varepsilon^{-1}\,\partial_t u(t), \quad u(t)|_{\partial\Omega} = 0 \tag{4.17}$$

for every fixed $t \geqslant T_0$, estimate (4.16) implies that

$$\big\|h(t)\big\|^2 = \big\|A_\varepsilon^{-1/2}\,\partial_t u(t)\big\|^2 \leqslant \mathrm{e}^{K_1 t} Q\big(\|u_0\|_{H^2(\Omega)}\big). \tag{4.18}$$

Multiplying (4.17) by $u(t)$, using the fact that $f(u)u \geqslant -c$ and integrating over Ω, we obtain

$$\big\|\nabla u(t)\big\|^2 \leqslant c\big(1 + \big\|h(t)\big\|^2\big). \tag{4.19}$$

Multiplying (4.17) by $\Delta u(t)$, using the fact that $f'(u) \geqslant -\kappa$ and integrating over Ω, we find

$$\big\|\Delta u(t)\big\|^2 \leqslant c\big(\big\|\nabla u(t)\big\|^2 + \big\|h(t)\big\|^2\big). \tag{4.20}$$

We deduce from estimates (4.18), (4.19) and (4.20) that

$$\big\|u(t)\big\|^2_{H^2(\Omega)} \leqslant c\big\|\Delta u(t)\big\|^2 \leqslant c\mathrm{e}^{K_1 t} Q\big(\|u_0\|_{H^2(\Omega)}\big) \tag{4.21}$$

and the lemma is proven. □

Corollary 1. *Let the above assumptions hold and let $u(t)$ be a solution of problem* (2.1). *Then*, *the following estimate holds*:

$$\left\|u(t)\right\|_{H^2(\Omega)} \leqslant Q\left(\left\|u(0)\right\|_{H^2(\Omega)}\right)\mathrm{e}^{Kt} \tag{4.22}$$

for all $t \geqslant 0$ and where the positive constant K and the monotonic function Q are independent of ε.

Proof. We have proved that estimate (4.22) is true for $t \leqslant T_0(\|u_0\|_{H^2(\Omega)})$ in Lemma 4.2 and for $t \geqslant T_0(\|u_0\|_{H^2(\Omega)})$ in Lemma 4.4, which completes the proof. $\square$

Lemma 4.5. *Let $u(t)$ be a solution of problem* (2.1). *Then*, *the following estimate is valid*, *uniformly with respect to ε*:

$$\left\|u(1)\right\|_{H^2(\Omega)} \leqslant Q\left(\|u_0\|^2\right) \tag{4.23}$$

for some monotonic function Q that is independent of ε.

Proof. Estimate (3.14) yields

$$\int_0^1 \left\|u(s)\right\|_{H^2(\Omega)} \mathrm{d}s \leqslant c\left\|u(0)\right\|^2 + c.$$

It follows that there exists a time $T \in [0, 1]$ such that

$$\left\|u(T)\right\|_{H^2(\Omega)} \leqslant c\left\|u(0)\right\|^2 + c. \tag{4.24}$$

Applying estimate (4.22) starting with the time $t = T$ instead of $t = 0$ and using (4.24), we obtain the desired inequality. $\square$

To complete the proof of Theorem 4.1, we proceed as follows. We have from estimate (3.14) that for every $t \geqslant 1$, there exists a time $t_* \in [t-1, t]$ such that:

$$\left\|\Delta u(t_*)\right\|^2 \leqslant c\left\|u(0)\right\|^2\mathrm{e}^{-\alpha t} + c.$$

Consequently, for $t' \in [0, 1]$ such that $t = t_* + t'$, estimate (4.22) yields:

$$\begin{aligned}\left\|\Delta u(t)\right\|^2 &= \left\|\Delta u\left(t_* + t'\right)\right\|^2 \leqslant Q\left(\left\|\Delta u(t_*)\right\|^2\right)\mathrm{e}^{Kt'} \leqslant cQ\left(\left\|\Delta u(t_*)\right\|^2\right)\mathrm{e}^{-\alpha t'} \\ &\leqslant cQ\left(c\left\|u(0)\right\|^2\mathrm{e}^{-\alpha t} + c\right) \leqslant Q\left(\left\|u(0)\right\|^2_{H^2(\Omega)}\right)\mathrm{e}^{-\alpha t} + c\end{aligned}$$

for some monotonic function Q. Theorem 4.1 is thus proven.

Remark 2. Theorem 4.1 is also true for every function f such that $f \in C^2(\mathbb{R})$, $f(0) = 0$, $f'(s) \geqslant -\kappa$ and $f(s)s \geqslant -c$, $s \in \mathbb{R}$ and $\kappa, c \geqslant 0$.

The next two lemmata give additional regularity for the time derivative of u.

Lemma 4.6. *Let the assumptions of Theorem* 4.1 *hold. Then, the solution* $u(t)$ *of problem* (2.1) *satisfies*

$$\|\partial_t u(t)\|^2 + \int_t^{t+1} \|\partial_t u(s)\|^2_{H^2(\Omega)} \, \mathrm{d}s \leqslant Q(\|u_0\|_{H^2(\Omega)}) \mathrm{e}^{-\alpha t} + c, \quad t \geqslant 1, \tag{4.25}$$

where the positive constants c, α *and the function* Q *are independent of* ε.

Proof. Multiplying (4.11) by $(t - T_0)A_\varepsilon \theta(t)$, where T_0 is the same as in Lemma 4.3 and $\theta = \partial_t u$, and integrating over Ω, we find

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left((t-T_0)\frac{\|\theta(t)\|^2}{2}\right) + \alpha'(t-T_0)\frac{\|\theta(t)\|^2}{2} + c_{\alpha'}(t-T_0)\|\Delta\theta(t)\|^2 + \varepsilon(t-T_0)\|\nabla\theta(t)\|^2 \\
&\quad \leqslant \big(1 + \kappa\varepsilon(t-T_0)\big)\|\theta(t)\|^2 + c(t-T_0)\|f'(u(t))\theta(t)\|^2 \\
&\quad \leqslant c(t-T_0)\big(\|\theta(t)\|^2 + \|f'(u(t))\theta(t)\|^2\big) := c(t-T_0)h_u(t)
\end{aligned} \tag{4.26}$$

for appropriate positive constants α' and c'_α. Applying Gronwall's lemma to (4.26) over (T_0, t), we obtain

$$(t-T_0)\|\theta(t)\|^2 \mathrm{e}^{\alpha'(t-T_0)} \leqslant c\int_{T_0}^t (s-T_0)h_u(s)\mathrm{e}^{\alpha'(s-T_0)}\,\mathrm{d}s \leqslant c(t-T_0)\int_{T_0}^t h_u(s)\mathrm{e}^{\alpha'(s-T_0)}\,\mathrm{d}s,$$

so that

$$\|\theta(t)\|^2 \leqslant c\int_{T_0}^t h_u(s)\mathrm{e}^{-\alpha'(t-T_0)}\mathrm{e}^{\alpha'(s-T_0)}\,\mathrm{d}s = c\int_{T_0}^t h_u(s)\mathrm{e}^{-\alpha'(t-s)}\,\mathrm{d}s. \tag{4.27}$$

To estimate $\int_{T_0}^t h_u(s)\mathrm{e}^{-\alpha'(t-s)}\,\mathrm{d}s$, we proceed as follows.

Using Theorem 4.1 and estimate (4.16), we find

$$\begin{aligned}
\int_t^{t+1} \|\partial_t u(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s &\leqslant Q\big(\|u(t-T_0)\|_{H^2(\Omega)}\big)\mathrm{e}^{K_1 T_0} \\
&\leqslant \mathrm{e}^{K_1 T_0}\big(Q(\|u_0\|_{H^2(\Omega)})\mathrm{e}^{-\alpha(t-T_0)} + c\big) \\
&\leqslant Q(\|u_0\|_{H^2(\Omega)})\mathrm{e}^{-\alpha t} + c,
\end{aligned} \tag{4.28}$$

and using the fact that $W \subset C(\overline{\Omega})$, we obtain

$$\begin{aligned}
\int_t^{t+1} h_u(s)\,\mathrm{d}s &\leqslant Q\Big(\sup_{s\in[t,t+1]}\|u(s)\|_{L^\infty(\Omega)}\Big)\int_t^{t+1}\|\partial_t u(s)\|^2_{H^1(\Omega)}\,\mathrm{d}s \\
&\leqslant Q(\|u_0\|_{H^2(\Omega)})\mathrm{e}^{-\alpha t} + c
\end{aligned} \tag{4.29}$$

for $t \geqslant T_0$ and for an appropriate function Q and constants $c, \alpha > 0$ which are independent of ε.

We have

$$\begin{aligned}
&\int_{T_0}^{t} h_u(s)\mathrm{e}^{-\alpha'(t-s)}\,\mathrm{d}s \\
&\quad= \mathrm{e}^{-\alpha' t}\int_{T_0}^{t} h_u(s)\mathrm{e}^{\alpha' s}\,\mathrm{d}s \\
&\quad\leqslant \mathrm{e}^{-\alpha' t}\biggl(\int_{T_0}^{1} h_u(s)\mathrm{e}^{\alpha' s}\,\mathrm{d}s + \int_{1}^{2} h_u(s)\mathrm{e}^{\alpha' s}\,\mathrm{d}s + \cdots + \int_{[t]-1}^{[t]} h_u(s)\mathrm{e}^{\alpha' s}\,\mathrm{d}s \\
&\qquad\quad + \int_{[t]}^{[t]+1} h_u(s)\mathrm{e}^{\alpha' s}\,\mathrm{d}s\biggr) \\
&\quad\leqslant \mathrm{e}^{-\alpha' t}\biggl(\mathrm{e}^{\alpha'}\int_{T_0}^{1} h_u(s)\,\mathrm{d}s + \mathrm{e}^{2\alpha'}\int_{1}^{2} h_u(s)\,\mathrm{d}s + \cdots + \mathrm{e}^{\alpha'[t]}\int_{[t]-1}^{[t]} h_u(s)\,\mathrm{d}s \\
&\qquad\quad + \mathrm{e}^{\alpha'([t]+1)}\int_{[t]}^{[t]+1} h_u(s)\,\mathrm{d}s\biggr) \\
&\quad\leqslant \mathrm{e}^{-\alpha' t}\bigl(\mathrm{e}^{\alpha'}\bigl(Q\bigl(\|u_0\|_{H^2(\Omega)}\bigr)\mathrm{e}^{-\alpha T_0} + c\bigr) + \mathrm{e}^{2\alpha'}\bigl(Q\bigl(\|u_0\|_{H^2(\Omega)}\bigr)\mathrm{e}^{-\alpha} + c\bigr) + \cdots \\
&\qquad\quad + \mathrm{e}^{\alpha'[t]}\bigl(Q\bigl(\|u_0\|_{H^2(\Omega)}\bigr)\mathrm{e}^{-\alpha([t]-1)} + c\bigr) + \mathrm{e}^{\alpha'([t]+1)}\bigl(Q\bigl(\|u_0\|_{H^2(\Omega)}\bigr)\mathrm{e}^{-\alpha[t]} + c\bigr)\bigr) \\
&\quad\leqslant \mathrm{e}^{-\alpha' t}\bigl(\mathrm{e}^{\alpha'}\bigl(Q\bigl(\|u_0\|_{H^2(\Omega)}\bigr) + c\bigr) + \mathrm{e}^{2\alpha'}\bigl(Q\bigl(\|u_0\|_{H^2(\Omega)}\bigr)\mathrm{e}^{-\alpha} + c\bigr) + \cdots \\
&\qquad\quad + \mathrm{e}^{\alpha'[t]}\bigl(Q\bigl(\|u_0\|_{H^2(\Omega)}\bigr)\mathrm{e}^{-\alpha([t]-1)} + c\bigr) + \mathrm{e}^{\alpha'([t]+1)}\bigl(Q\bigl(\|u_0\|_{H^2(\Omega)}\bigr)\mathrm{e}^{-\alpha[t]} + c\bigr)\bigr) \\
&\quad\leqslant \mathrm{e}^{-\alpha' t}\mathrm{e}^{\alpha'}Q\bigl(\|u_0\|_{H^2(\Omega)}\bigr)\left(\frac{\mathrm{e}^{(\alpha'-\alpha)([t]+1)} - 1}{\mathrm{e}^{(\alpha'-\alpha)} - 1}\right) + c\mathrm{e}^{-\alpha' t}\mathrm{e}^{\alpha'}\left(\frac{\mathrm{e}^{\alpha'([t]+1)} - 1}{\mathrm{e}^{\alpha'} - 1}\right) \\
&\quad\leqslant Q\bigl(\|u_0\|_{H^2(\Omega)}\bigr)\mathrm{e}^{-\alpha t} + c,
\end{aligned} \tag{4.30}$$

where $\alpha \leqslant \alpha'$.

From estimates (4.27) and (4.30), we deduce the lemma. □

Lemma 4.7. *Let the assumptions of Theorem* 4.1 *hold. Then*, *we have the following*:

$$\bigl\|\partial_t u(t)\bigr\|_{H^1(\Omega)}^2 + \int_t^{t+1}\bigl\|\partial_t^2 u(s)\bigr\|_{H^{-1}(\Omega)}^2\,\mathrm{d}s \leqslant Q\bigl(\bigl\|u(0)\bigr\|_{H^2(\Omega)}\bigr)\mathrm{e}^{-\alpha t} + c,$$

where $t \geqslant 2$ and the constants $c, \alpha > 0$ and the function Q are independent of ε.

Proof. Multiplying (4.11) by $(t - T_*)\,\partial_t\theta(t)$, where $\theta = \partial_t u$ and $t \geqslant T_* \geqslant 1$ is arbitrary, using (2.5) and integrating over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigl[(t - T_*)\bigl\|\nabla\theta(t)\bigr\|^2\bigr] + (t - T_*)\bigl\|\partial_t\theta(t)\bigr\|_{H^{-1}(\Omega)}^2 \\
&\quad= \frac{1}{2}\bigl\|\nabla\theta(t)\bigr\|^2 - (t - T_*)\bigl(f'\bigl(u(t)\bigr)\theta(t), \partial_t\theta(t)\bigr) := H_u(t).
\end{aligned} \tag{4.31}$$

We have

$$\begin{aligned}
&\big(f'\big(u(t)\big)\theta(t), \partial_t\theta(t)\big)\\
&\quad\leqslant \big\|f'\big(u(t)\big)\theta(t)\big\|_{H^1(\Omega)}\big\|\partial_t\theta(t)\big\|_{H^{-1}(\Omega)}\\
&\quad\leqslant \big(\big\|f'\big(u(t)\big)\big\|_{L^\infty(\Omega)}\big\|\nabla\theta(t)\big\| + \|\nabla f'\big(u(t)\big)\|_{L^4(\Omega)}\big\|\theta(t)\big\|_{L^4(\Omega)}\big)\big\|\partial_t\theta(t)\big\|_{H^{-1}(\Omega)}\\
&\quad\leqslant \big(\big\|f'\big(u(t)\big)\big\|_{L^\infty(\Omega)} + \big\|\nabla f'\big(u(t)\big)\big\|_{L^4(\Omega)}\big)\big\|\theta(t)\big\|_{H^1(\Omega)}\big\|\partial_t\theta(t)\big\|_{H^{-1}(\Omega)}\\
&\quad\leqslant \big(\big\|f'\big(u(t)\big)\big\|_{L^\infty(\Omega)} + \big\|f''\big(u(t)\big)\big\|_{L^\infty(\Omega)}\big\|u(t)\big\|_{H^2(\Omega)}\big)\big\|\theta(t)\big\|_{H^1(\Omega)}\big\|\partial_t\theta(t)\big\|_{H^{-1}(\Omega)}. \qquad (4.32)
\end{aligned}$$

It follows from Theorem 4.1 and estimates (4.25), (4.32) and from the fact that $W \subset C(\overline{\Omega})$ and $H^1(\Omega) \subset L^6(\Omega) \subset L^4(\Omega)$ ($n \leqslant 3$) that

$$\begin{aligned}
\int_{T_*}^{T_*+s} H_u(t)\,\mathrm{d}t &\leqslant (1+s)\big(Q\big(\big\|u(0)\big\|_{H^2(\Omega)}\big)\mathrm{e}^{-\alpha T_*} + c\big)\\
&\quad + \frac{1}{2}\int_{T_*}^{T_*+s}(t-T_*)\big\|\partial_t\theta(t)\big\|^2_{H^{-1}(\Omega)}\,\mathrm{d}t, \qquad (4.33)
\end{aligned}$$

where $s \in [0,2]$ and Q is an appropriate function independent of ε.

Integrating (4.31) with respect to $t \in [T_*, T_* + s]$ and using (4.33), we find

$$s\big\|\theta(T_*+s)\big\|^2_{H^1(\Omega)} + \int_{T_*}^{T_*+s}(t-T_*)\big\|\partial_t\theta(t)\big\|^2_{H^{-1}(\Omega)}\,\mathrm{d}t \leqslant Q\big(\big\|u(0)\big\|_{H^2(\Omega)}\big)\mathrm{e}^{-\alpha T_*} + c$$

for $T_* \geqslant 1$ and $s \in [0,2]$. Thus, for $T_* := t-1 \geqslant 1$ and $s = 1$, we deduce that

$$\begin{aligned}
\big\|\theta(t)\big\|^2_{H^1(\Omega)} + \int_{t-1}^{t}\big\|\partial_t\theta(s)\big\|^2_{H^{-1}(\Omega)}\,\mathrm{d}s &\leqslant Q\big(\big\|u(0)\big\|_{H^2(\Omega)}\big)\mathrm{e}^{-\alpha(t-1)} + c\\
&\leqslant cQ\big(\big\|u(0)\big\|_{H^2(\Omega)}\big)\mathrm{e}^{-\alpha t} + c. \qquad (4.34)
\end{aligned}$$

We thus deduce the lemma from estimates (4.34). □

5. Existence of the global attractor

In this section, we will study the long time behavior of the solutions of (2.1). We have the following lemma.

Lemma 5.1. *Problem* (2.1) *generates the following semigroup on the phase space* H:

$$\begin{aligned}
&S_\varepsilon(t): H \to H,\\
&\qquad u_0 \mapsto S_\varepsilon(t)u_0 = u(t), \quad t \geqslant 0,
\end{aligned}$$

where $u(t)$ is the unique solution of problem (2.1), *at time t, with initial datum u_0. Furthermore, this semigroup is Lipschitz continuous in the $H^{-1}(\Omega)$-topology,*

$$\begin{aligned}&\big\|A_\varepsilon^{-1/2}\big(S_\varepsilon(t)u_1 - S_\varepsilon(t)u_2\big)\big\|^2 + \int_t^{t+1} \big\|S_\varepsilon(s)u_1 - S_\varepsilon(s)u_2\big\|^2_{H^1(\Omega)}\,\mathrm{d}s \\ &\quad \leqslant c\mathrm{e}^{Ct}\big\|A_\varepsilon^{-1/2}(u_1 - u_2)\big\|^2 \end{aligned} \tag{5.1}$$

for any u_1, $u_2 \in H$, where c and C are positive constants independent of t and ε. Thus, $S_\varepsilon(t)$ can be uniquely extended, by continuity, to a semigroup, still denoted by $S_\varepsilon(t)$, acting on $H^{-1}(\Omega)$.

To prove the existence of the global attractor, we apply the following theorem.

Theorem 5.2 (See [15]). *If $S_\varepsilon(t)$ is dissipative in $H^{-1}(\Omega)$ and B is a compact absorbing set in $H^{-1}(\Omega)$, then $S_\varepsilon(t)$ has the global attractor $\mathcal{A}_\varepsilon = \omega(B)$.*

We have the following theorem.

Theorem 5.3. *The semigroup $S_\varepsilon(t)$ possesses the global attractor $\mathcal{A}_\varepsilon$ in $H^{-1}(\Omega)$ which is bounded in W.*

We divide the proof of this theorem into several propositions.

Proposition 1. *Problem* (2.1) *has an absorbing set in $H^{-1}(\Omega)$. More precisely, there exists a positive constant ρ_0 and a time $t_0 := t_0(\|u_0\|_{-1})$ such that*

$$\big\|A_\varepsilon^{-1/2}u(t)\big\|^2 \leqslant \rho_0^2 \quad \textit{for all } t \geqslant t_0.$$

Proof. We multiply (2.1) by u, integrate over Ω and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|A_\varepsilon^{-1/2}u\big\|^2 + \|\nabla u\|^2 + \big(f(u), u\big) = 0. \tag{5.2}$$

Using (1.3) with $\alpha = p - \frac{3}{2}$, we have

$$0 \leqslant \int_\Omega \big|f(u)\big|\,\mathrm{d}x \leqslant pb_{2p}\int_\Omega u^{2p}\,\mathrm{d}x - \frac{3}{2}b_{2p}\int_\Omega u^{2p}\,\mathrm{d}x + c|\Omega| \leqslant \big(f(u), u\big) - \frac{3}{2}b_{2p}\int_\Omega u^{2p}\,\mathrm{d}x + c.$$

Thus, using (1.3), we get

$$\|\nabla u\|^2 + \big(f(u), u\big) \geqslant \|\nabla u\|^2 + \frac{3}{2}b_{2p}\int_\Omega u^{2p}\,\mathrm{d}x - c \geqslant \frac{1}{2}\|\nabla u\|^2 + \int_\Omega F(u)\,\mathrm{d}x - c = J(u) - c.$$

Therefore, (5.2) yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|A_\varepsilon^{-1/2}u\big\|^2 + J(u) \leqslant c. \tag{5.3}$$

Using (1.3) and the fact that

$$\left\|A_\varepsilon^{-1/2}u\right\|^2 \leqslant c\|u\|^2 \leqslant c\|\nabla u\|^2 \quad \forall u \in V,$$

we have the following inequality:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|A_\varepsilon^{-1/2}u\right\|^2 + \frac{1}{2c}\left\|A_\varepsilon^{-1/2}u\right\|^2 + \frac{1}{2}b_{2p}\int_\Omega u^{2p}\,\mathrm{d}x \leqslant c.$$

By Gronwall's lemma, we find

$$\begin{aligned}\left\|A_\varepsilon^{-1/2}u(t)\right\|^2 &\leqslant \left\|A_\varepsilon^{-1/2}u(0)\right\|^2 \mathrm{e}^{-t/c} + c\left(1-\mathrm{e}^{-t/c}\right)\\ &\leqslant c\left\|(-\Delta)^{-1/2}u(0)\right\|^2 \mathrm{e}^{-t/c} + c\left(1-\mathrm{e}^{-t/c}\right).\end{aligned}$$

It follows that if $t \geqslant t_0(\|(-\Delta)^{-1/2}u(0)\|^2) \equiv c\ln(c\|(-\Delta)^{-1/2}u(0)\|^2)$, then

$$\left\|A_\varepsilon^{-1/2}u(t)\right\|^2 \leqslant \rho_0^2. \qquad \square$$

Proposition 2. *For an arbitrary constant $r > 0$, there exists a positive constant ρ_V depending on r and a time $t_1 := t_1(\|u_0\|_{-1}, r)$ such that*

$$\left\|\nabla u(t)\right\| \leqslant \rho_V(r) \quad \textit{and} \quad \left\|u(t)\right\|_{L^{2p}(\Omega)}^p \leqslant \frac{\rho_V(r)}{\sqrt{b_{2p}}} \tag{5.4}$$

for all $t \geqslant t_0 + r := t_1$.

Proof. Fixing a constant $r > 0$ and integrating (5.3) with respect to t, we obtain

$$\int_t^{t+r} J\left(u(s)\right)\mathrm{d}s \leqslant \frac{1}{2}\rho_0^2 + rc \quad \forall t \geqslant t_0, \tag{5.5}$$

where t_0 is as in Proposition 1. Since J decays along the trajectories, we conclude from (5.5) that

$$J\left(u(t+r)\right) \leqslant \frac{1}{2r}\rho_0^2 + c \quad \forall t \geqslant t_0.$$

By the definition of J and (1.3), we have

$$\left\|\nabla u(t)\right\|^2 + b_{2p}\int_\Omega u^{2p}(x,t)\,\mathrm{d}x \leqslant 2c|\Omega| + \frac{1}{r}\rho_0^2 + 2c := \rho_V^2 \tag{5.6}$$

for all $t \geqslant t_0 + r := t_1$. The proposition is thus proven. $\square$

Proposition 3. *For an arbitrary constant $r > 0$, there exists a positive constant ρ depending on r and a time $t_2 := t_2(\|u_0\|_{-1}, r)$ such that*

$$\left\|\Delta u(t)\right\| \leqslant \rho(r) \quad \textit{for all } t \geqslant t_1 + r := t_2.$$

Proof. Multiplying (2.1) by $\partial_t u$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F(u)\,\mathrm{d}x + \big\|A_\varepsilon^{-1/2}\,\partial_t u\big\|^2 = 0. \tag{5.7}$$

Using (5.4), we find

$$\begin{aligned}&\int_t^{t+r}\big\|A_\varepsilon^{-1/2}\,\partial_t u(s)\big\|^2\,\mathrm{d}s + \frac{1}{2}\big\|\nabla u(t+r)\big\|^2 + \frac{b_{2p}}{2}\int_\Omega u^{2p}(t+r)\,\mathrm{d}x\\ &\quad\leqslant \frac{1}{2}\big\|\nabla u(t)\big\|^2 + \frac{b_{2p}}{2}\int_\Omega u^{2p}(t)\,\mathrm{d}x + c \leqslant \frac{3}{2}\rho_V^2 + c \quad \forall t \geqslant t_1.\end{aligned} \tag{5.8}$$

Multiplying (4.11) by $\partial_t u$ and using the fact that $f'(u) \geqslant -\kappa$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|A_\varepsilon^{-1/2}\,\partial_t u\big\|^2 + \|\nabla\partial_t u\|^2 \leqslant \kappa\|\partial_t u\|^2 \leqslant c\big\|A_\varepsilon^{-1/2}\,\partial_t u\big\|\|\nabla\partial_t u\|,$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|A_\varepsilon^{-1/2}\,\partial_t u\big\|^2 \leqslant c\big\|A_\varepsilon^{-1/2}\,\partial_t u\big\|^2, \tag{5.9}$$

for some positive constant c. Using (5.8), (5.9) and the uniform Gronwall lemma, we deduce that

$$\big\|A_\varepsilon^{-1/2}\,\partial_t u(t)\big\|^2 \leqslant c(r) \quad \forall t \geqslant t_1 + r. \tag{5.10}$$

We rewrite (2.1) in the following form:

$$-\Delta u + f(u) = -A_\varepsilon^{-1}\,\partial_t u. \tag{5.11}$$

Multiplying (5.11) by $-\Delta u(t)$ and using the fact that $f'(u) \geqslant -\kappa$, we obtain

$$\begin{aligned}\big\|\Delta u(t)\big\|^2 &\leqslant \kappa\big\|\nabla u(t)\big\|^2 + \big(A_\varepsilon^{-1}\,\partial_t u(t), \Delta u(t)\big)\\ &\leqslant \kappa\big\|\nabla u(t)\big\|^2 + \big\|A_\varepsilon^{-1}\,\partial_t u(t)\big\|\big\|\Delta u(t)\big\|\\ &\leqslant \kappa\big\|\nabla u(t)\big\|^2 + c\big\|A_\varepsilon^{-1/2}\,\partial_t u(t)\big\|\big\|\Delta u(t)\big\|,\end{aligned}$$

which yields

$$\big\|\Delta u(t)\big\|^2 \leqslant 2\kappa\big\|\nabla u(t)\big\|^2 + c\big\|A_\varepsilon^{-1/2}\,\partial_t u(t)\big\|^2. \tag{5.12}$$

Using (5.4) and (5.10), estimate (5.12) gives

$$\big\|\Delta u(t)\big\| \leqslant \rho(r) \quad \forall t \geqslant t_2 := t_1 + r$$

for some positive constant $\rho(r)$ which depend on r and Proposition 3 is proven. □

Let

$$B := \{u \in W, \|\Delta u\|^2 \leqslant \rho\}, \tag{5.13}$$

where ρ is the same as in Proposition 3. Then, B is a compact absorbing set in $H^{-1}(\Omega)$ since it is an absorbing set in W and W is compactly embedded into $H^{-1}(\Omega)$. Applying Theorem 5.2, we deduce the existence of the global attractor $\mathcal{A}_\varepsilon = \omega(B)$ and Theorem 5.3 is thus proven.

6. Existence of an exponential attractor

To prove the existence of an exponential attractor, we apply the following theorem.

Theorem 6.1 (See [13]). *Let E and E_1 be two Hilbert spaces such that E_1 is compactly embedded into E and $S(t): X \to X$ be a semigroup acting on a closed subset X of E. We assume that*

(1) $\forall x_1, x_2 \in X, \forall t > 0,$

$$\|S(t)x_1 - S(t)x_2\|_{E_1} \leqslant h(t)\|x_1 - x_2\|_E,$$

where the function h is continuous;

(2) $(t, x) \mapsto S(t)x$ *is uniformly Hölder continuous in the topology of E on* $[0, T] \times B$, $\forall T > 0$, $\forall B \subset X$ *bounded.*

Then, $S(t)$ possesses an exponential attractor on X.

In order to apply this result to the semigroup $S_\varepsilon(t)$ associated with problem (2.1), we set $E = H^{-1}(\Omega)$, $E_1 = H^2(\Omega)$ and we consider the set

$$X = \overline{\bigcup_{t \geqslant t_2} S(t)B}^{H^{-1}(\Omega)},$$

where B is defined in (5.13). X is thus compact in $H^{-1}(\Omega)$, bounded in $H^2(\Omega)$ and positively invariant by $S_\varepsilon(t)$.

Let u_1 and u_2 be two solutions of (2.1) with initial data in X. We set $w = u_1 - u_2$. Then, w verifies

$$\begin{cases} A_\varepsilon^{-1}\, \partial_t w - \Delta w + l(t)w = 0, \\ w(0) = u_1(0) - u_2(0), \end{cases} \tag{6.1}$$

where $l(t) = \int_0^1 f'(su_1(t) + (1 - s)u_2(t))\, \mathrm{d}s$.

The next lemma gives the $H^{-1}(\Omega) \to H$-smoothing property for the difference of two solutions.

Lemma 6.2. *Let $u_1(t)$ and $u_2(t)$ be two solutions of* (2.1) *such that* $\|\Delta u_i(0)\| \leqslant \rho$, $i = 1, 2$. *Then, the following estimate is valid*:

$$t\|u_1(t) - u_2(t)\|^2 + \int_t^{t+1} \|\Delta u_1(s) - \Delta u_2(s)\|^2 \,\mathrm{d}s \leqslant R_\rho \mathrm{e}^{\alpha t}\|u_1(0) - u_2(0)\|_{-1}^2, \quad t > 0, \tag{6.2}$$

where the constants R_ρ, $\alpha > 0$ are independent of ε and R_ρ depends on ρ.

Proof. Multiplying (6.1) by $-t\Delta w$ and integrating over Ω, we find

$$\frac{1}{2}t\,\frac{\mathrm{d}}{\mathrm{d}t}\big(-\Delta A_\varepsilon^{-1}w, w\big) + t\|\Delta w\|^2 - t\big(l(t)w, \Delta w\big) = 0. \tag{6.3}$$

Setting $(-\Delta A_\varepsilon^{-1}w, w) = \|w\|_*^2$, where $\|\cdot\|_* \approx \|\cdot\|_{L^2(\Omega)}$, Eq. (6.3) yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(t\|w\|_*^2\big) + t\|\Delta w\|^2 &\leqslant \frac{1}{2}\|w\|_*^2 + t\big(l(t)w, \Delta w\big)\\ &\leqslant \frac{1}{2}\|w\|_*^2 + t\big\|l(t)\big\|_{L^\infty(\Omega)}\|w\|\|\Delta w\|\\ &\leqslant \frac{1}{2}\|w\|_*^2 + tc_\rho\|w\|\|\Delta w\|\\ &\leqslant \frac{1}{2}\|w\|_*^2 + \frac{t}{2}c_\rho^2\|w\|^2 + \frac{t}{2}\|\Delta w\|^2\\ &\leqslant \frac{1}{2}\|w\|_*^2 + \frac{t}{2}c_\rho^2\|\nabla w\|^2 + \frac{t}{2}\|\Delta w\|^2,\end{aligned} \tag{6.4}$$

where $\|l(t)\|_{L^\infty(\Omega)} \leqslant C(\|u_1(0)\|_{H^2(\Omega)}, \|u_2(0)\|_{H^2(\Omega)}) \leqslant c_\rho$ and c_ρ is a positive constant depending on ρ. Integrating with respect to $s \in [0, t]$, we obtain

$$\begin{aligned}t\big\|w(t)\big\|_*^2 + \int_0^t s\big\|\Delta w(s)\big\|^2\,\mathrm{d}s &\leqslant \int_0^t \big\|w(s)\big\|_*^2\,\mathrm{d}s + c_\rho^2\int_0^t s\big\|\nabla w(s)\big\|^2\,\mathrm{d}s\\ &\leqslant c\int_0^t \big\|\nabla w(s)\big\|^2\,\mathrm{d}s + c_\rho^2\int_0^t s\big\|\nabla w(s)\big\|^2\,\mathrm{d}s\\ &\leqslant R_\rho \mathrm{e}^{\alpha t}\big\|u_1(0) - u_2(0)\big\|_{-1}^2,\end{aligned} \tag{6.5}$$

where we have used the fact that $\|w\|_* \leqslant c\|\nabla w\|$ and estimates (2.5) and (3.3).

Now, multiplying (6.1) by $-\Delta w$ and integrating over Ω, we find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\|w\|_*^2\big) + \|\Delta w\|^2 \leqslant \big(l(t)w, \Delta w\big) \leqslant \frac{c_\rho^2}{2}\|\nabla w\|^2 + \frac{1}{2}\|\Delta w\|^2. \tag{6.6}$$

Integrating (6.6) over $(t, t+1)$, $t > 0$, we obtain

$$\int_t^{t+1}\big\|\Delta w(s)\big\|^2\,\mathrm{d}s \leqslant \big\|u_1(0) - u_2(0)\big\|_{-1}^2 R_\rho \mathrm{e}^{\alpha t}. \tag{6.7}$$

Thus, the lemma is an immediate consequence of estimates (6.5) and (6.7). □

Lemma 6.3. *Let the assumptions of Lemma* 6.2 *hold. Then*, *the following estimate holds*:

$$\begin{aligned}&\big\|u_1(t) - u_2(t)\big\|_{H^1(\Omega)}^2 + \int_t^{t+1}\big\|\partial_t u_1(s) - \partial_t u_2(s)\big\|_{-1}^2\,\mathrm{d}s\\ &\quad\leqslant R_\rho \mathrm{e}^{\alpha_\rho t}\big\|u_1(0) - u_2(0)\big\|_{-1}^2, \quad t \geqslant 2,\end{aligned} \tag{6.8}$$

where R_ρ and α_ρ are positive constants depending on ρ and independent of ε.

Proof. We have

$$\begin{aligned}\big\|\nabla\big(l(t)w\big)\big\| &\leqslant \big\|l(t)\big\|_{L^\infty(\Omega)}\|\nabla w\| + c\big\|\nabla l(t)\big\|_{L^4(\Omega)}\|w\|_{L^4(\Omega)} \\ &\leqslant c\big(\big\|l(t)\big\|_{L^\infty(\Omega)} + \big\|\nabla l(t)\big\|_{L^4(\Omega)}\big)\|\nabla w\|, \quad n \leqslant 3.\end{aligned}$$

Using the embeddings $W \subset C(\overline{\Omega})$ and $W \subset W^{1,4}(\Omega)$, we find

$$\big\|\nabla l(t)\big\|_{L^4(\Omega)} \leqslant Q\Big(\max_{i=1,2}\big\|u_i(t)\big\|_{L^\infty(\Omega)}\Big)\max_{i=1,2}\big\|u_i(t)\big\|_{W^{1,4}(\Omega)} \leqslant C_\rho,$$

where Q is an appropriate function and C_ρ is a positive constant depending on ρ which are independent of ε. Consequently, we obtain

$$\big\|\nabla\big(l(t)w\big)\big\| \leqslant c_\rho\|\nabla w\|,$$

where $c_\rho > 0$ depend on ρ and independent of ε.

Now, multiplying (6.1) by $(t - T_*)\,\partial_t w$, where $T_* \geqslant 1$ is arbitrary and such that $t - T_* \geqslant 0$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big((t-T_*)\|\nabla w\|^2\big) + c(t-T_*)\|\partial_t w\|_{-1}^2 \\ &\quad = \frac{1}{2}\|\nabla w\|^2 - (t-T_*)\big(l(t)w, \partial_t w\big) \\ &\quad \leqslant \frac{1}{2}\|\nabla w\|^2 + (t-T_*)\big\|\nabla\big(l(t)w\big)\big\|\|\partial_t w\|_{-1} \\ &\quad \leqslant \frac{1}{2}\|\nabla w\|^2 + (t-T_*)c_\rho\|\nabla w\|\|\partial_t w\|_{-1} \\ &\quad \leqslant \big(1 + c_\rho(t-T_*)\big)\frac{\|\nabla w\|^2}{2} + \frac{c}{2}(t-T_*)\|\partial_t w\|_{-1}^2.\end{aligned} \tag{6.9}$$

Integrating (6.9) with respect to $t \in [T_*, T_* + s]$, for $T_* \geqslant 1$ and $s \in [0, 2]$, and using estimate (3.3), we find

$$s\big\|\nabla w(T_* + s)\big\|^2 + c\int_{T_*}^{T_*+s}(t-T_*)\big\|\partial_t w(t)\big\|_{-1}^2\,\mathrm{d}t \leqslant \big\|u_1(0) - u_2(0)\big\|_{-1}^2 R_\rho \mathrm{e}^{\alpha_\rho T_*}. \tag{6.10}$$

Setting $T_* := t - 1 \geqslant 1$ and $s = 1$, estimate (6.10) yields

$$\begin{aligned}\big\|\nabla w(t)\big\|^2 + c\int_{t-1}^{t}\big\|\partial_t w(s)\big\|_{-1}^2\,\mathrm{d}s &\leqslant \big\|u_1(0) - u_2(0)\big\|_{-1}^2 R_\rho \mathrm{e}^{\alpha_\rho(t-1)} \\ &\leqslant \big\|u_1(0) - u_2(0)\big\|_{-1}^2 R_\rho \mathrm{e}^{\alpha_\rho t},\end{aligned} \tag{6.11}$$

and the lemma is thus proven. □

We now have a $H^{-1}(\Omega) \to H^2(\Omega)$-smoothing property.

Lemma 6.4. *Let $u_1(t)$ and $u_2(t)$ be two solutions of* (2.1) *such that $\|\Delta u_i(0)\| \leqslant \rho$, $i = 1, 2$. Then, the following estimate is valid*:

$$\left\|u_1(t) - u_2(t)\right\|_{H^2(\Omega)}^2 + \left\|\partial_t u_1(t) - \partial_t u_2(t)\right\|_{-1}^2 \leqslant R_\rho \mathrm{e}^{\alpha_\rho t}\left\|u_1(0) - u_2(0)\right\|_{-1}^2, \quad t \geqslant 3, \tag{6.12}$$

where R_ρ and α_ρ are positive constants depending on ρ and independent of ε.

Proof. We differentiate Eq. (6.1) with respect to t and set $\theta(t) = \partial_t w(t)$. This function satisfies the equation

$$A_\varepsilon^{-1}\,\partial_t\theta - \Delta\theta + l(t)\theta + \partial_t l(t) w = 0, \quad \theta|_{\partial\Omega} = 0.$$

Multiplying this equation by $(t-2)\theta(t)$, integrating over Ω and noting that

$$\begin{aligned}\left\|\partial_t l(t) w(t)\right\|^2 &\leqslant \max_{i=1,2}\left(\left\|u_i(t)\right\|_{L^\infty(\Omega)}\right)\left(\left\|\partial_t u_1(t)\right\|_{H^1(\Omega)}^2 + \left\|\partial_t u_2(t)\right\|_{H^1(\Omega)}^2\right)\left\|\nabla w(t)\right\|^2 \\ &\leqslant c_\rho\left\|\nabla w(t)\right\|^2 \quad \text{(using Lemma 4.7)},\end{aligned}$$

we obtain the following estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((t-2)\left\|A_\varepsilon^{-1/2}\theta\right\|^2\right) + (t-2)\|\nabla\theta\|^2 \leqslant c_\rho(t-1)\left\|A_\varepsilon^{-1/2}\theta\right\|^2 + c_\rho(t-2)\|\nabla w\|^2, \quad t \geqslant 3.$$

Applying Gronwall's Lemma, using the bound for $\|\nabla w\|^2$ and estimate (6.8), we prove that

$$\left\|\theta(t)\right\|_{-1}^2 + \int_t^{t+1}\left\|\nabla\theta(s)\right\|^2 \mathrm{d}s \leqslant R_\rho \mathrm{e}^{\alpha_\rho t}\left\|u_1(0) - u_2(0)\right\|_{-1}^2 \quad \forall t \geqslant 3.$$

Now, interpreting Eq. (6.1) as an elliptic equation,

$$\Delta w - l(t) w = A_\varepsilon^{-1}\,\partial_t w,$$

we have

$$\begin{aligned}\left\|\Delta w(t)\right\|^2 &= \left(A_\varepsilon^{-1}\,\partial_t w(t), \Delta w(t)\right) + \left(l(t)w(t), \Delta w(t)\right) \\ &\leqslant c\left\|\partial_t w(t)\right\|_{-1}^2 + \frac{1}{2}\left\|\Delta w(t)\right\|^2 + \left\|\nabla\left(l(t)w(t)\right)\right\|\left\|\nabla w(t)\right\| \\ &\leqslant c\left\|\partial_t w(t)\right\|_{-1}^2 + \frac{1}{2}\left\|\Delta w(t)\right\|^2 + c_\rho\left\|\nabla w(t)\right\|^2.\end{aligned}$$

Using the above estimates and (6.8), we find

$$\left\|\Delta w(t)\right\|^2 \leqslant R_\rho \mathrm{e}^{\alpha_\rho t}\left\|u_1(0) - u_2(0)\right\|_{-1}^2 \quad \forall t \geqslant 3. \qquad \square$$

Theorem 6.5. *Let $u_1(t)$ and $u_2(t)$ be two solutions of* (2.1) *such that* $\|\Delta u_i(0)\| \leqslant \rho$, $i = 1, 2$. *Then, the following estimate is valid*:

$$\left\|u_1(t_2) - u_2(t_2)\right\|^2_{H^2(\Omega)} \leqslant R_\rho \left\|u_1(0) - u_2(0)\right\|^2_{-1}, \tag{6.13}$$

where R_ρ is a positive constant depending on ρ and independent of ε.

Proof. Having estimate (6.12) for every $t \geqslant 3$, we rescale the time ($t \to \alpha t$) and we deduce that estimate (6.13) is valid for every $t \geqslant T'$, $T' > 0$ arbitrary. In particular, this holds for $t = t_2$. □

Lemma 6.6. *The semigroup $S_\varepsilon(t)$ is uniformly Hölder continuous on $[0, T] \times B$, having endowed B with the H^{-1}-topology, i.e.*,

$$\left\|A_\varepsilon^{-1/2}\left(S_\varepsilon(t_1)u_{01} - S_\varepsilon(t_2)u_{02}\right)\right\| \leqslant R_{\rho,T}\left(\left\|A_\varepsilon^{-1/2}(u_{01} - u_{02})\right\| + |t_1 - t_2|^{1/2}\right),$$

where $u_{0i} \in B$, $t_i \leqslant T$, $i = 1, 2$, and $R_{\rho,T}$ is a positive constant depending on ρ and T and independent of ε.

Proof. The Lipschitz continuity with respect to the initial conditions is an immediate consequence of estimate (3.3). In order to verify the Hölder continuity with respect to t, we integrate (5.7) from t_1 to t_2 and find

$$\int_{t_1}^{t_2} \left\|A_\varepsilon^{-1/2}\,\partial_t u(s)\right\|^2 \mathrm{d}s + \frac{1}{2}\left\|\nabla u(t_2)\right\|^2 + \int_\Omega F\left(u(t_2)\right)\mathrm{d}x = \frac{1}{2}\left\|\nabla u(t_1)\right\|^2 + \int_\Omega F\left(u(t_1)\right)\mathrm{d}x.$$

Using (1.3), we have

$$\begin{aligned}
&\int_{t_1}^{t_2} \left\|A_\varepsilon^{-1/2}\,\partial_t u(s)\right\|^2 \mathrm{d}s + \frac{1}{2}\left\|\nabla u(t_2)\right\|^2 + \frac{b_{2p}}{2}\int_\Omega u^{2p}(t_2)\,\mathrm{d}x \\
&\quad \leqslant 2c|\Omega| + \frac{1}{2}\left\|\nabla u(t_1)\right\|^2 + \frac{3}{2}b_{2p}\int_\Omega u^{2p}(t_1)\,\mathrm{d}x \\
&\quad \leqslant c(\rho) \quad \text{(using (5.8))}.
\end{aligned}$$

Consequently, we deduce that

$$\begin{aligned}
\left\|A_\varepsilon^{-1/2}\left(u(t_1) - u(t_2)\right)\right\| &\leqslant c\left\|u(t_1) - u(t_2)\right\|_{-1} = c\left\|\int_{t_1}^{t_2} \partial_t u(s)\,\mathrm{d}s\right\|_{H^{-1}(\Omega)} \\
&\leqslant c\int_{t_1}^{t_2} \left\|\partial_t u(s)\right\|_{H^{-1}(\Omega)} \mathrm{d}s \leqslant c|t_1 - t_2|^{1/2}\left(\int_{t_1}^{t_2} \left\|\partial_t u(s)\right\|^2_{H^{-1}(\Omega)} \mathrm{d}s\right)^{1/2} \\
&\leqslant c|t_1 - t_2|^{1/2}\left(\int_{t_1}^{t_2} \left\|A_\varepsilon^{-1/2}\,\partial_t u(s)\right\|^2 \mathrm{d}s\right)^{1/2} \leqslant c_\rho|t_1 - t_2|^{1/2},
\end{aligned}$$

where c_ρ is a positive constant depending on ρ. We have thus proved the Hölder continuity with respect to t. □

Hence, we deduce the existence of an exponential attractor for problem (2.1).

7. Construction of a robust family of exponential attractors

In this section, we will apply the following theorem to the study of the long time behavior of the solutions of (2.1).

Theorem 7.1 (See [4]). *Let E_1 and E be two Banach spaces such that the inclusion $E_1 \subset E$ is compact and let B be a bounded set of E. We assume that there exists a family of operators $S_\varepsilon : B \to B, \varepsilon \in [0, 1]$, which satisfies the following assumptions*:

(1) *For every $x_1, x_2 \in B$, the following estimate is valid*:

$$\|S_\varepsilon x_1 - S_\varepsilon x_2\|_{E_1} \leqslant L\|x_1 - x_2\|_E, \tag{7.1}$$

where the constant L is independent of ε.

(2) *For every $\varepsilon \in [0, 1]$, for $i \in \mathbb{N}$ and for $x \in B$, we have the estimate*

$$\left\|S_\varepsilon^i x - S_0^i x\right\|_E \leqslant K^i \varepsilon. \tag{7.2}$$

Then, for every $\varepsilon \in [0, \varepsilon_0]$, there exists an exponential attractor $\mathcal{M}_\varepsilon$ for the map S_ε in B. Moreover, the exponential attractor $\mathcal{M}_\varepsilon$ can be chosen such that the following estimate is valid:

$$\operatorname{dist}_{\mathrm{sym}}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \leqslant C_1 \varepsilon^\nu,$$

where the constant C_1 and the exponent $0 < \nu < 1$ can be calculated explicitly and $\operatorname{dist}_{\mathrm{sym}}$ *denotes the symmetric Hausdorff distance between two sets in E. Finally, the fractal dimension of the exponential attractors considered above is uniformly bounded with respect to $\varepsilon \in [0, \varepsilon_0]$,*

$$\dim_F(\mathcal{M}_\varepsilon, E) \leqslant C = C(L),$$

where the constant C is independent of ε and can be calculated explicitly.

In order to apply the theorem above, let $u^0(t)$ and $u^\varepsilon(t)$ be two solutions of (1.1) with respectively zero ($\varepsilon = 0$) and nonzero ($0 < \varepsilon \leqslant 1$) parameters,

$$\partial_t u^\varepsilon + A^2 u^\varepsilon + Af\left(u^\varepsilon\right) + \varepsilon A u^\varepsilon + \varepsilon f\left(u^\varepsilon\right) = 0$$

and

$$\partial_t u^0 + A^2 u^0 + Af\left(u^0\right) = 0,$$

where $A = -\Delta$.

We have the following result on the difference of the two solutions.

Theorem 7.2. *Let $\|u^0(0)\|_{H^2(\Omega)}$ and $\|u^\varepsilon(0)\|_{H^2(\Omega)} \leqslant \rho$, with ρ a positive constant. Then, the following estimate holds*:

$$\left\|u^0(t) - u^\varepsilon(t)\right\|_{-1}^2 \leqslant R_\rho \mathrm{e}^{\alpha t}\left(\left\|u^0(0) - u^\varepsilon(0)\right\|_{-1}^2 + \varepsilon^2\right), \tag{7.3}$$

where $t \geqslant 0$, α is a positive constant independent of ε and R_ρ is a positive constant depending on ρ and independent of ε.

Proof. Let $w^\varepsilon(t) = u^\varepsilon(t) - u^0(t)$. Then, w^ε satisfies the following equation:

$$A^{-1}\,\partial_t w^\varepsilon + Aw^\varepsilon + l_\varepsilon(t)w^\varepsilon + \varepsilon u^\varepsilon + \varepsilon A^{-1} f\big(u^\varepsilon\big) = 0, \tag{7.4}$$

where $l_\varepsilon(t) := \int_0^1 f'(su^\varepsilon(t) + (1-s)u^0(t))\,\mathrm{d}s$. Multiplying (7.4) by $w^\varepsilon(t)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|w^\varepsilon\big\|_{-1}^2 + \big\|\nabla w^\varepsilon\big\|^2 + \big(l_\varepsilon(t)w^\varepsilon, w^\varepsilon\big) + \varepsilon\big(u^\varepsilon, w^\varepsilon\big) + \varepsilon\big(A^{-1}f\big(u^\varepsilon\big), w^\varepsilon\big) = 0. \tag{7.5}$$

We have

$$\big(l_\varepsilon(t)w^\varepsilon, w^\varepsilon\big) \geqslant -\kappa\big\|w^\varepsilon\big\|^2 \geqslant -\frac{1}{4}\big\|\nabla w^\varepsilon\big\|^2 - c\big\|w^\varepsilon\big\|_{-1}^2,$$
$$\varepsilon\big(u^\varepsilon, w^\varepsilon\big) \leqslant \big\|w^\varepsilon\big\|^2 + \frac{\varepsilon^2}{4}\big\|u^\varepsilon\big\|^2 \leqslant \frac{1}{4}\big\|\nabla w^\varepsilon\big\|^2 + c\big\|w^\varepsilon\big\|_{-1}^2 + \frac{\varepsilon^2}{4}\big\|u^\varepsilon\big\|^2$$

and

$$\begin{aligned}
\varepsilon\big(A^{-1}f\big(u^\varepsilon\big), w^\varepsilon\big) &\leqslant \varepsilon\big\|f\big(u^\varepsilon\big)\big\|_{-1}\big\|w^\varepsilon\big\|_{-1} \\
&\leqslant \big\|w^\varepsilon\big\|_{-1}^2 + \frac{\varepsilon^2}{4}\big\|f\big(u^\varepsilon\big)\big\|_{-1}^2 \\
&\leqslant \big\|w^\varepsilon\big\|_{-1}^2 + \varepsilon^2 Q\big(\big\|u^\varepsilon\big\|_{H^2(\Omega)}\big) \quad \text{(using (4.5))} \\
&\leqslant \big\|w^\varepsilon\big\|_{-1}^2 + c_\rho\varepsilon^2 \quad \text{(using (4.2))},
\end{aligned}$$

where c_ρ depends on ρ and is independent of ε. Then, (7.5) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|w^\varepsilon\big\|_{-1}^2 + \big\|\nabla w^\varepsilon\big\|^2 \leqslant c\big\|w^\varepsilon\big\|_{-1}^2 + c\varepsilon^2 h_\varepsilon(t), \tag{7.6}$$

where $h_\varepsilon(t) := \|u^\varepsilon(t)\|^2 + c_\rho$. Applying Gronwall's lemma and using estimate (4.2), inequality (7.6) gives

$$\big\|w^\varepsilon(t)\big\|_{-1}^2 + \int_t^{t+1}\big\|\nabla w^\varepsilon(s)\big\|^2\,\mathrm{d}s \leqslant R_\rho \mathrm{e}^{\alpha t}\big(\big\|w^\varepsilon(0)\big\|_{-1}^2 + \varepsilon^2\big). \qquad \square \tag{7.7}$$

Theorem 7.3. *Let $\|u^0(0)\|_{H^2(\Omega)}$ and $\|u^\varepsilon(0)\|_{H^2(\Omega)} \leqslant \rho$. We have the following*:

$$t\big\|u^\varepsilon(t) - u^0(t)\big\|^2 \leqslant R_\rho \mathrm{e}^{\alpha_\rho t}\big(\big\|u^\varepsilon(0) - u^0(0)\big\|_{-1}^2 + \varepsilon^2\big), \quad t \geqslant 0, \tag{7.8}$$

where R_ρ and α_ρ are positive constants depending on ρ and independent of ε.

Proof. Multiplying (7.4) by tAw^ε and integrating over Ω, where $w^\varepsilon = u^\varepsilon - u^0$, we obtain

$$\frac{t}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w^\varepsilon\|^2 + t\|Aw^\varepsilon\|^2 + t\big(l_\varepsilon(t)w^\varepsilon, Aw^\varepsilon\big) + \varepsilon t\big(u^\varepsilon, Aw^\varepsilon\big) + \varepsilon t\big(f(u^\varepsilon), w^\varepsilon\big) = 0.$$

Thus,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(t\|w^\varepsilon\|^2\big) + t\|\Delta w^\varepsilon\|^2 \\
&\quad = t\big(l_\varepsilon(t)w^\varepsilon, \Delta w^\varepsilon\big) + \varepsilon t\big(u^\varepsilon, \Delta w^\varepsilon\big) - \varepsilon t\big(f(u^\varepsilon), w^\varepsilon\big) + \frac{1}{2}\|w^\varepsilon\|^2 \\
&\quad \leqslant t\|l_\varepsilon(t)\|_\infty\|w^\varepsilon\|\|\Delta w^\varepsilon\| + \varepsilon t\|u^\varepsilon\|\|\Delta w^\varepsilon\| + \varepsilon t\|f(u^\varepsilon)\|\|w^\varepsilon\| + \frac{1}{2}\|w^\varepsilon\|^2 \\
&\quad \leqslant tc_\rho^2\|w^\varepsilon\|^2 + \frac{t}{4}\|\Delta w^\varepsilon\|^2 + \varepsilon^2 t\|u^\varepsilon\|^2 + \frac{t}{4}\|\Delta w^\varepsilon\|^2 + \varepsilon^2 t\|f(u^\varepsilon)\|^2 + t\|w^\varepsilon\|^2 + \frac{1}{2}\|w^\varepsilon\|^2,
\end{aligned}$$

where $\|l_\varepsilon(t)\|_{L^\infty(\Omega)} \leqslant Q(\max(\|u^\varepsilon(t)\|_{L^\infty(\Omega)}, \|u^0(t)\|_{L^\infty(\Omega)})) \leqslant c_\rho$. Therefore, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big(t\|w^\varepsilon\|^2\big) + t\|\Delta w^\varepsilon\|^2 &\leqslant \big(c_\rho^2 + 1\big)t\|w^\varepsilon\|^2 + \varepsilon^2 t\big(\|u^\varepsilon\|^2 + \|f(u^\varepsilon)\|^2\big) + \|w^\varepsilon\|^2 \\
&\leqslant \big(c_\rho^2 + 1\big)t\|w^\varepsilon\|^2 + \varepsilon^2 t\big(\|u^\varepsilon\|^2 + c_\rho\big) + c\|\nabla w^\varepsilon\|^2,
\end{aligned}$$

where we have used (4.6). Applying Gronwall's lemma and using (4.2) and estimate (7.7), we deduce that

$$t\|u^\varepsilon(t) - u^0(t)\|^2 + \int_t^{t+1} s\|\Delta w^\varepsilon(s)\|^2\,\mathrm{d}s \leqslant R_\rho \mathrm{e}^{\alpha_\rho t}\big(\|u^\varepsilon(0) - u^0(0)\|_{-1}^2 + \varepsilon^2\big). \qquad \square \tag{7.9}$$

Theorem 7.4. *Let the assumptions of Theorem* 7.2 *hold. Then, the following estimate is valid*:

$$\begin{aligned}
&t\|u^\varepsilon(t) - u^0(t)\|_{H^1(\Omega)}^2 + \int_t^{t+1} s\|\partial_t u^\varepsilon(s) - \partial_t u^0(s)\|_{-1}^2\,\mathrm{d}s \\
&\quad \leqslant R_\rho \mathrm{e}^{\alpha_\rho t}\big(\|u^\varepsilon(0) - u^0(0)\|_{-1}^2 + \varepsilon^2\big), \quad t > 0,
\end{aligned} \tag{7.10}$$

where R_ρ and α_ρ are positive constants depending on ρ and independent of ε.

Proof. Multiplying (7.4) by $t\,\partial_t w^\varepsilon$, we find

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(t\|\nabla w^\varepsilon\|^2\big) + t\|\partial_t w^\varepsilon\|_{-1}^2 \\
&\quad = \frac{1}{2}\|\nabla w^\varepsilon\|^2 - t\big(l_\varepsilon(t)w^\varepsilon, \partial_t w^\varepsilon\big) - \varepsilon t\big(u^\varepsilon, \partial_t w^\varepsilon\big) - \varepsilon t\big(A^{-1}f(u^\varepsilon), \partial_t w^\varepsilon\big) \\
&\quad \leqslant \frac{1}{2}\|\nabla w^\varepsilon\|^2 + t\|\nabla\big(l_\varepsilon(t)w^\varepsilon\big)\|\|\partial_t w^\varepsilon\|_{-1} + \varepsilon t\|\nabla u^\varepsilon\|\|\partial_t w^\varepsilon\|_{-1} + \varepsilon t\|f(u^\varepsilon)\|_{-1}\|\partial_t w^\varepsilon\|_{-1} \\
&\quad \leqslant \frac{1}{2}\|\nabla w^\varepsilon\|^2 + c_\rho t\|\nabla w^\varepsilon\|^2 + \frac{t}{2}\|\partial_t w^\varepsilon\|_{-1}^2 + c_\rho \varepsilon^2 t\big(\|\nabla u^\varepsilon\|^2 + 1\big).
\end{aligned}$$

Applying Gronwall's lemma, using estimates (7.7), (7.9) and Theorem 4.1, we deduce estimate (7.10). □

Theorem 7.5. *Let* $\|u^0(0)\|_{H^2(\Omega)}$ *and* $\|u^\varepsilon(0)\|_{H^2(\Omega)} \leqslant \rho$. *We have the following*:

$$t\|u^\varepsilon(t) - u^0(t)\|^2_{H^2(\Omega)} + t\|\partial_t u^\varepsilon(t) - \partial_t u^0(t)\|^2_{-1} \leqslant R_\rho \mathrm{e}^{\alpha_\rho t}\left(\|u^\varepsilon(0) - u^0(0)\|^2_{-1} + \varepsilon^2\right), \quad t \geqslant 0, \tag{7.11}$$

where R_ρ *and* α_ρ *are positive constants depending on* ρ *and independent of* ε.

Proof. Differentiating (7.4) with respect to t and setting $\theta^\varepsilon = \partial_t w^\varepsilon$, we obtain

$$A^{-1}\,\partial_t\theta^\varepsilon + A\theta^\varepsilon + l_\varepsilon(t)\theta^\varepsilon + \partial_t l_\varepsilon(t) w^\varepsilon + \varepsilon\,\partial_t u^\varepsilon + \varepsilon A^{-1} f'\big(u^\varepsilon(t)\big)\,\partial_t u^\varepsilon = 0. \tag{7.12}$$

Multiplying (7.12) by $t\theta^\varepsilon$, using the fact that $l_\varepsilon(t) \geqslant -\kappa$, $\|\partial_t l_\varepsilon(t) w^\varepsilon\| \leqslant c_\rho\|\nabla w^\varepsilon\|$ and $\|f'(u^\varepsilon)\|_{L^\infty(\Omega)} \leqslant c_\rho$, we find

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(t\|\theta^\varepsilon\|^2_{-1}\right) + t\|\nabla\theta^\varepsilon\|^2 \\
&\quad\leqslant \frac{1}{2}\|\theta^\varepsilon\|^2_{-1} + \kappa t\|\theta^\varepsilon\|^2 - t\big(\partial_t l_\varepsilon(t) w^\varepsilon, \theta^\varepsilon\big) - \varepsilon t\big(\partial_t u^\varepsilon, \theta^\varepsilon\big) - \varepsilon t\big(A^{-1} f'\big(u^\varepsilon\big)\,\partial_t u^\varepsilon, \theta^\varepsilon\big) \\
&\quad\leqslant \frac{1}{2}\|\theta^\varepsilon\|^2_{-1} + ct\|\theta^\varepsilon\|_{-1}\|\nabla\theta^\varepsilon\| + t\|\partial_t l_\varepsilon(t) w^\varepsilon\|\|\theta^\varepsilon\| + \varepsilon t\|\partial_t u^\varepsilon\|_{-1}\|\nabla\theta^\varepsilon\| \\
&\qquad + \varepsilon t\|f'\big(u^\varepsilon\big)\,\partial_t u^\varepsilon\|_{-1}\|\theta^\varepsilon\|_{-1} \\
&\quad\leqslant c_\rho(t+1)\|\theta^\varepsilon\|^2_{-1} + c_\rho t\left(\|\nabla w^\varepsilon\|^2 + \varepsilon^2\|\partial_t u^\varepsilon\|^2_{-1}\right) + \frac{t}{2}\|\nabla\theta^\varepsilon\|^2.
\end{aligned}$$

Using estimates (4.10) and (7.10), we deduce that

$$t\|\partial_t u^\varepsilon(t) - \partial_t u^0(t)\|^2_{-1} \leqslant R_\rho \mathrm{e}^{\alpha_\rho t}\left(\|u^\varepsilon(0) - u^0(0)\|^2_{-1} + \varepsilon^2\right), \quad t \geqslant 0. \tag{7.13}$$

Interpreting Eq. (7.4) as an elliptic equation,

$$A w^\varepsilon(t) + l_\varepsilon(t) w^\varepsilon(t) = -A^{-1}\,\partial_t w^\varepsilon(t) - \varepsilon u^\varepsilon(t) - \varepsilon A^{-1} f\big(u^\varepsilon(t)\big) \tag{7.14}$$

and multiplying (7.14) by $Aw^\varepsilon(t)$, we obtain

$$\begin{aligned}
\|Aw^\varepsilon(t)\|^2 &= -\big(l_\varepsilon(t) w^\varepsilon(t), Aw^\varepsilon(t)\big) - \big(A^{-1}\,\partial_t w^\varepsilon(t), Aw^\varepsilon(t)\big) - \big(\varepsilon u^\varepsilon(t), Aw^\varepsilon(t)\big) \\
&\quad - \big(\varepsilon A^{-1} f\big(u^\varepsilon(t)\big), Aw^\varepsilon(t)\big) \\
&\leqslant c_\rho\|\nabla w^\varepsilon(t)\|\|Aw^\varepsilon(t)\| + \|\partial_t w^\varepsilon(t)\|_{-1}\|\nabla w^\varepsilon(t)\| + \varepsilon\|u^\varepsilon(t)\|\|Aw^\varepsilon(t)\| \\
&\quad + \varepsilon\|f\big(u^\varepsilon(t)\big)\|_{-1}\|\nabla w^\varepsilon(t)\|,
\end{aligned}$$

hence

$$\big\|Aw^\varepsilon(t)\big\|^2 \leqslant c_\rho\big(\big\|\partial_t w^\varepsilon(t)\big\|_{-1}^2 + \big\|\nabla w^\varepsilon(t)\big\|^2 + \varepsilon^2\big). \tag{7.15}$$

Using (7.13) and (7.15), we have the desired inequality. □

Theorem 7.6. *Let the above assumptions hold. Then, for every $\varepsilon \geqslant 0$, the semigroup $S_\varepsilon(t)$ generated by equation* (2.1) *possesses an exponential attractor $\mathcal{M}_\varepsilon \subset W$. Moreover, these exponential attractors can be chosen such that*

$$\dim_F(\mathcal{M}_\varepsilon, W) \leqslant C, \tag{7.16}$$

$$\mathrm{dist}_{\mathrm{sym},W}(\mathcal{M}_\varepsilon, \mathcal{M}_0) \leqslant C'\varepsilon^\eta, \tag{7.17}$$

where the constants $C, C' > 0$ and $0 < \eta < 1$ are independent of ε and can be calculated explicitly and where the rate of convergence to these attractors is also uniform with respect to ε, i.e., there exists a constant $\alpha > 0$ such that, for every bounded subset $B_0 \subset W$, there exists a constant $C'' = C''(B_0)$ such that

$$\mathrm{dist}_W\big(S_\varepsilon(t)B_0, \mathcal{M}_\varepsilon\big) \leqslant C''\mathrm{e}^{-\alpha t},$$

where C'' and α are also independent of ε.

Proof. We recall that the set

$$B = \big\{u_0 \in W, \|\Delta u_0\| \leqslant \rho\big\}$$

is a uniformly absorbing set for the semigroup $S_\varepsilon(t)$, $\varepsilon \in [0,1]$, i.e., for every bounded set $B_0 \subset H^2(\Omega)$, there exists a time T which is independent of ε, such that

$$S_\varepsilon(t)B_0 \subset B \quad \forall t \geqslant T,\ \forall \varepsilon \in [0,1].$$

Therefore, it is sufficient to construct exponential attractors $\mathcal{M}_\varepsilon$ on B only. We define a family of maps $S_\varepsilon := S_\varepsilon(t_2) : B \to B$, for $\varepsilon \in [0,1]$ and then construct exponential attractors $\mathcal{M}_\varepsilon^d$ for the discrete semigroups generated by these maps. Moreover, it is convenient to construct these attractors on B endowed first with the metric of the space $H^{-1}(\Omega)$. Then, we apply Theorem 7.1 with $E := H^{-1}(\Omega)$ and $E_1 := H^2(\Omega)$. Estimate (7.1) is an immediate consequence of Theorem 6.5 and estimate (7.2) is a corollary of Theorem 7.2. Thus, all assumptions of Theorem 7.1 are satisfied and, consequently, we have

$$\dim_F\big(\mathcal{M}_\varepsilon^d, H^{-1}(\Omega)\big) \leqslant C,$$

$$\mathrm{dist}_{\mathrm{sym},H^{-1}(\Omega)}\big(\mathcal{M}_\varepsilon^d, \mathcal{M}_0^d\big) \leqslant C'\varepsilon^\nu,$$

$$\mathrm{dist}_{H^{-1}(\Omega)}\big(S_\varepsilon^n B, \mathcal{M}_\varepsilon^d\big) \leqslant C''\mathrm{e}^{-\alpha n},$$

for appropriate constants C, C', C'', α and ν which are independent of ε. Finally, we set

$$\mathcal{M}_\varepsilon^c := \bigcup_{t \in [0,t_2]} S_\varepsilon(t)\mathcal{M}_\varepsilon^d.$$

Since, $(t,x) \to S_\varepsilon(t)x$ is uniformly Hölder continuous (Lemma 6.6) on $[0,t_2] \times B$, the exponential attractors $\mathcal{M}_\varepsilon^c$, $\varepsilon \in [0,1]$, satisfy

(1) $$\dim_F\big(\mathcal{M}_\varepsilon^c, H^{-1}(\Omega)\big) \leqslant C + 2, \tag{7.18}$$

(2) $$\operatorname{dist}_{H^{-1}(\Omega)}\big(S_\varepsilon(t)B, \mathcal{M}_\varepsilon^d\big) \leqslant C\mathrm{e}^{-\alpha t/t_2}, \tag{7.19}$$

(3) $$\operatorname{dist}_{\mathrm{sym},H^{-1}(\Omega)}\big(\mathcal{M}_\varepsilon^c, \mathcal{M}_0^c\big) \leqslant C\big(\operatorname{dist}_{\mathrm{sym},H^{-1}(\Omega)}\big(\mathcal{M}_\varepsilon^d, \mathcal{M}_0^d\big) + \varepsilon\big) \leqslant c_4' \varepsilon^{c_5'}. \tag{7.20}$$

Thus, we have constructed exponential attractors $\mathcal{M}_\varepsilon^c$ for the metric of $H^{-1}(\Omega)$. We note that it follows from its definition that the set $S_\varepsilon(1)\mathcal{M}_\varepsilon^c$ is also an exponential attractor. We finally set

$$\mathcal{M}_\varepsilon := S_\varepsilon(1)\mathcal{M}_\varepsilon^c.$$

Thus, we deduce estimates (7.16) and (7.17) from estimates (7.18), (7.20) and from Lemma 6.2 and Theorem 7.5 for $t = 1$. □

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