```
> # Identiifability of the Elk model when we include the starting values (N[a,1]) as parameters
>
>
> restart;
> with(LinearAlgebra) :
> Dmat := proc(se, pars)
  local DD1, i, j;
  description "Form the derivative matrix";
  with(LinearAlgebra) :
  DD1 := Matrix(1 ..Dimension(pars), 1 ..Dimension(se)) :
  for i from 1 to Dimension(pars) do
      for j from 1 to Dimension(se) do
          DD1[i, j] := diff(se[j], pars[i])
      end do
   end do;
  DD1;
   end proc:
> Estpar := proc(DD1, pars, ret)
  local r, d, alphapre, alpha, PDE, FF, i, j, ans;
  description "Finds the estimable set of parameters for derivative matrix DD1. If ret = 1 returns
       alpha, PDEs, estimable parameter combinations. Otherwise returns estimable parameter
       combinations";
  with(LinearAlgebra) :
  r := Rank(DD1); d := Dimension(pars) - r :
  alphapre := NullSpace(Transpose(DD1)) : α := Matrix(d, Dimension(pars)) : PDE :=
       Vector(d) :
  FF := f(seq(pars[i], i = 1 ..Dimension(pars))) :
  for i from 1 to d do
       α[i, 1 ..Dimension(pars)] := alphapre[i] :
       PDE[i] := add(diff(FF, pars[j]) · α[i, j], j = 1 ..Dimension(pars)) :
  end do:
   if ret = 1 then
         ans := ⟨pdsolve({seq(PDE[i] = 0, i = 1 ..d)}), {alpha}, {PDE}⟩ :
   else
        ans := pdsolve({seq(PDE[i] = 0, i = 1 ..d)}) :
   end if:
   ans :
  end proc:
>
> # Starting with 4 age categories:
> C := ⟨⟨ h[t]·r[t]|0|0|0⟩, ⟨ 0| h[t]·r[t]|0|0⟩, ⟨ 0|0| h[t]·r[t]|0⟩, ⟨ 0|0| 0| h[t]·r[t]⟩⟩;

  A := ⟨⟨0|f[t − 1]|f[t − 1]|f[t − 1]⟩, ⟨(1 − h[t − 1])·s[t − 1]|0|0|0⟩, ⟨0|(1 − h[t − 1])·s[t
       − 1]|0|0⟩, ⟨0 |0|(1 − h[t − 1])·s[t − 1]|0⟩⟩; x1 := ⟨N[1, 1], N[1, 2], N[1, 3], N[1, 4]⟩;
```

$$C := \begin{bmatrix} h_t r_t & 0 & 0 & 0 \\ 0 & h_t r_t & 0 & 0 \\ 0 & 0 & h_t r_t & 0 \\ 0 & 0 & 0 & h_t r_t \end{bmatrix}$$

$$A := \begin{bmatrix} 0 & f_{t-1} & f_{t-1} & f_{t-1} \\ (1-h_{t-1})\, s_{t-1} & 0 & 0 & 0 \\ 0 & (1-h_{t-1})\, s_{t-1} & 0 & 0 \\ 0 & 0 & (1-h_{t-1})\, s_{t-1} & 0 \end{bmatrix}$$

$$x1 := \begin{bmatrix} N_{1,1} \\ N_{1,2} \\ N_{1,3} \\ N_{1,4} \end{bmatrix} \quad \textbf{(1)}$$

> #` `Components of the exhaustive summary

> # $C_1 x_1$

> $kappa11 := MatrixMatrixMultiply(eval(C, t=1), x1);$

$$\kappa 11 := \begin{bmatrix} h_1 r_1 N_{1,1} \\ h_1 r_1 N_{1,2} \\ h_1 r_1 N_{1,3} \\ h_1 r_1 N_{1,4} \end{bmatrix} \quad \textbf{(2)}$$

> #` `$C_2 A_2 x_1$

> $kappa12 := MatrixMatrixMultiply(eval(C, t=2), MatrixMatrixMultiply(eval(A, t=2), x1));$

$$\kappa 12 := \begin{bmatrix} h_2 r_2 \left(f_1 N_{1,2} + f_1 N_{1,3} + f_1 N_{1,4}\right) \\ h_2 r_2 \left(1-h_1\right) s_1 N_{1,1} \\ h_2 r_2 \left(1-h_1\right) s_1 N_{1,2} \\ h_2 r_2 \left(1-h_1\right) s_1 N_{1,3} \end{bmatrix} \quad \textbf{(3)}$$

> #$C_3 A_3 A_2 x_1$

> $kappa13 := MatrixMatrixMultiply(eval(C, t=3), MatrixMatrixMultiply(eval(A, t=3), MatrixMatrixMultiply(eval(A, t=2), x1)));$

(4)

$$\kappa 13 := \begin{bmatrix} h_3 r_3 \left(f_2 \left(1-h_1\right) s_1 N_{1,1} + f_2 \left(1-h_1\right) s_1 N_{1,2} + f_2 \left(1-h_1\right) s_1 N_{1,3}\right) \\ h_3 r_3 \left(1-h_2\right) s_2 \left(f_1 N_{1,2} + f_1 N_{1,3} + f_1 N_{1,4}\right) \\ h_3 r_3 \left(1-h_2\right) s_2 \left(1-h_1\right) s_1 N_{1,1} \\ h_3 r_3 \left(1-h_2\right) s_2 \left(1-h_1\right) s_1 N_{1,2} \end{bmatrix} \quad (4)$$

```
> #C3A4A3A2x1
```

```
> kappa14 := MatrixMatrixMultiply(eval(C, t=4), MatrixMatrixMultiply(eval(A, t=4),
      MatrixMatrixMultiply(eval(A, t=3), MatrixMatrixMultiply(eval(A, t=2), x1))));
```

$$\kappa 14 := \left[\left[h_4 r_4 \left(f_3 \left(1-h_2\right) s_2 \left(f_1 N_{1,2} + f_1 N_{1,3} + f_1 N_{1,4}\right) + f_3 \left(1-h_2\right) s_2 \left(1-h_1\right) s_1 N_{1,1} + f_3 \left(1-h_2\right) s_2 \left(1-h_1\right) s_1 N_{1,2}\right)\right], \left[h_4 r_4 \left(1-h_3\right) s_3 \left(f_2 \left(1-h_1\right) s_1 N_{1,1} + f_2 \left(1-h_1\right) s_1 N_{1,2} + f_2 \left(1-h_1\right) s_1 N_{1,3}\right)\right], \left[h_4 r_4 \left(1-h_3\right) s_3 \left(1-h_2\right) s_2 \left(f_1 N_{1,2} + f_1 N_{1,3} + f_1 N_{1,4}\right)\right], \left[h_4 r_4 \left(1-h_3\right) s_3 \left(1-h_2\right) s_2 \left(1-h_1\right) s_1 N_{1,1}\right]\right] \quad (5)$$

```
> #Building exhaustive summary up one component at a time
> kappa1 := <kappa11>;
```

$$\kappa 1 := \begin{bmatrix} h_1 r_1 N_{1,1} \\ h_1 r_1 N_{1,2} \\ h_1 r_1 N_{1,3} \\ h_1 r_1 N_{1,4} \end{bmatrix} \quad (6)$$

```
> indets(kappa1)
```

$$\{N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, h_1, r_1\} \quad (7)$$

```
> pars1 := <N1,1, N1,2, N1,3, N1,4, h1, r1>;
```

$$pars1 := \begin{bmatrix} N_{1,1} \\ N_{1,2} \\ N_{1,3} \\ N_{1,4} \\ h_1 \\ r_1 \end{bmatrix} \quad (8)$$

```
> D1 := Dmat(convert(kappa1, Vector), pars1);
```

(9)

$$D1 := \begin{bmatrix} h_1 r_1 & 0 & 0 & 0 \\ 0 & h_1 r_1 & 0 & 0 \\ 0 & 0 & h_1 r_1 & 0 \\ 0 & 0 & 0 & h_1 r_1 \\ r_1 N_{1,1} & r_1 N_{1,2} & r_1 N_{1,3} & r_1 N_{1,4} \\ h_1 N_{1,1} & h_1 N_{1,2} & h_1 N_{1,3} & h_1 N_{1,4} \end{bmatrix} \quad (9)$$

> $nopars := Dimension(pars1); rr := Rank(D1); d := Dimension(pars1) - rr;$

$$nopars := 6$$
$$rr := 4$$
$$d := 2 \quad (10)$$

> $Estpar(D1, pars1, 0);$

$$\left\{ f\left(N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, h_1, r_1\right) = \_F1\left(\frac{N_{1,2}}{N_{1,1}}, \frac{N_{1,3}}{N_{1,1}}, \frac{N_{1,4}}{N_{1,1}}, h_1 r_1 N_{1,1}\right) \right\} \quad (11)$$

> $kappa1 := \langle kappa11, kappa12 \rangle;$

$$\kappa 1 := \begin{bmatrix} h_1 r_1 N_{1,1} \\ h_1 r_1 N_{1,2} \\ h_1 r_1 N_{1,3} \\ h_1 r_1 N_{1,4} \\ h_2 r_2 \left(N_{1,2} f_1 + N_{1,3} f_1 + N_{1,4} f_1\right) \\ h_2 r_2 \left(1 - h_1\right) s_1 N_{1,1} \\ h_2 r_2 \left(1 - h_1\right) s_1 N_{1,2} \\ h_2 r_2 \left(1 - h_1\right) s_1 N_{1,3} \end{bmatrix} \quad (12)$$

> $indets(kappa1)$

$$\left\{N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, f_1, h_1, h_2, r_1, r_2, s_1\right\} \quad (13)$$

> $pars1 := \langle N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, f_1, h_1, h_2, r_1, r_2, s_1 \rangle;$

(14)

$$pars1 := \begin{bmatrix} N_{1,1} \\ N_{1,2} \\ N_{1,3} \\ N_{1,4} \\ f_1 \\ h_1 \\ h_2 \\ r_1 \\ r_2 \\ s_1 \end{bmatrix} \tag{14}$$

> $D1 := Dmat(convert(kappa1, Vector), pars1);$

$$\begin{aligned} D1 := & [[h_1 r_1, 0, 0, 0, 0, h_2 r_2 (1-h_1) s_1, 0, 0], \\ & [0, h_1 r_1, 0, 0, h_2 r_2 f_1, 0, h_2 r_2 (1-h_1) s_1, 0], \\ & [0, 0, h_1 r_1, 0, h_2 r_2 f_1, 0, 0, h_2 r_2 (1-h_1) s_1], \\ & [0, 0, 0, h_1 r_1, h_2 r_2 f_1, 0, 0, 0], \\ & [0, 0, 0, 0, h_2 r_2 (N_{1,2} + N_{1,3} + N_{1,4}), 0, 0, 0], \\ & [r_1 N_{1,1}, r_1 N_{1,2}, r_1 N_{1,3}, r_1 N_{1,4}, 0, -h_2 r_2 s_1 N_{1,1}, -h_2 r_2 s_1 N_{1,2}, -h_2 r_2 s_1 N_{1,3}], \\ & [0, 0, 0, 0, r_2 (N_{1,2} f_1 + N_{1,3} f_1 + N_{1,4} f_1), r_2 (1-h_1) s_1 N_{1,1}, r_2 (1-h_1) s_1 N_{1,2}, r_2 (1 \\ & -h_1) s_1 N_{1,3}], \\ & [h_1 N_{1,1}, h_1 N_{1,2}, h_1 N_{1,3}, h_1 N_{1,4}, 0, 0, 0, 0], \\ & [0, 0, 0, 0, h_2 (N_{1,2} f_1 + N_{1,3} f_1 + N_{1,4} f_1), h_2 (1-h_1) s_1 N_{1,1}, h_2 (1-h_1) s_1 N_{1,2}, \\ & h_2 (1-h_1) s_1 N_{1,3}], \\ & [0, 0, 0, 0, 0, h_2 r_2 (1-h_1) N_{1,1}, h_2 r_2 (1-h_1) N_{1,2}, h_2 r_2 (1-h_1) N_{1,3}]] \end{aligned} \tag{15}$$

> $nopars := Dimension(pars1); rr := Rank(D1); d := Dimension(pars1) - rr;$

$$\begin{aligned} nopars &:= 10 \\ rr &:= 6 \\ d &:= 4 \end{aligned} \tag{16}$$

> $Estpar(D1, pars1, 0);$

$$\left\{ f(N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, f_1, h_1, h_2, r_1, r_2, s_1) = \_F1\left(\frac{N_{1,2}}{N_{1,1}}, \frac{N_{1,3}}{N_{1,1}}, \frac{N_{1,4}}{N_{1,1}}, h_1 r_1 N_{1,1}, h_2 r_2 f_1 N_{1,1}, \frac{s_1 (-1+h_1)}{f_1} \right) \right\} \tag{17}$$

> $kappa1 := \langle kappa11, kappa12, kappa13 \rangle;$

$$\kappa 1 := \begin{bmatrix} h_1 r_1 N_{1,1} \\ h_1 r_1 N_{1,2} \\ h_1 r_1 N_{1,3} \\ h_1 r_1 N_{1,4} \\ h_2 r_2 \left(N_{1,2} f_1 + N_{1,3} f_1 + N_{1,4} f_1\right) \\ h_2 r_2 \left(1 - h_1\right) s_1 N_{1,1} \\ h_2 r_2 \left(1 - h_1\right) s_1 N_{1,2} \\ h_2 r_2 \left(1 - h_1\right) s_1 N_{1,3} \\ h_3 r_3 \left(f_2 \left(1 - h_1\right) s_1 N_{1,1} + f_2 \left(1 - h_1\right) s_1 N_{1,2} + f_2 \left(1 - h_1\right) s_1 N_{1,3}\right) \\ h_3 r_3 \left(1 - h_2\right) s_2 \left(N_{1,2} f_1 + N_{1,3} f_1 + N_{1,4} f_1\right) \\ \vdots \end{bmatrix} \quad (18)$$

12 × 1 Matrix

> *indets*(*kappa1*)

$$\{N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2\} \quad (19)$$

> *pars1* := $\langle N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2 \rangle$:

> *D1* := *Dmat*(*convert*(*kappa1*, *Vector*), *pars1*) :

> *nopars* := *Dimension*(*pars1*); *rr* := *Rank*(*D1*); *d* := *Dimension*(*pars1*) − *rr*;

$$nopars := 14$$
$$rr := 8$$
$$d := 6 \quad (20)$$

> *Estpar*(*D1*, *pars1*, 0);

$$\left\{f\left(N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2\right) = \_F1\left(\frac{N_{1,2}}{N_{1,1}}, \frac{N_{1,3}}{N_{1,1}}, \frac{N_{1,4}}{N_{1,1}}, h_1 r_1 N_{1,1}, h_2 r_2 f_1 N_{1,1}, h_3 r_3 f_2 f_1 N_{1,1}, \frac{s_1 \left(-1 + h_1\right)}{f_1}, \frac{s_2 \left(-1 + h_2\right)}{f_2}\right)\right\} \quad (21)$$

> # *Clear pattern*

> #*Estimable parameter combinations are* $r_1 h_1 N_{1,1}$, $r_2 h_2 f_1 N_{1,1}$, $r_3 h_3 f_1 f_2 N_{1,1}$, ..., $r_n h_n f_1 f_2 ... f_n N_{1,1}$,

$$\frac{s_1 \left(1 - h_1\right)}{f_1}, \frac{s_2 \left(1 - h_2\right)}{f_2}, \ldots, \frac{s_{n-1} \left(1 - h_{n-1}\right)}{f_{n-1}}, \frac{N_{1,2}}{N_{1,1}}, \frac{N_{1,3}}{N_{1,1}}, \ldots, \frac{N_{1,n}}{N_{1,1}}$$

> *pp* := $4 \cdot n - 2 + a$;

$$pp := 4\,n - 2 + a$$
$$14 \quad (22)$$

> *rr* := $2 \cdot n + a - 2$;

$$rr := 2\,n + a - 2$$
$$8 \quad (23)$$

```
> dd := pp - rr;
```

$$dd := 2\,n \tag{24}$$

```
> # Apply extension Theomrem in two directions
> # First involves adding extra age classes. Would need to first reparameterise in terms of
    estimable parameter combinations. .
> # Second involves adding extra years. Also involves reparamerising in terms of the estimable
    parameter combinations.
>
> # Hunter Survey data
> kappa2 := <r[1]·a[1], r[2]·a[2], r[3]·a[3]>;
```

$$\kappa 2 := \begin{bmatrix} r_1\,a_1 \\ r_2\,a_2 \\ r_3\,a_3 \end{bmatrix} \tag{25}$$

```
> pars2 := <r[1], r[2], r[3]>;
```

$$pars2 := \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \tag{26}$$

```
> D2 := Dmat(kappa2, pars2);
```

$$D2 := \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \tag{27}$$

```
> nopars := Dimension(pars2); rr := Rank(D2); d := Dimension(pars2) - rr;
```

$$nopars := 3$$
$$rr := 3$$
$$d := 0 \tag{28}$$

```
>
>
> # Radio Tracking data
> kappa3 := <h[1]·v[1], (1 - h[1])·(1 - s[1])·v[1], h[2]·v[2], (1 - h[2])·(1 - s[2])·v[2],
    h[3]·v[3], (1 - h[3])·(1 - s[3])·v[3]>;
```

$$\kappa 3 := \begin{bmatrix} h_1\,v_1 \\ (1-h_1)\,(1-s_1)\,v_1 \\ h_2\,v_2 \\ (1-h_2)\,(1-s_2)\,v_2 \\ h_3\,v_3 \\ (1-h_3)\,(1-s_3)\,v_3 \end{bmatrix} \tag{29}$$

> $pars3 := \langle h_1, h_2, h_3, s_1, s_2, s_3 \rangle;$

$$pars3 := \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \tag{30}$$

> $D3 := Dmat(kappa3, pars3);$

$$D3 := \begin{bmatrix} v_1 & -(1-s_1)\,v_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 & -(1-s_2)\,v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_3 & -(1-s_3)\,v_3 \\ 0 & -(1-h_1)\,v_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-h_2)\,v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1-h_3)\,v_3 \end{bmatrix} \tag{31}$$

> $nopars := Dimension(pars3); rr := Rank(D3); d := Dimension(pars3) - rr;$

$$nopars := 6$$
$$rr := 6$$
$$d := 0 \tag{32}$$

>

> # Considering different combinations of data sets:

> $kappajoin := convert(\langle kappa1, kappa2 \rangle, Vector) :$

> $indets(kappajoin)$

$$\{N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, a_1, a_2, a_3, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2\} \tag{33}$$

> $pars := \langle N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2 \rangle :$

> $Djoin := Dmat(kappajoin, pars) :$

> $nopars := Dimension(pars); rr := Rank(Djoin); d := Dimension(pars) - rr;$

$$nopars := 14$$
$$rr := 11$$
$$d := 3 \tag{34}$$

> $Estpar(Djoin, pars, 0);$

$$\left\{ f\left(N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2\right) = \_F1\left(h_1\, N_{1,1}, \frac{N_{1,2}}{N_{1,1}}, \frac{N_{1,3}}{N_{1,1}}, \frac{N_{1,4}}{N_{1,1}}, \right.\right. \tag{35}$$
$$\left.\left. f_1\, h_2\, N_{1,1}, f_2\, h_3 f_1\, N_{1,1}, r_1, r_2, r_3, -\frac{s_1\,(-1+h_1)}{f_1}, -\frac{s_2\,(-1+h_2)}{f_2}\right)\right\}$$

> $pp := 4 \cdot n + a - 2;$

(36)

$$pp := 4\,n - 2 + a \tag{36}$$

> $rr := 3\cdot n + a - 2;$

$$rr := 3\,n + a - 2 \tag{37}$$

> $dd := pp - rr;$

$$dd := n \tag{38}$$

> $kappajoin := convert(\langle kappa1, kappa3\rangle, Vector)$:

> $indets(kappajoin)$

$$\{N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3, v_1, v_2, v_3\} \tag{39}$$

> $pars := \langle N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3\rangle$:

> $Djoin := Dmat(kappajoin, pars)$:

> $nopars := Dimension(pars); rr := Rank(Djoin); d := Dimension(pars) - rr;$

$$nopars := 15$$
$$rr := 14$$
$$d := 1 \tag{40}$$

> $Estpar(Djoin, pars, 0);$

$$\left\{f\left(N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3\right) = \_F1\left(\frac{N_{1,2}}{N_{1,1}}, \frac{N_{1,3}}{N_{1,1}}, \frac{N_{1,4}}{N_{1,1}}, f_1, f_2, h_1, h_2, h_3, r_1\,N_{1,1}, N_{1,1}\,r_2, N_{1,1}\,r_3, s_1, s_2, s_3\right)\right\} \tag{41}$$

> $kappajoin := convert(\langle kappa2, kappa3\rangle, Vector)$:

> $indets(kappajoin)$

$$\{a_1, a_2, a_3, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3, v_1, v_2, v_3\} \tag{42}$$

> $pars := \langle h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3\rangle$:

> $Djoin := Dmat(kappajoin, pars)$:

> $nopars := Dimension(pars); rr := Rank(Djoin); d := Dimension(pars) - rr;$

$$nopars := 9$$
$$rr := 9$$
$$d := 0 \tag{43}$$

> $kappajoin := convert(\langle kappa1, kappa2, kappa3\rangle, Vector)$:

> $indets(kappajoin)$

$$\{N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, a_1, a_2, a_3, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3, v_1, v_2, v_3\} \tag{44}$$

> $pars := \langle N_{1,1}, N_{1,2}, N_{1,3}, N_{1,4}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3\rangle$:

> $Djoin := Dmat(kappajoin, pars)$:

> $nopars := Dimension(pars); rr := Rank(Djoin); d := Dimension(pars) - rr;$

$$nopars := 15$$
$$rr := 15$$
$$d := 0 \tag{45}$$

>