Fast and Stable Schemes for Phase Fields Models

M. Brachet* J.-P. Chehab†

July 24, 2017

Abstract

Contents

*Institut Elie Cartan de Lorraine, Université de Lorraine, Site de Metz, Bât. A Ile du Saucy, F-57045 Metz Cedex 1, matthieu.brachet@math.univ-metz.fr

†Laboratoire Amienois de Mathématiques Fondamentales et Appliquées (LAMFA), UMR 7352, Université de Picardie Jules Verne, 33 rue Saint Leu, 80039 Amiens France , (Jean-Paul.Chehab@u-picardie.fr).

1 Introduction

Phase fields equations, such as Allen-Cahn's or Cahn-Hilliard's, play an important role in applications since they allow to model natural phenomena; let us cite [1, 2, 16, 26] in material science, [5, 6, 7, 18, 20, 22, 23] in image processing, [22, 25] in chemistry or [10, 19] in ecology and in medecine, just to cite but a few, the list being non-exhaustive. There is also a important interest fort these models in the mathematical analysis point of view, see [14, 15]. The simulation of Phase fields models is then an important issue.

The numerical integration of such reaction-diffusion equations can be a delicate task: it needs to recover at the discrete level intrinsic properties of the solution (Energy diminishing, maximum principle) and the presence of small parameter (typically, the interphase length) can generate practical difficulties in the iterations processes with a hard time step restriction, even for fully-implicit schemes ; this is due on the way the fixed points problems are solved at each iteration.

The construction of a robust (stable) and efficient (fast) scheme lies on the balance between the advantages and the drawbacks of implicit (stable but costly) and of Explicit (fast but with often stability condition) times-marching schemes. For instance, the simple Forward Euler's, can be used only for small time steps; this restriction can be very important, e.g., when considering heat-equation the basic linear part of reaction-diffusion equations. This restriction allows to prevent the expansion of high mode components, the ones that lead to the divergence of the scheme. A way to enhance the stability region is to introduce an approximation to an unconditionally stable Scheme. Consider, e.g., Backward Euler's applied to the discretized Heat equation:

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + Au^{(k+1)} = 0, \tag{1}$$

where A is the stiffness matrix, $\Delta t > 0$, the time step; here $u^{(k)}$ is the approximation of the solution at time $t = k\Delta t$ in the spatial approximation space. To simplify the linear system that must be solved at each step, one replaces $Au^{(k+1)}$ by $\tau B(u^{(k+1)} - u^{(k)}) + Au^{(k)}$, where $\tau \geq 0$

and where B is a pre-conditioner of A. This leads to the so-called RSS scheme

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \tau B(u^{(k+1)} - u^{(k)}) + Au^{(k)} = 0. \tag{2}$$

This stabilization procedure, also called RSS scheme (Residual Smoothing Scheme), was introduced independently by [3] and [11] (in the multilevel case), see also [8] for recent developments. It allows to take large time steps while simplifying the linear problem to solve at each step: in that way the stability is enhanced and at the same a save of computation time can be obtained as respect to the classical backward Euler's scheme. Of course the stabilization procedure can be applied to a large variety of schemes that are used for reaction diffusion equations:

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \tau B(u^{(k+1)} - u^{(k)}) + Au^{(k)} + f(u^{(k)}) = 0, \tag{3}$$

that corresponds to stabilized semi-implicit Euler scheme for, e.g., Allen-Cahn equations, and

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \tau B(\mu^{(k+1)} - \mu^{(k)}) + A\mu^{(k)} = 0, \tag{4}$$

$$\mu^{(k+1)} = \tau B(u^{(k+1)} - u^{(k)}) + Au^{(k)} + f(u^{(k)}) = 0, \tag{5}$$

which can be considered for high order or coupled problems such as Cahn-Hillard's. It must be noticied that this stabilzation procedure allows to recover the same steady states as teh original scheme, this s an important property when considering, e.g., inpainting or image segmentation problems.

The aim of this article is to propose and analyse fast finite differences schemes for phase fields models such as Allen Cahn's or Cahn-Hilliard's equations, when the space discretization is realized with compact schemes. The new methods combine high order compact finite difference scheme for the discretization in space together with a stabilization of explicit time schemes implemented by using low coast pre-conditioners of the linear term.

The article is organized as follows: in Section 2 we recall the principle of the stabilization (RRS- scheme) and derive stability results for a number of time schemes that will be used in the non linear case. After that, in Section 3, then in Section 4 ,we introduce and study new stabilized schemes for Allen-Cahn's (then Cahn-Hillard's) equation, we give in particular conditions to obtain energy diminishing schemes. In Section 5 we present numerical illustrations on pattern dynamics, image segmentation and inpainting.

2 Stabilized schemes in the linear case

We first give here stability results for stabilized-Schemes derived from time marching method in the linear case; these schemes will be used to build new methods for solving nonlinear time dependent problems, as presented in Sections 3 and 4.

2.1 Explicit Schemes and stabilization

We consider the Heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0, \quad x \in \Omega,\ t > 0, \tag{6}$$

$$u = 0 \quad x \in \partial\Omega, , t > 0, \tag{7}$$

$$u(x, 0) = u_0(x), \qquad x \in \Omega. \tag{8}$$

They are several way to express the stability of a scheme, depending on the norm one considers to measure the boundness of the sequence of time approximations of the solution. We will focus on two following stability notions

- Stability in Energy (a consequence of energy time-diminishing):

$$\sum_{|\alpha|=0}^{m} \aleph_\alpha \|D^\alpha u(t)\|^2_{L^2\Omega)} \leq \sum_{|\alpha|=0}^{m} \aleph_\alpha \|D^\alpha u(t')\|^2_{L^2\Omega},\ \forall t > t'.$$

  with $\aleph_\alpha \geq 0,\ \sum \aleph_\alpha > 0.$

- L^∞ Stability (a consequence of the maximum principle):

$$\exists L > 0 / \|u(t)\|_{L^\infty} \leq L, \forall t \geq 0.$$

The space discretization of (6) leads to the differential system

$$\frac{du}{dt} + Au = 0, \quad x \in \Omega,\ t > 0, \tag{9}$$

$$u(0) = u_0, \tag{10}$$

Here A is the stiffness matrix. The time numerical integration of (9) produces a sequence of vectors $u^k \simeq u(k\Delta t)$, and we define the stability of the schemes as

- Stability in Energy (discrete Energy diminishing), e.g.,

$$\frac{1}{2} < Au^{k+1}, u^{k+1} > \leq \frac{1}{2} < Au^k, u^k >, \forall k$$

- L^∞ Stability

$$\exists L > 0, / \max_i |u_i^k| \leq L, \forall k.$$

Remark 2.1 *These notions of stability will be used also in the nonlinear case, especially for Allen-Cahn's equation.*

Let us first recall a simple but useful result, [8]. Let A and B be two $n \times n$ symmetric positive definite matrices. We assume that there exist two strictly positive constant α and β such that

$$\alpha < Bu, u > \leq < Au, u > \leq \beta < Bu, u >,\ \forall u \in I\!R^n \tag{11}$$

The simplest stabilized scheme is obtained from Backward Euler's as

$$\frac{u^{(k+1)}-u^{(k)}}{\Delta t}+\tau B(u^{(k+1)}-u^{(k)})+Au^{(k)}=0, \qquad (12)$$

where $\tau \geq 0$; Forward Euler is recovered for $\tau=0$ while Backward Euler's is obtained for $\tau=1$ and $B=A$. We have the stability conditions, see [8],

Proposition 2.2 *Assume that A and B are two SPD matrices. Under hypothesis (11), we have the following stability conditions:*

- *If $\tau \geq \frac{\beta}{2}$, the schemes (18) and (21) are unconditionally stable (i.e. stable $\forall\ \Delta t>0$),*
- *If $\tau < \frac{\beta}{2}$, then the scheme is stable for $0<\Delta t<\dfrac{2}{\left(1-\frac{2\tau}{\beta}\right)\rho(A)}$.*

We have considered here the simple Backward scheme to build the new method. Of course we can consider second order schemes such as Gear's or Crank Nicolson's and apply the same stabilization procedure. We can establish the following result:

Proposition 2.3 *Consider the stabilized scheme derived from Gear's method*

$$\frac{1}{2\Delta t}(3u^{(k+1)}-4u^{(k)}+u^{(k-1)})+\tau B(u^{(k+1)}-u^{(k)})+Au^{k}=0$$

We have the following stability conditions

- *If $\tau \geq \frac{\beta}{2}$, then (1) is unconditionally stable*
- *If $\tau < \frac{\beta}{2}$, then (1) is stable when*

$$0<\Delta t<\frac{2}{\rho(A)(1-\frac{2\tau}{\beta})}$$

Proof. We start from the identity

$$\begin{aligned}< 3u^{(k+1)}-4u^{(k)}+u^{(k-1)}, u^{(k+1)}-u^{(k)} > \quad &= 2\|u^{(k+1)}-u^{(k)}\|^2+\tfrac{1}{2}(\|u^{(k+1)}-u^{(k)}\|^2\\ &-\|u^{(k)}-u^{(k-1)}\|^2+\|u^{(k+1)}-2u^{(k)}+u^{(k-1)}\|^2)\end{aligned}$$

We now take the scalar product of each term of (1) with $u^{(k+1)}-u^{(k)}$ and obtain

$$\begin{aligned}&2\|u^{(k+1)}-u^{(k)}\|^2+\tfrac{1}{2}(\|u^{(k+1)}-u^{(k)}\|^2-\|u^{(k)}-u^{(k-1)}\|^2+\|u^{(k+1)}-2u^{(k)}+u^{(k-1)}\|^2)\\ &+2\Delta t\left(\tau< B(u^{(k+1)}-u^{(k)}), u^{(k+1)}-u^{(k)} > + < Au^{(k)}, u^{(k+1)}-u^{(k)} >\right)=0\end{aligned}$$

Using the parallelogram identity on the last term, we find

$$\begin{aligned}&2\|u^{(k+1)}-u^{(k)}\|^2+\tfrac{1}{2}(\|u^{(k+1)}-u^{(k)}\|^2-\|u^{(k)}-u^{(k-1)}\|^2+\|u^{(k+1)}-2u^{(k)}+u^{(k-1)}\|^2)\\ &+2\Delta t\left(\tau< B(u^{(k+1)}-u^{(k)}), u^{(k+1)}-u^{(k)} >\right.\\ &+\tfrac{1}{2}(< Au^{(k+1)}, u^{(k+1)} > - < Au^{(k)}, u^{(k)} > - < A(u^{(k+1)}-u^{(k)}), u^{(k+1)}-u^{(k)} >=0\end{aligned}$$

Now, we let $E^{k+1} = \frac{1}{2} < Au^{(k+1)}, u^{(k+1)} > + \frac{1}{2}\|u^{(k+1)} - u^{(k)}\|^2$ and we obtain,

$$
\begin{aligned}
&2\|u^{(k+1)} - u^{(k)}\|^2 + \frac{1}{2}\|u^{(k+1)} - 2u^{(k)} + u^{(k-1)}\|^2 + 2\Delta t(E^{k+1} - E^k) \\
&+2\Delta t(\tau < B(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} > -\frac{1}{2} < A(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} >)
\end{aligned}
$$

The stability is obtained when $E^{k+1} < E^k$, hence the conditions. The classical Gear's method is obtained taking $\tau = 0$. ■

The stabilized Gear's scheme can be implemented as follows:

Algorithm 1 : Stabilized-Gear

for $k = 0, 1, \cdots$ until convergence **do**

  Solve $(3.Id + \tau 2\Delta t B)\delta = -2\Delta t(u^{(k)} - u^{(k-1)} + Au^{(k)})$

  Set $u^{(k+1)} = u^{(k)} + \delta$

end for

Consider now the stabilzation of Crank-Nicolson scheme:

$$
\begin{aligned}
&\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \frac{1}{2}(Au^{(k+1)} + Au^{(k)}) = 0, \qquad (13) \\
& \qquad (14)
\end{aligned}
$$

This scheme can be rewritten as

$$
\begin{aligned}
&\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \frac{1}{2}(Au^{(k+1)} - Au^{(k)}) = -Au^{(k)}, \qquad (15) \\
& \qquad (16)
\end{aligned}
$$

and implemented following the two steps:

- Solve $(Id + \frac{\Delta t}{2}A)\delta = -\Delta t Au^{(k)}$
- Set $u^{(k+1)} = u^{(k)} + \delta$

The stabilization scheme consists in replacing the implicit part by a simplified one using a preconditioning matrix B of A, namely

- Solve $(Id + \tau\frac{\Delta t}{2}B)\delta = -\Delta t Au^{(k)}$
- Set $u^{(k+1)} = u^{(k)} + \delta$

The implementation of the stabilzed-simplified CN scheme reads as

Algorithm 2 : Stabilized Crank-Nicolson

for $k = 0, 1, \cdots$ until convergence **do**

  Solve $(Id + \frac{\tau\Delta t}{2}B)\delta = -\Delta t Au^{(k)}$

  Set $u^{(k+1)} = u^{(k)} + \delta$

end for

This scheme is consistent with the classical CN one since, for $\tau = 1$ and $B = A$ the original method is recovered. We can easily derive the following stability result for this scheme:

Proposition 2.4 *Assume that A and B are two SPD matrices. Under hypothesis (11), we have the following stability conditions:*

- *If $\tau \geq \beta$, the schemes (18) and (21) are unconditionally stable (i.e. stable $\forall\ \Delta t > 0$)*
- *If $\tau < \beta$, then the scheme is stable for $0 < \Delta t < \dfrac{2}{\left(1 - \frac{\tau}{\beta}\right)\rho(A)}$.*

Proof. It suffices to replace τ by $\frac{\tau}{2}$ in Proposition (2.2). ■

Another important stability property, crucial in a number of applications, is the L^∞ stability garanted by a maximum principle. We have the

Proposition 2.5 *Assume that $f \geq 0$ and $u^{(0)} \geq 0$. Assume in addition that A and B are M-matrices. Set $D = -A$ and $I = \{i \in \{1, \cdots, n\}/D_{ii} < 0\}$. If*

$$D_{i,j} \geq 0, \forall i, j = 1, \cdots, N, i \neq j \text{ and } 0 < \Delta t < \frac{1}{\max_{i\in I} \mid D_{ii} \mid},\ i \in I,$$

then $u^{(k)} \geq 0, \forall k$.

Proof. Let k be fixed and assume that $u^{(k)} \geq 0$. We have

$$(Id + \tau\Delta tB)u^{(k+1)} = (Id + \Delta t(\tau B - A))u^{(k)} + \delta tf$$

The matrix $Id + \tau\Delta tB$ is a M-matrix since B is also one, as a direct consequence. Hence, a sufficient condition to have $u^{(k+1)} \geq 0$ is $(Id + \Delta t(\tau B - A))u^{(k)} \geq 0$, thist is garanted when the matrix $R = Id + \Delta t(\tau B - A) = Id + \Delta tD$ has all positive entries, say

$$\begin{cases} 1 + \Delta t(\tau B_{ii} - A_{ii}) \geq 0 & i = 1, \cdots, n \\ \\ \tau B_{ij} - A_{ij} \geq 0 & i = 1, \cdots, n\, i \neq j \end{cases}$$

Hence the result by a simple induction. ■

TRY TO RECOVER CLASSICAL RESULTS FOR PARTICULAR CHOICES OF A, B AND τ ($A = B$: Euler's ($\tau = 1$, Crank Nicolson's $\tau = 1/2$)...)

2.2 ADI Stabilized Scheme

A important issue for a fast simulation of parabolic equations is the us of splitting methods. We give here stabilized versions of classical ADI schemes. Consider the linear differential system

$$\frac{dU}{dt} + AU = 0$$

with $A = A_1 + A_2$. Let B_1 and B_2 be pre-conditioners of A_1 and A_2 respectively and $\tau_1,\ \tau_2$ two positive real numbers. All the matrices are supposed to be symmetric definite positive. We introduce the stabilized ADI-schemes

$$\frac{u^{(k+1/2)} - u^{(k)}}{\Delta t} + \tau_1 B_1(u^{(k+1/2)} - u^{(k)}) = -A_1 u^{(k)}, \tag{17}$$

$$\frac{u^{(k+1)} - u^{(k+1/2)}}{\Delta t} + \tau_2 B_2(u^{(k+1)} - u^{(k+1/2)}) = -A_2 u^{(k+1/2)}, \tag{18}$$

and the Strang's Splitting

$$\frac{u^{(k+1/3)} - u^{(k)}}{\Delta t/2} + \tau_1 B_1(u^{(k+1/3)} - u^{(k)}) = -A_1 u^{(k)}, \tag{19}$$

$$\frac{u^{(k+2/3)} - u^{(k+1/3)}}{\Delta t} + \tau_2 B_2(u^{(k+2/3)} - u^{(k+1/3)}) = -A_2 u^{(k+1/3)}, \tag{20}$$

$$\frac{u^{(k+1)} - u^{(k+2/3)}}{\Delta t/2} + \tau_1 B_1(u^{(k+1)} - u^{(k+2/3)}) = -A_1 u^{(k+2/3)}, \tag{21}$$

considering $A = \sum_{i=1}^{m} A_i$ and $B = \sum_{i=1}^{m} B_i$ and the splitting

$$\frac{u^{(k+i/m)} - u^{(k+(i-1)/m)}}{\Delta t} + \tau_i B_i(u^{(k+i/m)} - u^{(k+(i-1)/m)}) = -A_i u^{(k+(i-1)/m)}, \tag{22}$$

We recall that

As a direct consequence of proposition 2.2, we can prove the following result

Proposition 2.6 *Under hypothesis (11), we have the following stability conditions:*

- *If* $\tau_i \geq \frac{\beta_i}{2}$,$i = 1, 2$ *the scheme (18) is unconditionally stable (i.e. stable* $\forall\ \Delta t > 0$*)*
- *If* $\tau_i < \frac{\beta_i}{2}$, $i = 1, 2$, *then the scheme is stable for*

$$0 < \Delta t < Min(\frac{2}{\left(1 - \frac{2\tau_1}{\beta_1}\right)\rho(A_1)}, \frac{2}{\left(1 - \frac{2\tau_2}{\beta_2}\right)\rho(A_2)}).$$

Proof. It suffices to apply proposition 2.2 to each system. ∎

2.3 Numerical illustrations

Before presenting the stabilized schemes for phase fields models, we give hereafter some numerical illustrations on linear problems when discretized in space by finite differences compact schemes, focusing on Neumann boundary conditions. We propose as in [8] to use a (lower) second order discretization matrix for preconditioning the underlining matrices. We first consider the Elliptic Neumann problem then the Heat equation.

2.3.1 Finite difference Preconditioning for compact schemes and the Neumann problem

We will use finite differences compact schemes for the discretization in space: in two words these schemes are nonlocal (they have an implicit part), and they allows to reach an accuracy comparable to the spectral one; we refer to the classical book of Collatz [12] and to the seminal paper of Lele [21]; the details of the building of Poisson problems with Homogeneous Dirichlet or Neumann boundary conditions can be found also in [8].

The 2D and 3D reaction-diffusion problems we will consider (Allen-Cahn or Cahn Hilliard's equations) are completed with Neumann Boundary Conditions:

$$\alpha u - \Delta u = f \quad \text{in } \Omega =]0,1[^{2,3}, \tag{23}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega, \tag{24}$$

when discretized by fourth order compact schemes. The preconditioning matrix the usual five point scheme finite difference schemes. A random r.h.s `b=1-2*rand(N,1)` is chosen so in order to have the presence of a late number of frequencies. The initial data is $u = 0$, the tolerance parameter 10^{-10}, we took $\alpha = \beta = 1$ but the results are very close for other values of these parameters. The result we report is the maximum number of external iterations to convergence, on 5 independent numerical resolutions, the number of discretization point per direction n is reported as (n). The stiffness matrices are of respective sizes $n^2 \times n^2$ (2D problem) and $n^3 \times n^3$ (3D problem). The preconditioning systems are solved using the cosine FFT.

Problem	# it. (n)	# it. (n))	# it. (n)	# it. (n)	#it. (n)	#it. (n)
Poisson 2D	16 (n=15)	15 (n=31)	14 (n=63)	13 (n=127)	12 (n=255)	(n=511)
Poisson 3D	30 (n=15)	30 (n=31)	(n=63)			

Table 1: Solutions of 2D and 3D Neumann problem with GMRES and second order preconditioner

Of course, due the implicity of the scheme, the fourth order discretization matrix A of $-\Delta$ is nonsymmetric while the preconditioning matrix B is. However, in practice the RSS method is still efficient. This is due to the small size of the skew-symmetric part of A, see [8].

2.3.2 Heat Equation

For the 2D and the 3D heat equation, we here compare different RSS schemes presented above (ADI, Gear's, CN and Euler's) in order to illustrate the stability results and to point out the owl of the stabilization parameter τ.

USE THE MATLAB CODES IN THE DIRECTORY ADI_MATLAB. see program `chaleur_2D_splitting.m`

The error is clearly in Δt.

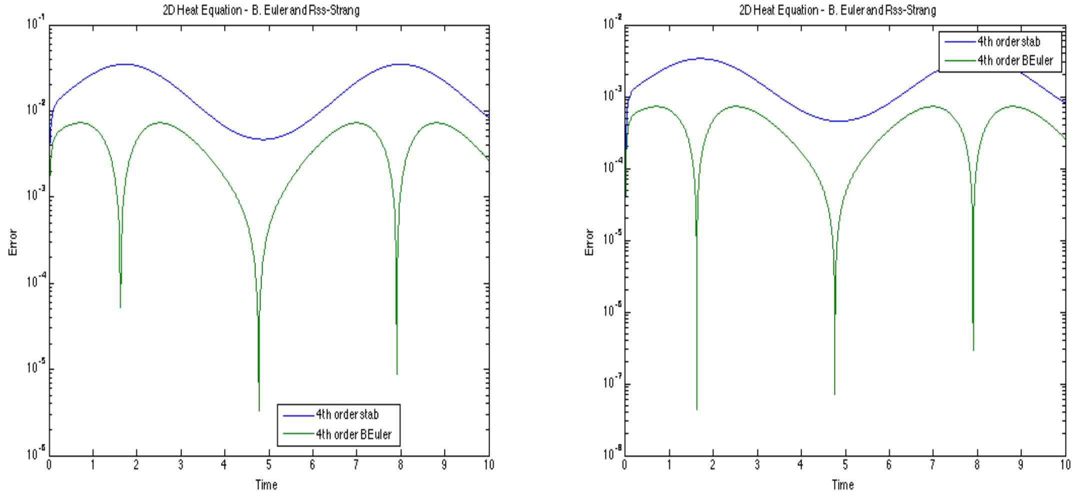

Figure 1: Solution of the heat equation with $\Delta t = 0.01$, (left) and $\Delta t = 0.001$ (right) $n = 31$, $\tau = 1$

We now consider the 3D Heat equation, with Homogeneous Neumann BC.
The error is clearly in Δt.

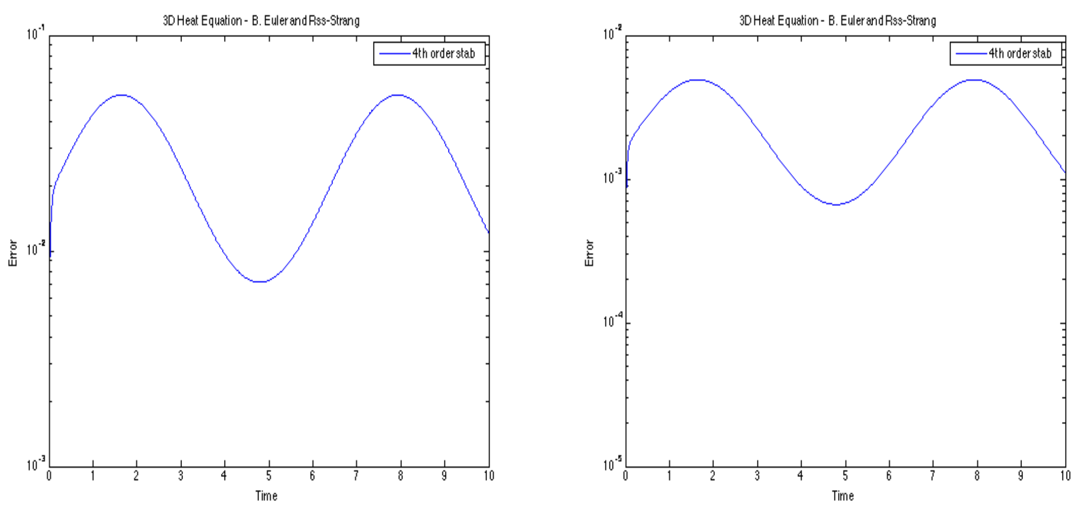

Figure 2: Solution of the 3D heat equation with $\Delta t = 0.01$, (left) and $\Delta t = 0.001$ (right) $n = 31$, $\tau = 1$

- Solution of the Neumann Problem with Preconditioned GMRES `solve Neum 3D CS.m`
- Neumann Problem `test dct neumann.m`
- Heat equation `test chaleur.m`
-

3 Allen-Cahn's equation

Let $\Omega \subset I\!R^n, n = 2, 3$ a regular open bounded set. We here consider the simple Allen-Cahn equation

$$\frac{\partial u}{\partial t} + M(-\Delta u + \frac{1}{\epsilon^2} f(u)) = 0, \quad x \in \Omega, t > 0 \tag{25}$$

$$\frac{\partial u}{\partial n} = 0 \qquad t > 0, \tag{26}$$

$$u(0, x) = u_0(x), \qquad x \in \Omega. \tag{27}$$

which describes the process of phase separation in iron alloys [1, 2], including order-disorder transitions: M is the mobilty (taken to be 1 for simplicity), $F = \displaystyle\int_{-\infty}^{u} f(v)dv$ is the free energy, u is the (non-conserved) order parameter, ϵ is the interface length. The homogenous Neumann boundary condition implies that there is not a loss of mass outside the domain Ω. It is important to note that here is a competition between the potential term and the diffusion term: regularization in phase transition. Two important properties are satisfied by the solution and must be captured by the numerical scheme (intrinsicly or numerically):

- the energy diminishing: Allen-Cahn equation is a gradient flow for the energy

  $$E(u) = \frac{1}{2} \int_{\Omega} \|\nabla u\|^2 dx + \frac{1}{\epsilon^2} F(u) dx,$$

  so $E(u(t)) \leq E(u(t')), \forall t \geq t'$,
- the maximum principle: $|u(., t)|_{L^\infty} \leq L, \forall t > 0$.

3.1 Energy diminishing schemes

We set $E(u) = \frac{1}{2} < Au, u > + \frac{1}{\epsilon^2} < F(u), \mathbf{1} >$, where F is a primitive of f that we choose such that $F(0) = 0$; $\mathbf{1}$ is the vector whose components are all equal to 1. We say that the scheme is energy decreasing if

$$E(u^{(k+1)}) < E(u^{(k)}).$$

We here first recall somme schemes and their stability conditions, see also [8].

The semi-implicit Scheme applied to the pattern evolution A-C equation:

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + Au^{(k+1)} + \frac{1}{\epsilon^2} f(u^{(k)}) = 0 \tag{28}$$

is energy diminishing under the time step restriction

$$0 < \Delta t < \frac{\epsilon^2 L}{2}$$

where $| f' |_\infty \leq L$, see [27].

The stabilization of its linear part is

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \tau B(u^{(k+1)} - u^{(k)}) + \frac{1}{\epsilon^2} f(u^{(k)}) = -Au^{(k)}. \tag{29}$$

We have the stability result, [8]:

Theorem 3.1 *Assume that f is $\mathcal{C}^1$ and $| f' |_\infty \leq L$. We have the following stability conditions*

- *If $\tau \geq \frac{\beta}{2}$ then*
  - *if $\left(\frac{\tau}{\beta} - \frac{1}{2}\right) \lambda_{min} - \frac{L}{2\epsilon^2} \geq 0$ then the scheme is unconditionally stable,*
  - *if $\left(\frac{\tau}{\beta} - \frac{1}{2}\right) \lambda_{min} - \frac{L}{2\epsilon^2} < 0$ then the scheme is stable for*

$$0 < \Delta t < \frac{1}{\frac{L}{2\epsilon^2} - \left(\frac{\tau}{\beta} - \frac{1}{2}\right) \lambda_{min}},$$

- *If $\tau < \frac{\beta}{2}$ then the scheme is stable for*

$$0 < \Delta t < \frac{1}{\frac{L}{2\epsilon^2} - \left(\frac{\tau}{\beta} - \frac{1}{2}\right) \rho(A)}.$$

A more stable way to overcome the stability restiriction is to consider directly AC equation as a gradient system with a natural diminishing energy property. A first unconditionnally stable scheme is ([14, 15])

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + Au^{(k+1)} + \frac{1}{\epsilon^2} DF(u^{(k)}, u^{(k+1)}) = 0, \tag{30}$$

where

$$DF(u, v) = \begin{cases} \dfrac{F(u) - F(v)}{u - v} & \text{if } u \neq v, \\ f(u) & \text{if } u = v. \end{cases}$$

In [8] it was introduced the RSS-scheme

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \tau B(u^{(k+1)} - u^{(k)}) + DF(u^{(k+1)}, u^{(k)}) = -Au^{(k)} \tag{31}$$

which enjoys of the following stability condition, see [8] for the proof.

Proposition 3.2 *Under hypothsesis $\mathcal{H}$*

- *if $\tau \geq \frac{\beta}{2}$, the RSS scheme is unconditionally stable,*

- *if $\tau < \frac{\beta}{2}$, the RSS scheme is stable under condition*

$$0 < \Delta t < \frac{\beta}{\rho(A)(\frac{\beta}{2} - \tau)}.$$

Finally, unconditionally stable scheme is to use the so-called convex splitting, [17, 13]. These schemes are based on a proper splitting of the free energy term

$$F(u) = F_c(u) - F_e(u),$$

where $F_* \in \mathcal{C}^2(I\!R^n, I\!R)$, $* = c$ or $* = e$.

ii. F_* is strictly convex in $I\!R^n$, $* = c$ or $* = e$.

iii. $< [\nabla F_e(u)]u, u > \geq -\lambda$, $\forall u \in I\!R^n$.

The scheme reads as

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + Au^{(k+1)} + \nabla F_c(u^{(k+1)}) = +\nabla F_e(u^{(k)}), \tag{32}$$

Its sabilization reads as

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \tau B(u^{(k+1)} - u^{(k)}) + \nabla F_c(u^{(k+1)}) = -Au^{(k)} + \nabla F_e(u^{(k)}), \tag{33}$$

and we can prove the

Theorem 3.3
- *If $(\tau\beta - \frac{1}{2}\rho(A) + \left(\hat{\lambda} - |\lambda|\right) > 0$ then the scheme is unconditionally stable*
- *Else it is stable under condition*

$$0 < \Delta t < \frac{1}{(\frac{1}{2} - \tau\beta)\rho(A) + |\lambda| - \hat{\lambda}}.$$

3.2 Splitting schemes

We follow [22] who proposed for the Double Well potential case the scheme ($F(u) = \frac{1}{4}(1 - u^2)^2$) the splitting scheme

$$\frac{u^* - u^{(k)}}{\Delta t} + \frac{1}{2}(Au^* + Au^{(k)}) = 0, \tag{34}$$

$$\frac{u^{(k+1)} - u^*}{\Delta t} = \frac{u^{(k+1)} - (u^{(k+1)})^3}{\epsilon^2} \tag{35}$$

The last equation ca be simplified since it correspond to a one-step approximation by backward Euler's to the differential equation

$$\frac{du}{dt} = \frac{u - u^3}{\epsilon^3} \tag{36}$$

whose the solution is

$$u(t) = \frac{u(0)}{\sqrt{e^{-2\frac{t}{\epsilon^2}} + u(0)^2(1 - e^{-2\frac{t}{\epsilon^2}})}}$$

Hence the simplified scheme

$$\frac{u^* - u^{(k)}}{\Delta t} + \frac{1}{2}(Au^* + Au^{(k)}) = 0, \tag{37}$$

$$u^{(k+1)} = \frac{u^*}{\sqrt{e^{-2\frac{\Delta t}{\epsilon^2}} + (u^*)^2(1 - e^{-2\frac{\Delta t}{\epsilon^2}})}} \tag{38}$$

is rather considered and have nice stability properties, see [22].

We get give here a first stability result that will be useful to prove

Theorem 3.4 *Let* $\mathbf{1} = (1, 1, \cdots, 1)^T \in I\!R^N$*. Assume that* $A\mathbf{1} = 0$ *and that* $Id + \Delta t A$ *enjoys of the discrete maximum principle. Assume that* $|u_i^{(0)}| \leq 1, i = 1, \cdots N$*. Then the sequence* $u^{(k)}$ *defined by*

$$\frac{u^* - u^{(k)}}{\Delta t} + Au^* = 0, \tag{39}$$

$$u^{(k+1)} = \frac{u^*}{\sqrt{e^{-2\frac{\Delta t}{\epsilon^2}} + (u^*)^2(1 - e^{-2\frac{\Delta t}{\epsilon^2}})}} \tag{40}$$

satisfies $|u_i^{(k)}| \leq 1, i = 1, \cdots N$

Proof. We proceed by induction. First of all we show that if $|u^{(k)}| \leq 1$ then $|u^{(*)}| \leq 1$. We have

$$(Id + \Delta t A)(u^* - \mathbf{1}) = (u^{(k)} - \mathbf{1})$$

hence, by the maximum principle, if $u^{(k)} - \mathbf{1} \leq 0$ then $u^{(*)} - \mathbf{1} \leq 0$. Replacing $(u^{(k)} - \mathbf{1})$ (resp. $(u^* - \mathbf{1})$) by $(u^{(k)} + \mathbf{1})$ (resp. $(u^* + \mathbf{1})$) we find that $u^{(*)} + \mathbf{1} \geq 0$ and conclude that $-1 \leq u^* \leq 1$. Now, to conclude, it suffices to show that

$$\left|\frac{x}{\sqrt{e^{-2\frac{\Delta t}{\epsilon^2}} + (x)^2(1 - e^{-2\frac{\Delta t}{\epsilon^2}})}}\right| \leq 1 \forall x \in [-1, 1], \forall \Delta t > 0, \forall \epsilon^2 > 0.$$

We set for convenience $\gamma = e^{-2\frac{\Delta t}{\epsilon^2}} \in [0, 1]$. We start from

$$x^2 \leq 1 \iff \gamma x^2 \leq \gamma \iff x^2 \leq (1 - \gamma)x^2 + \gamma \iff \frac{x^2}{(1 - \gamma)x^2 + \gamma} \leq 1$$

The result is obtained by taking the squareroot of this last expression. ■ We can define a stabilized version of this splitting scheme as

$$\frac{u^* - u^{(k)}}{\Delta t} + \tau B(u^* - u^{(k)}) = -Au^{(k)}, \tag{41}$$

$$u^{(k+1)} = \frac{u^*}{\sqrt{e^{-2\frac{\Delta t}{\epsilon^2}} + (u^*)^2(1 - e^{-2\frac{\Delta t}{\epsilon^2}})}} \tag{42}$$

To implement RRS-like version of this splitting scheme it then suffices to replace the first step by a RSS-CN scheme as proposed in section 2. We then obtain the RSS-splitting scheme

Algorithm 3 : RSS splitting dor Allen Cahn

1: **for** $k = 0, 1, \cdots$ **do**
2: **Solve** $(Id + \frac{\tau\Delta t}{2}\epsilon B)\delta = -\Delta t A u^{(k)}$
3: **Set** $u^{(*)} = u^{(k)} + \delta$
4: **Set** $u^{(k+1)} = \dfrac{u^*}{\sqrt{e^{-2\frac{\Delta t}{\epsilon^2}} + (u^*)^2(1 - e^{-2\frac{\Delta t}{\epsilon^2}})}}$
5: **end for**

SEE Program `RSSsplittingAC3DCS.m`.

The proof for the L^∞-stability of the classical θ scheme with a second order FD matrix is given in ([24]) p-33, but is based on a point-wize analysis.

Theorem 3.5 *Let* $\mathbf{1} = (1, 1, \cdots, 1)^T \in I\!R^N$. *Assume that* $A\mathbf{1} = 0$, $B\mathbf{1} = 0$ *and* $(Id + \tau\Delta t B)^{-1}(Id + \Delta t(\tau B - A))$ *enjoys of the discrete maximum principle. Assume that* $|u_i^{(0)}| \leq 1, i = 1, \cdots N$. *Then the sequence* $u^{(k)}$ *defined by (41)-(42) satisfies* $|u_i^{(k)}| \leq 1, i = 1, \cdots N$

Proof. According to the proof of Theorem 3.4, it suffices to show that if $|u^{(k)}| \leq 1$ then $|u^{(*)}| \leq 1$. This is automatically provided by the assumption: $(Id + \tau\Delta t B)^{-1}(Id + \Delta t(\tau B - A))$ statisfies the maximum principe property. ■

4 Cahn-Hilliard equation

We here present briefly Cahn-Hillard (CH) equations used for Phase transition and for image inpainting. We introduce new stabilized schemes and establish stability properties.

4.1 The models

4.1.1 Cahn-Hilliard and Patterns

The CH equation describes the process of phase separation, by which the two components of a binary fluid spontaneously separate and form domains pure in each component. It writes as

$$\frac{\partial u}{\partial t} - \Delta(-\Delta u + \frac{1}{\epsilon^2} f(u)) = 0, \tag{43}$$

$$\frac{\partial u}{\partial n} = 0, \tag{44}$$

$$\frac{\partial}{\partial n}\left(\Delta u - \frac{1}{\epsilon^2} f(u)\right) = 0, \tag{45}$$

$$u(0,x) = u_0(x) \tag{46}$$

This equation enjoys of the following properties

- Conservation of the mass: $\bar{u} = \int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx$
- Decay of the energy in time

$$\frac{\partial E(u)}{\partial t} = -\int_\Omega |\nabla(-\Delta u + \frac{1}{\epsilon^2} f(u))|^2 dx \leq 0$$

A classical way to study and to simulate CH is to decouple the equation as follows:

$$\frac{\partial u}{\partial t} - \Delta\mu = 0, \qquad \text{in } \Omega, t > 0 \tag{47}$$

$$\mu = -\Delta u + \frac{1}{\epsilon^2} f(u), \qquad \text{in } \Omega, tt > 0 \tag{48}$$

$$\frac{\partial u}{\partial n} = 0, \frac{\partial \mu}{\partial n} = 0, \qquad \text{on } \partial\Omega, t > 0 \tag{49}$$

$$u(0,x) = u_0(x) \qquad \text{in } \Omega \tag{50}$$

4.1.2 The inpainting problem

Cahn hilliard equations allow here to in paint a tagged picture. Let g be the original image and $D \subset \Omega$ the region of Ω in which the image is deterred. The idea is to add a penalty term that forces the image to remain unchanged in $\Omega \setminus D$ and to reconnect the fields of g inside D. Let $\lambda >> 1$

$$\frac{\partial u}{\partial t} - \Delta(-\epsilon\Delta u + \frac{1}{\epsilon} f(u)) \quad +\lambda\chi_{\Omega\setminus D}(x)(u-g) = 0, \tag{51}$$

$$\underbrace{\text{Cahn-Hilliard equation}} \qquad \underbrace{\text{Fidelity term}} \tag{52}$$

$$\frac{\partial u}{\partial n} = 0 \quad \frac{\partial}{\partial n}\left(\Delta u - \frac{1}{\epsilon^2} f(u)\right) = 0, \tag{53}$$

$$u(0,x) = u_0(x) \tag{54}$$

Here $\chi_{\Omega \setminus D}(x) = \begin{cases} 1 & \text{if } x \in \Omega \setminus D, \\ 0 & \text{else} \end{cases}$

- The presence of the penalization term $\lambda \chi_{\Omega\setminus D}(x)(u-g)$ forces the solution to be close to g in $\Omega \setminus D$ when $\lambda >> 1$
- The Cahn-Hilliard flow has as effect to connect the fields inside D
- here ϵ will play the role of the "contrast". A post-processing is possible using a thresholding procedure.

4.2 The Stabilized Scheme

The semi-implicit scheme

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + A\mu^{(k+1)} = 0, \tag{55}$$

$$\mu^{(k+1)} = \epsilon A u^{(k+1)} + \frac{1}{\epsilon} f(u^{(k)}), \tag{56}$$

suffers from a hard time step restriction, its energy stability is guaranteed for

$$0 < \Delta t < \epsilon^2$$

see [27] We derive the Stabilized-Scheme from the backward Euler's (55)-(56) by replacing $Az^{(k+1)}$ by $\tau B(z^{(k+1)} - z^{(k)}) + Az^{(k)}$ for $z = u$ or $z = \mu$. We obtain

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \tau B(\mu^{(k+1)} - \mu^{(k)}) + A\mu^{(k)} = 0, \tag{57}$$

$$\mu^{(k+1)} = \epsilon\tau B(u^{(k+1)} - u^{(k)}) + \epsilon A u^{(k)} + \frac{1}{\epsilon} f(u^{(k)}). \tag{58}$$

We remark that this scheme preserves the steady state. We now address a stability analysis.

Theorem 4.1 *We have the following stability conditions in the linear and in the nonlinear case:*

- *Linear case $f \equiv 0$: If $\tau > \beta$, then the scheme (57)-(58) is unconditionally stable.*
- *Nonlinear case: Assume that there exists $\theta \in]0,1[$ such that*
  - $\tau(1-\theta) \geq \beta$
  - $\tau \geq (1+\beta)$

  Then, the stability is obtained when

- $0 < \Delta t < \dfrac{4\epsilon^2}{L^2}$
- $0 < \Delta t < \dfrac{\epsilon(\tau - \beta/2)}{\frac{\tau}{2\theta} + \frac{\beta}{2\eta}}$ *for some* $\eta > 0$

Proof. We first consider the linear case ($f \equiv 0$).

We take the scalar product of (57) with $\mu^{(k+1)}$ and of (58) with $u^{(k+1)} - u^{(k)}$. After the use of the parallelogram identity and usual simplifications, we obtain, on the one hand

$$\begin{aligned}
&< u^{(k+1)} - u^{(k)}, \mu^{(k+1)} > + \frac{\Delta t \tau}{2} \big(< B\mu^{(k+1)}, \mu^{(k+1)} > - < B\mu^{(k)}, \mu^{(k)} > \\
&+ < B(\mu^{(k+1)} - \mu^{(k)}), \mu^{(k+1)} - \mu^{(k)} > \\
&+ \frac{\Delta t}{2} \big(< A\mu^{(k+1)}, \mu^{(k+1)} > + < A\mu^{(k)}, \mu^{(k)} > \\
&- < A(\mu^{(k+1)} - \mu^{(k)}), \mu^{(k+1)} - \mu^{(k)} > = 0,
\end{aligned}$$

and on the other hand

$$\begin{aligned}
< u^{(k+1)} - u^{(k)}, \mu^{(k+1)} > \quad &= \tau\epsilon < B(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} > \\
&+ \frac{1}{2}\epsilon \big(< Au^{(k+1)}, u^{(k+1)} > - < Au^{(k)}, u^{(k)} > - < A(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} > \big)
\end{aligned}$$

Taking the difference of the last two identities, we obtain

$$\begin{aligned}
&\epsilon\{\tau < B(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} > - \frac{1}{2} < A(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} >\} \\
&+ \Delta t\{\frac{\tau}{2} < B(\mu^{(k+1)} - \mu^{(k)}), \mu^{(k+1)} - \mu^{(k)} > - \frac{1}{2} < A(\mu^{(k+1)} - \mu^{(k)}), \mu^{(k+1)} - \mu^{(k)} >\} \\
&+ \frac{\Delta t}{2} \big(\tau < B\mu^{(k)}, \mu^{(k)} > - < A\mu^{(k)}, \mu^{(k)} > \big) \\
&+ \frac{\Delta t}{2} \big(< A\mu^{(k+1)}, \mu^{(k+1)} > + \tau < B\mu^{(k+1)}, \mu^{(k+1)} > \big) + R^{k+1} - R^k = 0,
\end{aligned}$$

where

$$R^{k+1} = \frac{1}{2}\epsilon < Au^{(k+1)}, u^{(k+1)} >$$

The scheme is then stable when $R^{k+1} < R^k$. Now using (11), we obtain

$$\begin{aligned}
&\epsilon\{\tau < B(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} > - \frac{1}{2} < A(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} >\} \\
&+ \Delta t\{\frac{\tau}{2} < B(\mu^{(k+1)} - \mu^{(k)}), \mu^{(k+1)} - \mu^{(k)} > - \frac{1}{2} < A(\mu^{(k+1)} - \mu^{(k)}), \mu^{(k+1)} - \mu^{(k)} >\} \\
&+ \frac{\Delta t}{2} \big(\tau < B\mu^{(k)}, \mu^{(k)} > - < A\mu^{(k)}, \mu^{(k)} > \big) \\
&\geq \\
&\epsilon(\tau - \frac{\beta}{2}) B(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} > \\
&+ \frac{\Delta t}{2}(\tau - \beta) < B(\mu^{(k+1)} - \mu^{(k)}), \mu^{(k+1)} - \mu^{(k)} > \\
&+ \frac{\Delta t}{2}(\tau - \beta) < B\mu^{(k)}, \mu^{(k)} >
\end{aligned}$$

Hence the sufficient stability conditions.

Consider now the general case $f \neq 0$. First of all, as in [27], we take the Taylor expansion of the term $\frac{1}{\epsilon} f(u^k) = \dfrac{1}{\epsilon} < F(u^{(k+1))} - F(u^{(k)}), 1 > + \dfrac{1}{2\epsilon} < f'(\xi^{(k)})(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} >$. Hence, we deduce from the previous inequality

$$\begin{array}{ll} \epsilon(\tau - \frac{\beta}{2})B(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} > & \\ + \frac{\Delta t}{2}(\tau - \beta) < B(\mu^{(k+1)} - \mu^{(k)}), \mu^{(k+1)} - \mu^{(k)} > & \\ + \frac{\Delta t}{2}(\tau - \beta) < B\mu^{(k)}, \mu^{(k)} > & \leq \frac{L}{2\epsilon}\|u^{(k+1)} - u^{(k)}\|^2 \\ + \frac{\Delta t}{2}\left(< A\mu^{(k+1)}, \mu^{(k+1)} > + \tau < B\mu^{(k+1)}, \mu^{(k+1)} >\right) + E^{k+1} - E^k & \end{array}$$

where $L = \|f(u)\|_\infty$ and $E^{k+1} = \frac{1}{2}\epsilon < Au^{(k+1)}, u^{(k+1)} > + \frac{1}{\epsilon} < F(u^{(k+1)} - F(u^{(k)}, 1 >$.

Finally have, taking the scalar product of (57) with $\sqrt{\Delta t}(u^{(k+1)} - u^{(k)})$,

$$\frac{\|u^{(k+1)} - u^{(k)}\|^2}{\sqrt{\Delta t}} \leq \tau\sqrt{\Delta t}| < B(\mu^{(k+1)} - \mu^{(k)}), u^{(k+1)} - u^{(k)} > | + \sqrt{\Delta t}| < A\mu^{(k)}, u^{(k+1)} - u^{(k)} > |,$$

Then, using Young's inequality

$$\begin{array}{rl} \dfrac{\|u^{(k+1)} - u^{(k)}\|^2}{\sqrt{\Delta t}} & \leq \tau\frac{\Delta t}{2}\left(\theta < B(\mu^{(k+1)} - \mu^{(k)}), \mu^{(k+1)} - \mu^{(k)} > + \frac{1}{\theta} < B(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} >\right) \\ & + \frac{\Delta t}{2}\left(\eta < A\mu^{(k)}, \mu^{(k)} > + \frac{1}{\eta} < A(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} >\right) \end{array}$$

We take the sum of these inequalities to get

$$\begin{array}{ll} (\frac{1}{\sqrt{\Delta t}} - \frac{L}{2\epsilon})\|u^{(k+1)} - u^{(k)}\|^2 & \\ (\epsilon(\tau - \frac{\beta}{2}) - \frac{\tau\Delta t}{2\theta} - \beta\frac{\Delta t}{2\eta}) < B(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} > & \\ + \frac{\Delta t}{2}(\tau - \beta - \tau\theta) < B(\mu^{(k+1)} - \mu^{(k)}), \mu^{(k+1)} - \mu^{(k)} > & \\ + \frac{\Delta t}{2}(\tau - \beta - \beta\eta) < B\mu^{(k)}, \mu^{(k)} > & \leq 0 \\ + \frac{\Delta t}{2}\left(< A\mu^{(k+1)}, \mu^{(k+1)} > + \tau < B\mu^{(k+1)}, \mu^{(k+1)} >\right) + E^{k+1} - E^k & \end{array}$$

We can now deduce stability conditions. First of all we assume that

- $\tau(1 - \theta) \geq \beta$
- $\tau \geq (1 + \beta)$

Then, the stability is obtained when each factor is positive, say

- $0 < \Delta t < \dfrac{4\epsilon^2}{L^2}$
- $0 < \Delta t < \dfrac{\epsilon(\tau - \beta/2)}{\frac{\tau}{2\theta} + \frac{\beta}{2\eta}}$

■

Remark 4.2 *Another stability condition can be obtained, infering that* $< Bv, v > \leq \rho(B)\|v\|^2$ *so we use the minoration*

$$\frac{1}{\sqrt{\Delta t}}\|u^{(k+1)} - u^{(k)}\|^2$$
$$+(\epsilon(\tau - \frac{\beta}{2}) - \rho(B)\frac{L}{2\epsilon}) - \frac{\tau\Delta t}{2\theta} - \beta\frac{\Delta t}{2\eta}) < B(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} >$$
$$\leq$$
$$(\frac{1}{\sqrt{\Delta t}} - \frac{L}{2\epsilon})\|u^{(k+1)} - u^{(k)}\|^2$$
$$+(\epsilon(\tau - \frac{\beta}{2}) - \frac{\tau\Delta t}{2\theta} - \beta\frac{\Delta t}{2\eta}) < B(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} >$$

Finally, we get

- $0 < \Delta t < \dfrac{\epsilon(\tau - \beta/2 - \rho(B)\frac{L}{2\epsilon^2})}{\frac{\tau}{2\theta} + \frac{\beta}{2\eta}}$

which is a inconditionnal stability condition satisfied for τ *large enough.*

We can now give other stability results for nonlinear RSS schemes. We first consider

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \tau B(\mu^{(k+1)} - \mu^{(k)}) + A\mu^{(k)} = 0, \tag{59}$$

$$\mu^{(k+1)} = \epsilon\tau B(u^{(k+1)} - u^{(k)}) + \epsilon A u^{(k)} + \frac{1}{\epsilon} DF(u^{(k+1)}, u^{(k)}). \tag{60}$$

where $DF(u^{(k+1)}, u^{(k)})$ is defined in Section 3.2.

Theorem 4.3 *If* $\tau > \beta$, *then the scheme (59)-(60) is unconditionally stable.*

Proof. We proceed exactly as in Theorem 4.1. We obtain after the usual simplifications

$$\epsilon(\tau - \frac{\beta}{2}) < B(u^{(k+1)} - u^{(k)}), u^{(k+1)} - u^{(k)} >$$
$$+\frac{\Delta t}{2}(\tau - \beta) < B(\mu^{(k+1)} - \mu^{(k)}), \mu^{(k+1)} - \mu^{(k)} >$$
$$+\frac{\Delta t}{2}(\tau - \beta) < B\mu^{(k)}, \mu^{(k)} > + \frac{\Delta t}{2}\left(< A\mu^{(k+1)}, \mu^{(k+1)} > + \tau < B\mu^{(k+1)}, \mu^{(k+1)} >\right) + E^{k+1} - E^k \leq 0,$$

where

$$E^{k+1} = \frac{1}{\epsilon} < F(u^{(k+1)}, 1 > + \frac{1}{2}\epsilon < A u^{(k+1)}, u^{(k+1)} >$$

The scheme is then stable when $R^{k+1} < R^k$. Hence the sufficient stability conditions. ∎

Remark 4.4 *Stabilization of semi-implicit scheme for Cahn-Hilliard equations have been considered, e.g. in [6, 7, 18] for inpainting problems,*

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + A\mu^{(k+1)} + c_1 A(u^{(k+1)} - u^{(k)}) + c_2(u^{(k+1)} - u^{(k)}) + \lambda_0 D(u^{(k)} - g) = 0, \tag{61}$$

$$\mu^{(k+1)} = \epsilon A u^{(k+1)} + \frac{1}{\epsilon} f(u^{(k)}). \tag{62}$$

Here c_1 *and* c_2 *are positive constants, they play the role of stabilization parameters. Large values of* c_1 *and* c_2 *allow to take large time step, however it deteriores the dynamics. Our approach is here different.*

We now describe the practical solution. We can write

$$\begin{pmatrix} Id & \tau\Delta t B \\ -\epsilon\tau B & Id \end{pmatrix}\begin{pmatrix} u^{(k+1)} - u^{(k)} \\ \mu^{(k+1)} \end{pmatrix} = \begin{pmatrix} -\Delta t A u^{(k)} \\ \epsilon A u^{(k)} + \frac{1}{\epsilon} f(u^{(k)}) \end{pmatrix}$$

The matrix of the system can be factorized as Block LU

$$M = \begin{pmatrix} Id & \tau\Delta t B \\ -\epsilon\tau B & Id \end{pmatrix} = \begin{pmatrix} Id & 0 \\ -\epsilon\tau B & Id \end{pmatrix}\begin{pmatrix} Id & \tau\Delta t B \\ 0 & S \end{pmatrix}$$

where $S = Id + \tau^2\Delta t\epsilon B^2$ is the Schur complement. We have to solve the coupled linear system

$$\begin{cases} X_1 + \tau\Delta t B X_2 = F1, \\ -\tau\epsilon B X_1 + X_2 = F_2. \end{cases}$$

Hence

$$(Id + \tau^2\Delta t\epsilon B^2)X_2 = F_2 + \epsilon\tau B F1$$

Then,

$$X_1 = F_1 - \tau\Delta t B X_2$$

We can resume the solution as

Algorithm 4 : RSS Cahn-Hilliard

for $k = 0, 1, \cdots$ until convergence **do**
  Set $F_1 = -\Delta t A\mu^{(k)}$ and $F_2 = -\mu^{(k)} + \epsilon A u^{(k)} + \frac{1}{\epsilon} f(u^{(k)})$
  Solve $(Id + \tau^2\Delta t\epsilon B^2)\delta\mu = F_2 + \tau\epsilon B F_1$
  Set $\mu^{(k+1)} = \mu^{(k)} + \delta\mu$
  Set $u^{(k+1)} = u^{(k)} - \tau\Delta t B\delta\mu$
end for

When considering the inpainting model, the RSS scheme can be written as

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \tau B(\mu^{(k+1)} - \mu^{(k)}) + A\mu^{(k)} + \lambda_0 D(u^{(k+1)} - g) = 0, \tag{63}$$

$$\mu^{(k+1)} = \epsilon\tau B(u^{(k+1)} - u^{(k)}) + \epsilon A u^{(k)} + \frac{1}{\epsilon} f(u^{(k)}). \tag{64}$$

say in the matricidal form

$$\begin{pmatrix} Id + \delta t\lambda_0 D & \tau\Delta t B \\ -\epsilon\tau B & Id \end{pmatrix}\begin{pmatrix} u^{(k+1)} - u^{(k)} \\ \mu^{(k+1)} - \mu^{(k)} \end{pmatrix} = \begin{pmatrix} \Delta t(\lambda_0 D(g - u^{(k)}) - A u^{(k)}) \\ \epsilon A u^{(k)} + \frac{1}{\epsilon} f(u^{(k)}) - \mu^{(k)} \end{pmatrix}$$

The implementation of the scheme reads as

Algorithm 5 : RSS Cahn-Hilliard for implainting

for $k = 0, 1, \cdots$until convergence **do**
  Set $F_1 = \Delta t(\lambda_0 D(g - u^{(k)}) - Au^{(k)})$
  Set $F_2 = -\mu^{(k)} + \epsilon Au^{(k)} + \frac{1}{\epsilon} f(u^{(k)})$
  Solve $(Id + \lambda_0 D + \tau^2 \Delta t \epsilon B^2)\delta = F_1 - \tau \Delta t B F_2$
  Set $\delta\mu = F_2 + \epsilon\tau B\delta$
  Set $u^{(k+1)} = u^{(k)} + \delta$
  Set $\mu^{(k+1)} = \mu^{(k)} + \delta\mu$
end for

5 Numerical Results

5.1 Implementation

The applications we are interested with are Allen-Cahn and Cahn-Hilliard equations to which homogeneous Neumann boundary conditions are associated. We proceed as in [8] and we first discretize in space the equation with high order finite difference compact schemes; the matrix A corresponds then to the laplacien with Homogenous Neumann BC (HNBC). Matrix B is the (sparse) second order laplacian matrix with HNBC. For a fast solution of linear systems in the RSS, we will use the cosine-fft to solve the Neumann problems with matrix $Id + \tau\Delta t B$. `test_Neumann_2D.m` is a (non RSS) solver that uses cosine-FFT for 2D neumann problem on the square

5.2 Allen-Cahn equation

`Allen_Cahn_fft.m` runs (a non RSS) Allen-Cahn with semi-implicit scheme and cos-fft, see directory AC_CH: this seems correct
Also`Allen_Cahn_fft_3D.m` runs (a non RSS) Allen-Cahn with semi-implicit scheme and cosine-FFT, see directory AC_CH: To check

- ALLEN CAHN + SPLITTING + CN-RSS ON THE LINEAR PART `RSS splitting AC3D CS.m` (WITH FFT)
- ALLEN CAHN 3D + EULER-RSS WITH SPLITTING IN EVERY DIRECTION `AC 3D RSS.m` (WITHOUT FFT)
- ALLEN CAHN 2D + FFT Allen Cahn fft.m (WITHOUT CS)

APPLICATIONS : PATTERNS, IMAGE RESTAURATION (???) see `http://fr.mathworks.com/discove`
see [22] p 1604.

Method	N	ϵ	Δt	τ	$[0,T]$	$\|error\|_\infty$	CPU factor
RSS	$N=64$	0.5	10^{-3}	5	$[0,1]$	0.0194	1
RSS	$N=64$	0.5	10^{-3}	2	$[0,1]$	0.0084	1
Classic	$N=64$	0.5	10^{-3}		$[0,1]$	0.0047	226
RSS	$N=64$	0.5	10^{-2}	2.2	$[0,1]$	0.0773	1
Classic	$N=64$	0.5	10^{-2}		$[0,1]$	0.0486	226

Table 2: 2D Allen-Cahn equation: simulation of patterns - RSS-semi-implicit scheme vs classic semi-implicit scheme, exact solution is $u(x,y,t)=\cos(\pi x)\cos(\pi y)\exp(sin(3\pi t))$, $\Omega=[0,1]^2$

5.2.1 Patterns dynamics

Method	N	ϵ	Δt	τ	$[0,T]$	$\|error\|_\infty$	CPU factor
RSS	$N=32$	0.5	10^{-3}	5	$[0,1]$	5.960^{-2}	1
RSS	$N=32$	0.5	10^{-3}	2	$[0,1]$	$3.03\ 10^{-2}$	1
Classic	$N=32$	0.5	10^{-3}		$[0,1]$	$2.1\ 10^{-2}$	2.22
RSS	$N=32$	0.5	10^{-2}	2	$[0,1]$	0.3123	1
RSS	$N=32$	0.5	10^{-2}	1.9	$[0,1]$	0.3066	1
Classic	$N=32$	0.5	10^{-2}		$[0,1]$	0.2586	2.22

Table 3: 3D Allen-Cahn equation: simulation of patterns - RSS-Lie splitting scheme vs classic Lie -splitting scheme, exact solution is $u(x,y,z,t)=\cos(\pi x)\cos(\pi y)\cos(\pi z)\exp(sin(3\pi t))$, $\Omega=[0,1]^3$

5.2.2 Image Segmentation

Image Segmentation consists in labelling pixels in digital images in such a way the image becomes easier to analyse; in particular it allows to locate objects and boundaries. Allen-Cahn equations were proposed as a tool in image segmentation as follows, [22]

$$\frac{\partial \phi}{\partial t}-\Delta\phi+\frac{F'}{\epsilon^2}+\lambda\left((1+\phi)(f_0-c_1)^2-(1-\Phi)(f_0-c_2)^2\right), \quad x\in\Omega \tag{65}$$

$$\frac{\partial \Phi}{\partial n}=0, \quad \partial\Omega \tag{66}$$

If C is the segmenting curve, then the phase ϕ corresponds to the situations

$$\phi(x) = \begin{cases} > 0 & \text{if } x \text{ is inside } C, \\ = 0 & \text{if } x \in C, \\ < 0 & \text{if } x \text{ is outside } C, \end{cases}$$

Here $\epsilon > 0$, $F'(\phi) = \phi(\phi^2 - 1)$, λ is a nonnegative parameter, f_0 is the given image.The terms c_1 and c_2 are the averages of f_0 in the regions $(\phi \geq 0)$ and $(\phi < 0)$, say

$$c_1 = \frac{\int_\Omega f_0(x)(1+\phi(x))dx}{\int_\Omega (1+\phi(x))dx} \text{ and } c_2 = \frac{\int_\Omega f_0(x)(1-\phi(x))dx}{\int_\Omega (1-\phi(x))dx}$$

These constants will be updated at each time step. In our numerical experiments, the given image f is normalized with $f_0 = \dfrac{f - f_{min}}{f_{max} - f_{min}}$, where f_{max} and $f_{min)}$ are the maximum and the minimum values of the given image, respectively, so we have $f_0 \in [0,1]$. The initial condition is $\phi = 2f_0 - 1$ and $\Omega = [0,1]^2$.

We can now write the RSS method

Algorithm 6 : RSS splitting for Image segmentation with Allen Cahn

1: **for** $k = 0, 1, \cdots$ **do**

2: **Set**

$$c_1^{(k)} = \frac{\int_\Omega f_0(x)(1+\phi^{(k)}(x))dx}{\int_\Omega (1+\phi^{(k)}(x))dx},$$
$$c_2^{(k)} = \frac{\int_\Omega f_0(x)(1-\phi^{(k)}(x))dx}{\int_\Omega (1-\phi^{(k)}(x))dx}$$

3: **Solve** $\dfrac{\phi^{(k+1/3)} - \phi^{(k)}}{\Delta t} = -\lambda \left((1 + \phi^{(k+1/3)})(f_0 - c_1^{(k)})^2 - (1 - \phi^{(k+1/3)})(f_0 - c_2^{(k)})^2 \right)$

4: **Solve** $(Id + \tau \Delta t B)\delta\phi = -\Delta t A \phi^{(k+/3)}$

5: **Set** $\phi^{(k+2/3)} = \phi^{(k+1/3)} + \delta\phi$

6: **Set** $\phi^{(k+1)} = \dfrac{\phi^{(k+2/3)}}{\sqrt{e^{-2\frac{\Delta t}{\epsilon^2}} + (\phi^{(k+2/3)})^2 (1 - e^{-2\frac{\Delta t}{\epsilon^2}})}}$

7: **end for**

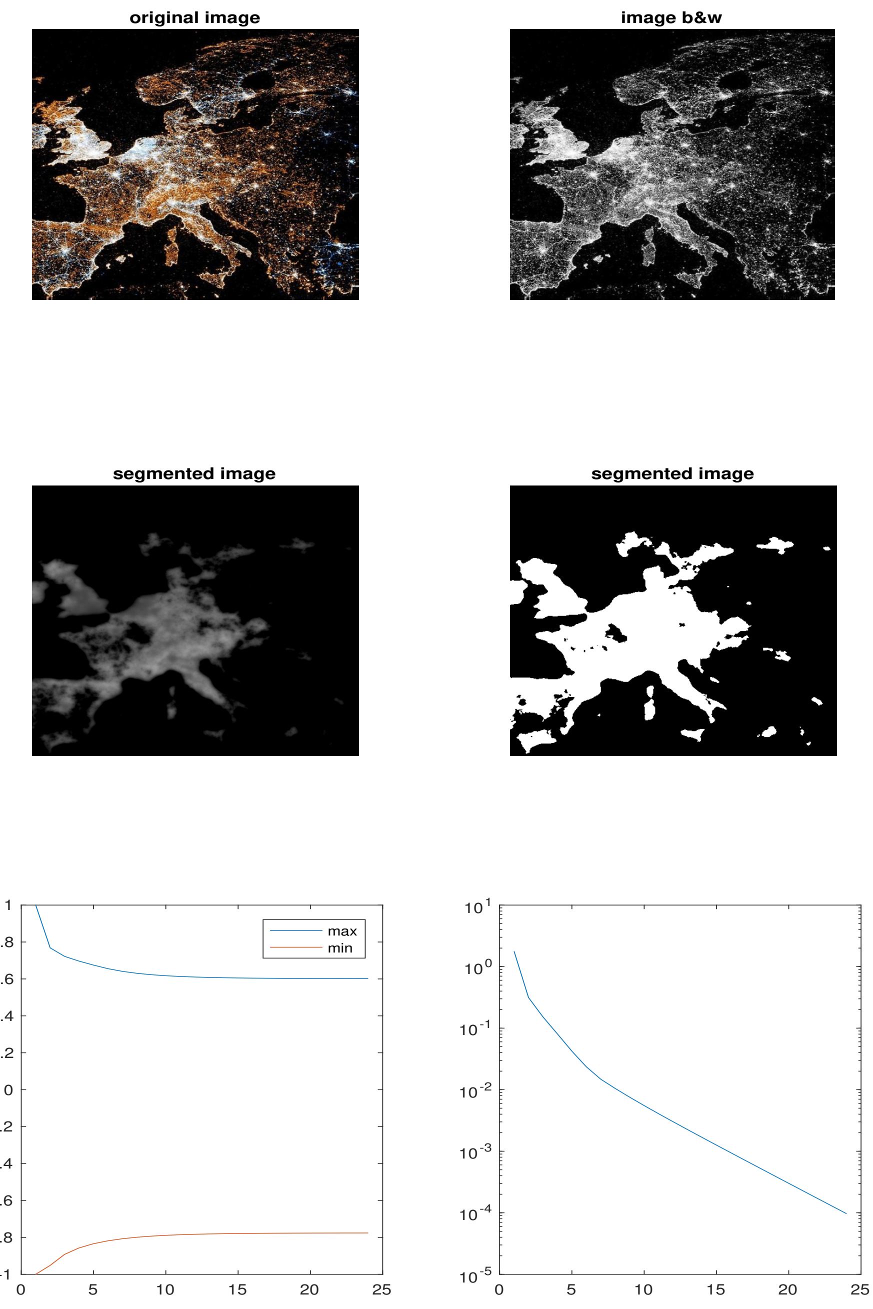

Figure 3: Segmentation (classical scheme) $\Delta t = 5.e-5$, $\epsilon = 0.04$, $\tau = 1$, $A = B$, $\lambda = 750000$

5.3 Cahn-Hilliard equation

USE THE (NON RSS BUT STABILIZED AS IN BERTOZZI PAPER) CODEs IN THE DIRECTORY CH_INPAINTING

- PATTERNS COMPUTATION: IMPLICIT EXPLIT SCHEME WITH FFT 3DCahn Hilliard fft.m (NON RSS)
- INPAINTING 2D test CH2D RSS solver1.m (without fft), see [9], p22

APPLICATIONS: 2D (3D) PATTERNS,
2D, 3D INPAINTING (SEE WHATEN et al)

5.3.1 Patterns dynamics

Method	N	ϵ	Δt	τ	$[0,T]$	$\|error\|_\infty$	CPU
RSS	$N=64$	0.5	10^{-3}	5	$[0,1]$	0.0194	7.358853
RSS	$N=64$	0.5	10^{-3}	2	$[0,1]$	0.0084	7.196025
Classic	$N=64$	0.5	10^{-3}		$[0,1]$	0.0047	1661.661410
RSS	$N=64$	0.5	10^{-2}	2.2	$[0,1]$	0.0773	0.795797
Classic	$N=64$	0.5	10^{-2}		$[0,1]$	0.0486	157.812224

Table 4: 2D Allen-Cahn equation: simulation of patterns - RSS-semi-implicit scheme vs classic semi-implicit scheme, exact solution is $u(x,y,t) = \cos(\pi x)\cos(\pi y)\exp(sin(3\pi t))$, $\Omega = [0,1]^2$ pgm Allen Cahn fft RSS.m

Method	N	ϵ	Δt	τ	$[0,T]$	$\|error\|_\infty$	CPU
RSS	$N=32$	0.5	10^{-3}	5	$[0,1]$	5.960^{-2}	94.874912
RSS	$N=32$	0.5	10^{-3}	2	$[0,1]$	$3.03\ 10^{-2}$	94.874912
Classic	$N=32$	0.5	10^{-3}		$[0,1]$	$2.1\ 10^{-2}$	210.545565
RSS	$N=32$	0.5	10^{-2}	2	$[0,1]$	0.3123	9.495254
RSS	$N=32$	0.5	10^{-2}	1.9	$[0,1]$	0.3066	9.819845
Classic	$N=32$	0.5	10^{-2}		$[0,1]$	0.2586	21.360027

Table 5: 3D Allen-Cahn equation: simulation of patterns - RSS-Lie splitting scheme vs classic Lie -splitting scheme, exact solution is $u(x,y,z,t) = \cos(\pi x)\cos(\pi y)\cos(\pi z)\exp(sin(3\pi t))$, $\Omega = [0,1]^3$ pgm Allen Cahn fft.m

5.3.2 2D Inpainting

Method	N	ϵ	Δt	τ	$[0, T]$	quality	CPU factor (iterations)
RSS	$N = 64$	0.05	10^{-3}	1.4	$[0, 0.1]$	EX	1
Classic	$N = 64$	0.05	10^{-3}		$[0, 0.1]$	EX	> 10
RSS	$N = 64$	0.05	5.10^{-3}	1.5	$[0, 0.1]$	EX	1
Classic	$N = 64$	0.05	5.10^{-3}		$[0, 0.1]$	EX	> 10
RSS	$N = 64$	0.05	10^{-2}	2.8	$[0, 0.1]$	middle	1
Classic	$N = 64$	0.5	10^{-2}		$[0, 0.1]$	middle	> 10

Table 6: 2D Cahn-Hilliard Inpainting equation, the triangle example: , $\Omega = [0, 1]^2$, $\lambda = 90000$

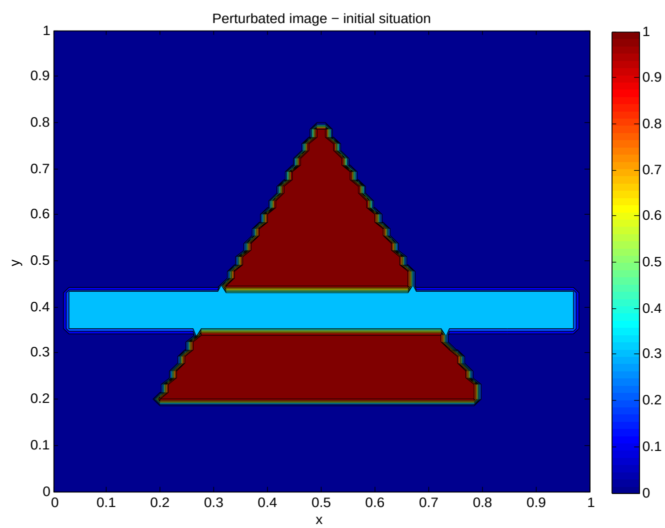

Figure 4: Inpainting with C-H. $\Delta t = 0.001$, $\epsilon = 0.05$, $N = 64$ - Initial inpainted image

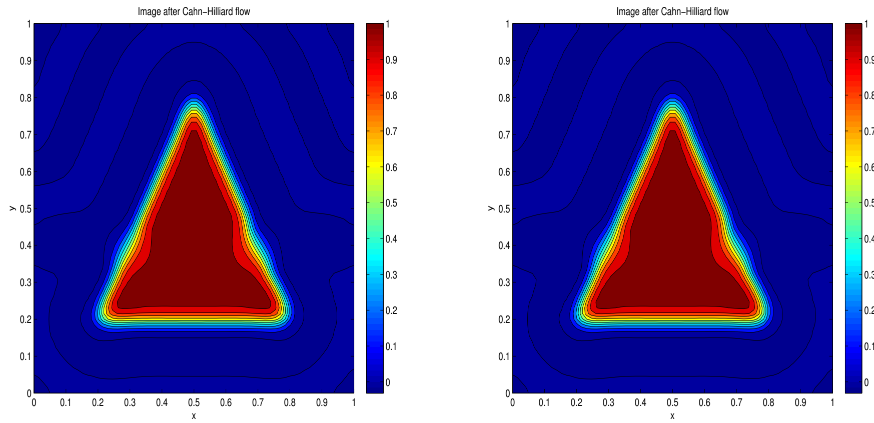

Figure 5: Inpainting with C-H. $\Delta t = 0.001$, $\epsilon = 0.05$, $N = 64$ - Restored triangle at $T = 0.1$, classical (left) RSS method (right)

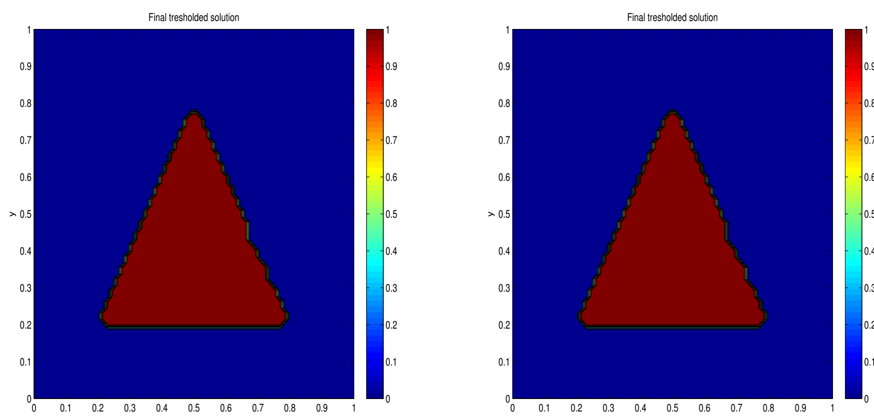

Figure 6: Inpainting with C-H. $\Delta t = 0.001$, $\epsilon = 0.05$, $N = 64$ - Restored triangle with thresholding at $T = 0.1$, classical (left) RSS method (right)

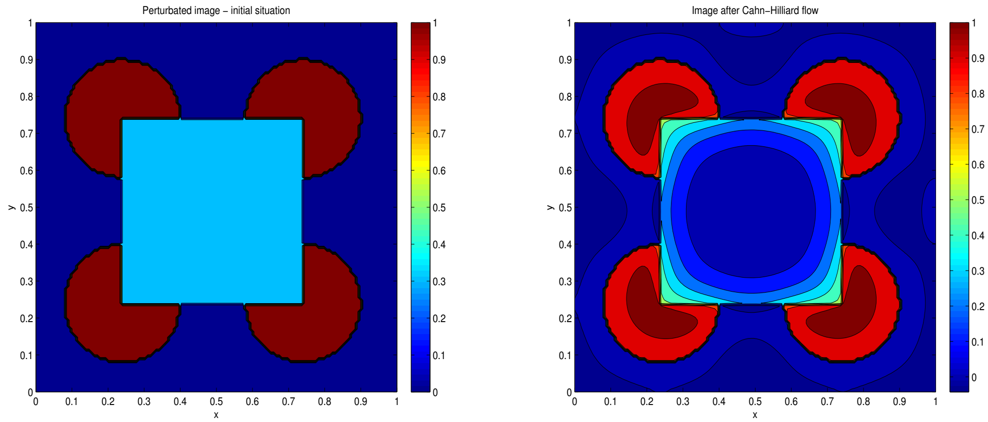

Figure 7: Inpainting with C-H. $\Delta t = 0.001$, $\epsilon = 0.05$, $N = 128$ - Initial inpainted image, left ($t = 0$), right ($t = 0.005$)

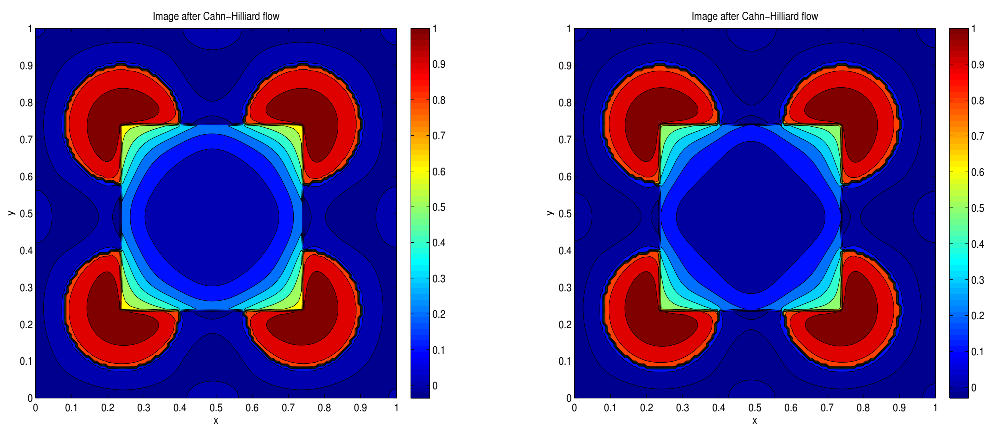

Figure 8: Inpainting with C-H. $\Delta t = 0.001$, $\epsilon = 0.05$, $N = 128$ - image at $t = 0.008$ (left) and at $t = 0.01$ (right)

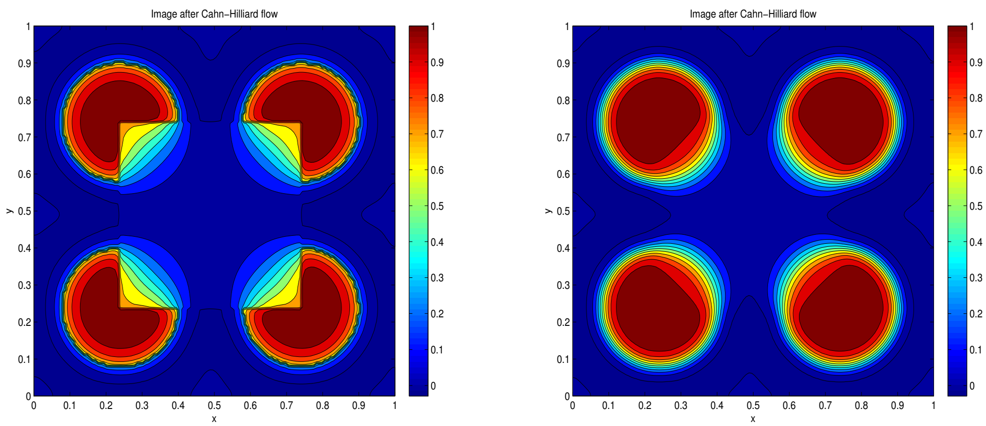

Figure 9: Inpainting with C-H. $\Delta t = 0.001$, $\epsilon = 0.05$, $N = 128$ - image at $t = 0.02$ (left) and at $t = 0.1$ (right). RSS cCheme

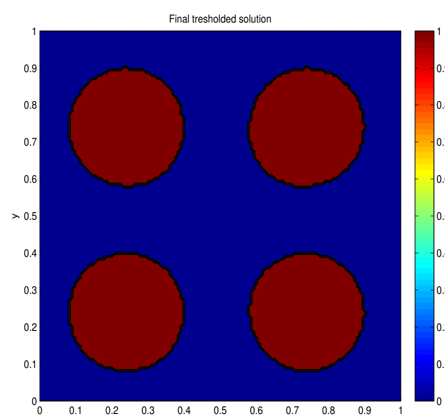

Figure 10: Inpainting with C-H. $\Delta t = 0.001$, $\epsilon = 0.05$, $N = 128$ - thresholded image at $t = 0.1. RSSScheme$

5.3.3 3D Inpainting

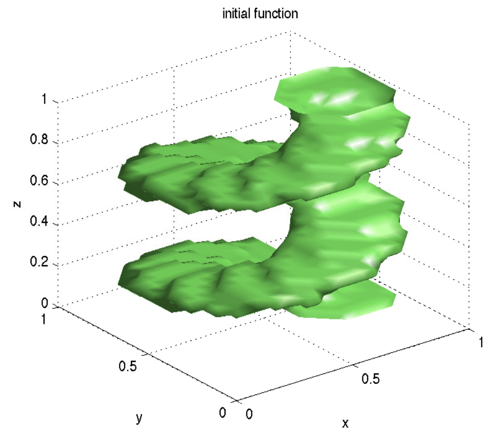

Figure 11: Inpainting 3D with C-H. $\Delta t = 1.e-7$, $\lambda = 100000$, $\epsilon = 0.05$, $N = 20$ -

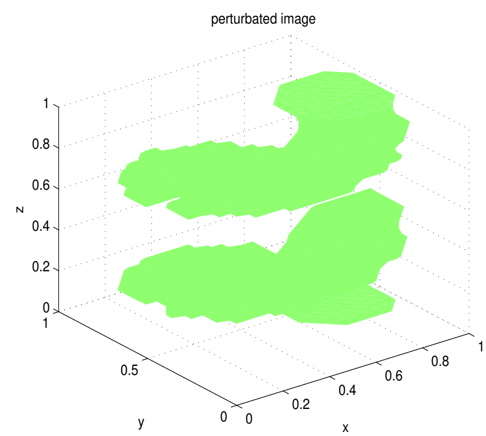

Figure 12: Inpainting 3D with C-H. $\Delta t = 1.e-7$, $\lambda = 100000$, $\epsilon = 0.05$, $N = 20$

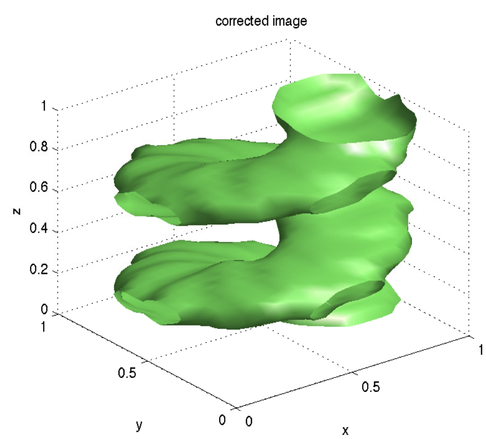

Figure 13: Inpainting 3D with C-H. $\Delta t = 1.e - 7$, $\lambda = 100000$, $\epsilon = 0.05$, $N = 20$

6 Concluding Remarks

We have introduced stabilized finite differences semi-implicit schemes that allows to simulate fastly high accurate solution of phase fields problems since the main effort of the computation lies on the efficient solution of sparse linear systems.

References

[1] S. M. Allen, J. W. Cahn. *Ground State Structures in Ordered Binary Alloys with Second Neighbor Interactions.* Acta Met. 20, 423 (1972).

[2] S. M. Allen, J. W. Cahn. *A Correction to the Ground State of FCC Binary Ordered Alloys with First and Second Neighbor Pairwise Interactions.* Scripta Met. 7, 1261 (1973).

[3] Amir Averbuch, Albert Cohen et Moshe Israeli : A stable and accurate explicit scheme for parabolic evolution equations. rapport LAN 1998, unpublished

[4] S. Bartels, Numerical Methods for Nonlinear Partial Differential Equations, Springer Series in Computational Mathematics 47, 015

[5] M. Benes, V. Chalupecky, K. Mikula, Geometrical image segmentation by the Allen-Cahn equation, Applied Numerical Mathematics 51, (2004), 187-205.

[6] A. Bertozzi, S. Esedoglu, and A. Gillette, Analysis of a two-scale Cahn–Hilliard model for binary image inpainting, Multiscale Model. Simul. 6 (2007), 913–936.

[7] A. Bertozzi, S. Esedoglu, and A. Gillette, Inpainting of binary images using the Cahn-Hilliard equation, IEEE Trans. Image Proc. (2007), 285–291.

[8] Matthieu Brachet, Jean-Paul Chehab, Stabilized Times Schemes for High Accurate Finite Differences Solutions of Nonlinear Parabolic Equations, of Scientific Computing, 69 (3), 946–982, 2016

[9] J. Bosch, D. Kay, M. Stoll, and A.J. Wathen, Fast solvers for Cahn–Hilliard inpainting, SIAM J. Imag. Sci. 7 (2013), 67–97.

[10] J.-P. Chehab, A.A. Franco and Y. Mammeri, Boundary Control of the number of the interfaces for the one-dimensional Allen-Cahn equation, DiscR. Cont. dyn. Syst. Serie S, Volume 10, Number 1, pp 87–100, 2017

[11] B. Costa. L. Dettori, D. Gottlieb and R. Temam, Time marching techniques for the nonlinear Galerkin method, SIAM J. SC. comp., 23, (2001), 1, 46-65.

[12] L. Collatz, The Numerical Treatment of Differential Equations, 3rd Edition, Springer-Verlag, 1966.

[13] A. E. Diegel, Numerical Analysis of Convex Splitting Schemes for Cahn-Hilliard and Coupled Cahn-Hilliard- Fluid-Flow Equations, PhD thesis, may 2015, University of Tennessee - Knoxville

[14] C.M. Elliott, The Chan-Hilliard Model for the Kinetics of Phase Separation, *in* Mathematical Models for Phase Change Problems, International Series od Numerical Mathematics, Vol. 88, (1989) Birkhäuser.

[15] C.M. Elliott and A. Stuart The global dynamics of discrete semilinear parabolic equations. SIAM J. Numer. Anal. 30 (1993) 1622–1663.

[16] H. Emmerich, The Diffuse Interface Approach in Materials Science Thermodynamic. Concepts and Applications of Phase-Field Models. Lecture Notes in Physics Monographs, Springer, Heidelberg, 2003.

[17] D. J. Eyre, Unconditionallly Stable One-step Scheme for Gradient Systems, June 1998, unpublished, http://www.math.utah.edu/eyre/research/methods/stable.ps.

[18] H. Fakih, *Etude mathematique et numerique de quelques generalisations de l'equation de Cahn-Hilliard : Applications a la retouche d'images et a la biologie*, Doctoral Thesis, University of Poitier, december 2014

[19] J. Jiang, J. Shi. *Bistability Dynamics in Structured Ecological Models. Spatial Ecology*, Stephen Cantrell, Chris Cosner, Shigui Ruan ed., CRC Press, 2009

[20] D. Lee , J-Y Huh , D. Jeong , J. Shin, A.Yun a J. Kim, *Physical, mathematical, and numerical derivations of the Cahn-Hilliard equation.* Computational Materials Science 81 (2014) 216–225

[21] S. Lele, Compact Difference Schemes with Spectral Like resolution, J. Comp. Phys., 103, (1992), 16–2

[22] Y. Li, H. G. Lee, D. Jeong, J. Kim, An unconditionally stable hybrid numerical method for solving the Allen-Cahn equation, Computers and Mathematics with Applications 60 (2010) 1591–1606

[23] Y. Li, D. Jeong, J. Choi, S. Lee, J. Kim, Fast local image inpainting based on the Allen-Cahn model, Digital SignalProcessing37(2015)6574

[24] K.M. Morton and D.F. Mayers, Numerical Solution of Partial Differential Equations: An Introduction, Cambridge University Press, second Edition (2005)

[25] M. Pierre and A. Rougirel, Stationary solutions to phase field crystal equations, Math. Methods Appl. Sci., 34 (2011), no. 3, pp. 278–308

[26] Nikolas Provatas and Ken Elder, Phase-Field Methods in Material Science and Engineering, Wiley-VCH

[27] J. Shen, X. Yang, Numerical Approximations of Allen-Cahn and Cahn-Hilliard Equations. DCDS, Series A, (28), (2010), pp 1669–1691.

[28] G. Tierra, F. Guillén-Gonzàlez, *Numerical methods for solving the Cahn-Hilliard equation and its applicability to related Energy-based models.* Neĉas Center for Mathematical Modeling, Preprint no. 2013-035.