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ON THE VISCOUS CAHN-HILLIARD-NAVIER-STOKES EQUATIONS WITH DYNAMIC BOUNDARY CONDITIONS

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Abstract. In the present article we study the viscous Cahn-Hilliard-Navier-Stokes model, endowed with dynamic boundary conditions, from the theoretical and numerical point of view. We start by deducing results on the existence, uniqueness and regularity of the solutions for the continuous problem. Then we propose a space semi-discrete finite element approximation of the model and we study the convergence of the approximate scheme. We also prove the stability and convergence of a fully discretized scheme, obtained using the semi-implicit Euler scheme applied to the space semi-discretization proposed previously. Numerical simulations are also presented to illustrate the theoretical results.

1. **Setting of the problem.** In this paper we consider the coupling between the incompressible Navier-Stokes equations with the viscous Cahn-Hilliard equations and we endow the Cahn-Hilliard equations with dynamic boundary conditions. This coupling of equations models the motion of isothermal mixture of two confined immiscible and incompressible with comparable (equal) densities and viscosities; when the two fluids have comparable constant densities the temperature differences are neglectable and the diffusive interface between the two phases has a small, non-zero thickness. The model reads as follows:

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + \lambda\varphi\nabla w = \mathbf{h} & \text{in } \Omega_T = \Omega\times(0,T), \\ \text{div } \mathbf{u} = 0, & \text{in } \Omega_T, \\ \varphi_t + \mathbf{u}\cdot\nabla\varphi - \gamma\Delta w = 0 & \text{in } \Omega_T, \\ w = -\Delta\varphi + f(\varphi) + \varepsilon\varphi_t, & \text{in } \Omega_T, \\ \mathbf{u} = 0, \quad \partial_n w = 0 & \text{on } \Gamma\times(0,T), \\ \varphi_t = \delta\Delta_\Gamma\varphi - \lambda_s\varphi - g_s(\varphi) - \partial_n\varphi \quad t>0, & \text{on } \Gamma\times(0,T), \end{cases} \tag{1}$$

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with $\varepsilon > 0$ a viscosity parameter, $\delta > 0$, $\lambda_s > 0$, $\gamma > 0$ the elastic relaxation time, $\lambda > 0$ the surface tension and Ω a bi-dimensionnal domain with smooth boundary Γ. Here $\mathbf{u}$ and p respectively denote the average velocity and the pressure of the fluid mixture and the scalar φ is an order parameter indicating the phases of the fluid. The two fluids are supposed to have the same density, taken here equal to 1, and the same viscosity, denoted by ν. In (1), Δ_Γ is the Laplace-Beltrami operator on Γ, $\mathbf{n}$ and ∂_n are respectively the outer unit normal and the associated normal derivative on Γ.

The evolution boundary value problem (1) is completed by the initial condition $u(0) = u_0$ and $\varphi(0) = \varphi_0$.

The functions f and g_s are polynomials of degree p and q respectively, with $p \geq 3$ odd, $q \geq 1$ odd and the dominant coefficients positive. Typical choices are

$$f(v) = v^3 - v \qquad \text{and} \qquad g_s(v) = a_\Gamma v - b_\Gamma, \quad v \in \mathbb{R},\ a_\Gamma > 0,\ b_\Gamma \in \mathbb{R}. \tag{2}$$

We will also consider F and G_s to be two antiderivatives of the functions f and g_s.

The Cahn-Hilliard-Navier-Stokes (CHNS) model with classical (non-dynamical) boundary conditions was highly studied from the theoretical and mathematical point of view. The model was first introduced in physical papers as [6], [8] (see also [20, 24] where the model was derived using mathematical arguments). The mathematical study of the problem, meaning the existence and uniqueness of solutions for the two and three dimensional cases, as well as the asymptotic stability of the model was established in [3]. In [15] the authors studied the asymptotics of the CHNS model proving the existence of a global finite dimensional attractor those fractal dimension is estimated; the convergence of each trajectory to an equilibrium is established. The case of a CHNS model with degenerate mobility for the Cahn-Hilliard equation was also studied in [1]. The study of pullback attractors for Cahn-Hilliard-Navier-Stokes models can be found in [2, 26].

For the numerical studies, there is also a lot of litterature where different kind of approximate models are considered. We refer the interested reader to [11, 7, 12, 13, 21, 22], where different finite element type of approximations for interface models are considered.

The novelty in this paper is that we consider the case of a coupling between the viscous Cahn-Hilliard equations endowed with dynamic boundary conditions and the Navier-Stokes equations. To the best of our knowledge, only one work concerns the same subject, see [16], where the authors consider the theoretical study of the problem, meaning the well posedness of the model. A study of bounded liquid-gas flows with different dynamic boundary conditions can be found in [9].

The paper is structured as follows: the first two sections address the theoretical study of the model while the rest of the paper proposes discrete approximations, first obtained by the discretisation of the space variable using a finite element approximation and then by introducing a fully discrete model. Thus, in Section 2 we derive a priori estimates on which the existence, uniqueness and regularity of the solutions are proved. In Section 3 the existence of a weak solution is rigorously derived. In Section 4 we propose a space semi-discrete finite element approximation for the continuous problem and in Section 5 we derive error estimates and we prove the convergence of the sequence of approximate solutions to the exact solution. Section 6 is devoted to the study of a fully discretized scheme, obtained using the backward Euler method for the discretization of the time derivative in the space semi-discrete scheme studied previously. The existence of a unique solution for the fully discretized model, the conditional stability and the convergence of the method are established. In the end we also present some numerical simulations that illustrate the results presented above.

2. A priori estimates.

2.1. Energy estimates. We start by introducing the following typical function spaces for our model:
$\dot{L}^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q dx = 0\}$, $\mathbf{W}_0 = \{\mathbf{v} \in (H_0^1(\Omega))^2\,;\ \nabla\cdot\mathbf{v} = 0\,\}$,
$H^m(\Omega,\Gamma) = \{\phi \in H^m(\Omega)$ such that $\phi|_\Gamma \in H^m(\Gamma)\}$, with $m \geq 1$. We denote by $|\cdot|_\Omega$ the classical $L^2(\Omega)$ (or $(L^2(\Omega))^2$) norm, $|\cdot|_\Gamma$ the norm on $L^2(\Gamma)$ while $(\cdot,\cdot)_\Omega$ and $(\cdot,\cdot)_\Gamma$ are the corresponding scalar products. The norm on $H^m(\Omega,\Gamma)$ is given by:

$$\|\cdot\|^2_{H^m(\Omega,\Gamma)} = \|\cdot\|^2_{H^m(\Omega)} + \|^2_{H^m(\Gamma)}.$$

We multiply $(1)_1$ by $\mathbf{v}$, $(1)_3$ by ψ and $(1)_4$ by χ, and integrate over Ω. We obtain, after elementary simplifications, the following variational formulation:

Find $\mathbf{u} \in \mathbf{W}_0$, $\varphi \in H^1(\Omega,\Gamma)$ and $w \in H^1(\Omega)$ such that:

$$\begin{cases} (\mathbf{u}_t,\mathbf{v})_\Omega + \nu(\nabla\mathbf{u},\nabla\mathbf{v})_\Omega + b(\mathbf{u},\mathbf{u},\mathbf{v}) + \lambda(\varphi\nabla w,\mathbf{v})_\Omega = (\mathbf{h},\mathbf{v})_\Omega, \\ (\varphi_t,\psi)_\Omega + (\mathbf{u}\cdot\nabla\varphi,\psi)_\Omega + \gamma(\nabla w,\nabla\psi)_\Omega = 0, \\ \varepsilon(\varphi_t,\chi)_\Omega + (\varphi_t,\chi)_\Gamma + (\nabla\varphi,\nabla\chi)_\Omega + \delta(\nabla_\Gamma\varphi,\nabla_\Gamma\chi)_\Gamma + (f(\varphi),\chi)_\Omega \\ \qquad\qquad + (g_s(\varphi),\chi)_\Gamma + \lambda_s(\varphi,\chi)_\Gamma - (w,\chi)_\Omega = 0, \end{cases} \tag{3}$$

for all $\mathbf{v} \in \mathbf{W}_0$, $\psi \in H^1(\Omega)$ and $\chi \in H^1(\Omega,\Gamma)$.

In (3) we denote by b the trilinear continuous form on $\mathbf{W}_0\times\mathbf{W}_0\times\mathbf{W}_0$ with values in $\mathbb{R}$ defined by $b(\mathbf{u},\mathbf{v},\mathbf{w}) = ((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})_\Omega$. We frequently use the orthogonality property of b, meaning the fact that $b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$ for all $\mathbf{u},\mathbf{v} \in \mathbf{W}_0$; for more details and properties for b, we refer the interested reader to [27]. We also introduce A, the Stokes operator from $\mathbf{W}_0$ into $(\mathbf{W}_0)'$, defined by $(A\mathbf{u},\mathbf{v}) = (\nabla\mathbf{u},\nabla\mathbf{v})_\Omega \quad \forall\mathbf{v} \in \mathbf{W}_0$ (see [28]).

Remark 1. We note that $(3)_1$ can also be written as

$$(\mathbf{u}_t,\mathbf{v})_\Omega+\nu(\nabla\mathbf{u},\nabla\mathbf{v})_\Omega+b(\mathbf{u},\mathbf{u},\mathbf{v})-(p,\nabla\cdot v)_\Omega+\lambda(\varphi\nabla w,\mathbf{v})_\Omega = (\mathbf{h},\mathbf{v})_\Omega \forall\mathbf{v} \in (H_0^1(\Omega))^2. \tag{4}$$

Taking $\psi = 1$ in $(3)_2$ and using the fact that $(\mathbf{u}\cdot\nabla\varphi,1)_\Omega = 0$ since $\mathbf{u} \in \mathbf{W}_0$, we obtain the mass conservation property:

$$(\varphi_t,1)_\Omega = 0, \quad i.e. \quad \langle\varphi\rangle := \frac{1}{|\Omega|}(\varphi,1)_\Omega = \text{ const } \ \forall t \geq 0. \tag{5}$$

We also take $\mathbf{v} = \mathbf{u}$ in $(3)_1$, $\psi = \lambda w$ in $(3)_2$, and $\chi = \lambda\varphi_t$ in $(3)_3$ and sum the three resulting equations. After simplifications and using the fact that $\mathbf{u} \in \mathbf{W}_0$ implies $(\varphi\nabla w,\mathbf{u})_\Omega + (\mathbf{u}\cdot\nabla\varphi,w)_\Omega = 0$, we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}J(\mathbf{u},\varphi) + \varepsilon\lambda|\varphi_t|^2_\Omega + \lambda|\varphi_t|^2_\Gamma + \lambda\gamma|\nabla w|^2_\Omega + \frac{\nu}{2}|\nabla\mathbf{u}|^2_\Omega \leq c|\mathbf{h}|^2_\Omega, \tag{6}$$

with

$$J(\mathbf{u},\varphi) = |\mathbf{u}|^2_\Omega+\delta\lambda|\nabla_\Gamma\varphi|^2_\Gamma+\lambda_s\lambda|\varphi|^2_\Gamma+\lambda|\nabla\varphi|^2_\Omega+2\lambda\int_\Gamma G_s(\varphi)d\Gamma+2\lambda\int_\Omega F(\varphi)d\Omega. \tag{7}$$

Lemma 2.1. *We assume that $\mathbf{h}$ belongs to $L^2(0,T;(L^2(\Omega))^2)$, $\mathbf{u}_0$ belongs to $(L^2(\Omega))^2$ and $\varphi_0 \in H^1(\Omega,\Gamma)$. Then, the solution of (3) satisfies, for all $T > 0$:*

- $\varphi \in L^\infty(0,T;H^1(\Omega,\Gamma))$;
- $\varphi_t \in L^2(0,T;L^2(\Omega)\times L^2(\Gamma))$;
- $\mathbf{u} \in L^\infty(0,T;(L^2(\Omega))^2)\cap L^2(0,T;(H_0^1(\Omega))^2)$;
- $w \in L^2(0,T;H^1(\Omega))$;
- $ess\ sup_{t\in[0,T]} |\int_0^t p(s)ds|_\Omega \le c$,

with c a constant depending on the initial conditions and the forcing term.

Proof. Integrating (6) over $(0,t)$ with $t\in[0,T]$, we obtain:

$$\begin{aligned} esssup_{t\in[0,T]}\, \{|\mathbf{u}|_\Omega^2 + \delta\lambda|\nabla_\Gamma\varphi|_\Gamma^2 + \lambda_s\lambda|\varphi|_\Gamma^2 + \lambda|\nabla\varphi|_\Omega^2 + 2\lambda\int_\Gamma G_s(\varphi)d\Gamma \\ + 2\lambda\int_\Omega F(\varphi)d\Omega\} \le C, \end{aligned} \tag{8}$$

and

$$\nu\int_0^T |\nabla\mathbf{u}|_\Omega^2\,dt + \lambda\gamma\int_0^T |\nabla w|_\Omega^2\,dt + \lambda\int_0^T |\varphi_t|_\Gamma^2\,dt + \varepsilon\lambda\int_0^T |\varphi_t|_\Omega^2\,dt \le C, \tag{9}$$

where $C = C(T, |\mathbf{u}_0|_\Omega, \|\varphi_0\|_{H^1(\Omega,\Gamma)}, |\mathbf{h}|_{L^2(0,T;(L^2(\Omega))^2)})$.

Using the fact that $F(\varphi)\ge -c_1$ and $G_s(\varphi)\ge -c_2$, where c_1, c_2 are positive, absolute constants, the first three lines of Lemma 2.1 are direct consequences of (8) and (9). We also infer that $\nabla w\in L^2(0,T;L^2(\Omega))$. Thus, we only need an estimate on $\langle w\rangle$ in order to be able to conclude that the H^1- norm of w remains bounded. From $(3)_3$, taking $\chi=1$ and recalling that $\langle\varphi_t\rangle=0$, we obtain:

$$\begin{aligned} \int_\Omega w\,d\Omega = \lambda_s\int_\Gamma \varphi d\Gamma + \int_\Gamma g_s(\varphi)d\Gamma + \int_\Gamma \varphi_t d\Gamma + \int_\Omega f(\varphi)d\Omega, \\ \le c(|\varphi|_\Gamma + \|\varphi\|_{H^1(\Gamma)}^q + |\varphi_t|_\Gamma + \|\varphi\|_{H^1(\Omega)}^p + 1). \end{aligned} \tag{10}$$

Using (10), we conclude that $\langle w\rangle$ belongs to $L^2(0,T)$ and thus $w\in L^2(0,T,H^1(\Omega))$.

It remains to detail the estimate for p. To this aim, we integrate (4) with respect to $t\in[0,\tau]$, $\tau\le T$. This yields

$$\begin{aligned} (\int_0^\tau p dt, \nabla\cdot\mathbf{v})_\Omega \ &= (\mathbf{u}(\tau)-\mathbf{u}(0),\mathbf{v})_\Omega + \nu\int_0^\tau (\nabla\mathbf{u},\nabla\mathbf{v})_\Omega dt + \int_0^\tau ((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v})_\Omega dt \\ &\quad + \lambda\int_0^\tau (\varphi\nabla w,\mathbf{v})_\Omega dt - \int_0^\tau (\mathbf{h},\mathbf{v})_\Omega dt \\ &\le C\Big(|u_0|_\Omega + \|\mathbf{h}\|_{L^2(\Omega_T)} + \|\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{u}\|_{L^2(0,T;H^1(\Omega))}^2 \\ &\quad + \|\varphi\|_{L^2(0,T;H^1(\Omega))}\,\|\nabla w\|_{L^2(\Omega_T)} + 1\Big)\ \|\mathbf{v}\|_{H^1(\Omega)} \\ &\le C\|\mathbf{v}\|_{H^1(\Omega)}, \end{aligned}$$

thanks to (8) and (9). Recalling the Inf-Sup condition (see [18]):

$$\sup_{v\in H_0^1}\frac{(q,\nabla\cdot v)_\Omega}{\|v\|_{H^1(\Omega)}} \ge c|q|_\Omega \quad \forall q\in \dot{L}^2(\Omega),$$

we get

$$|\int_0^\tau p(t)dt|_\Omega \le C, \quad \forall\tau\in[0,T].$$

This completes the proof of the lemma. ∎

2.2. **Uniqueness of the weak solution.**

Let $(\mathbf{u}_1, \varphi_1, w_1)$ and $(\mathbf{u}_2, \varphi_2, w_2)$ be two solutions to (3) departing from the same initial data (i.e. $\mathbf{u}_1(0) = \mathbf{u}_2(0)$, $\varphi_1(0) = \varphi_2(0)$). Then, $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$, $\varphi = \varphi_1 - \varphi_2$, $w = w_1 - w_2$ satisfy the following variational formulation:

$$\left\{\begin{array}{l} (\mathbf{u}_t, \mathbf{v})_\Omega + \nu(\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega + ((\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2, \mathbf{v})_\Omega + \lambda(\varphi_1 \nabla w_1, \mathbf{v})_\Omega \\ \qquad\qquad -\lambda(\varphi_2 \nabla w_2, \mathbf{v})_\Omega = 0, \\ (\varphi_t, \psi)_\Omega + (\mathbf{u}_1 \cdot \nabla\varphi_1, \psi)_\Omega - (\mathbf{u}_2 \cdot \nabla\varphi_2, \psi)_\Omega + \gamma(\nabla w, \nabla\psi)_\Omega = 0, \\ \varepsilon(\varphi_t, \chi)_\Omega + (\varphi_t, \chi)_\Gamma + (\nabla\varphi, \nabla\chi)_\Omega + \delta(\nabla_\Gamma\varphi, \nabla_\Gamma\chi)_\Gamma + (f(\varphi_1) - f(\varphi_2), \chi)_\Omega \\ \qquad\qquad +(g_s(\varphi_1) - g_s(\varphi_2), \chi)_\Gamma + \lambda_s(\varphi, \chi)_\Gamma - (w, \chi)_\Omega = 0, \end{array}\right. \tag{11}$$

for all $\mathbf{v} \in \mathbf{W}_0$, $\psi \in H^1(\Omega)$ and $\chi \in H^1(\Omega, \Gamma)$.

We take $\mathbf{v} = \mathbf{u}$ in $(11)_1$. Using the fact that

$$((\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2, \mathbf{u})_\Omega \quad = \quad ((\mathbf{u} \cdot \nabla)\mathbf{u}_1, \mathbf{u})_\Omega,$$

and

$$|((\mathbf{u} \cdot \nabla)\mathbf{u}_1, \mathbf{u})_\Omega| \leq |\mathbf{u}|^2_{L^4(\Omega)} |\nabla \mathbf{u}_1|_\Omega \;\leq\; c|\mathbf{u}|_\Omega |\nabla\mathbf{u}|_\Omega |\nabla\mathbf{u}_1|_\Omega \;\leq\; \frac{\nu}{2}|\nabla\mathbf{u}|^2_\Omega + c|\nabla\mathbf{u}_1|^2_\Omega |\mathbf{u}|^2_\Omega,$$

we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}|\mathbf{u}|^2_\Omega + \frac{\nu}{2}|\nabla\mathbf{u}|^2_\Omega + \lambda(\varphi_1 \cdot \nabla w_1, \mathbf{u})_\Omega - \lambda(\varphi_2 \cdot \nabla w_2, \mathbf{u})_\Omega \;\leq\; c|\nabla\mathbf{u}_1|^2_\Omega\, |\mathbf{u}|^2_\Omega. \tag{12}$$

Now we take $\psi = w$ in $(11)_2$ and $\chi = \varphi_t$ in $(11)_3$ and sum the resulting equations. This yields to

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\{|\nabla\varphi|^2_\Omega + \delta|\nabla_\Gamma\varphi|^2_\Gamma + \lambda_s|\varphi|^2_\Gamma\} + |\varphi_t|^2_\Gamma + \varepsilon|\varphi_t|^2_\Omega + \gamma|\nabla w|^2_\Omega + (g_s(\varphi_1) - g_s(\varphi_2), \varphi_t)_\Gamma \\ +(f(\varphi_1) - f(\varphi_2), \varphi_t)_\Omega + (\mathbf{u}_1 \cdot \nabla\varphi_1 - \mathbf{u}_2 \cdot \nabla\varphi_2, w)_\Omega \;= 0. \end{aligned} \tag{13}$$

Using the fact that f and g_s are polynomials, we can write:

$$\begin{aligned} (f(\varphi_1) - f(\varphi_2), \varphi_t)_\Omega \leq & c|\varphi|_{L^4(\Omega)} |\varphi_t|_\Omega \,(|\varphi_1|^{p-1}_{L^{4(p-1)}(\Omega)} + |\varphi_2|^{p-1}_{L^{4(p-1)}(\Omega)} + 1) \\ \leq & \frac{\varepsilon}{2}|\varphi_t|^2_\Omega + c\|\varphi\|^2_{H^1(\Omega)} (|\varphi_1|^{2(p-1)}_{L^{4(p-1)}(\Omega)} + |\varphi_2|^{2(p-1)}_{L^{4(p-1)}(\Omega)} + 1) \\ \leq & \frac{\varepsilon}{2}|\varphi_t|^2_\Omega + c(|\nabla\varphi|^2_\Omega + \lambda_s|\varphi|^2_\Gamma)\,(\|\varphi_1\|^{2(p-1)}_{H^1(\Omega)} + \|\varphi_2\|^{2(p-1)}_{H^1(\Omega)} + 1), \end{aligned}$$

with the constant c depending on $1/\varepsilon$.

We also have:

$$\begin{aligned} (g_s(\varphi_1) - g_s(\varphi_2), \varphi_t)_\Gamma \leq & c\,|\varphi\,(1 + |\varphi_1|^{q-1} + |\varphi_2|^{q-1})|_\Gamma\, |\varphi_t|_\Gamma \\ \leq & c(1 + |\varphi_1|^{q-1}_{L^\infty(\Gamma)} + |\varphi_2|^{q-1}_{L^\infty(\Gamma)})\, |\varphi|_\Gamma\, |\varphi_t|_\Gamma \\ \leq & c(1 + \|\varphi_1\|^{q-1}_{H^1(\Gamma)} + \|\varphi_2\|^{q-1}_{H^1(\Gamma)})\, |\varphi|_\Gamma\, |\varphi_t|_\Gamma \\ \leq & \frac{1}{2}|\varphi_t|^2_\Gamma + c(1 + \|\varphi_1\|^{2(q-1)}_{H^1(\Gamma)} + \|\varphi_2\|^{2(q-1)}_{H^1(\Gamma)})\, |\varphi|^2_\Gamma. \end{aligned}$$

Moreover, since $\nabla \cdot \mathbf{u}_i = 0$ for $i = 1, 2$, we have:

$$(\mathbf{u}_1 \cdot \nabla\varphi_1 - \mathbf{u}_2 \cdot \nabla\varphi_2, w)_\Omega = -(\varphi\nabla w, \mathbf{u}_1)_\Omega - (\varphi_2 \nabla w, \mathbf{u})_\Omega, \tag{14}$$

and we bound the terms from the right hand side of (14) as follows:

$$\begin{aligned}|(\varphi\nabla w, \mathbf{u}_1)_\Omega| \leq& |\nabla w|_\Omega\,|\varphi|_{L^4(\Omega)}|\mathbf{u}_1|_{L^4(\Omega)}\\ \leq& c|\nabla w|_\Omega\,\|\varphi\|_{H^1(\Omega)}\,|\nabla\mathbf{u}_1|_\Omega\\ \leq& \frac{\gamma}{8}|\nabla w|^2_\Omega + c\|\varphi\|^2_{H^1(\Omega)}|\nabla\mathbf{u}_1|^2_\Omega\\ \leq& \frac{\gamma}{8}|\nabla w|^2_\Omega + c(|\nabla\varphi|^2_\Omega + \lambda_s|\varphi|^2_\Gamma)|\nabla\mathbf{u}_1|^2_\Omega,\end{aligned}$$

and

$$\begin{aligned}|(\varphi_2\nabla w, \mathbf{u})_\Omega| \leq& |\nabla w|_\Omega\,|\varphi_2|_{L^4(\Omega)}\,|\mathbf{u}|_{L^4(\Omega)}\\ \leq& c|\nabla w|_\Omega\,\|\varphi_2\|_{H^1(\Omega)}\,|\mathbf{u}|^{1/2}_\Omega\,|\nabla\mathbf{u}|^{1/2}_\Omega\\ \leq& \frac{\gamma}{8}|\nabla w|^2_\Omega + c\|\varphi_2\|^2_{H^1(\Omega)}\,|\mathbf{u}|_\Omega|\nabla\mathbf{u}|_\Omega\\ \leq& \frac{\gamma}{8}|\nabla w|^2_\Omega + \frac{\nu}{16}|\nabla\mathbf{u}|^2_\Omega + c\|\varphi_2\|^4_{H^1(\Omega)}\,|\mathbf{u}|^2_\Omega.\end{aligned}$$

Similarly, we compute:

$$(\varphi_1\nabla w_1, \mathbf{u})_\Omega - (\varphi_2\nabla w_2, \mathbf{u})_\Omega \;=\; (\varphi\nabla w_1, \mathbf{u})_\Omega + (\varphi_2\nabla w, \mathbf{u})_\Omega,$$

with

$$\begin{aligned}|(\varphi\nabla w_1, \mathbf{u})_\Omega| \leq& |\nabla w_1|_\Omega\,|\varphi|_{L^4(\Omega)}\,|\mathbf{u}|_{L^4(\Omega)}\\ \leq& c\,|\nabla w_1|_\Omega\,\|\varphi\|_{H^1(\Omega)}\,|\nabla\mathbf{u}|_\Omega\\ \leq& \frac{\nu}{16}|\nabla\mathbf{u}|^2_\Omega + c|\nabla w_1|^2_\Omega\,(|\nabla\varphi|^2_\Omega + \lambda_s|\varphi|^2_\Gamma),\end{aligned}$$

and

$$\begin{aligned}|(\varphi_2\nabla w, \mathbf{u})_\Omega| \leq& c|\nabla w|_\Omega\|\varphi_2\|_{H^1(\Omega)}\,|\mathbf{u}|^{1/2}_\Omega|\nabla\mathbf{u}|^{1/2}_\Omega\\ \leq& \frac{\gamma}{8}|\nabla w|^2_\Omega + \frac{\nu}{16}|\nabla\mathbf{u}|^2_\Omega + c\|\varphi_2\|^4_{H^1}\,|\mathbf{u}|^2_\Omega.\end{aligned}$$

Hence, setting

$$H(t) = |\mathbf{u}|^2_\Omega + |\nabla\varphi|^2_\Omega + \delta|\nabla_\Gamma\varphi|^2_\Gamma + \lambda_s|\varphi|^2_\Gamma,$$

we infer from (12), (13) and the estimates above that the following estimate holds:

$$\frac{\mathrm{d}}{\mathrm{dt}}H(t) + \frac{\nu}{2}|\nabla\mathbf{u}|^2_\Omega + |\varphi_t|^2_\Gamma + \varepsilon|\varphi_t|^2_\Omega + \gamma|\nabla w|^2_\Omega \leq\; h(t)\,H(t),$$

where the function h belongs to $L^1(0,T)$.

Thus, we deduce from the Gronwall Lemma that:

$$H(t) \leq cH(0) = 0 \quad \forall t \geq 0 \qquad (\text{since } \varphi(0) = 0 \text{ and } \mathbf{u}(0) = 0).$$

In particular, we infer that, for every $t \geq 0$,

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|^2_\Omega + |\nabla\varphi_1(t) - \nabla\varphi_2(t)|^2_\Omega + \delta|\nabla_\Gamma\varphi_1(t) - \nabla_\Gamma\varphi_2(t)|^2_\Gamma + \lambda_s|\varphi_1(t) - \varphi_2(t)|^2_\Gamma = 0, \tag{15}$$

which implies the uniqueness of a weak solution.

2.3. **More regularity results.** In what follows we prove the following higher order regularity results for the solution of (1):

Lemma 2.2. *We assume that* $\mathbf{u}_0 \in (H_0^1(\Omega))^2, \varphi_0 \in H^1(\Omega, \Gamma)$. *Then the solution to* (1) *satisfies, for all* $T > 0$*:*

- $\varphi \in L^2(0,T; H^2(\Omega,\Gamma))$;
- $\mathbf{u} \in L^\infty(0,T; (H_0^1(\Omega))^2) \cap L^2(0,T; (H^2(\Omega))^2)$;
- $w \in L^2(0,T; H^2(\Omega))$.

We first prove the H^2- regularity for φ. To this aim, we rewrite the fourth and sixth equations of (1) as

$$\begin{cases} -\Delta\varphi + f(\varphi) = w - \varepsilon\varphi_t, & \text{in } \Omega_T, \\ -\delta\Delta_\Gamma\varphi + \lambda_s\varphi + g_s(\varphi) + \partial_n\varphi = -\varphi_t, & \text{on } \Gamma \times (0,T), \end{cases}$$

with the right hand side of these equations belonging respectively to $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;L^2(\Gamma))$. Then, we apply the Maximum Principle introduced in [25] and obtain:

$$\|\varphi\|^2_{L^\infty(\Omega)} + \|\varphi\|^2_{L^\infty(\Gamma)} \le c\,(1 + |w|^2_\Omega + \varepsilon^2|\varphi_t|^2_\Omega + |\varphi_t|^2_\Gamma).$$

Now, having the L^∞- estimates for φ and $\varphi_{|\Gamma}$, we can interpret the nonlinearities $f(\varphi)$ and $g_s(\varphi)$ as external forces as well and apply the H^2- regularity to the linear elliptic system

$$\begin{cases} -\Delta\varphi = w - \varepsilon\varphi_t - f(\varphi), \\ -\delta\Delta_\Gamma\varphi + \lambda_s\varphi + \partial_n\varphi = -\varphi_t - g_s(\varphi), \end{cases}$$

and obtain that φ is bounded in $L^2(0,T;H^2(\Omega,\Gamma))$ (for more details, see [25]).

Now we focus on the regularity for w. We write $(1)_3$ as:

$$\begin{cases} -\gamma\Delta w + w = -\varphi_t - \mathbf{u}\cdot\nabla\varphi \ + w, & \text{in } \Omega, \\ \partial_n w = 0, & \text{on } \Gamma. \end{cases} \tag{16}$$

Noticing that $\|\mathbf{u}\cdot\nabla\varphi\|_{L^{3/2}(\Omega)} \le |\mathbf{u}|_\Omega\,|\nabla\varphi|_{L^6(\Omega)}$, we obtain

$$\int_0^T |\mathbf{u}\cdot\nabla\varphi|^2_{L^{3/2}(\Omega)} \le c\|\mathbf{u}\|^2_{L^\infty(0,T;L^2(\Omega))}\,\|\varphi\|^2_{L^2(0,T;H^2(\Omega))} < +\infty,$$

in view of Lemma 2.1 and of the above proved regularity for φ. Using the fact that $\varphi_t + \mathbf{u}\cdot\nabla\varphi$ and w are in $L^2(0,T,L^{3/2}(\Omega))$, we apply the H^2-regularity for an elliptic linear system and we obtain $w \in L^2(0,T;W^{2,\frac{3}{2}}(\Omega))$.

Now we focus on the regularity for $\mathbf{u}$. We take $\mathbf{v} = A\mathbf{u}$ in $(3)_1$, where A is the Stokes operator (see [28]). We obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}|\nabla\mathbf{u}|^2_\Omega + \nu|A\mathbf{u}|^2_\Omega + b(\mathbf{u},\mathbf{u},A\mathbf{u}) + \lambda(\varphi\nabla w, A\mathbf{u})_\Omega = (\mathbf{h}, A\mathbf{u})_\Omega. \tag{17}$$

We can write:

$$\begin{aligned} |b(\mathbf{u},\mathbf{u},A\mathbf{u})| =& |\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{u}]\,A\mathbf{u}| \le |\mathbf{u}|_{L^4(\Omega)}|\nabla\mathbf{u}|_{L^4(\Omega)}|A\mathbf{u}|_\Omega \\ \le& c|\mathbf{u}|^{\frac{1}{2}}_\Omega\,|\nabla\mathbf{u}|_\Omega\,|A\mathbf{u}|^{\frac{3}{2}}_\Omega \le \frac{\nu}{6}|A\mathbf{u}|^2_\Omega + c|\mathbf{u}|^2_\Omega|\nabla\mathbf{u}|^4_\Omega \end{aligned}$$

and, using the fact that $W^{1,\frac{3}{2}}(\Omega) \hookrightarrow L^6(\Omega)$,

$$\begin{aligned}
\lambda|(\varphi\nabla w, A\mathbf{u})_\Omega| \leq & c|\varphi|_{L^4(\Omega)}\, |\nabla w|_{L^4(\Omega)} |A\mathbf{u}|_\Omega \\
\leq & \frac{\nu}{6}|A\mathbf{u}|^2_\Omega + c\|\varphi\|^2_{H^1(\Omega)}\|\nabla w\|^2_{L^6(\Omega)} \\
\leq & \frac{\nu}{6}|A\mathbf{u}|^2_\Omega + c\|\varphi\|^2_{H^1(\Omega)}\|w\|^2_{W^{2,\frac{3}{2}}(\Omega)} \\
|(\mathbf{h}, A\mathbf{u})_\Omega| \leq & \frac{\nu}{6}|A\mathbf{u}|^2_\Omega + c|\mathbf{h}|^2_\Omega.
\end{aligned}$$

Gathering all the estimates, we obtain the following energy estimate:

$$\frac{\mathrm{d}}{\mathrm{dt}}|\nabla\mathbf{u}|^2_\Omega + \nu|A\mathbf{u}|^2_\Omega \leq c|\mathbf{u}|^2_\Omega|\nabla\mathbf{u}|^4_\Omega + c\|\varphi\|^2_{H^1(\Omega)}\|w\|^2_{W^{2,\frac{3}{2}}(\Omega)} + c|\mathbf{h}|^2_\Omega. \tag{18}$$

Since $t \mapsto |\mathbf{u}|^2_\Omega\, |\nabla\mathbf{u}|^2_\Omega$ and $t \mapsto c\|\varphi(t)\|^2_{H^1(\Omega)}\, \|w(t)\|^2_{W^{2,\frac{3}{2}}(\Omega)} + c|\mathbf{h}(t)|^2_\Omega$ are in $L^1(0,T)$, we can apply the Gronwall lemma to (18) and we obtain that $\mathbf{u} \in L^\infty(0,T,(H^1_0(\Omega))^2) \cap L^2(0,T,(H^2(\Omega))^2)$.

Considering again problem (16), we now have $\varphi_t + \mathbf{u}\cdot\nabla\varphi$ in $L^2(0,T;L^2(\Omega))$. Then, applying one more time the H^2- regularity, this implies $w \in L^2(0,T;H^2(\Omega))$.

Lemma 2.3. *Let us suppose moreover that $\varphi_t(0) \in H^1(\Omega,\Gamma)$, $\mathbf{u}_t(0) \in L^2(\Omega)$ and $\mathbf{h}_t \in L^2(\Omega_T)$, the solution to (1) satisfies, for all $T > 0$:*

- $\varphi \in L^\infty(0,T;H^2(\Omega,\Gamma)) \cap \mathcal{C}^0([0,T];H^1(\Omega))$,
- $\varphi_t \in L^\infty(0,T;H^1(\Omega,\Gamma))$,
- $\mathbf{u} \in \mathcal{C}^0([0,T];(H^1_0(\Omega))^2)$,
- $\mathbf{u}_t \in L^\infty(0,T;(L^2(\Omega))^2) \cap L^2(0,T;(H^1_0(\Omega))^2)$,
- $\varphi_{tt} \in L^2(0,T;L^2(\Omega)\times L^2(\Gamma))$,
- $w \in L^\infty(0,T;H^2(\Omega)) \cap \mathcal{C}^0([0,T];H^{\frac{3}{2}}(\Omega))$, $\quad w_t \in L^2(0,T;H^1(\Omega))$.

Proof. We differentiate (1) with respect to t and find that $(\mathbf{u}_t, p_t, \varphi_t, w_t)$ satisfies the following system:

$$\left\{\begin{array}{l}
\mathbf{u}_{tt} - \nu\Delta\mathbf{u}_t + (\mathbf{u}_t\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}_t + \nabla p_t + \lambda\varphi_t\nabla w + \lambda\varphi\nabla w_t = \mathbf{h}_t \quad \text{in } \Omega_T, \\
\text{div } \mathbf{u}_t = 0, \qquad \text{in } \Omega_T, \\
\varphi_{tt} + \mathbf{u}_t\cdot\nabla\varphi + \mathbf{u}\cdot\nabla\varphi_t - \gamma\Delta w_t = 0, \qquad \text{in } \Omega_T, \\
w_t = -\Delta\varphi_t + f'(\varphi)\varphi_t + \varepsilon\varphi_{tt}, \qquad \text{in } \Omega_T, \\
\mathbf{u}_t = 0, \quad \partial_n w_t = 0, \qquad \text{on } \Gamma\times(0,T), \\
\varphi_{tt} = \delta\Delta_\Gamma\varphi_t - \lambda_s\varphi_t - g'_s(\varphi)\varphi_t - \partial_n\varphi_t, \qquad \text{on } \Gamma\times(0,T).
\end{array}\right. \tag{19}$$

We multiply the first equation of (19) by $\mathbf{u}_t$ and integrate over Ω. It yields:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}|\mathbf{u}_t|^2_\Omega + \nu|\nabla\mathbf{u}_t|^2_\Omega + ((\mathbf{u}_t\cdot\nabla)\mathbf{u},\mathbf{u}_t)_\Omega + \lambda(\varphi_t\nabla w,\mathbf{u}_t)_\Omega + \lambda(\varphi\nabla w_t,\mathbf{u}_t)_\Omega = (h_t,\mathbf{u}_t)_\Omega. \tag{20}$$

We first notice from (5) that $\langle\varphi_t\rangle = 0$. Multiplying $(19)_3$ by w_t, $(19)_4$ by φ_{tt}, integrating over Ω and summing the resulting equations, we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\{|\nabla\varphi_t|^2_\Omega + \delta|\nabla_\Gamma\varphi_t|^2_\Gamma + \lambda_s|\varphi_t|^2_\Gamma\right\} + |\varphi_{tt}|^2_\Gamma + \varepsilon|\varphi_{tt}|^2_\Omega + \gamma|\nabla w_t|^2_\Omega \\
&\qquad + (f'(\varphi)\varphi_t,\varphi_{tt})_\Omega + (g'_s(\varphi)\varphi_t,\varphi_{tt})_\Gamma + (\mathbf{u}_t\cdot\nabla\varphi, w_t)_\Omega + (\mathbf{u}\cdot\nabla\varphi_t, w_t)_\Omega = 0.
\end{aligned} \tag{21}$$

Now we notice that (since $\nabla \cdot \mathbf{u}_t = 0$ and $u_t = 0$ on Γ)

$$(\mathbf{u}_t \cdot \nabla\varphi, w_t)_\Omega = \sum_{i=1}^{2} \int_\Omega u_{t,i} \frac{\partial \varphi}{\partial x_i} w_t \, d\Omega = -\sum_{i=1}^{2} \int_\Omega \varphi u_{t,i} \frac{\partial w_t}{\partial x_i} d\Omega = -(\varphi \nabla w_t, \mathbf{u}_t)_\Omega.$$

Therefore, we sum (20) to equation (21) multiplied by λ, and use the following bounds:

$$\begin{aligned}
\lambda|(g_s'(\varphi)\varphi_t, \varphi_{tt})_\Gamma| &\le c|g_s'(\varphi)|_{L^\infty(\Gamma)} \, |\varphi_t|_\Gamma |\varphi_{tt}|_\Gamma \\
&\le c(|\varphi|^{q-1}_{L^\infty(\Gamma)} + 1) \, |\varphi_t|_\Gamma |\varphi_{tt}|_\Gamma \\
&\le c(\|\varphi\|^{q-1}_{H^1(\Gamma)} + 1) \, |\varphi_t|_\Gamma |\varphi_{tt}|_\Gamma \\
&\le \frac{\lambda}{4} |\varphi_{tt}|^2_\Gamma + c(\|\varphi\|^{2q-2}_{H^1(\Gamma)} + 1) \, |\varphi_t|^2_\Gamma, \\
\lambda|(f'(\varphi)\varphi_t, \varphi_{tt})_\Omega| &\le c|f'(\varphi)|_{L^4(\Omega)} \, |\varphi_t|_{L^4(\Omega)} \, |\varphi_{tt}|_\Omega \\
&\le c(|\varphi|^{p-1}_{L^{4(p-1)}} + 1) \, |\nabla \varphi_t|_\Omega \, |\varphi_{tt}|_\Omega \\
&\le \frac{\lambda\varepsilon}{4} |\varphi_{tt}|^2_\Omega + c(\|\varphi\|^{2(p-1)}_{H^1(\Omega)} + 1) \, |\nabla\varphi_t|^2_\Omega, \\
|((\mathbf{u}_t \cdot \nabla)\mathbf{u}, \mathbf{u}_t)_\Omega| &\le c|\mathbf{u}_t|^2_{L^4(\Omega)} \, |\nabla \mathbf{u}|_\Omega \\
&\le c|\mathbf{u}_t|_\Omega |\nabla \mathbf{u}_t|_\Omega \, |\nabla \mathbf{u}|_\Omega \\
&\le \frac{\nu}{8} |\nabla \mathbf{u}_t|^2_\Omega + c|\mathbf{u}_t|^2_\Omega \, |\nabla \mathbf{u}|^2_\Omega, \\
\lambda|(\varphi_t \nabla w, \mathbf{u}_t)_\Omega| &\le \lambda \, |\nabla w|_\Omega \, |\varphi_t|_{L^4(\Omega)} \, |\mathbf{u}_t|_{L^4(\Omega)} \\
&\le c \, |\nabla w|_\Omega \, |\nabla \varphi_t|_\Omega \, |\nabla \mathbf{u}_t|_\Omega \\
&\le \frac{\nu}{8} |\nabla \mathbf{u}_t|^2_\Omega + c|\nabla w|^2_\Omega \, |\nabla \varphi_t|^2_\Omega, \\
\lambda|(\mathbf{u} \cdot \nabla\varphi_t, w_t)_\Omega| &= \lambda|(\mathbf{u} \cdot \nabla w_t, \varphi_t)_\Omega| \\
&\le \lambda |\nabla w_t|_\Omega \, |\mathbf{u}|_{L^4} \, |\varphi_t|_{L^4} \\
&\le \frac{\lambda\gamma}{4} |\nabla w_t|^2_\Omega + c|\nabla \mathbf{u}|^2_\Omega |\nabla \varphi_t|^2_\Omega,
\end{aligned}$$

and

$$|(\mathbf{h}_t, \mathbf{u}_t)_\Omega| \le |\mathbf{h}_t|_\Omega \, |\mathbf{u}_t|_\Omega \le |\mathbf{h}_t|^2_\Omega + |\mathbf{u}_t|^2_\Omega.$$

Hence we obtain:

$$\frac{\mathrm{d}}{\mathrm{dt}} \mathcal{E} + \nu |\nabla \mathbf{u}_t|^2_\Omega + \lambda |\varphi_{tt}|^2_\Gamma + \lambda\varepsilon |\varphi_{tt}|^2_\Omega + \lambda\gamma |\nabla w_t|^2_\Omega \le \Lambda(t)\mathcal{E} + |\mathbf{h}_t|^2_\Omega, \tag{22}$$

with

$$\mathcal{E}(t) = |\mathbf{u}_t|^2_\Omega + \lambda\lambda_s |\varphi_t|^2_\Gamma + \lambda\delta |\nabla_\Gamma \varphi_t|^2_\Gamma + \lambda |\nabla \varphi_t|^2_\Omega, \tag{23}$$

and

$$\Lambda(t) = C(1 + \|\varphi\|^{2(p-1)}_{H^1(\Omega)} + \|\varphi\|^{2(q-1)}_{H^1(\Gamma)} + |\nabla \mathbf{u}|^2_\Omega + |\nabla w|^2_\Omega). \tag{24}$$

Applying Lemma 2.1, we infer that Λ defined in (24) belongs to $L^1(0,T)$. Hence, we can apply Gronwall Lemma to get the announced regularity for $\mathbf{u}_t$, φ_t and φ_{tt}. Since $\mathbf{u}_t$, $\mathbf{u} \in L^2(0,T;(H^1_0(\Omega))^2)$, then using Lemma 1.2 in [23] and eventually modifying $\mathbf{u}$ on a set of zero measure in $(0,T)$, we can conclude that $\mathbf{u} \in \mathcal{C}^0([0,T];(H^1_0(\Omega))^2)$.

We also obtain that $\nabla w_t \in L^2(0,T;(L^2(\Omega))^2)$. As previously, we have

$$\int_\Omega w_t\, d\Omega = \int_\Gamma \varphi_{tt} d\Gamma + \lambda_s \int_\Gamma \varphi_t d\Gamma + \int_\Gamma g_s'(\varphi)\varphi_t d\Gamma + \int_\Omega f'(\varphi)\varphi_t\, d\Omega,$$
$$\leq c(|\varphi_{tt}|_\Gamma + |\varphi_t|_\Gamma + |\varphi_t|_\Gamma(\|\varphi\|^{2(q-1)}_{H^1(\Gamma)} + 1) + |\varphi_t|_\Omega(\|\varphi\|^{2(p-1)}_{H^1(\Omega)} + 1)),$$

which means that $\langle w_t\rangle \in L^2(0,T)$ and $w_t \in L^2(0,T;H^1(\Omega))$. Thus, using the fact that $w \in L^2(0,T;H^1(\Omega))$, Lemma 1.2 in [23] implies that $w \in \mathcal{C}^0([0,T];H^1(\Omega))$ after an eventual modification of w on a set of zero measure in $[0,T]$.

It remains to prove the $L^\infty(H^2(\Omega,\Gamma))$- regularity for φ. To this aim, we proceed as in Lemma 2.2 and rewrite the fourth and sixth equations of (1) as

$$\begin{cases} -\Delta\varphi + f(\varphi) = w - \varepsilon\varphi_t, & \text{in } \Omega_T, \\ -\delta\Delta_\Gamma\varphi + \lambda_s\varphi + g_s(\varphi) + \partial_n\varphi = -\varphi_t, & \text{on } \Gamma\times(0,T), \end{cases}$$

remarking this time that the right hand side of these equations belongs respectively to $L^\infty(0,T;L^2(\Omega))$ and $L^\infty(0,T;L^2(\Gamma))$. Then, we apply the Maximum principle introduced in [25] and arguing as previously in Lemma 2.2, we obtain that φ is bounded in $L^\infty(0,T;H^2(\Omega,\Gamma))$.

Now, using the facts that $w \in L^\infty(0,T,L^2(\Omega))$ and $-\varphi_t - \mathbf{u}\cdot\nabla\varphi \in L^\infty(0,T;L^2(\Omega))$, we apply the H^2- regularity for the elliptic linear equation (16) and obtain $w \in L^\infty(0,T;H^2(\Omega))$. Using the fact that $w \in L^\infty(0,T;H^2(\Omega))$ and $w_t \in L^2(0,T;H^1(\Omega))$, we also obtain that $w \in \mathcal{C}([0,T];H^{3/2}(\Omega))$. □

Remark 2. We emphasize that the constant C appearing in (24) depends on $1/\varepsilon$. Hence we will not be able to obtain results for the zero viscosity Navier-Stokes-Cahn-Hilliard problem (i.e. $\varepsilon = 0$).

Lemma 2.4. *With the assumption of Lemma 2.3, and assuming* $\mathbf{h} \in \mathcal{C}^0([0,T];(L^2(\Omega)^2)$ *and* $\mathbf{h}_t \in L^2(0,T;\mathbf{W}_0')$*, the solution to* (1) *satisfies*

- $\mathbf{u} \in \mathcal{C}^0([0,T];(H^2(\Omega))^2)$
- $\mathbf{u}_t \in \mathcal{C}^0([0,T];(L^2(\Omega))^2)$.

Proof. First, we derive the regularity for $\mathbf{u}_t$. We introduce the operator $B(\mathbf{u},\mathbf{v}) = (\mathbf{u}\cdot\nabla)\mathbf{v}$ on $\mathbf{W}_0 \times \mathbf{W}_0$ and write $(19)_1$ as:

$$\mathbf{u}_{tt} + \nu A\mathbf{u}_t + B(\mathbf{u}_t,\mathbf{u}) + B(\mathbf{u},\mathbf{u}_t) + \lambda\varphi_t\nabla w + \lambda\varphi\nabla w_t = \mathbf{h}_t \quad \text{in } \mathbf{W}_0'$$

Since $\mathbf{u}_t \in L^2(0,T;\mathbf{W}_0)$ and $\mathbf{u} \in L^\infty(0,T;\mathbf{W}_0)$ we easily check that $A\mathbf{u}_t$ and $B(\mathbf{u}_t,\mathbf{u}) + B(\mathbf{u},\mathbf{u}_t)$ belongs to $L^2(0,T;\mathbf{W}_0')$.

Using $w \in \mathcal{C}^0([0,T];H^{\frac{3}{2}}(\Omega))$, we find that

$$\nabla w \in \mathcal{C}^0([0,T];(H^{\frac{1}{2}}(\Omega))^2) \subset \mathcal{C}^0([0,T];(L^4(\Omega))^2).$$

It follows from $\varphi_t \in L^\infty(0,T;H^1(\Omega))$ that $\varphi_t\nabla w \in L^\infty(0,T;L^2(\Omega)) \subset L^\infty(0,T;\mathbf{W}_0')$.

Recalling that $\varphi \in L^\infty(0,T;L^\infty(\Omega))$, we get that $\varphi\nabla w_t \in L^2(0,T;(L^2(\Omega))^2) \subset L^2(0,T,\mathbf{W}_0')$.

Then, it yields that $\mathbf{u}_{tt} \in L^2(0,T,\mathbf{W}_0')$, and, consequently $\mathbf{u}_t \in \mathcal{C}^0([0,T];(L^2(\Omega))^2)$.

Let us now focus on the regularity for $\mathbf{u}$. We rewrite (1) as

$$\nu A\mathbf{u} = -\mathbf{u}_t - B(\mathbf{u},\mathbf{u}) - \lambda\varphi\nabla w + \mathbf{h}.$$

Arguing as before, we easily infer that $-\mathbf{u}_t - \lambda\varphi\nabla w + \mathbf{h}$ is in $\mathcal{C}^0([0,T];(L^2(\Omega))^2)$. Applying a result in [27] with $\mathbf{u} \in \mathcal{C}^0([0,T];(H^1(\Omega))^2)$, we obtain $B(\mathbf{u},\mathbf{u}) \in \mathcal{C}([0,T],(H_0^{\frac{1}{2}}(\Omega))')$. Hence, using the regularizing effect of A, we obtain $\mathbf{u} \in$

$\mathcal{C}^0([0,T];(H^{\frac{3}{2}}(\Omega))^2)$. Applying again the lemma in [27] but now with $\mathbf{u}\in\mathcal{C}^0([0,T];(H^{\frac{3}{2}}(\Omega))^2)$, we now obtain $B(\mathbf{u},\mathbf{u})\in\mathcal{C}^0([0,T];(L^2(\Omega))^2)$ which leads to $A\mathbf{u}\in\mathcal{C}^0([0,T];(L^2(\Omega))^2)$ and $\mathbf{u}\in\mathcal{C}^0([0,T];(H^2(\Omega))^2)$. $\square$

3. **Existence of solutions.**

We consider the following problem, in order to obtain the Galerkin approximation for $\mathbf{u}$:

$$(\nabla\mathbf{u},\nabla\mathbf{v})_\Omega=(f,\mathbf{v})\quad\forall\mathbf{v}\in\mathbf{W}_0.$$

This problem defines the operator A_1 such that $A_1\mathbf{u}=f$. Then, $A_1^{-1}:(H^{-1}(\Omega))^2\to\mathbf{W}_0\subset(H_0^1(\Omega))^2\subset\subset(L^2(\Omega))^2$, A_1^{-1} is a compact, self-adjoint operator. Hence there exists $(\mathbf{f_j})_j\subset\mathbf{W}_0$ a sequence of orthonormal eigenfunctions such that $A_1\mathbf{f}_j=\lambda_j\mathbf{f}_j$, $\lambda_j>0$; the family $(\mathbf{f}_j)_j$ spans $(L^2(\Omega))^2$ and satisfies, for all $i,\,k\in\mathbb{N}$:

$$\begin{cases}(\mathbf{f}_i,\mathbf{f}_k)_\Omega=\delta_{i,k},\\(\nabla\mathbf{f}_i,\nabla\mathbf{f}_k)_\Omega=\lambda_i\,\delta_{i,k}.\end{cases}$$

We also introduce the operator $\mathcal{N}$ which is the inverse of the Laplacian operator $(-\Delta)$, where $-\Delta$ is endowed with Neumann boundary conditions, imposing zero average over the domain Ω. We take $\{e_n\}_n$ orthonormal in $L^2(\Omega)$, $e_1=const$ $(\beta_1=0)$ and $\mathcal{N}e_i=\frac{1}{\beta_i}e_i$, $\beta_i>0$, for all $i\geq 2$. Note that $\langle e_i\rangle=0\ \forall i\geq 2$.

We introduce the following finite dimensional approximate problem:

Search for $\mathbf{u}^n=\sum_{j=1}^n u_j(t)\,\mathbf{f_j}(x),\quad w^n=\sum_{j=1}^n w_j(t)e_j(x)\,,\quad \varphi^n=\sum_{j=1}^n\varphi_j(t)e_j(x)$ *such that:*

$$\begin{cases}(\mathbf{u}_t^n,\mathbf{v})_\Omega+\nu(\nabla\mathbf{u}^n,\nabla\mathbf{v})_\Omega+((\mathbf{u}^n\cdot\nabla)\mathbf{u}^n,\mathbf{v})_\Omega+\lambda(\varphi^n\nabla w^n,\mathbf{v})_\Omega=(P_n\mathbf{h},\mathbf{v})_\Omega,\\(\varphi_t^n,\psi)_\Omega+(\mathbf{u}^n\cdot\nabla\varphi^n,\psi)_\Omega+\gamma(\nabla w^n,\nabla\psi)_\Omega=0,\\\varepsilon(\varphi_t^n,\chi)_\Omega+(\varphi_t^n,\chi)_\Gamma+(\nabla\varphi^n,\nabla\chi)_\Omega+\delta(\nabla_\Gamma\varphi^n,\nabla_\Gamma\chi)_\Gamma+(f(\varphi^n),\chi)_\Omega\\+(g_s(\varphi^n),\chi)_\Gamma+\lambda_s(\varphi^n,\chi)_\Gamma-(w^n,\chi)_\Omega=0,\end{cases}\tag{25}$$

for all $\mathbf{v}\in\mathbf{W}_n=Span\{\mathbf{f}_1,\mathbf{f}_2,\dots,\mathbf{f}_n\}$, $\psi,\chi\in V_n=Span\{e_1,e_2,\dots,e_n\}$, *where* P_n *is the projection operator from* $(L^2(\Omega))^2$ *into* $\mathbf{W}_n$.

Now we take $\mathbf{v}=\mathbf{f}_j$, $j=1\dots n$ and $\chi=\psi=e_j$, $j=1\dots n$. We obtain:

$$\begin{cases}u_j'(t)+\nu\displaystyle\sum_{k=1}^n u_k(t)(\nabla\mathbf{f}_k,\nabla\mathbf{f}_j)_\Omega+\sum_{k,l=1}^n u_k(t)u_l(t)((\mathbf{f}_k\cdot\nabla)\mathbf{f}_l,\mathbf{f}_j)_\Omega\\\qquad\qquad+\lambda\displaystyle\sum_{k,l=1}^n\varphi_k(t)w_l(t)(e_k\nabla e_l,\mathbf{f}_j)_\Omega=(P_n\mathbf{h},\mathbf{f}_j)_\Omega,\\\varphi_j'(t)+\displaystyle\sum_{k,l=1}^n u_k(t)\varphi_l(t)(\mathbf{f}_k\cdot\nabla e_l,e_j)_\Omega+\gamma\sum_{k=1}^n w_k(t)(\nabla e_k,\nabla e_j)_\Omega=0,\\\varepsilon\varphi_j'(t)+\displaystyle\sum_{k=1}^n\varphi_k'(t)(e_k,e_j)_\Gamma+\sum_{k=1}^n\varphi_k(t)(\nabla e_k,\nabla e_j)_\Omega+\delta\sum_{k=1}^n\varphi_k(t)(\nabla_\Gamma e_k,\nabla_\Gamma e_j)_\Gamma\\+(f(\displaystyle\sum_{k=1}^n\varphi_k(t)e_k),e_j)_\Omega+(g_s(\sum_{k=1}^n\varphi_k(t)e_k),e_j)_\Gamma+\lambda_s\sum_{k=1}^n\varphi_k(t)(e_k,e_j)_\Gamma-w_j(t)=0.\end{cases}\tag{26}$$

We define the matrices:

$$(M_\Omega)_{ij} = (\nabla e_i, \nabla e_j)_\Omega, \quad (M_\Gamma)_{ij} = (\nabla_\Gamma e_i, \nabla_\Gamma e_j)_\Gamma,$$
$$(A_\Gamma)_{ij} = (e_i, e_j)_\Gamma \text{ and } D_{i,j} = (\nabla \mathbf{f}_i, \nabla \mathbf{f}_j)_\Omega, \quad 1 \le i, j \le n,$$

the vectors

$$U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad \Phi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix}, \quad W = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix},$$

and the functions

$$F(\Phi) = \begin{pmatrix} (f(\varphi^n), e_1)_\Omega \\ \vdots \\ (f(\varphi^n), e_n)_\Omega \end{pmatrix}, \quad G(\Phi) = \begin{pmatrix} (\lambda_s \varphi^n + g(\varphi^n), e_1)_\Gamma \\ \vdots \\ (\lambda_s \varphi^n + g(\varphi^n), e_n)_\Gamma \end{pmatrix}.$$

We also introduce the vector functions defined by:

$$(H_1(U(t), U(t)))_j = \sum_{k,l=1}^n u_k(t) u_l(t)((\mathbf{f}_k \cdot \nabla)\mathbf{f}_l, \mathbf{f}_j)_\Omega, \quad \forall\, j = 1, \dots, n,$$

$$(H_2(\Phi, W))_j = \sum_{k,l=1}^n \varphi_k(t) w_l(t)(e_k \nabla e_l, \mathbf{f}_j)_\Omega, \quad \forall\, j = 1, \dots, n,$$

$$(H_3(U, \Phi))_j = \sum_{k,l=1}^n u_k(t) \varphi_l(t)(\mathbf{f}_k \cdot \nabla e_l, e_j)_\Omega, \quad \forall\, j = 1, \dots, n,$$

$$(\tilde{H})_j = (P_n \mathbf{h}, f_j)_\Omega, \quad \forall\, j = 1, \dots, n.‘$$

Then, problem (26) becomes the following system of nonlinear ordinary differential equations:

$$\begin{cases} U'(t) + \nu D U(t) + H_1(U(t), U(t)) + \lambda H_2(\Phi, W) = \tilde{H}, \\ \Phi'(t) + \gamma M_\Omega W + H_3(U, \Phi) = 0, \\ \varepsilon \Phi'(t) + A_\Gamma \Phi'(t) + M_\Omega \Phi(t) + \delta M_\Gamma \Phi(t) + F(\Phi(t)) + G(\Phi(t)) = W. \end{cases} \tag{27}$$

Combining the last two equations in (27), we obtain:

$$\begin{aligned} (I_M + \varepsilon\gamma M_\Omega + \gamma M_\Omega A_\Gamma)\Phi'(t) + \gamma M_\Omega^2 \Phi(t) + \gamma\delta M_\Omega M_\Gamma \Phi(t) \\ + \gamma M_\Omega F(\Phi(t)) + \gamma M_\Omega G(\Phi(t)) + H_3(U, \Phi(t)) = 0. \end{aligned} \tag{28}$$

In order to be able to apply the Cauchy-Lipschitz theorem to $(27)_1$ and (28), we need to prove that the matrix $N = I_M + \varepsilon\gamma M_\Omega + \gamma M_\Omega A_\Gamma$ is invertible. Since $e_1 = const$, we emphasize that M_Ω has all elements from the first line and the first column equal to zero. As a consequence, $M_\Omega A_\Gamma$ has its first line equal to zero, and we are lead to prove that $\dot{N}$ is invertible, where $\dot{N}$ is the $(M-1) \times (M-1)$ matrix obtained, deleting the first line and the first column of N. The matrix $\dot{M}_\Omega$ is diagonal, with all terms on the diagonal strictly positive. In particular, $\dot{M}_\Omega$ is positive definite, and so is the matrix $\dot{M}_\Omega^{-1} + \varepsilon\gamma I_{M-1}$. We also notice that:

$$\dot{N} = I_{M-1} + \varepsilon\gamma \dot{M}_\Omega + \gamma \dot{M}_\Omega \dot{A}_\Gamma = \dot{M}_\Omega(\dot{M}_\Omega^{-1} + \varepsilon\gamma I_{M-1} + \gamma \dot{A}_\Gamma).$$

Since $\dot{A}_\Gamma$ is positive semi definite, we conclude that $\dot{N}$ is invertible. Thus we can solve the system in (Φ, U). By the Cauchy-Lipschitz theorem, problem (25) has a unique maximal solution $(\mathbf{u}^n, \varphi^n, w^n)$ in $\mathcal{C}^1([0, T^+); \mathbf{W}^n \times V^n \times V^n)$.

We take $\mathbf{v} = \mathbf{u}^n$, $\psi = \lambda w^n$ and $\chi = \lambda\varphi_t^n$ in (25) and repeat the computations performed for the a priori estimates. We obtain the following energy estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}J(u^n,\varphi^n)+\varepsilon\lambda|\varphi_t^n|_\Omega^2+\lambda|\varphi_t^n|_\Gamma^2+\lambda\gamma|\nabla w^n|_\Omega^2+\frac{\nu}{2}|\nabla\mathbf{u}^n|_\Omega^2 \le c|P_n\mathbf{h}|_\Omega^2 \le c\,|\mathbf{h}|_\Omega^2, \quad (29)$$

with J defined in (7).

Then, integrating (29) in time, we get:

$$|\mathbf{u}^n|_\Omega^2+\delta\lambda|\nabla_\Gamma\varphi^n|_\Gamma^2+\lambda_s\lambda|\varphi^n|_\Gamma^2+\lambda|\nabla\varphi^n|_\Omega^2+2\lambda\int_\Gamma G_s(\varphi^n)d\Gamma+2\lambda\int_\Omega F(\varphi^n)d\Omega \le C, \quad (30)$$

and

$$\nu\int_0^{T^+}|\nabla\mathbf{u}^n|_\Omega^2dt+\lambda\gamma\int_0^{T^+}|\nabla w^n|_\Omega^2dt+\lambda\int_0^{T^+}|\varphi_t^n|_\Gamma^2\,dt+\varepsilon\lambda\int_0^{T^+}|\varphi_t^n|_\Omega^2\,dt \le C, \quad (31)$$

where $C = C(T^+, |\mathbf{u}_0|_\Omega, \|\varphi_0\|_{H^1(\Omega,\Gamma)}, |\mathbf{h}|_{L^2(0,T;(L^2(\Omega))^2)})$.

Thus, we obtain the boundedness of φ^n in $L^\infty(0,T^+;H^1(\Omega,\Gamma))$, φ_t^n in $L^2(0,T^+;L^2(\Omega)\times L^2(\Gamma))$, $\mathbf{u}^n$ in $L^\infty(0,T^+;(L^2(\Omega))^2)\cap L^2(0,T^+;(H_0^1(\Omega))^2)$ and ∇w^n in $L^2(0,T^+;(L^2(\Omega))^2)$, and we conclude that $T^+=T$.

In particular, we obtain $\varphi^n\to\varphi$ weakly-star in $L^\infty(0,T;H^1(\Omega))$, $\varphi^n_{|\Gamma}\to\varphi_{|\Gamma}$ weakly-star in $L^\infty(0,T;H^1(\Gamma))$, $\varphi_t^n\to\varphi_t$ weakly in $L^2(0,T;L^2(\Omega))$, $\varphi_{t|\Gamma}^n\to\varphi_{t|\Gamma}$ weakly in $L^2(0,T;L^2(\Gamma))$. Hence, applying Aubin compactness Theorem, we infer that $\varphi^n\to\varphi$ strongly in $L^\infty(0,T;L^2(\Omega))$ and ${\varphi^n}_{|\Gamma}\to\varphi_{|\Gamma}$ strongly in $L^\infty(0,T;L^2(\Gamma))$.

We also test $(25)_3$ by $\chi=1$ since $1\in V_n$ ($e_1 = const$). We obtain (using the Sobolev embedding $H^1(\Omega)\subset L^p(\Omega)$ for Ω bounded in $\mathbb{R}^2$):

$$|\langle w^n\rangle| \le c(|\varphi_t^n|_\Omega+|\varphi_t^n|_\Gamma+\|\varphi^n\|_{H^1(\Omega)}^p+\|\varphi^n\|_{H^1(\Gamma)}^q+|\varphi^n|_\Gamma+1).$$

And we conclude that $(w^n)_n$ is bounded in $L^2(0,T;H^1(\Omega))$, thus $w^n\to w$ weakly in $L^2(0,T;H^1(\Omega))$. Since $\mathbf{u}^n\to\mathbf{u}$ weakly in $L^2(0,T;\mathbf{W}_0)$ and $\mathbf{u}^n\to\mathbf{u}$ weakly-star in $L^\infty(0,T;(L^2(\Omega))^2)$ we have the following convergence result for the nonlinear term (see [26] for more details):

Lemma 3.1. *If $\mathbf{u}^n$ converges to $\mathbf{u}$ weakly in $L^2(0,T;\mathbf{W}_0)$ and strongly in $L^2(0,T;(L^2(\Omega))^2)$, then, for every vector $\mathbf{v}$ in $\mathcal{C}^1([0,T]\times\bar\Omega)$, we have:*

$$\int_0^T((\mathbf{u}^n(t)\cdot\nabla)\mathbf{u}^n(t),\mathbf{v}(t))_\Omega \to \int_0^T((\mathbf{u}(t)\cdot\nabla)\mathbf{u}(t),\mathbf{v}(t))_\Omega\,dt.$$

Similar results are true for the other nonlinear terms, using the same kind of arguments, while for the nonlinear terms in the Cahn-Hilliard part of the model, the convergence is standard. This implies that we can pass to the limit $n\to+\infty$ in (25) and conclude the proof for the existence of a weak solution for (1). We thus proved the following result:

Theorem 3.2. *Let us consider $\mathbf{u}_0\in(L^2(\Omega))^2$, $\varphi_0\in H^1(\Omega,\Gamma)$ and $\mathbf{h}\in L^2(0,T;(L^2(\Omega))^2)$. Then problem (3) has a unique solution satisfying the following:*

$$\varphi\in L^\infty(0,T;H^1(\Omega,\Gamma)),\quad \varphi_t\in L^\infty(0,T;L^2(\Omega)\times L^2(\Gamma)),\quad w\in L^2(0,T;H^1(\Omega)),$$
$$\mathbf{u}\in L^\infty(0,T;(L^2(\Omega))^2)\cap L^2(0,T;(H_0^1(\Omega))^2).$$

Remark 3. Once a unique solution for problem (3) is known, we can recover the pressure p from (1) by classical arguments (see [27] or [14]).

4. **Finite element approximation.** In this section and all the following sections, the domain Ω will be considered to be a $2d$ slab, meaning $\Omega = (0, L_1)\times(\mathbb{R}/L_2\mathbb{Z})$ with smooth boundary $\Gamma = \{0, L_1\}\times(\mathbb{R}/L_2\mathbb{Z})$. The purpose of this section is to propose a finite element approximation for the exact solution of (1), where all the unknowns in (1) are searched to be periodic in the x_2 direction, while the Dirichlet boundary condition for $\mathbf{u}$ and the dynamic boundary condition for (φ, w) are imposed on $x_1 = 0$ and $x_1 = L_1$.

Starting with this section, the typical function spaces are:

$$H^m_p(\Omega,\Gamma) = \{\phi \in H^m_p(\Omega); \phi_{|\Gamma} \in H^m_{per}(\Gamma)\}, \quad \mathbf{W}_{0,p} = \{\mathbf{v} \in (H^1_{0,p}(\Omega))^2; \nabla\cdot\mathbf{v} = 0\}$$

where by H^m_p we understand the functions that belong to $H^m(\Omega)$ and which are periodic in the x_2-direction, while $H^m_{0,p}(\Omega) = \{\mathbf{v} \in H^m_p(\Omega); \mathbf{v} = 0 \text{ at } x_1 = 0 \text{ and } x_1 = L_1\}$. We remark here that all the results proved previously still remain valid when we consider the domain with periodic boundary conditions in one direction.

Let J^h be a quasi-uniform partitioning of Ω into disjoint open simplices K. We set $h_K =$ diam (K) and $h = \max\limits_{K\in J^h} h_K$. The mesh J^h is assumed to be weakly acute, that is for any pair of adjacent triangles the sum of the opposite angles relative to the common side does not exceed π. We also consider $J^{h/2}$ to be the mesh obtained from J^h by refining each simplex K into four similar triangles obtained joining the midpoints of each edge of K.

We set $V^h = \{v \in \mathcal{C}(\bar\Omega),\ v_{/K} \text{ linear } \forall K \in J^h\} \subset H^1_p(\Omega)$ and $V^{h/2} = \{v \in \mathcal{C}(\bar\Omega),$ $v_{/K}$ linear $\forall K \in J^{h/2}\} \subset H^1_p(\Omega)$. The triangulation J^h of $\bar\Omega$ induces in a natural way a triangulation Γ^h of Γ. Note that for every $v^h \in V^h$, the restriction $v^h|_\Gamma$ on the boundary is a P^1- finite element on the one dimensional domain Γ. Thus, V^h is also used for the discretisation of $H^1_p(\Omega,\Gamma)$ (see [4] for more details).

We now define the projections $Q^1_h : (L^2(\Omega))^2 \to (V^{h/2})^2$, such that:

$$|(I - Q^1_h)\mathbf{f}|_\Omega + h|\nabla(I - Q^1_h)\mathbf{f}|_\Omega \le Ch|\mathbf{f}|_{1,\Omega} \quad \forall\, \mathbf{f} \in (H^1_p(\Omega))^2, \tag{32}$$

and $Q^2_h : L^2(\Omega) \to V^h$, such that:

$$|(I - Q^2_h)\eta|_\Omega + h|\nabla(I - Q^2_h)\eta|_\Omega \le Ch|\eta|_{1,\Omega} \quad \forall\, \eta \in H^1_p(\Omega). \tag{33}$$

We introduce the elliptic projections of φ and w on V^h:

$$\begin{cases} (\nabla\tilde\varphi^h, \nabla\phi)_\Omega + \delta(\nabla_\Gamma\tilde\varphi^h, \nabla_\Gamma\phi)_\Gamma + \lambda_s(\tilde\varphi^h, \phi)_\Gamma = (\nabla\varphi, \nabla\phi)_\Omega \\ \qquad\qquad + \delta(\nabla_\Gamma\varphi, \nabla_\Gamma\phi)_\Gamma + \lambda_s(\varphi, \phi)_\Gamma, \\ (\nabla\tilde w^h, \nabla\phi)_\Omega = (\nabla w, \nabla\phi)_\Omega, \\ (\tilde w^h, 1)_\Omega = (w, 1)_\Omega, \end{cases} \tag{34}$$

for all $\phi \in V^h$.

Then, the following result holds (see [5] for more details):

Lemma 4.1. *For all $\varphi \in H^2_p(\Omega,\Gamma)$ and for all $w \in H^2_p(\Omega)$, the functions $(\tilde\varphi^h, \tilde w^h) \in V^h \times V^h$ defined by (34), satisfy:*

$$|\tilde\varphi^h - \varphi|_\Omega + |\tilde\varphi^h - \varphi|_\Gamma + h|\nabla(\tilde\varphi^h - \varphi)|_\Omega + h|\nabla_\Gamma(\tilde\varphi^h - \varphi)|_\Gamma \le ch^2(|\varphi|_{2,\Omega} + |\varphi|_{2,\Gamma}), \tag{35}$$

and

$$|\tilde w^h - w|_\Omega + h|\nabla(\tilde w^h - w)|_\Omega \le ch^2|w|_{2,\Omega}, \tag{36}$$

where $|\cdot|_{m,\Omega}$ and $|\cdot|_{m,\Gamma}$ are the seminorms associated with respectively the $H^m_p(\Omega)$ and $H^m_p(\Gamma)$ norms.

We now introduce the following finite element spaces:

$$\begin{aligned}
&\mathbf{W}^h = \{\mathbf{v} \in (V^{h/2})^2, \quad (\nabla\cdot\mathbf{v},\chi)_\Omega = 0 \quad \forall\,\chi\in V^h\},\\
&\mathbf{W}_0^h = \{\mathbf{v}\in\mathbf{W}^h;\ \mathbf{v}=0 \ \text{ on } \Gamma\},
\end{aligned}$$

and the elliptic projection of $\mathbf{u}$ on $\mathbf{W}_0^h$ which is defined by:

$$(\nabla\tilde{\mathbf{v}}^h,\nabla\mathbf{z})_\Omega = (\nabla\mathbf{v},\nabla\mathbf{z})_\Omega \quad \forall\mathbf{z}\in\mathbf{W}_0^h. \tag{37}$$

We emphasize that the space $\mathbf{W}_0^h$ is continuously embedded in $(H^1_{0,p}(\Omega))^2$, but not in $\mathbf{W}_{0,p}$. The following result is classical:

Lemma 4.2. *For all $\mathbf{v}\in\mathbf{W}_{0,p}$, the projection $\tilde{\mathbf{v}}^h$ defined by (37) satisfies:*

$$|\tilde{\mathbf{v}}^h-\mathbf{v}|_\Omega + h|\nabla(\tilde{\mathbf{v}}^h-\mathbf{v})|_\Omega \,\le\, Ch^s|\mathbf{v}|_{s,\Omega} \quad \forall\mathbf{v}\in(H^s_p(\Omega))^2\cap\mathbf{W}_{0,p}, \quad s=2,\dots$$

The semi-discrete scheme reads (we take $\lambda=1$):
Find $(\mathbf{u}^h,\varphi^h,w^h)\ \in\mathbf{W}_0^h\times V^h\times V^h$ satisfying:

$$\left\{\begin{aligned}
&(\mathbf{u}^h_t,\mathbf{v})_\Omega+\nu(\nabla\mathbf{u}^h,\nabla\mathbf{v})_\Omega+b(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v})+(\varphi^h\nabla w^h,\mathbf{v})_\Omega=(Q_h^1\mathbf{h},\mathbf{v})_\Omega,\\
&(\varphi^h_t,\psi)_\Omega+(\mathbf{u}^h\cdot\nabla\varphi^h,\psi)_\Omega+\gamma(\nabla w^h,\nabla\psi)_\Omega=0,\\
&\varepsilon(\varphi^h_t,\chi)_\Omega+(\varphi^h_t,\chi)_\Gamma+(\nabla\varphi^h,\nabla\chi)_\Omega+\delta(\nabla_\Gamma\varphi^h,\nabla_\Gamma\chi)_\Gamma+(f(\varphi^h),\chi)_\Omega\\
&\qquad\qquad+(g_s(\varphi^h),\chi)_\Gamma+\lambda_s(\varphi^h,\chi)_\Gamma-(w^h,\chi)_\Omega=0,
\end{aligned}\right. \tag{38}$$

for all $\mathbf{v}\in\mathbf{W}_0^h$ and $\psi,\chi\in V^h$.

Taking $\psi=1$ in $(38)_2$, we obtain:

$$(\varphi^h_t,1)_\Omega+(\mathbf{u}^h\cdot\nabla\varphi^h,1)_\Omega=0$$

and

$$(\mathbf{u}^h\cdot\nabla\varphi^h,1)_\Omega=\int_\Omega\mathbf{u}^h\cdot\nabla\varphi^h\,d\Omega=\int_\Gamma\mathbf{u}^h\cdot\mathbf{n}\,\varphi^h\,d\Gamma-\int_\Omega\varphi^h\,\nabla\cdot\mathbf{u}^h\,d\Omega=0,$$

since $\mathbf{u}^h\in\mathbf{W}_0^h$. We thus obtain the mass conservation for the approximation φ^h:

$$(\varphi^h,1)_\Omega \;=\; \text{const} \quad\ \forall t\ge 0.$$

We take $\mathbf{v}=\mathbf{u}^h$ in $(38)_1$, $\psi=w_h$ in $(38)_2$ and $\chi=\varphi^h_t$ in $(38)_3$ and we get:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}J(\mathbf{u}^h,\varphi^h)+\varepsilon|\varphi^h_t|^2_\Omega+|\varphi^h_t|^2_\Gamma+|\nabla w^h|^2_\Omega+\frac{\nu}{2}|\nabla\mathbf{u}^h|^2_\Omega\le c|Q^1_h\mathbf{h}|^2_\Omega,$$

with J as in (7).

This gives the following lemma:

Lemma 4.3. *The semi-discrete scheme satisfies the following stability estimate:*

$$\begin{aligned}
J(\mathbf{u}^h,\varphi^h)(T)+\int_0^T\{\varepsilon|\varphi^h_t|^2_\Omega+|\varphi^h_t|^2_\Gamma+|\nabla w^h|^2_\Omega+\frac{\nu}{2}|\nabla\mathbf{u}^h|^2_\Omega\}\,dt\\
\le\ J(\mathbf{u}^h,\varphi^h)(0)+\int_0^T c|Q^1_h\mathbf{h}|^2_\Omega\ \le C.
\end{aligned}$$

The existence of the solution for the discrete problem (38) follows from Lemma 4.3, using the theory of ordinary differential equations.

5. **Error estimates for the semi-discrete scheme.** In order to estimate the error between the exact solution of the variational problem (3) and the approximate solution given by (38), we use a splitting method introduced by [10] (see also [17], [5] for the application of the splitting method in different contexts). We write:

$$\begin{aligned} E_{\mathbf{u}} &:= \mathbf{u}^h - \mathbf{u} = \theta^{\mathbf{u}} + \rho^{\mathbf{u}} && \text{with} \quad \theta^{\mathbf{u}} = \mathbf{u}^h - \tilde{\mathbf{u}}^h, \quad \rho^{\mathbf{u}} = \tilde{\mathbf{u}}^h - \mathbf{u}, \\ E_{\varphi} &:= \varphi^h - \varphi = \theta^{\varphi} + \rho^{\varphi} && \text{with} \quad \theta^{\varphi} = \varphi^h - \tilde{\varphi}^h, \quad \rho^{\varphi} = \tilde{\varphi}^h - \varphi, \\ E_w &:= w^h - w = \theta^w + \rho^w && \text{with} \quad \theta^w = w^h - \tilde{w}^h, \quad \rho^w = \tilde{w}^h - w, \end{aligned}$$

where $(\tilde{\mathbf{u}}^h, \tilde{w}^h, \tilde{\varphi}^h)$ are the elliptic projections of $(\mathbf{u}, w, \varphi)$ defined by (37), (34).

We also define the discrete inverse Laplacian $T^h : \dot{L}^2(\Omega) \to \dot{V}^h$ $(\dot{V}^h = \dot{L}^2(\Omega) \cap V^h)$ by $T^h f = v^h$ where for a given $f \in \dot{L}^2(\Omega)$, v^h is the unique solution of the problem:

$$(\nabla v^h, \nabla \chi^h)_\Omega = (f, \chi^h)_\Omega \quad \forall \chi^h \in \dot{V}^h. \tag{39}$$

Note that (39) still holds for $\chi^h \in V^h$ since f is a function of zero average.

We also define the discrete negative seminorm:

$$|v|_{-1,h} := (T^h v, v)_\Omega^{1/2} = |\nabla T^h v|_\Omega \quad \forall v \in \dot{L}^2(\Omega).$$

Taking into account Lemma 4.1 and 4.2, we already have estimates on $\rho^{\mathbf{u}}$, ρ^{φ}, ρ^w.

It remains to estimate $\theta^{\mathbf{u}}$, θ^{φ}, θ^w; in order to do so we write the equations that they satisfy:

Equation for $\theta^{\mathbf{u}} = \mathbf{u}^h - \tilde{\mathbf{u}}^h$

Using (4) and (38), we obtain the following equation for $\theta^{\mathbf{u}}$:

$$\begin{aligned} (\theta^{\mathbf{u}}_t, \mathbf{v})_\Omega + \nu(\nabla\theta^{\mathbf{u}}, \nabla\mathbf{v})_\Omega = &-(\rho^{\mathbf{u}}_t, \mathbf{v})_\Omega - \nu(\nabla\rho^{\mathbf{u}}, \nabla\mathbf{v})_\Omega - b(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) \\ &-(p, \nabla\cdot\mathbf{v})_\Omega - (\varphi^h \nabla w^h - \varphi\nabla w, \mathbf{v})_\Omega + (Q^1_h \mathbf{h} - \mathbf{h}, \mathbf{v})_\Omega, \end{aligned} \tag{40}$$

$\forall\, \mathbf{v} \in \mathbf{W}^h_0 \subset (H^1_0(\Omega))^2$.

In order to deduce a priori estimates on $\theta^{\mathbf{u}}$, we take $\mathbf{v} = \theta^{\mathbf{u}}$ in (40) and obtain:

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}|\theta^{\mathbf{u}}|^2_\Omega + \nu|\nabla\theta^{\mathbf{u}}|^2_\Omega = &-(\rho^{\mathbf{u}}_t, \theta^{\mathbf{u}})_\Omega - \nu(\nabla\rho^{\mathbf{u}}, \nabla\theta^{\mathbf{u}})_\Omega - b(\mathbf{u}^h, \mathbf{u}^h, \theta^{\mathbf{u}}) + b(\mathbf{u}, \mathbf{u}, \theta^{\mathbf{u}}) \\ &-(p, \nabla\cdot\theta^{\mathbf{u}})_\Omega - (\varphi^h\nabla w^h - \varphi\nabla w, \theta^{\mathbf{u}})_\Omega + (Q^1_h\mathbf{h} - \mathbf{h}, \theta^{\mathbf{u}})_\Omega. \end{aligned} \tag{41}$$

We need to estimate the terms from the right hand side of (41). We have:

$$|(\rho^{\mathbf{u}}_t, \theta^{\mathbf{u}})_\Omega| \le c\|\rho^{\mathbf{u}}_t\|_{-1,\Omega}\, |\nabla\theta^{\mathbf{u}}|_\Omega \le \frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|^2_\Omega + c\|\rho^{\mathbf{u}}_t\|^2_{-1,\Omega} \le \frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|^2_\Omega + ch^2|\mathbf{u}_t|^2_\Omega,$$

where we used $\|\cdot\|_{-1,\Omega}$ to denote the norm on $H^{-1}(\Omega) = (H^1_0(\Omega))'$. We also used that $\|\rho^{\mathbf{u}}_t\|_{-1,\Omega} \le ch\,|\mathbf{u}_t|_\Omega$.

We also know that:

$$|\nu(\nabla\rho^{\mathbf{u}}, \nabla\theta^{\mathbf{u}})_\Omega| \le \frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|^2_\Omega + c|\nabla\rho^{\mathbf{u}}|^2_\Omega \le \frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|^2_\Omega + ch^2|\mathbf{u}|^2_{2,\Omega},$$

and

$$(p, \nabla\cdot\theta^{\mathbf{u}})_\Omega = (p - Q^2_h p, \nabla\cdot\theta^{\mathbf{u}})_\Omega + (Q^2_h p, \nabla\cdot\theta^{\mathbf{u}})_\Omega = (p - Q^2_h p, \nabla\cdot\theta^{\mathbf{u}})_\Omega,$$

since $Q^2_h p \in V^h$ and $\theta^{\mathbf{u}} \in W^h_0$, and we can write:

$$\begin{aligned} |(p - Q^2_h p, \nabla\cdot\theta^{\mathbf{u}})_\Omega| &\le c|p - Q^2_h p|_\Omega\, |\nabla\theta^{\mathbf{u}}|_\Omega \le ch|p|_{1,\Omega}|\nabla\theta^{\mathbf{u}}|_\Omega \\ &\le \frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|^2_\Omega + ch^2|p|^2_{1,\Omega}. \end{aligned}$$

For the nonlinear terms in b, we proceed as follows:

$$|b(\mathbf{u}^h,\mathbf{u}^h,\theta^{\mathbf{u}})-b(\mathbf{u},\mathbf{u},\theta^{\mathbf{u}})|=|b(\theta^{\mathbf{u}}+\rho^{\mathbf{u}},\mathbf{u}^h,\theta^{\mathbf{u}})+b(\mathbf{u},\theta^{\mathbf{u}}+\rho^{\mathbf{u}},\theta^{\mathbf{u}})|.$$

Thanks to the orthogonality property $b(\mathbf{u},\mathbf{v},\mathbf{v})=0$, we obtain:

$$\begin{aligned}|b(\mathbf{u}^h,\mathbf{u}^h,\theta^{\mathbf{u}})-b(\mathbf{u},\mathbf{u},\theta^{\mathbf{u}})| =&|b(E_{\mathbf{u}},\mathbf{u}^h,\theta^{\mathbf{u}})+b(\mathbf{u},\rho^{\mathbf{u}},\theta^{\mathbf{u}})|\\ \leq&|b(E_{\mathbf{u}},\theta^{\mathbf{u}},\theta^{\mathbf{u}})+b(E_{\mathbf{u}},\tilde{\mathbf{u}}^h,\theta^{\mathbf{u}})|+|b(\mathbf{u},\rho^{\mathbf{u}},\theta^{\mathbf{u}})|,\\ \leq&|b(E_{\mathbf{u}},\tilde{\mathbf{u}}^h,\theta^{\mathbf{u}})|+|b(\mathbf{u},\rho^{\mathbf{u}},\theta^{\mathbf{u}})|.\end{aligned}$$

The terms from the right hand side of the latter inequality are estimated as follows:

$$\begin{aligned}|b(\mathbf{u},\rho^{\mathbf{u}},\theta^{\mathbf{u}})| \leq&|\mathbf{u}|_{L^4(\Omega)}|\nabla\rho^{\mathbf{u}}|_\Omega|\theta^{\mathbf{u}}|_{L^4(\Omega)}\\ \leq&c|\mathbf{u}|_\Omega^{1/2}|\nabla\mathbf{u}|_\Omega^{1/2}|\nabla\rho^{\mathbf{u}}|_\Omega|\theta^{\mathbf{u}}|_\Omega^{1/2}|\nabla\theta^{\mathbf{u}}|_\Omega^{1/2}\\ \leq&\frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|_\Omega^2+ch^2|\mathbf{u}|_{2,\Omega}^2+c|\theta^{\mathbf{u}}|_\Omega^2,\end{aligned}$$

and

$$\begin{aligned}|b(E_{\mathbf{u}},\tilde{\mathbf{u}}^h,\theta^{\mathbf{u}})| \leq&|E_{\mathbf{u}}|_{L^4(\Omega)}|\nabla\tilde{\mathbf{u}}^h|_\Omega|\theta^{\mathbf{u}}|_\Omega^{1/2}|\nabla\theta^{\mathbf{u}}|_\Omega^{1/2}\\ \leq&c|\rho^{\mathbf{u}}|_\Omega^{1/2}|\nabla\rho^{\mathbf{u}}|_\Omega^{1/2}|\nabla\tilde{\mathbf{u}}^h|_\Omega|\theta^{\mathbf{u}}|_\Omega^{1/2}|\nabla\theta^{\mathbf{u}}|_\Omega^{1/2}+c|\nabla\tilde{\mathbf{u}}^h|_\Omega|\theta^{\mathbf{u}}|_\Omega|\nabla\theta^{\mathbf{u}}|_\Omega\\ \leq&\frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|_\Omega^2+c|\theta^{\mathbf{u}}|_\Omega^2+ch^3|\mathbf{u}|_{2,\Omega}^2,\end{aligned}$$

since, using the fact that $\mathbf{u}\in\mathcal{C}^0([0,T],(H_p^2(\Omega))^2)$, the term $|\nabla\tilde{\mathbf{u}}^h|_\Omega$ can be estimated as follows:

$$|\nabla\tilde{\mathbf{u}}^h|_\Omega\leq|\nabla(\tilde{\mathbf{u}}^h-\mathbf{u})|_\Omega+|\nabla\mathbf{u}|_\Omega\leq|\nabla\mathbf{u}|_\Omega+ch|\mathbf{u}|_{2,\Omega}\ \leq\ C. \tag{42}$$

We also estimate:

$$\begin{aligned}|(\varphi^h\nabla w^h-\varphi\nabla w,\theta^{\mathbf{u}})_\Omega| \leq&|(\theta^\varphi\nabla w^h,\theta^{\mathbf{u}})_\Omega|+|(\rho^\varphi\nabla w^h,\theta^{\mathbf{u}})_\Omega|\\ &+|(\varphi\nabla\theta^w,\theta^{\mathbf{u}})_\Omega|+|(\varphi\nabla\rho^w,\theta^{\mathbf{u}})_\Omega|,\end{aligned}$$

and for each term we proceed as follows:

$$\begin{aligned}|(\theta^\varphi\nabla w^h,\theta^{\mathbf{u}})_\Omega| \leq&c|\theta^\varphi|_{L^4(\Omega)}|\nabla w^h|_\Omega|\theta^{\mathbf{u}}|_\Omega^{\frac12}|\nabla\theta^{\mathbf{u}}|_\Omega^{\frac12}\\ \leq&c(|\nabla\theta^\varphi|_\Omega+\lambda_s|\theta^\varphi|_\Gamma)|\nabla w^h|_\Omega|\theta^{\mathbf{u}}|_\Omega^{\frac12}|\nabla\theta^{\mathbf{u}}|_\Omega^{\frac12}\\ \leq&\frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|_\Omega^2+c|\theta^{\mathbf{u}}|_\Omega^2+c(|\nabla\theta^\varphi|_\Omega^2+\lambda_s|\theta^\varphi|_\Gamma^2)|\nabla w^h|_\Omega^2,\\ |(\rho^\varphi\nabla w^h,\theta^{\mathbf{u}})_\Omega| \leq&c\|\rho^\varphi\|_{H^1(\Omega)}^{\frac12}|\rho^\varphi|_\Omega^{\frac12}|\nabla w^h|_\Omega|\theta^{\mathbf{u}}|_\Omega^{\frac12}|\nabla\theta^{\mathbf{u}}|_\Omega^{\frac12}\\ \leq&\frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|_\Omega^2+c|\theta^{\mathbf{u}}|_\Omega^2+ch^3|\nabla w^h|_\Omega^2,\\ |(\varphi\nabla\theta^w,\theta^{\mathbf{u}})_\Omega| \leq&\|\varphi\|_{L^\infty(\Omega)}|\nabla\theta^w|_\Omega|\theta^{\mathbf{u}}|_\Omega\ \leq c\,\|\varphi\|_{H^2(\Omega)}|\nabla\theta^w|_\Omega|\theta^{\mathbf{u}}|_\Omega\\ \leq&\frac{\gamma}{12}|\nabla\theta^w|_\Omega^2+c|\theta^{\mathbf{u}}|_\Omega^2,\end{aligned}$$

and similarly,

$$|(\varphi\nabla\rho^w,\theta^{\mathbf{u}})_\Omega|\leq|\nabla\rho^w|_\Omega^2+c|\theta^{\mathbf{u}}|_\Omega^2,$$

where we used that $\varphi\in L^\infty(0,T;H^2(\Omega))$.

The last term in (41) is estimated as follows:

$$|(Q_h^1\mathbf{h}-\mathbf{h},\theta^{\mathbf{u}})_\Omega|\leq\frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|_\Omega^2+c|Q_h^1\mathbf{h}-\mathbf{h}|_\Omega^2\leq\frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|_\Omega^2+ch^2|\mathbf{h}|_{1,\Omega}^2.$$

Equation for $\theta^\varphi=\varphi^h-\tilde{\varphi}^h$

Combining (3), (38) and (34), we find:

$$
\begin{aligned}
\varepsilon(\varphi_t^h,\chi)_\Omega &+ (\varphi_t^h,\chi)_\Gamma + (\nabla\theta^\varphi,\nabla\chi)_\Omega + \delta(\nabla_\Gamma\theta^\varphi,\nabla_\Gamma\chi)_\Gamma + (f(\varphi^h),\chi)_\Omega \\
&+(g_s(\varphi^h),\chi)_\Gamma + \lambda_s(\theta^\varphi,\chi)_\Gamma \\
&= (w^h,\chi)_\Omega - (\nabla\varphi,\nabla\chi)_\Omega - \delta(\nabla_\Gamma\varphi,\nabla_\Gamma\chi)_\Gamma - \lambda_s(\varphi,\chi)_\Gamma \\
&= (w^h,\chi)_\Omega + \varepsilon(\varphi_t,\chi)_\Omega + (\varphi_t,\chi)_\Gamma + (f(\varphi),\chi)_\Omega + (g_s(\varphi),\chi)_\Gamma - (w,\chi)_\Omega,
\end{aligned}
$$

for all $\chi \in V^h$, which yields to the following equation for θ^φ:

$$
\begin{aligned}
&\varepsilon(\theta_t^\varphi,\chi)_\Omega + (\theta_t^\varphi,\chi)_\Gamma + (\nabla\theta^\varphi,\nabla\chi)_\Omega + \delta(\nabla_\Gamma\theta^\varphi,\nabla_\Gamma\chi)_\Gamma + (f(\varphi^h)-f(\varphi),\chi)_\Omega \\
&+\lambda_s(\theta^\varphi,\chi)_\Gamma + (g_s(\varphi^h)-g_s(\varphi),\chi)_\Gamma - (\theta^w,\chi)_\Omega = (\rho^w,\chi)_\Omega - \varepsilon(\rho_t^\varphi,\chi)_\Omega - (\rho_t^\varphi,\chi)_\Gamma,
\end{aligned} \tag{43}
$$

for all $\chi \in V^h$.

Equation for $\theta^w = w^h - \tilde{w}^h$

The equation for θ^w is obtained by (3), (38) and (34), and it reads as:

$$
(\theta_t^\varphi,\phi)_\Omega + \gamma(\nabla\theta^w,\nabla\phi)_\Omega + (\rho_t^\varphi,\phi)_\Omega + (\mathbf{u}^h\cdot\nabla\varphi^h - \mathbf{u}\cdot\nabla\varphi,\phi)_\Omega = 0 \quad \forall\phi\in V^h. \tag{44}
$$

We need to obtain estimates on θ^φ and θ^w. For that purpose, we take $\chi = \theta_t^\varphi$ in (43), $\phi = \theta^w$ in (44) and combining the resulting equations, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\{\delta|\nabla_\Gamma\theta^\varphi|_\Gamma^2 + |\nabla\theta^\varphi|_\Omega^2 + \lambda_s|\theta^\varphi|_\Gamma^2\right\} + \varepsilon|\theta_t^\varphi|_\Omega^2 + |\partial_t\theta^\varphi|_\Gamma^2 + \gamma|\nabla\theta^w|_\Omega^2 \\
&+(f(\varphi^h)-f(\varphi),\theta_t^\varphi)_\Omega + (g_s(\varphi^h)-g_s(\varphi),\theta_t^\varphi)_\Gamma + (\rho_t^\varphi,\theta^w)_\Omega \\
&+(\mathbf{u}^h\cdot\nabla\varphi^h - \mathbf{u}\cdot\nabla\varphi,\theta^w)_\Omega = (\rho^w,\theta_t^\varphi)_\Omega - \varepsilon(\rho_t^\varphi,\theta_t^\varphi)_\Omega - (\rho_t^\varphi,\theta_t^\varphi)_\Gamma.
\end{aligned} \tag{45}
$$

The following Remark will help us estimate the terms in f and g_s:

Remark 4. Since $\varphi \in L^\infty(0,T,H_p^2(\Omega,\Gamma))$, we assume that $\sup_{t\in[0,T]}\|\varphi(t)\|_{\mathcal{C}^0(\bar{\Omega})} < R$, $\|\varphi^h(0)\|_{\mathcal{C}^0(\bar{\Omega})} < R$ for some constant $R < +\infty$. Let $\hat{T}^h$ be the maximal time such that $\|\varphi^h(t)\|_{L^\infty(\Omega)} \leq R$ for all $t \in [0,\hat{T}^h]$. Then, we have:

$$
|f(\varphi^h) - f(\varphi)|_\Omega \leq L_f|\varphi^h - \varphi|_\Omega,
$$

and

$$
|g_s(\varphi^h) - g_s(\varphi)|_\Gamma \leq L_{g_s}|\varphi^h - \varphi|_\Gamma \text{ on } [0,\hat{T}^h],
$$

where L_f and L_{g_s} are respectively the Lipschitz constants of f and g_s on $[-R,R]$.

Thus, applying Remark 4, we obtain:

$$
\begin{aligned}
|(f(\varphi^h)-f(\varphi),\theta_t^\varphi)_\Omega| &\leq \frac{\varepsilon}{12}|\theta_t^\varphi|_\Omega^2 + c|f(\varphi^h)-f(\varphi)|_\Omega^2, \\
&\leq \frac{\varepsilon}{12}|\theta_t^\varphi|_\Omega^2 + c|\theta^\varphi|_\Omega^2 + c|\rho^\varphi|_\Omega^2, \\
&\leq \frac{\varepsilon}{12}|\theta_t^\varphi|_\Omega^2 + ch^4|\varphi|_{2,\Omega}^2 + c(|\nabla\theta^\varphi|_\Omega^2 + \lambda_s|\theta^\varphi|_\Gamma^2),
\end{aligned}
$$

where here and below the constant c can depend increasingly in ε^{-1}. Similarly,

$$
|(g_s(\varphi^h)-g_s(\varphi),\theta_t^\varphi)_\Gamma| \leq \frac{1}{12}|\theta_t^\varphi|_\Gamma^2 + c|\theta^\varphi|_\Gamma^2 + ch^4|\varphi|_{2,\Gamma}^2.
$$

We also have:

$$
|(\rho^w,\theta_t^\varphi)_\Omega| \leq \frac{\varepsilon}{12}|\theta_t^\varphi|_\Omega^2 + ch^4|w|_{2,\Omega}^2,
$$

$$\varepsilon|(\rho_t^\varphi, \theta_t^\varphi)_\Omega| \le \frac{\varepsilon}{12}|\theta_t^\varphi|_\Omega^2 + ch^2|\varphi_t|_{1,\Omega}^2,$$

$$|(\rho_t^\varphi, \theta_t^\varphi)_\Gamma| \le \frac{1}{12}|\theta_t^\varphi|_\Gamma^2 + ch^2|\varphi_t|_{1,\Gamma}^2.$$

Before continuing to bound the remaining terms in (45), we need to estimate $\langle\theta^w\rangle = \frac{1}{|\Omega|}(\theta^w, \mathbf{1})_\Omega$. We choose $\chi = \mathbf{1}$ in (43) and obtain

$$\begin{aligned}\varepsilon(\theta_t^\varphi, \mathbf{1})_\Omega + (\theta_t^\varphi, \mathbf{1})_\Gamma + (f(\varphi^h) - f(\varphi), \mathbf{1})_\Omega + (g_s(\varphi^h) - g_s(\varphi), \mathbf{1})_\Gamma - (\theta^w, \mathbf{1})_\Omega \\ +\lambda_s(\theta^\varphi, \mathbf{1})_\Gamma = (\rho^w, \mathbf{1})_\Omega - \varepsilon(\rho_t^\varphi, \mathbf{1})_\Omega - (\rho_t^\varphi, \mathbf{1})_\Gamma.\end{aligned}$$

Since $(\varphi_t^h, \mathbf{1})_\Omega = 0$ and $(\varphi_t, \mathbf{1})_\Omega = 0$, then we have:

$$(\theta_t^\varphi, \mathbf{1})_\Omega + (\rho_t^\varphi, \mathbf{1})_\Omega = ((E_\varphi)_t, \mathbf{1})_\Omega = 0,$$

and

$$\begin{aligned}|\langle\theta^w\rangle| \le& c(|(\theta_t^\varphi, \mathbf{1})_\Gamma + |(f(\varphi^h) - f(\varphi), \mathbf{1})_\Omega| + |(g_s(\varphi^h) - g_s(\varphi), \mathbf{1})_\Gamma| + \lambda_s|(\theta^\varphi, \mathbf{1})_\Gamma| \\ &+ |(\rho^w, \mathbf{1})_\Omega| + |(\rho_t^\varphi, \mathbf{1})_\Gamma|) \\ \le& c(|\theta_t^\varphi|_\Gamma + |f(\varphi^h) - f(\varphi)|_\Omega + |g_s(\varphi^h) - g_s(\varphi)|_\Gamma + |\theta^\varphi|_\Gamma + |\rho^w|_\Omega + |\rho_t^\varphi|_\Gamma) \\ \le& c(|\theta_t^\varphi|_\Gamma + |\theta^\varphi|_\Omega + |\theta^\varphi|_\Gamma) + h(|\varphi|_{2,\Omega} + |\varphi|_{2,\Gamma} + |w|_{2,\Omega} + |\varphi_t|_{1,\Gamma}),\end{aligned}$$

where we used again Remark 4.

We can now estimate the remaining terms in (45):

$$\begin{aligned}|(\rho_t^\varphi, \theta^w)_\Omega| \le& |\rho_t^\varphi|_\Omega|\theta^w|_\Omega \le c_0|\theta^w|_\Omega^2 + c|\rho_t^\varphi|_\Omega^2 \\ \le& c_0c(\gamma|\nabla\theta^w|_\Omega^2 + |m(\theta^w)|^2) + c|\rho_t^\varphi|_\Omega^2 \\ \le& c_0c(\gamma|\nabla\theta^w|_\Omega^2 + |\theta_t^\varphi|_\Gamma^2) + c_0c(|\nabla\theta^\varphi|_\Omega^2 + \lambda_s|\theta^\varphi|_\Gamma^2) \\ &+ ch^2(|\varphi|_{2,\Omega}^2 + |\varphi|_{2,\Gamma}^2 + |w|_{2,\Omega}^2 + |\varphi_t|_{1,\Gamma}^2 + |\varphi_t|_{1,\Omega}^2),\end{aligned}$$

and we choose the positive constant c_0 such that $cc_0 \le \frac{1}{12}$.

For the last term from the left hand side of (45), we write:

$$\begin{aligned}&|(\mathbf{u}^h \cdot \nabla\varphi^h - \mathbf{u} \cdot \nabla\varphi, \theta^w)_\Omega| \le |((\mathbf{u}^h - \mathbf{u}) \cdot \nabla\varphi^h, \theta^w)_\Omega| + |(\mathbf{u} \cdot \nabla(\varphi^h - \varphi), \theta^w)_\Omega| \\ &\le |(\theta^{\mathbf{u}} \cdot \nabla\varphi^h, \theta^w)_\Omega| + |(\rho^{\mathbf{u}} \cdot \nabla\varphi^h, \theta^w)_\Omega| + |(\mathbf{u} \cdot \nabla\theta^\varphi, \theta^w)_\Omega| + |(\mathbf{u} \cdot \nabla\rho^\varphi, \theta^w)_\Omega|\end{aligned} \tag{46}$$

and we now need to bound the terms from the right hand side of (46). For the first term, we write:

$$\begin{aligned}|(\theta^{\mathbf{u}} \cdot \nabla\varphi^h, \theta^w)_\Omega| \le& c|\theta^{\mathbf{u}}|_\Omega^{\frac{1}{2}}|\nabla\theta^{\mathbf{u}}|_\Omega^{\frac{1}{2}}|\nabla\varphi^h|_\Omega(|\nabla\theta^w|_\Omega + |\langle\theta^w\rangle|) \\ \le& \frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|_\Omega^2 + c_0\{\gamma|\nabla\theta^w|_\Omega^2 + |\langle\theta^w\rangle|^2\} + c|\nabla\varphi^h|_\Omega^4|\theta^{\mathbf{u}}|_\Omega^2 \\ \le& \frac{\nu}{16}|\nabla\theta^{\mathbf{u}}|_\Omega^2 + cc_0\{\gamma|\nabla\theta^w|_\Omega^2 + |\theta_t^\varphi|_\Gamma^2 + |\nabla\theta^\varphi|_\Omega^2 + \lambda_s|\theta^\varphi|_\Gamma^2\} \\ &+ c|\theta^{\mathbf{u}}|_\Omega^2 + ch^2(|\varphi|_{2,\Omega}^2 + |\varphi|_{2,\Gamma}^2 + |w|_{2,\Omega}^2 + |\varphi_t|_{1,\Gamma}^2),\end{aligned}$$

where we used that φ^h is bounded in $L^\infty(0, T; H_p^1(\Omega, \Gamma))$, thanks to Lemma 4.3 and we choose the positive constant c_0 small enough such that $cc_0 < 1/12$.

We proceed similarly for the second term in (46) and we obtain:

$$\begin{aligned}|\,(\rho^{\mathbf{u}}\cdot\nabla\varphi^h,\theta^w)_\Omega| \leq& |\rho^{\mathbf{u}}|_\Omega^{\frac12}|\nabla\rho^{\mathbf{u}}|_\Omega^{\frac12}|\nabla\varphi^h|_\Omega(|\nabla\theta^w|_\Omega+|\langle\theta^w\rangle|)\\ \leq& c_0(\gamma|\nabla\theta^w|_\Omega^2+|\langle\theta^w\rangle|^2)+c|\rho^{\mathbf{u}}|_\Omega|\nabla\rho^{\mathbf{u}}|_\Omega|\nabla\varphi^h|_\Omega^2\\ \leq& cc_0(\gamma|\nabla\theta^w|_\Omega^2+|\theta_t^\varphi|_\Gamma^2)+cc_0(|\nabla\theta^\varphi|_\Omega^2+\lambda_s|\theta^\varphi|_\Gamma^2)\\ &+ch^2(|\varphi|_{2,\Omega}^2+|\varphi|_{2,\Gamma}^2+|w|_{2,\Omega}^2+|\varphi_t|_{2,\Gamma}^2+|\mathbf{u}|_{2,\Omega}^2).\end{aligned}$$

We choose c_0 small enough such that $cc_0<1/12$.

The third term in the right hand side of (46) is bounded as follows:

$$\begin{aligned}|(\mathbf{u}\cdot\nabla\theta^\varphi,\theta^w)_\Omega| \leq& c|\mathbf{u}|_\Omega^{\frac12}|\nabla\mathbf{u}|_\Omega^{\frac12}|\nabla\theta^\varphi|_\Omega\|\theta^w\|_{H^1(\Omega)}\\ \leq& c|\mathbf{u}|_\Omega^{\frac12}|\nabla\mathbf{u}|_\Omega^{\frac12}|\nabla\theta^\varphi|_\Omega(|\nabla\theta^w|_\Omega+|\langle\theta^w\rangle|)\\ \leq& cc_0(\gamma|\nabla\theta^w|_\Omega^2+|\theta_t^\varphi|_\Gamma^2)+cc_0(|\nabla\theta^\varphi|_\Omega^2+\lambda_s|\theta^\varphi|_\Gamma^2)+c|\nabla\theta^\varphi|_\Omega^2\\ &+ch^2(|\varphi|_{2,\Omega}^2+|\varphi|_{2,\Gamma}^2+|w|_{2,\Omega}^2+|\varphi_t|_{1,\Gamma}^2),\end{aligned}$$

with c_0 such that $cc_0<1/12$. We also have:

$$\begin{aligned}|(\mathbf{u}\cdot\nabla\rho^\varphi,\theta^w)_\Omega| \leq& c|\mathbf{u}|_\Omega^{\frac12}|\nabla\mathbf{u}|_\Omega^{\frac12}|\nabla\rho^\varphi|_\Omega\|\theta^w\|_{H^1(\Omega)}\\ \leq& c|\mathbf{u}|_\Omega|\nabla\mathbf{u}|_\Omega|\nabla\rho^\varphi|_\Omega^2+c_0(\gamma|\nabla\theta^w|_\Omega^2+|\langle\theta^w\rangle|^2)\\ \leq& c_0c(\gamma|\nabla\theta^w|_\Omega^2+|\theta_t^\varphi|_\Gamma^2)+c_0c(|\nabla\theta^\varphi|_\Omega^2+\lambda_s|\theta^\varphi|_\Gamma^2)\\ &+ch^2(|\varphi|_{2,\Omega}^2+|\varphi|_{2,\Gamma}^2+|w|_{2,\Omega}^2+|\varphi_t|_{1,\Gamma}^2),\end{aligned}$$

and c_0 is taken again such that $cc_0<1/12$.

Gathering all the estimates on θ^φ, θ^w and $\theta^{\mathbf{u}}$ deduced above, we obtain the following inequality:

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{dt}}\{|\theta^{\mathbf{u}}|_\Omega^2+\delta|\nabla_\Gamma\theta^\varphi|_\Gamma^2+|\nabla\theta^\varphi|_\Omega^2+\lambda_s|\theta^\varphi|_\Gamma^2\}+\nu|\nabla\theta^{\mathbf{u}}|_\Omega^2+\varepsilon|\theta_t^\varphi|_\Omega^2+|\theta_t^\varphi|_\Gamma^2+\gamma|\nabla\theta^w|_\Omega^2\\ &\leq\ \frac12\{\nu|\nabla\theta^{\mathbf{u}}|_\Omega^2+\varepsilon|\theta_t^\varphi|_\Omega^2+|\theta_t^\varphi|_\Gamma^2+\gamma|\nabla\theta^w|_\Omega^2\}\\ &\quad+c(1+|\nabla w^h|_\Omega^2)\left(|\theta^{\mathbf{u}}|_\Omega^2+\delta|\nabla_\Gamma\theta^\varphi|_\Gamma^2+|\nabla\theta^\varphi|_\Omega^2+\lambda_s|\theta^\varphi|_\Gamma^2\right)\\ &\quad+ch^2(|\varphi|_{2,\Omega}^2+|\varphi|_{2,\Gamma}^2+|w|_{2,\Omega}^2+|\varphi_t|_{1,\Omega}^2+|\varphi_t|_{1,\Gamma}^2+|\mathbf{u}|_{2,\Omega}^2+|\mathbf{u}_t|_\Omega^2+|p|_{1,\Omega}^2+|\mathbf{h}|_{1,\Omega}^2).\end{aligned}\tag{47}$$

Setting $\mathcal{N}(t)=|\theta^{\mathbf{u}}|_\Omega^2+\delta|\nabla_\Gamma\theta^\varphi|_\Omega^2+|\nabla\theta^\varphi|_\Omega^2+\lambda_s|\theta^\varphi|_\Gamma^2$, we obtain the following differential inequality:

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{N}\ \leq\ f\mathcal{N}+g,\tag{48}$$

where we denoted by f the following $L^1(0,T)$-function:

$$f(t)=c(1+|\nabla w^h|_\Omega^2),$$

and where g is the function :

$$g(t)=ch^2\left(|\varphi|_{2,\Omega}^2+|\varphi|_{2,\Gamma}^2+|w|_{2,\Omega}^2+|\varphi_t|_{1,\Omega}^2+|\varphi_t|_{1,\Gamma}^2+|\mathbf{u}|_{2,\Omega}^2+|\mathbf{u}_t|_\Omega^2+|p|_{1,\Omega}^2+|\mathbf{h}|_{1,\Omega}^2\right).$$

Hence, applying the classical Gronwall Lemma to (48), we conclude that:

$$\mathcal{N}(t)\ \leq c_1\mathcal{N}(0)+c_2h^2.\tag{49}$$

We thus proved the following theorem:

Theorem 5.1. *Let the forcing term* $\mathbf{h}$ *be sufficiently regular and let* $(\mathbf{u}, p, \varphi, w)$ *be the solution of the Viscous-Cahn-Hilliard-Navier-Stokes system considered such that*

$$\mathbf{u} \in \mathcal{C}^0([0,T];(H_p^2(\Omega))^2),\ \mathbf{u}_t \in \mathcal{C}^0([0,T];(L^2(\Omega))^2),\ \varphi \in\ L^\infty(0,T;H_p^2(\Omega,\Gamma)),$$
$$w \in L^2(0,T;H_p^2(\Omega)),\ \varphi_t\ \in L^2(0,T;H_p^1(\Omega,\Gamma)),\ p \in L^2(0,T;H_p^1(\Omega)),$$

and let $(\mathbf{u}^h, \varphi^h, w^h)$ *be a solution of* (38) *satisfying the conditions of Remark 4. If* $\theta^{\mathbf{u}}(0) = 0$ *and* $\theta^\varphi(0) = 0$*, then*

$$\sup_{[0,T]} \left(|\mathbf{u}^h - \mathbf{u}|_\Omega + |\nabla(\varphi^h - \varphi)|_\Omega + |\nabla_\Gamma(\varphi^h - \varphi)|_\Gamma + |\varphi^h - \varphi|_\Gamma\right)\ \leq\ Ch,$$
$$\left(\int_0^T |\nabla(\mathbf{u}^h - \mathbf{u})|_\Omega^2 + |\varphi_t^h - \varphi_t|_\Omega^2 + |\varphi_t^h - \varphi_t|_\Gamma^2 + |\nabla(w^h - w)|_\Omega^2\right)^{\frac{1}{2}}\ \leq Ch.$$

Proof. The proof follows from (49), using $\mathcal{N}(0) = 0$ and the same kind of arguments as in [5]. For the details on the proof, we refer the interested reader to [5]. □

6. **Fully discrete scheme.**

In what follows, we consider the semi-implicit Euler scheme applied to the space semi-discrete scheme studied in the previous section. The purpose of this section is to prove the convergence of the fully discrete scheme introduced below. The space discretization is as in the previous section and let $0 = t_0 < t_1 < \cdots < t_N = T$ be a partitioning of the time interval $[0,T]$ with time step $\delta t > 0$ (we can also consider variable time steps $\tau_n := t_n - t_{n-1}$ for $n = 1, \ldots, N$ and in that case $\delta t = \max_n \tau_n$; without loss of generality, in this section we consider the time step τ_n to be constant). For simplicity of notation, we set $\bar{\partial}$ to be the operator which to a sequence $(v^n)_{n\geq 0}$ associates the sequence defined by

$$\bar{\partial} v^n = \frac{v^n - v^{n-1}}{\delta t},\ n \geq 0.$$

In this section f and g_s are as in (2) and we set $\tilde{g}_s(\sigma) = g_s(\sigma) + \lambda_s\sigma = \tilde{a}_s\sigma - b_\Gamma\ \forall\, \sigma \in \mathbb{R}$ and similarly we denote by $\tilde{G}_s$ the antiderivative of $\tilde{g}_s$, $\tilde{G}_s(\sigma) = G_s(\sigma) + \frac{\lambda_s}{2}\sigma^2$. We also set $\hat{f}(\varphi_h^n) = (\varphi_h^n)^3 - \varphi_h^{n-1}$.

In this section we consider without loss of generality that the forcing term in $(50)_1$ is zero. The fully discrete problem reads as:
Let $(\mathbf{u}_h^0,\ \varphi_h^0)\ \in \mathbf{W}_0^h \times V^h$. For $n \geq 1$, *find* $\{\mathbf{u}_h^n, \varphi_h^n, w_h^n\} \in\ \mathbf{W}_0^h \times V^h \times V^h$ *such that:*

$$\left\{\begin{array}{l} (\bar{\partial}\mathbf{u}_h^n, \mathbf{v})_\Omega + \nu(\nabla\mathbf{u}_h^n, \nabla\mathbf{v})_\Omega + b(\mathbf{u}_h^{n-1}, \mathbf{u}_h^n, \mathbf{v}) + (\varphi_h^{n-1}\nabla w_h^n, \mathbf{v})_\Omega = 0 \\ (\bar{\partial}\varphi_h^n, \psi)_\Omega - (\varphi_h^{n-1}, \mathbf{u}_h^{n-1}\cdot\nabla\psi)_\Omega + \gamma(\nabla w_h^n, \nabla\psi)_\Omega = 0 \\ \varepsilon(\bar{\partial}\varphi_h^n, \chi)_\Omega + (\bar{\partial}\varphi_h^n, \chi)_\Gamma + (\nabla\varphi_h^n, \nabla\chi)_\Omega + \delta(\nabla_\Gamma\varphi_h^n, \nabla_\Gamma\chi)_\Gamma + (\hat{f}(\varphi_h^n), \chi)_\Omega \\ \qquad\qquad\qquad + (\tilde{g}_s(\varphi_h^n), \chi)_\Gamma - (w_h^n, \chi)_\Omega = 0, \end{array}\right. \tag{50}$$

for all $\mathbf{v} \in \mathbf{W}_0^h, \psi \in V^h, \chi \in V^h$. In general we can take $\mathbf{u}_h^0 = \tilde{P}_h^1\mathbf{u}_0$ and $\varphi_h^0 = \tilde{P}_h^2\varphi_0$, with $\tilde{P}_h^i, i = 1, 2$ respectively the projections from $\mathbf{W}_{0,p}$ to $\mathbf{W}_0^h$ and from $H_p^1(\Omega, \Gamma)$ to V^h.

6.1. **Well-posedness.**

In what follows we prove the following result on the well-posedness of the fully discrete problem (50):

Theorem 6.1. *For every* $\mathbf{u}_h^0 \in \mathbf{W}_0^h$ *and* $\varphi_h^0 \in V^h$*, there exists a unique sequence* $(\mathbf{u}_h^n, \varphi_h^n, w_h^n)_{n\geq 1}$ *generated by* (50).

Proof. In order to prove the existence of a unique solution for (50), we first show the existence of a unique $(\varphi_h^n, w_h^n) \in V^h \times V^h$. We consider the variational problem:

$$J^h(\tilde{\varphi}) = \inf_{\varphi \in K^h} J^h(\varphi), \tag{51}$$

where

$$J^h(\varphi) = \frac{1}{2}|\nabla\varphi|_\Omega^2 + \frac{\delta}{2}|\nabla_\Gamma\varphi|_\Gamma^2 + \int_\Omega \hat{F}(\varphi)d\Omega + \int_\Gamma \tilde{G}_s(\varphi)d\Gamma + \frac{\varepsilon}{2\delta t}|\varphi - \varphi_h^{n-1}|_\Omega^2$$
$$+\frac{1}{2\delta t}|\varphi - \varphi_h^{n-1}|_\Gamma^2 + \frac{1}{2\delta t\gamma}|\varphi - \varphi_h^{n-1}|_{-1,h}^2 + \frac{1}{\gamma}(T^h(\nabla\cdot(\varphi_h^{n-1}\mathbf{u}_h^{n-1})), \varphi)_\Omega$$

and $K^h = \{\varphi \in V^h,\ (\varphi - \varphi_h^{n-1}, 1)_\Omega = 0\}$. Here, $\hat{F}(\varphi) = \frac{1}{4}\varphi^4 - \varphi\varphi_h^{n-1}$ and T^h is the discrete inverse Laplacian introduced in the previous section.

Taking into account the fact that $\tilde{F}$ and $\tilde{G}_s$ are polynomials, we can easily show that:

$$J^h(\varphi) \geq \frac{1}{4}(|\nabla\varphi|_\Omega^2 + \delta|\nabla_\Gamma\varphi|_\Gamma^2 + \lambda_s|\varphi|_\Gamma^2) - c(\Omega, \Gamma, \mathbf{u}_h^{n-1}, \varphi_h^{n-1}) \quad \forall\varphi \in V^h.$$

Since J^h is continuous, problem (51) has a unique solution that we denote by $\tilde{\varphi}$, satisfying the Euler- Lagrange equation:

$$(\nabla\tilde{\varphi}, \nabla\chi)_\Omega + \delta(\nabla_\Gamma\tilde{\varphi}, \nabla_\Gamma\chi)_\Gamma + (\hat{f}(\tilde{\varphi}), \chi)_\Omega + (\tilde{g}_s(\tilde{\varphi}), \chi)_\Gamma + \frac{\varepsilon}{\delta t}(\tilde{\varphi} - \varphi_h^{n-1}, \chi)_\Omega$$
$$+\frac{1}{\delta t}(\tilde{\varphi} - \varphi_h^{n-1}, \chi)_\Gamma + \frac{1}{\gamma\delta t}(T^h(\tilde{\varphi} - \varphi_h^{n-1}), \chi)_\Omega + \frac{1}{\gamma}(T^h(\nabla\cdot(\varphi_h^{n-1}\mathbf{u}_h^{n-1})), \chi)_\Omega$$
$$-\lambda(1, \chi)_\Omega = 0,$$

for all $\chi \in V^h$, where λ is the Lagrange multiplier for the constraint $(\varphi - \varphi_h^{n-1}, 1)_\Omega = 0$. We set $\varphi_h^n = \tilde{\varphi}$ and $w_h^n = \lambda - \frac{1}{\gamma\delta t}T^h(\tilde{\varphi} - \varphi_h^{n-1}) - \frac{1}{\gamma}T^h(\nabla\cdot(\varphi_h^{n-1}\mathbf{u}_h^{n-1}))$ and we see that:

$$(\nabla w_h^n, \nabla\chi)_\Omega = -\frac{1}{\gamma\delta t}(\varphi_h^n - \varphi_h^{n-1}, \chi)_\Omega - \frac{1}{\gamma}(\nabla\cdot(\varphi_h^{n-1}\mathbf{u}_h^{n-1}), \chi)_\Omega.$$

Thus,

$$\frac{1}{\delta t}(\varphi_h^n - \varphi_h^{n-1}, \chi)_\Omega - (\nabla\cdot(\varphi_h^{n-1}\mathbf{u}_h^{n-1}), \chi)_\Omega + \gamma(\nabla w_h^n, \nabla\chi)_\Omega = 0,$$

and φ_h^n satisfies:

$$\frac{\varepsilon}{\delta t}(\varphi_h^n - \varphi_h^{n-1}, \chi)_\Omega + \frac{1}{\delta t}(\varphi_h^n - \varphi_h^{n-1}, \chi)_\Gamma + (\nabla\varphi_h^n, \nabla\chi)_\Omega + \delta(\nabla_\Gamma\varphi_h^n, \nabla_\Gamma\chi)_\Gamma$$
$$+(\hat{f}(\varphi_h^n), \chi)_\Omega + (\tilde{g}_s(\varphi_h^n), \chi)_\Gamma - (w_h^n, \chi)_\Omega = 0 \quad \forall\chi \in V^h.$$

We obtain that (φ_h^n, w_h^n) is a solution of $(50)_{2,3}$.

The uniqueness of (φ_h^n, w_h^n) follows classically: we take $(\varphi_h^{n,1}, w_h^{n,1})$ and $(\varphi_h^{n,2}, w_h^{n,2})$ two solutions for $(50)_{2,3}$ obtained from the same $(\varphi_h^{n-1}, w_h^{n-1}, \mathbf{u}_h^{n-1})$. We denote $\theta^\varphi = \varphi_h^{n,1} - \varphi_h^{n,2}$ and $\theta^w = w_h^{n,1} - w_h^{n,2}$, and we write the corresponding equation for θ^φ, θ^w:

$$\begin{cases} \frac{1}{\delta t}(\theta^\varphi, \psi)_\Omega + \gamma(\nabla\theta^w, \nabla\psi)_\Omega = 0 \\ \frac{\varepsilon}{\delta t}(\theta^\varphi, \chi)_\Omega + \frac{1}{\delta t}(\theta^\varphi, \chi)_\Gamma + (\nabla\theta^\varphi, \nabla\chi)_\Omega + \delta(\nabla_\Gamma\theta^\varphi, \nabla_\Gamma\chi)_\Gamma \\ \qquad +((\varphi_h^{n,1})^3 - (\varphi_h^{n,2})^3, \chi)_\Omega + \tilde{a}_s(\theta^\varphi, \chi)_\Gamma - (\theta^w, \chi)_\Omega = 0, \end{cases} \tag{52}$$

for all $\psi, \chi \in V^h$.

Taking $\psi = \delta t \theta^w$ in $(52)_1$ and $\chi = \theta^\varphi$ in $(52)_2$ and adding the resulting equations, we find:

$$\gamma\delta t|\nabla\theta^w|^2_\Omega + \frac{\varepsilon}{\delta t}|\theta^\varphi|^2_\Omega + \frac{1}{\delta t}|\theta^\varphi|^2_\Gamma + |\nabla\theta^\varphi|^2_\Omega + \delta|\nabla_\Gamma\theta^\varphi|^2_\Gamma + ((\varphi_h^{n,1})^3 - (\varphi_h^{n,2})^3, \theta^\varphi)_\Omega + \tilde{a}_s|\theta^\varphi|^2_\Gamma = 0.$$

Since $((\varphi_h^{n,1})^3 - (\varphi_h^{n,2})^3, \theta^\varphi)_\Omega = \int_\Omega (\theta^\varphi)^2 \left[(\varphi_h^{n,1})^2 + \varphi_h^{n,1}\varphi_h^{n,2} + (\varphi_h^{n,2})^2\right] d\Omega \geq 0$, it implies that $\theta^\varphi = 0$ and $\theta^w = 0$.

Once we know w_h^n and φ_h^n, we can pass to finding the solution $\mathbf{u}_h^n$ of $(50)_1$. The existence and uniqueness of $\mathbf{u}_h^n$ can be found using classical methods for the stationary Navier- Stokes equation (see [27]). The theorem is thus proved. $\square$

6.2. **Stability.** Let $(\mathbf{u}_h^0, \varphi_h^0) \in \mathbf{W}_0^h \times V_h$ be finite element approximations of $(\mathbf{u}_0, \varphi_0)$. The fully discrete scheme (50) allows us to construct the sequence of approximations $\{\mathbf{u}_h^n, \varphi_h^n, w_h^n\} \in \mathbf{W}_0^h \times V^h \times V^h$.

Taking $\mathbf{v} = \mathbf{u}_h^n \in (50)_1$, we obtain:

$$\frac{1}{\delta t}(\mathbf{u}_h^n - \mathbf{u}_h^{n-1}, \mathbf{u}_h^n)_\Omega + \nu|\nabla\mathbf{u}_h^n|^2_\Omega + (\varphi_h^{n-1}\nabla w_h^n, \mathbf{u}_h^n)_\Omega = 0.$$

Using $(a-b)a = a^2 - ab = \frac{1}{2}\{a^2 - b^2 + (a-b)^2\}$, we find:

$$\frac{1}{2\delta t}\{|\mathbf{u}_h^n|^2_\Omega - |\mathbf{u}_h^{n-1}|^2_\Omega + |\mathbf{u}_h^n - \mathbf{u}_h^{n-1}|^2_\Omega\} + \nu|\nabla\mathbf{u}_h^n|^2_\Omega + (\varphi_h^{n-1}\nabla w_h^n, \mathbf{u}_h^n)_\Omega = 0. \quad (53)$$

Taking $\psi = \delta t w_h^n$ in $(50)_2$ and $\chi = \varphi_h^n - \varphi_h^{n-1}$ in $(50)_3$, we obtain:

$$(\varphi_h^n - \varphi_h^{n-1}, w_h^n)_\Omega - \delta t(\varphi_h^{n-1}, \mathbf{u}_h^{n-1}\cdot\nabla w_h^n)_\Omega + \gamma\delta t|\nabla w_h^n|^2_\Omega = 0, \quad (54)$$

and

$$\begin{aligned}
&\frac{\varepsilon}{\delta t}|\varphi_h^n - \varphi_h^{n-1}|^2_\Omega + \frac{1}{\delta t}|\varphi_h^n - \varphi_h^{n-1}|^2_\Gamma + (\nabla\varphi_h^n, \nabla(\varphi_h^n - \varphi_h^{n-1}))_\Omega \\
&\quad + \delta(\nabla_\Gamma\varphi_h^n, \nabla_\Gamma(\varphi_h^n - \varphi_h^{n-1}))_\Gamma + (\hat{f}(\varphi_h^n), \varphi_h^n - \varphi_h^{n-1})_\Omega \\
&\quad + (\tilde{g}_s(\varphi_h^n), \varphi_h^n - \varphi_h^{n-1})_\Gamma - (w_h^n, \varphi_h^n - \varphi_h^{n-1})_\Omega = 0.
\end{aligned} \quad (55)$$

Next, we multiply (53) by δt and add to (54) and (55) in order to obtain:

$$\begin{aligned}
&\frac{1}{2}\left\{|\mathbf{u}_h^n|^2_\Omega + |\nabla\varphi_h^n|^2_\Omega + \delta|\nabla_\Gamma\varphi_h^n|^2_\Gamma + |\mathbf{u}_h^n - \mathbf{u}_h^{n-1}|^2_\Omega + |\nabla(\varphi_h^n - \varphi_h^{n-1})|^2_\Omega\right. \\
&\quad \left. + \delta|\nabla(\varphi_h^n - \varphi_h^{n-1})|^2_\Gamma - |\mathbf{u}_h^{n-1}|^2_\Omega - |\nabla\varphi_h^{n-1}|^2_\Omega - \delta|\nabla_\Gamma\varphi_h^{n-1}|^2_\Gamma\right\} \\
&\quad + \delta t\left\{(\varphi_h^{n-1}\nabla w_h^n, \mathbf{u}_h^n)_\Omega - (\varphi_h^{n-1}, \mathbf{u}_h^{n-1}\cdot\nabla w_h^n)_\Omega\right\} + \gamma\delta t|\nabla w_h^n|^2_\Omega \\
&\quad + \frac{\varepsilon}{\delta t}|\varphi_h^n - \varphi_h^{n-1}|^2_\Omega + \frac{1}{\delta t}|\varphi_h^n - \varphi_h^{n-1}|^2_\Gamma + (\hat{f}(\varphi_h^n), \varphi_h^n - \varphi_h^{n-1})_\Omega \\
&\quad + \nu\delta t|\nabla\mathbf{u}_h^n|^2_\Omega + (\tilde{g}_s(\varphi_h^n), \varphi_h^n - \varphi_h^{n-1})_\Gamma = 0.
\end{aligned} \quad (56)$$

We need to evaluate:

$$\begin{aligned}
(\varphi_h^{n-1}\nabla w_h^n, \mathbf{u}_h^n)_\Omega - (\varphi_h^{n-1}, \mathbf{u}_h^{n-1}\cdot\nabla w_h^n)_\Omega &= \int_\Omega (\varphi_h^{n-1}\nabla w_h^n\cdot\mathbf{u}_h^n - \varphi_h^{n-1}\nabla w_h^n\cdot\mathbf{u}_h^{n-1})d\Omega \\
&= \int_\Omega \varphi_h^{n-1}\nabla w_h^n\cdot(\mathbf{u}_h^n - \mathbf{u}_h^{n-1})\,d\Omega,
\end{aligned}$$

which yields to:

$$\begin{aligned}\delta t\Big|\int_\Omega \varphi_h^{n-1}\nabla w_h^n\cdot(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})\,d\Omega\Big| &\le \delta t|\varphi_h^{n-1}|_{L^4(\Omega)}|\nabla w_h^n|_\Omega|\mathbf{u}_h^n-\mathbf{u}_h^{n-1}|_{L^4(\Omega)}\\ &\le \gamma\frac{\delta t}{2}|\nabla w_h^n|^2_\Omega+\frac{\delta t}{2\gamma}|\varphi_h^{n-1}|^2_{L^4(\Omega)}|\mathbf{u}_h^n-\mathbf{u}_h^{n-1}|^2_{L^4(\Omega)}\end{aligned}$$

Using the inverse inequality $|\chi|_{L^4(\Omega)} \le ch^{-\frac{1}{2}}|\chi|_\Omega \quad \forall\chi\in V^h$, we find:

$$\delta t\Big|\int_\Omega \varphi_h^{n-1}\nabla w_h^n\cdot(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})d\Omega\Big| \le \frac{\gamma\delta t}{2}|\nabla w_h^n|^2_\Omega+\frac{c\delta t}{2\gamma}|\varphi_h^{n-1}|^2_{L^4}h^{-1}|\mathbf{u}_h^n-\mathbf{u}_h^{n-1}|^2_\Omega.$$

Then, choosing δt small enough such that $\delta t\le \frac{\gamma h}{2c|\varphi_h^{n-1}|^2_{L^4(\Omega)}}$, we obtain:

$$\delta t\Big|\int_\Omega \varphi_h^{n-1}\nabla w_h^n\cdot(\mathbf{u}_h^n-\mathbf{u}_h^{n-1})d\Omega\Big| \le \frac{\gamma\delta t}{2}|\nabla w_h^n|^2_\Omega+\frac{1}{4}|\mathbf{u}_h^n-\mathbf{u}_h^{n-1}|^2_\Omega. \tag{57}$$

We also have that:

$$\begin{aligned}(\tilde g_s(\varphi_h^n),\varphi_h^n-\varphi_h^{n-1})_\Gamma &= \tilde a_s(\varphi_h^n,\varphi_h^n-\varphi_h^{n-1})_\Gamma-b_\Gamma(\mathbf{1},\varphi_h^n-\varphi_h^{n-1})_\Gamma\\ &=\frac{\tilde a_s}{2}\{|\varphi_h^n|^2_\Gamma-|\varphi_h^{n-1}|^2_\Gamma+|\varphi_h^n-\varphi_h^{n-1}|^2_\Gamma\}-b_\Gamma(\mathbf{1},\varphi_h^n-\varphi_h^{n-1})_\Gamma.\end{aligned}\tag{58}$$

Using that $(a^3-b)(a-b)=\frac14[(a^2-1)^2-(b^2-1)^2]+\frac14(a^2-b^2)^2+\frac12(a(a-b))^2+\frac12(a-b)^2$, we can also infer that:

$$\begin{aligned}(\hat f(\varphi_h^n),\varphi_h^n-\varphi_h^{n-1})_\Omega =&\frac14[|(\varphi_h^n)^2-1|^2_\Omega-|(\varphi_h^{n-1})^2-1|^2_\Omega]+\frac14|(\varphi_h^n)^2-(\varphi_h^{n-1})^2|^2_\Omega\\ &+\frac12|\varphi_h^n(\varphi_h^n-\varphi_h^{n-1})|^2_\Omega+\frac12|\varphi_h^n-\varphi_h^{n-1}|^2_\Omega.\end{aligned}\tag{59}$$

We define

$$\mathcal{E}(\mathbf{u},\varphi)=|\mathbf{u}|^2_\Omega+|\nabla\varphi|^2_\Omega+\delta|\nabla_\Gamma\varphi|^2_\Gamma+\frac12|\varphi^2-\mathbf{1}|^2_\Omega+\tilde a_s|\varphi|^2_\Gamma-2b_\Gamma(\mathbf{1},\varphi)_\Gamma$$

and using (57), (58), (59) into (56), we find the following energy estimate for the discrete solution:

$$\begin{aligned}&\mathcal{E}(\mathbf{u}_h^n,\varphi_h^n)+\frac12|\mathbf{u}_h^n-\mathbf{u}_h^{n-1}|^2_\Omega+|\nabla(\varphi_h^n-\varphi_h^{n-1})|^2_\Omega+\delta|\nabla_\Gamma(\varphi_h^n-\varphi_h^{n-1})|^2_\Gamma+\gamma\delta t|\nabla w_h^n|^2_\Omega\\ &+2\nu\delta t|\nabla\mathbf{u}_h^n|^2_\Omega+\frac{2\varepsilon}{\delta t}|\varphi_h^n-\varphi_h^{n-1}|^2_\Omega+\frac{2}{\delta t}|\varphi_h^n-\varphi_h^{n-1}|^2_\Gamma+\frac12|(\varphi_h^n)^2-(\varphi_h^{n-1})^2|^2_\Omega\\ &+|\varphi_h^n(\varphi_h^n-\varphi_h^{n-1})|^2_\Omega+|\varphi_h^n-\varphi_h^{n-1}|^2_\Omega+\tilde a_s|\varphi_h^n-\varphi_h^{n-1}|^2_\Gamma \le \mathcal{E}(\mathbf{u}_h^{n-1},\varphi_h^{n-1}).\end{aligned}\tag{60}$$

Summing (60) from 1 to n, we find:

$$\begin{aligned}&\mathcal{E}(\mathbf{u}_h^n,\varphi_h^n)+\frac12\sum_{k=1}^n|\mathbf{u}_h^k-\mathbf{u}_h^{k-1}|^2_\Omega+(1+\frac{2\varepsilon}{\delta t})\sum_{k=1}^n|\varphi_h^k-\varphi_h^{k-1}|^2_\Omega\\ &+(\frac{2}{\delta t}+\tilde a_s)\sum_{k=1}^n|\varphi_h^k-\varphi_h^{k-1}|^2_\Gamma+2\nu\delta t\sum_{k=1}^n|\nabla\mathbf{u}_h^k|^2_\Omega+\gamma\delta t\sum_{k=1}^n|\nabla w_h^k|^2_\Omega\\ &+\frac12\sum_{k=1}^n|(\varphi_h^k)^2-(\varphi_h^{k-1})^2|^2_\Omega+\sum_{k=1}^n|\varphi_h^k(\varphi_h^k-\varphi_h^{k-1})|^2_\Omega+\sum_{k=1}^n|\nabla(\varphi_h^k-\varphi_h^{k-1})|^2_\Omega\\ &+\delta\sum_{k=1}^n|\nabla_\Gamma(\varphi_h^k-\varphi_h^{k-1})|^2_\Gamma \le \mathcal{E}(\mathbf{u}_h^0,\varphi_h^0).\end{aligned}\tag{61}$$

Hence we proved the following Lemma:

Theorem 6.2. *Let* $(\mathbf{u}_h^{n-1}, \varphi_h^{n-1}, w_h^{n-1}) \in \mathbf{W}_0^h \times V^h \times V^h$ *be given. For all* $h > 0$ *and all time partitions such that* $\delta t \leq \frac{\gamma h}{2c|\varphi_h^{n-1}|^2_{L^4(\Omega)}}$ *the solution* $(\mathbf{u}_h^n, \varphi_h^n, w_h^n)$ *of problem* (50) *satisfies* (61).

Remark 5. We note that the bound $\delta t \leq \frac{\gamma h}{2c|\varphi_h^{n-1}|^2_{L^4(\Omega)}}$ is attainable since from (61) we prove that φ_h^n is bounded in $H^1(\Omega)$, and thus in $L^4(\Omega)$, independently of h.

6.3. **Convergence.** We can now study the convergence of the sequence of solutions given by (50), as h and δt tend to 0. For each fixed N, we associate to $\mathbf{u}_h^1, \cdots, \mathbf{u}_h^N$ the following approximate functions:

$$\mathbf{u}_{h,\tau} : [0,T] \to W_0^h, \text{ with } \mathbf{u}_{h,\tau}(t) = \frac{t - t_{n-1}}{\tau}\mathbf{u}_h^n + \frac{t_n - t}{\tau}\mathbf{u}_h^{n-1}, \; t \in [t_{n-1}, t_n], \tag{62}$$

where $\tau = \delta t$.The function $\mathbf{u}_{h,\tau}$ is a piecewise linear interpolant in time of the fully discrete solution $\{\mathbf{u}_h^n\}_n$. We also define

$$\mathbf{u}_{h,\tau}^+(t) = \mathbf{u}_h^n \;\text{ and }\; \mathbf{u}_{h,\tau}^-(t) = \mathbf{u}_h^{n-1} \quad \forall t \in [t_{n-1}, t_n], \quad \forall n.$$

The same notations are introduced for φ_h^n and w_h^n.

Remark 6. We notice that:

$$\partial_t \mathbf{u}_{h,\tau} = \frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\tau} = \bar{\partial}\mathbf{u}_h^n \quad \forall t \in (t_{n-1}, t_n).$$

With these above notations, we can restate (50) as:

Find $\{\mathbf{u}_{h,\tau}, \varphi_{h,\tau}, w_{h,\tau}\} \in L^2(0,T;\mathbf{W}_0^h) \times L^2(0,T;V^h) \times L^2(0,T;V^h)$ *such that:*

$$\begin{cases} \displaystyle\int_0^T \Big\{(\partial_t \mathbf{u}_{h,\tau}, \mathbf{v})_\Omega + \nu(\nabla \mathbf{u}_{h,\tau}^+, \nabla \mathbf{v})_\Omega + b(\mathbf{u}_{h,\tau}^-, \mathbf{u}_{h,\tau}^+, \mathbf{v}) + (\varphi_{h,\tau}^- \nabla w_{h,\tau}^+, \mathbf{v})_\Omega\Big\}\, dt = 0, \\ \displaystyle\int_0^T \Big\{(\partial_t \varphi_{h,\tau}, \psi)_\Omega - (\varphi_{h,\tau}^-, \mathbf{u}_{h,\tau}^- \cdot \nabla\psi)_\Omega + \gamma(\nabla w_{h,\tau}^+, \nabla\psi)_\Omega\Big\}\, dt = 0, \\ \displaystyle\int_0^T \Big\{\varepsilon(\partial_t \varphi_{h,\tau}, \chi)_\Omega + (\partial_t \varphi_{h,\tau}, \chi)_\Gamma + (\nabla\varphi_{h,\tau}^+, \nabla\chi)_\Omega + \delta(\nabla_\Gamma \varphi_{h,\tau}^+, \nabla_\Gamma \chi)_\Gamma \\ \qquad\qquad + ((\varphi_{h,\tau}^+)^3 - \varphi_{h,\tau}^-, \chi)_\Omega + (\tilde{g}_s(\varphi_{h,\tau}^+), \chi)_\Gamma - (w_{h,\tau}^+, \chi)_\Omega\Big\}\, dt = 0, \end{cases} \tag{63}$$

for all $\mathbf{v} \in L^2(0,T;W_0^h)$, $\quad \forall \psi, \chi \in L^2(0,T;V^h)$.

From (61), we know:

$$\begin{aligned} &\sup\nolimits_{n=1,\ldots,N} |\mathbf{u}_h^n|_\Omega \leq c(\mathbf{u}_h^0, \varphi_h^0) \leq C, \\ &\sup\nolimits_{n=1,\ldots,N} |\varphi_h^n|_\Gamma \leq c(\mathbf{u}_h^0, \varphi_h^0) \leq C, \; \sup\nolimits_{n=1,\ldots,N} |\nabla_\Gamma \varphi_h^n|_\Gamma \leq c(\mathbf{u}_h^0, \varphi_h^0) \leq C, \\ &\sup\nolimits_{n=1,\ldots,N} |\nabla\varphi_h^n|_\Omega \leq c(\mathbf{u}_h^0, \varphi_h^0) \leq C, \end{aligned} \tag{64}$$

where by $c(\mathbf{u}_h^0, \varphi_h^0)$ we denote a constant depending on the initial conditions $|\mathbf{u}_h^0|_\Omega$, $|\varphi_h^0|_{H^1(\Omega,\Gamma)}$ and C is a constant independent of h but depending on $\mathbf{u}_0$ and φ_0. We note that $c(\mathbf{u}_h^0, \varphi_h^0) \leq C$ thanks to the choice of initial conditions, more exactly thanks to the fact that projections $\tilde{P}_h^1$ and $\tilde{P}_h^2$ satisfy the same inequalities as in (32) and (33). From (64), we infer that:

$$\begin{aligned} &|\mathbf{u}_{h,\tau}^\pm|_{L^\infty(0,T,L^2(\Omega))} \leq C, \\ &|\varphi_{h,\tau}^\pm|_{L^\infty(0,T,H^1(\Omega))} \leq C, \quad |\varphi_{h,\tau}^\pm|_{L^\infty(0,T,H^1(\Gamma))} \leq C. \end{aligned} \tag{65}$$

From (61), we also find:

$$\frac{\varepsilon}{\delta t}\sum_{k=1}^{N}|\varphi_h^k-\varphi_h^{k-1}|_\Omega^2 \le C, \quad \frac{1}{\delta t}\sum_{k=1}^{N}|\varphi_h^k-\varphi_h^{k-1}|_\Gamma^2 \le C, \tag{66}$$

and

$$\delta t\sum_{k=1}^{N}|\nabla \mathbf{u}_h^k|_\Omega^2 \le C, \qquad \delta t\sum_{k=1}^{N}|\nabla w_h^k|_\Omega^2 \le C,$$

which imply

$$\int_0^T |\nabla \mathbf{u}_{h,\tau}^{\pm}|_\Omega^2 = |\nabla \mathbf{u}_{h,\tau}^{\pm}|^2_{L^2(0,T;L^2(\Omega))} \le C, \quad |\nabla w_{h,\tau}^{\pm}|_{L^2(0,T;L^2(\Omega))} \le C. \tag{67}$$

Since $\partial_t\varphi_{h,\tau} = \frac{1}{\delta t}(\varphi_h^k-\varphi_h^{k-1}) \quad \forall t\in(t_{k-1},t_k)$, we obtain:

$$\frac{1}{\delta t^2}|\varphi_h^k-\varphi_h^{k-1}|_\Omega^2 = \frac{1}{\delta t}\int_{t_{k-1}}^{t_k}|\partial_t\varphi_{h,\tau}|_\Omega^2 dt,$$

and summing for a k varying from 1 to N, we obtain:

$$\frac{1}{\delta t}\sum_{k=1}^{N}|\varphi_h^k-\varphi_h^{k-1}|_\Omega^2 = |\partial_t\varphi_{h,\tau}|^2_{L^2(0,T;L^2(\Omega))}.$$

Thus, from (66) and (63) we get:

$$\varepsilon|\partial_t\varphi_{h,\tau}|^2_{L^2(0,T;L^2(\Omega))} \le C, \quad |\partial_t\varphi_{h,\tau}|^2_{L^2(0,T;L^2(\Gamma))} \le C, \quad |\langle w_{h,\tau}^{\pm}\rangle| \le C. \tag{68}$$

We also notice that:

$$\begin{aligned}\varphi_{h,\tau}-\varphi_{h,\tau}^{+} &= \frac{t-t_n}{\delta t}(\varphi_h^n-\varphi_h^{n-1}) \quad \forall t\in[t_{n-1},t_n],\\ \varphi_{h,\tau}-\varphi_{h,\tau}^{-} &= \frac{t-t_{n-1}}{\delta t}(\varphi_h^n-\varphi_h^{n-1}) \quad \forall t\in[t_{n-1},t_n],\end{aligned}$$

and thus we deduce the following bound on $\varphi_{h,\tau}$:

$$\begin{aligned}\|\varphi_{h,\tau}-\varphi_{h,\tau}^{+}\|^2_{L^2(0,T;L^2(\Omega))} &= \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}|\varphi_{h,\tau}-\varphi_{h,\tau}^{+}|_\Omega^2\,dt\\ = \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}\left(\frac{t-t_n}{\delta t}\right)^2|\varphi_h^n-\varphi_h^{n-1}|_\Omega^2 dt &= \sum_{n=1}^{N}\frac{\delta t}{3}|\varphi_h^n-\varphi_h^{n-1}|_\Omega^2 \le C\delta t.\end{aligned}$$

Similarly, we get that (using (61))

$$\begin{aligned}&\|\varphi_{h,\tau}-\varphi_{h,\tau}^{\pm}\|^2_{L^2(0,T;L^2(\Omega))} \le C\delta t, \quad \|\nabla(\varphi_{h,\tau}-\varphi_{h,\tau}^{\pm})\|^2_{L^2(0,T;L^2(\Omega))} \le C\delta t,\\ &\|\varphi_{h,\tau}-\varphi_{h,\tau}^{\pm}\|^2_{L^2(0,T;L^2(\Gamma))} \le C\delta t, \quad \|\nabla(\varphi_{h,\tau}-\varphi_{h,\tau}^{\pm})\|^2_{L^2(0,T;L^2(\Gamma))} \le C\delta t,\\ &\|\mathbf{u}_{h,\tau}-\mathbf{u}_{h,\tau}^{\pm}\|^2_{L^2(0,T;L^2(\Omega))} \le C\delta t.\end{aligned} \tag{69}$$

In order to prove the convergence of the sequence of approximate solutions to the solution of the continuous problem, we proceed as follows:

Step 1. Extraction of convergent subsequences. From the bounds we proved, we find that we can extract subsequences satisfying, as h, δt tend to 0 :

$$\begin{cases} \varphi^{\pm}_{h,\tau} \to \varphi \text{ weak-star in } L^\infty(0,T;H^1(\Omega)) \text{ and } L^\infty(0,T;H^1(\Gamma)),\\ \varphi_{h,\tau},\ \varphi^{\pm}_{h,\tau} \text{ have the same limit in } L^2(0,T;H^1(\Omega)) \text{ and } L^2(0,T;H^1(\Gamma)),\\ \mathbf{u}^{\pm}_{h,\tau} \to \mathbf{u} \text{ weakly in } L^2(0,T;H^1(\Omega)) \text{ and } \mathbf{u}^{\pm}_{h,\tau} \to \mathbf{u} \text{ weak-star in } L^\infty(0,T;L^2(\Omega)),\\ \mathbf{u}_{h,\tau},\ \mathbf{u}^{\pm}_{h,\tau} \text{ have the same limit in } L^2(0,T;L^2(\Omega)),\\ w^{\pm}_{h,\tau} \to w \text{ weakly in } L^2(0,T;H^1(\Omega)),\\ \partial_t\varphi_{h,\tau} \to \partial_t\varphi \text{ weakly in } L^2(0,T;L^2(\Omega)) \text{ and weakly in } L^2(0,T;L^2(\Gamma)). \end{cases} \tag{70}$$

From $(70)_1$, $(70)_2$, $(70)_6$, we obtain that $\varphi^{\pm}_{h,\tau}$, $\varphi_{h,\tau}$ converge strongly to φ in $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;L^2(\Gamma))$. In order to be able to pass to the limit in (63), we also need to prove a strong convergence result for $\mathbf{u}_{h,\tau}$, $\mathbf{u}^{\pm}_{h,\tau}$.

Lemma 6.3. *The sequences $\mathbf{u}_{h,\tau}$ and $\mathbf{u}^{\pm}_{h,\tau}$ converge strongly to $\mathbf{u}$ in $L^2(0,T;(L^2(\Omega))^2)$.*

Proof. The proof follows exactly the same lines as Lemma 4.4 in [21] and will be omitted. □

Step 2. The passage to the limit follows exactly as in [21] (even easier, since for $\partial_t\varphi_{h,\tau}$ we already have stated the weak convergence in $L^2(0,T,L^2(\Omega))$ and $L^2(0,T,L^2(\Gamma))$. We obtain the following Theorem:

Theorem 6.4. *There exists a subsequence of $(\mathbf{u}_{h,\tau},\varphi_{h,\tau},w_{h,\tau}) \in L^2(0,T;\mathbf{W}^h_0) \times L^2(0,T;V^h) \times L^2(0,T;V^h)$ solving (63) and the functions $(\mathbf{u},\varphi,w)$ such that the convergences (70) and Lemma (6.3) hold. Furthermore, $\mathbf{u}(0)=\mathbf{u}_0$, $\varphi(0)=\varphi_0$ and satisfy:*

$$\begin{cases} -\displaystyle\int_0^T (\mathbf{u},\partial_t\mathbf{v})_\Omega\,dt + \int_0^T (\nabla\mathbf{u},\nabla\mathbf{v})_\Omega\,dt + \int_0^T b(\mathbf{u},\mathbf{u},\mathbf{v})\,dt + \int_0^T (\varphi\nabla w,\mathbf{v})_\Omega\,dt\\ \qquad = (\mathbf{u}_0,\mathbf{v}(0))_\Omega,\\ \displaystyle\int_0^T (\partial_t\varphi,\psi)_\Omega\,dt + \int_0^T (\varphi,\mathbf{u}\cdot\nabla\psi)_\Omega\,dt + \gamma\int_0^T (\nabla w,\nabla\psi)_\Omega dt = 0,\\ \varepsilon\displaystyle\int_0^T (\partial_t\varphi,\chi)_\Omega dt + \int_0^T (\partial_t\varphi,\chi)_\Gamma dt + \int_0^T (\nabla\varphi,\nabla\chi)_\Omega\,dt + \delta\int_0^T (\nabla_\Gamma\varphi,\nabla_\Gamma\chi)_\Gamma\,dt\\ \qquad + \displaystyle\int_0^T (f(\varphi),\chi)_\Omega\,dt + \int_0^T (\tilde{g}_s(\varphi),\chi)_\Gamma\,dt - \int_0^T (w,\chi)_\Omega\,dt = 0, \end{cases} \tag{71}$$

$\forall \mathbf{v} \in \mathcal{C}^1([0,T];\mathbf{W}_{0,p})$ *with* $\mathbf{v}(T)=0$, $\forall\psi \in L^2(0,T;H^1_p(\Omega))$, $\forall\chi \in L^2(0,T;H^1_p(\Omega,\Gamma))$.

Remark 7. The whole sequence $(\mathbf{u}_{h,\tau},\varphi_{h,\tau},w_{h,\tau})$ converges if the weak solution to problem (71) is unique.

7. **Numerical results.** In this section, we present several numerical simulations, using the FreeFem++ software [19]. We take $\Omega=[0,1]\times[0,1]$, $\Gamma_p=\{0,1\}\times[0,1]$

and $\Gamma_d = [0,1] \times \{0,1\}$, and consider the problem:

$$\begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + \lambda\varphi\nabla w = \mathbf{h} & \text{in } \Omega_T = \Omega\times(0,T), \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega_T, \\ \varphi_t + \mathbf{u}\cdot\nabla\varphi - \gamma\Delta w = 0 & \text{in } \Omega_T, \\ w = -\alpha\Delta\varphi + f(\varphi) + \varepsilon\varphi_t, & \text{in } \Omega_T, \\ \mathbf{u},\varphi,w \quad \text{periodic in } x_2 & \text{on } \Gamma_p\times(0,T), \\ \mathbf{u} = 0, \quad \partial_n w = 0 & \text{on } \Gamma_d\times(0,T), \\ \varphi_t = \delta\Delta_\Gamma\varphi - \tilde{g}_s(\varphi) - \partial_n\varphi & \text{on } \Gamma_d\times(0,T). \end{cases} \tag{72}$$

For simplicity, the problem is solved with a slightly different discretization (semi-implicit scheme for time discretization, combined with a finite element for the space discretization: P_1- finite element for φ, w and p, and P_2 for $\mathbf{u}$).

The mesh is obtained by dividing the domain into 150 rectangles, each rectangle being divided among the same diagonal into two triangles. The functions f and $\tilde{g}_s$ we consider in our tests are:

$$f(v) = v^3 - v, \qquad \tilde{g}_s(v) = 2v - h_s.$$

We use the following parameters (in view of [21]): $\delta t = 0.05$, $\gamma = 0.001$, $\nu = 0.01$, $\lambda = 1$, $\alpha = 1/120^2$, $\delta = 1$. Moreover, we choose $\varepsilon = 0.01$ and:

$$\mathbf{u}_0 = \left(-\sin^2(\pi x)\sin(2\pi y), \sin^2(\pi y)\sin(2\pi x)\right), \qquad \mathbf{h} = \frac{1}{2}\mathbf{u}_0. \tag{73}$$

In the figures below, the negative and the positive values of the solution φ are respectively displayed in yellow and red.

In our first test, we take u_0 and φ_0 as in Figure 1. Then, Figures 2 and 3 illustrate different values of φ, corresponding to the evolution of the order parameter with respect to time: $t = 0.5$, $t = 1$, $t = 2.5$ (in Figure 2), $t = 3$, $t = 4$ and $t = 7.5$ (in Figure 3). In Figure 4, we show the solution $\mathbf{u}$ and p corresponding to $t = 3$.

For our second test, we take φ_0 randomly distributed between -1 and 1, and $\mathbf{u}_0$ and $\mathbf{h}$ are again defined as in (73). We compare the solution φ at $t = 1$ (Figure 5) and $t = 3$ (Figure 6), but with different kind of boundary conditions on the horizontal sides of Ω, namely: dynamic boundary conditions with $\tilde{g}_s = 2v + 1$ (left column), dynamic boundary conditions with $\tilde{g}_s = 2v - 1$ (middle column) and Neumann boundary conditions (right column).

Figure 7 is obtained taking again φ_0 randomly distributed between -1 and 1, and $\mathbf{u}_0$ and $\mathbf{h}$ defined as in (73), and we choose the surface potential $\tilde{g}_s = 2v + 1$. Then we compare the solutions φ at the same time $t = 3$, but with three different values for the viscosity coefficient: $\varepsilon = 0.5$, $\varepsilon = 0.1$ and $\varepsilon = 0.01$.

We also remark that checking the first two columns of Figure 5 and 6, corresponding to the case of dynamic boundary conditions, the contact angle to the boundary is not $\pi/2$, contrary to the Neumann case. Nevertheless, in the present work we did not control the contact angle; this question will be addressed elsewhere, eventually with different dynamic boundary conditions.

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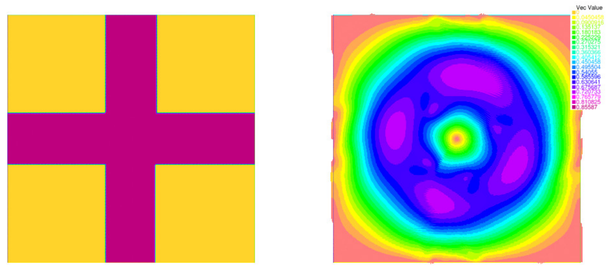

FIGURE 1. Initial conditions φ_0 and $\mathbf{u}_0$

FIGURE 2. Solution φ with $\tilde{g}_s = 2v - 1$ at time t=0.5 (left) , t=1.5 (middle), t=2.5 (right)

FIGURE 3. Solution φ with $\tilde{g}_s = 2v - 1$ at time t=3 (left) , t= 5 (middle), t=7.5 (right)

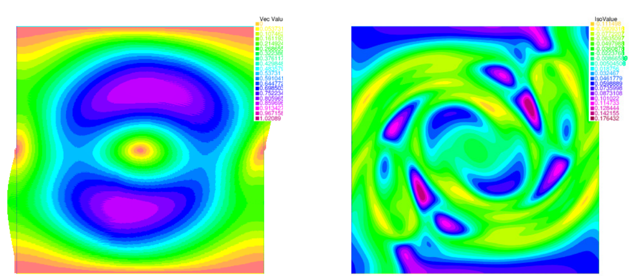

FIGURE 4. Solution $\mathbf{u}$ (left) and p (right) at time $t = 3$

FIGURE 5. Solution φ at time $t = 1$, $\tilde{g}_s = 2v+1$ (left) , $\tilde{g}_s = 2v-1$ (middle), Neumann + periodic (right)

FIGURE 6. Solution φ at time $t = 3$, $\tilde{g}_s = 2v+1$ (left) , $\tilde{g}_s = 2v-1$ (middle), Neumann + periodic (right)

FIGURE 7. Solution φ at time $t = 3$, with $\tilde{g}_s = 2v + 1$ and with viscosity coefficient $\varepsilon = 0.5$ (left) , $\varepsilon = 0.1$ (middle), $\varepsilon = 0.01$ (right)

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