Stabilized Splitting Schemes for Allen Cahn and Cahn Hillard equations

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Abstract

1 Introduction

2 The RSS-ADI schemes

Let A and B be two $n \times n$ symmetric positive definite matrices. We assume that there exist two strictly positive constant α and *beta* such that

$$\alpha < Bu, u > \leq < Au, u > \leq \beta < u, u >, \ \forall u \in I\!R^n \tag{1}$$

Consider the linear differential system

$$\frac{dU}{dt} + AU = 0$$

with $A = A_1 + A_2$. Let B_1 and B_2 be preconditioners of A_1 and A_2 respectively and τ_1, τ_2 two positive real numbers. All the matrices are supposed to be symmetric definite positive. We introduce the ADI-RSS schemes

$$\frac{u^{(k+1/2)} - u^{(k)}}{\Delta t} + \tau_1 B_1 (u^{(k+1/2)} - u^{(k)}) = -A_1 u^{(k)}, \tag{2}$$

$$\frac{u^{(k+1)} - u^{(k+1/2)}}{\Delta t} + \tau_2 B_2 (u^{(k+1)} - u^{(k+1/2)}) = -A_2 u^{(k+1/2)}, \tag{3}$$

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and the Strang's Splitting

$$\frac{u^{(k+1/3)} - u^{(k)}}{\Delta t/2} + \tau_1 B_1 (u^{(k+1/3)} - u^{(k)}) = -A_1 u^{(k)}, \tag{4}$$

$$\frac{u^{(k+2/3)} - u^{(k+1/3)}}{\Delta t} + \tau_2 B_2 (u^{(k+2/3)} - u^{(k+1/3)}) = -A_2 u^{(k+1/3)}, \tag{5}$$

$$\frac{u^{(k+1)} - u^{(k+2/3)}}{\Delta t/2} + \tau_1 B_1 (u^{(k+1)} - u^{(k+2/3)}) = -A_1 u^{(k+2/3)}, \tag{6}$$

Of course these approach can be applied in more general situations, eg considering $A = \sum_{i=1}^{m} A_i$ and $B = \sum_{i=1}^{m} B_i$ and the splitting

$$\frac{u^{(k+i/m)} - u^{(k+(i-1)/m)}}{\Delta t} + \tau_i B_i (u^{(k+i/m)} - u^{(k+(i-1)/m)}) = -A_i u^{(k+(i-1)/m)}, \tag{7}$$

We recall that

Proposition 2.1 *Under hypothesis (1, we have the following stability conditions:*

- *If $\tau \geq \frac{\beta}{2}$, the schemes (3) and (6) are unconditionally stable (i.e. stable $\forall\ \Delta t > 0$)*
- *If $\tau < \frac{\beta}{2}$, then the scheme is stable for $0 < \Delta t < \dfrac{2}{\left(1 - \frac{2\tau}{\beta}\right)\rho(A)}$.*

As a direct consequence, we can prove the following result

Proposition 2.2 *Under hypothesis (1), we have the followig stability conditions:*

- *If $\tau_i \geq \frac{\beta_i}{2}$,$i = 1, 2$ the scheme (3)is unconditionally stable (i.e. stable $\forall\ \Delta t > 0$)*
- *If $\tau_i < \frac{\beta_i}{2}$, $i = 1, 2$, then the scheme is stable for $0 < \Delta t < Min(\dfrac{2}{\left(1 - \frac{2\tau_1}{\beta_1}\right)\rho(A_1)}, \dfrac{2}{\left(1 - \frac{2\tau_2}{\beta_2}\right)\rho(A_2)})$.*

Proof. It suffices to apply proposition 2.1 to each system. ■

3 Numerical results

3.1 2D Heat Equation

see program `chaleur_2D_splitting.m`

The error is clearly in Δt.

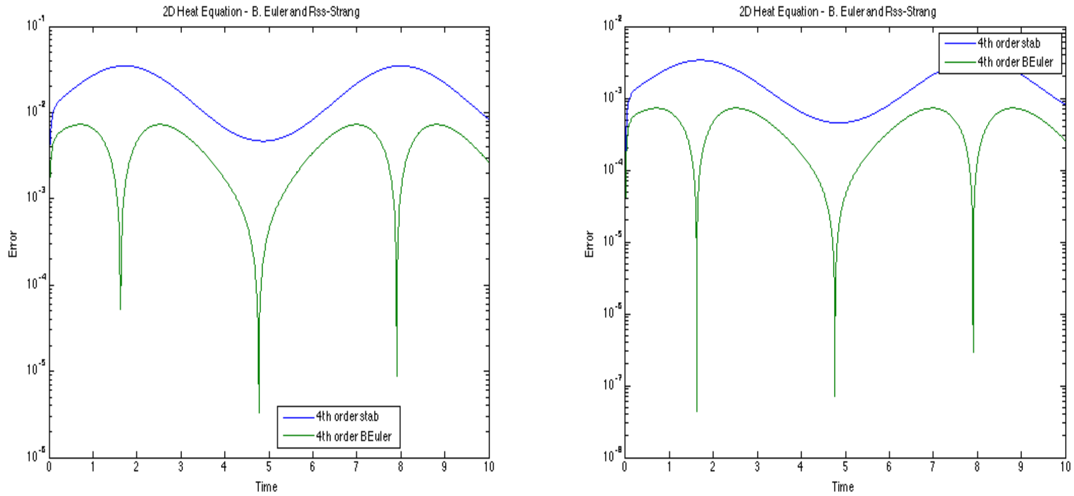

Figure 1: Solution of the heat equation with $\Delta t = 0.01$, (left) and $\Delta t = 0.001$ (right) $n = 31$, $\tau = 1$

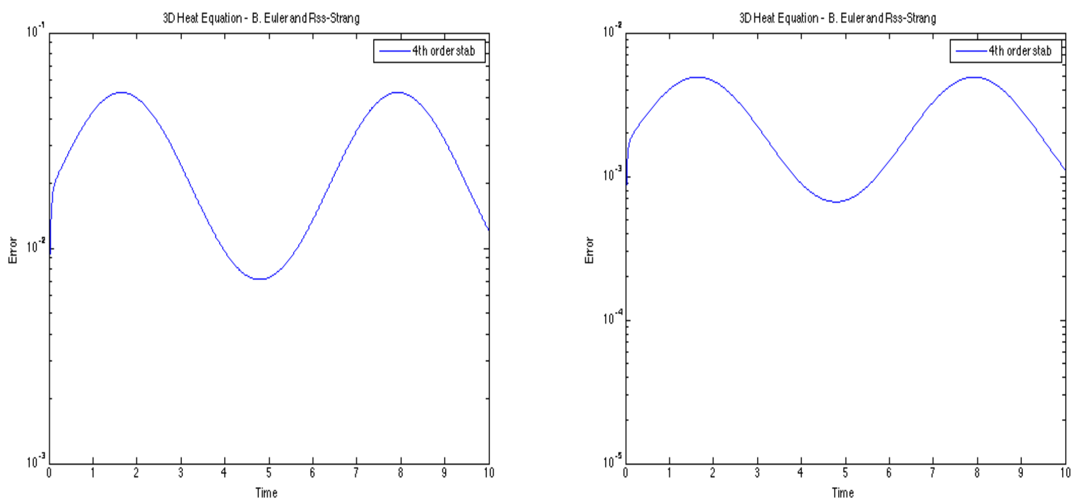

Figure 2: Solution of the 3D heat equation with $\Delta t = 0.01$, (left) and $\Delta t = 0.001$ (right) $n = 31$, $\tau = 1$

3.2 3D Heat Equation

The error is clearly in Δt.

4 Application to the numerical solution of Phase-Fields problems

4.1 Allen-Cahn's equation

4.2 Cahn-Hilliard's equation

4.2.1 The RSS-Scheme

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + A\mu^{(k)+1} = 0,$$
$$\mu^{(k+1)} = \epsilon A u^{(k+1)} + \frac{1}{\epsilon} f(u^{(k)},$$

We derive the RSS-Scheme from the backward Euler's (**??**) by replacing $Az^{(k+1)}$ by $\tau B(z^{(k+1)} - z^{(k)}) + Az^{(k)}$ for $z = u$ or $z = \mu$. We obtain

$$\frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \tau B(\mu^{(k+1)} - \mu^{(k)}) + A\mu^{(k)} = 0,$$
$$\mu^{(k+1)} = \epsilon\tau B(u^{(k+1)} - u^{(k)}) + \epsilon A u^{(k)} + \frac{1}{\epsilon} f(u^{(k)}).$$