```
> # Identifiability of Elk using 2 state model
>
> restart;
> with(LinearAlgebra) :
> Dmat :=proc(se, pars)
local DD1, i, j;
description "Form the derivative matrix";
with(LinearAlgebra) :
DD1 := Matrix(1 ..Dimension(pars), 1 ..Dimension(se)) :
for i from 1 to Dimension(pars)  do
     for j from 1 to Dimension(se) do
          DD1[i, j] := diff(se[j], pars[i])
     end do
 end do;
DD1;
 end proc:
> Estpar :=proc(DD1, pars, ret)
local r, d, alphapre, alpha, PDE, FF, i, j, ans;
description "Finds the estimable set of parameters for derivative matrix DD1. If ret = 1 returns
      alpha, PDEs, estimable parameter combinations. Otherwise returns estimable parameter
      combinations";
with(LinearAlgebra) :
r := Rank(DD1); d := Dimension(pars) - r :
alphapre := NullSpace(Transpose(DD1)) : α := Matrix(d, Dimension(pars)) : PDE :=
     Vector(d) :
FF := f(seq(pars[i], i = 1 ..Dimension(pars))) :
for i from 1 to d do
        α[i, 1 ..Dimension(pars)] := alphapre[i] :
        PDE[i] := add(diff(FF, pars[j])·α[i, j], j = 1 ..Dimension(pars)) :
end do:
 if ret = 1 then
           ans := ⟨pdsolve({seq(PDE[i] = 0, i = 1 ..d)}), {alpha}, {PDE}⟩ :
 else
          ans := pdsolve({seq(PDE[i] = 0, i = 1 ..d)}) :
 end if:
 ans :
end proc:
> C := ⟨⟨h[t]·r[t]|0⟩, ⟨0| h[t]·r[t]⟩⟩;
 A := ⟨⟨0 | f[t − 1]⟩, ⟨(1 − h[t − 1])·s[t − 1]|(1 − h[t − 1])·s[t − 1]⟩⟩;
 x1 := ⟨N[1, 1], N[1, 2]⟩;
```

$$C := \begin{bmatrix} h_t r_t & 0 \\ 0 & h_t r_t \end{bmatrix}$$

$$A := \begin{bmatrix} 0 & f_{t-1} \\ \left(1-h_{t-1}\right) s_{t-1} & \left(1-h_{t-1}\right) s_{t-1} \end{bmatrix}$$

(1)

$$x1 := \begin{bmatrix} N_{1,1} \\ N_{1,2} \end{bmatrix} \tag{1}$$

```
> 
> # Components of the exhaustive summary
```

> # $C_1 x_1$

> $kappa11 := MatrixMatrixMultiply(eval(C, t=1), x1);$

$$\kappa11 := \begin{bmatrix} h_1\, r_1\, N_{1,1} \\ h_1\, r_1\, N_{1,2} \end{bmatrix} \tag{2}$$

> # $C_2 A_2 x_1$

> $kappa12 := MatrixMatrixMultiply(eval(C, t=2), MatrixMatrixMultiply(eval(A, t=2), x1));$

$$\kappa12 := \begin{bmatrix} h_2\, r_2\, f_1\, N_{1,2} \\ h_2\, r_2\, \left(\left(1-h_1\right) s_1\, N_{1,1} + \left(1-h_1\right) s_1\, N_{1,2}\right) \end{bmatrix} \tag{3}$$

> #$C_3 A_3 A_2 x_1$

> $kappa13 := MatrixMatrixMultiply(eval(C, t=3), MatrixMatrixMultiply(eval(A, t=3), MatrixMatrixMultiply(eval(A, t=2), x1)));$

$$\kappa13 := \begin{bmatrix} h_3\, r_3\, f_2\, \left(\left(1-h_1\right) s_1\, N_{1,1} + \left(1-h_1\right) s_1\, N_{1,2}\right) \\ h_3\, r_3\, \left(\left(1-h_2\right) s_2\, f_1\, N_{1,2} + \left(1-h_2\right) s_2\, \left(\left(1-h_1\right) s_1\, N_{1,1} + \left(1-h_1\right) s_1\, N_{1,2}\right)\right) \end{bmatrix} \tag{4}$$

> #$C_3 A_4 A_3 A_2 x_1$

> $kappa14 := MatrixMatrixMultiply(eval(C, t=4), MatrixMatrixMultiply(eval(A, t=4), MatrixMatrixMultiply(eval(A, t=3), MatrixMatrixMultiply(eval(A, t=2), x1))));$

$$\kappa14 := \left[\left[h_4\, r_4\, f_3\, \left(\left(1-h_2\right) s_2\, f_1\, N_{1,2} + \left(1-h_2\right) s_2\, \left(\left(1-h_1\right) s_1\, N_{1,1} + \left(1-h_1\right) s_1\, N_{1,2}\right)\right)\right], \left[h_4\, r_4\, \left(\left(1-h_3\right) s_3\, f_2\, \left(\left(1-h_1\right) s_1\, N_{1,1} + \left(1-h_1\right) s_1\, N_{1,2}\right) + \left(1-h_3\right) s_3\, \left(\left(1-h_2\right) s_2\, f_1\, N_{1,2} + \left(1-h_2\right) s_2\, \left(\left(1-h_1\right) s_1\, N_{1,1} + \left(1-h_1\right) s_1\, N_{1,2}\right)\right)\right)\right]\right] \tag{5}$$

> # Building exhaustive summary up one component at a time

> $kappa1 := \langle kappa11 \rangle;$

$$\kappa1 := \begin{bmatrix} h_1\, r_1\, N_{1,1} \\ h_1\, r_1\, N_{1,2} \end{bmatrix} \tag{6}$$

> $indets(kappa1)$

$$\{N_{1,1}, N_{1,2}, h_1, r_1\} \tag{7}$$

> $pars1 := \langle h_1, r_1 \rangle;$

$$pars1 := \begin{bmatrix} h_1 \\ r_1 \end{bmatrix} \tag{8}$$

> $D1 := Dmat(convert(kappa1, Vector), pars1);$

$$D1 := \begin{bmatrix} r_1 N_{1,1} & r_1 N_{1,2} \\ h_1 N_{1,1} & h_1 N_{1,2} \end{bmatrix} \tag{9}$$

> $nopars := Dimension(pars1); rr := Rank(D1); d := Dimension(pars1) - rr;$

$$nopars := 2$$
$$rr := 1$$
$$d := 1 \tag{10}$$

> $Estpar(D1, pars1, 0);$

$$\{f(h_1, r_1) = \_F1(r_1 h_1)\} \tag{11}$$

> $kappa1 := \langle kappa11, kappa12 \rangle;$

$$\kappa 1 := \begin{bmatrix} h_1 r_1 N_{1,1} \\ h_1 r_1 N_{1,2} \\ h_2 r_2 f_1 N_{1,2} \\ h_2 r_2 \left(\left(1 - h_1\right) s_1 N_{1,1} + \left(1 - h_1\right) s_1 N_{1,2} \right) \end{bmatrix} \tag{12}$$

> $indets(kappa1)$

$$\{N_{1,1}, N_{1,2}, f_1, h_1, h_2, r_1, r_2, s_1\} \tag{13}$$

> $pars1 := \langle f_1, h_1, h_2, r_1, r_2, s_1 \rangle;$

$$pars1 := \begin{bmatrix} f_1 \\ h_1 \\ h_2 \\ r_1 \\ r_2 \\ s_1 \end{bmatrix} \tag{14}$$

> $D1 := Dmat(convert(kappa1, Vector), pars1);$

$$D1 := \begin{bmatrix} 0 & 0 & h_2 r_2 N_{1,2} & 0 \\ r_1 N_{1,1} & r_1 N_{1,2} & 0 & h_2 r_2 \left(-s_1 N_{1,1} - s_1 N_{1,2}\right) \\ 0 & 0 & r_2 f_1 N_{1,2} & r_2 \left(\left(1 - h_1\right) s_1 N_{1,1} + \left(1 - h_1\right) s_1 N_{1,2} \right) \\ h_1 N_{1,1} & h_1 N_{1,2} & 0 & 0 \\ 0 & 0 & h_2 f_1 N_{1,2} & h_2 \left(\left(1 - h_1\right) s_1 N_{1,1} + \left(1 - h_1\right) s_1 N_{1,2} \right) \\ 0 & 0 & 0 & h_2 r_2 \left(\left(1 - h_1\right) N_{1,1} + \left(1 - h_1\right) N_{1,2} \right) \end{bmatrix} \tag{15}$$

> $nopars := Dimension(pars1); rr := Rank(D1); d := Dimension(pars1) - rr;$

$$nopars := 6$$
$$rr := 3$$

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$$d := 3 \tag{16}$$

> $Estpar(D1, pars1, 0);$

$$\left\{ f\left(f_1, h_1, h_2, r_1, r_2, s_1\right) = \_F1\left(r_1\, h_1, h_2\, r_2 f_1, \frac{s_1\left(h_1 - 1\right)}{f_1}\right) \right\} \tag{17}$$

> $kappa1 := \langle kappa11, kappa12, kappa13 \rangle;$

$$\kappa 1 := \begin{bmatrix} h_1\, r_1\, N_{1,\,1} \\ h_1\, r_1\, N_{1,\,2} \\ h_2\, r_2 f_1\, N_{1,\,2} \\ h_2\, r_2\left(\left(1 - h_1\right) s_1\, N_{1,\,1} + \left(1 - h_1\right) s_1\, N_{1,\,2}\right) \\ h_3\, r_3 f_2\left(\left(1 - h_1\right) s_1\, N_{1,\,1} + \left(1 - h_1\right) s_1\, N_{1,\,2}\right) \\ h_3\, r_3\left(\left(1 - h_2\right) s_2 f_1\, N_{1,\,2} + \left(1 - h_2\right) s_2\left(\left(1 - h_1\right) s_1\, N_{1,\,1} + \left(1 - h_1\right) s_1\, N_{1,\,2}\right)\right) \end{bmatrix} \tag{18}$$

> $indets(kappa1)$

$$\left\{N_{1,\,1}, N_{1,\,2}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2\right\} \tag{19}$$

> $pars1 := \left\langle f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2 \right\rangle :$

> $D1 := Dmat(convert(kappa1, Vector), pars1) :$

> $nopars := Dimension(pars1); rr := Rank(D1); d := Dimension(pars1) - rr;$

$$nopars := 10$$
$$rr := 5$$
$$d := 5 \tag{20}$$

> $Estpar(D1, pars1, 0);$

$$\left\{ f\left(f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2\right) = \_F1\left(r_1\, h_1, h_2\, r_2 f_1, r_3\, h_3 f_2 f_1, \frac{s_1\left(h_1 - 1\right)}{f_1}, \frac{s_2\left(-1 + h_2\right)}{f_2}\right) \right\} \tag{21}$$

> # Clear pattern

> #Estimable parameter combinations are $r_1\, h_1,\ r_2 h_2 f_1,\ r_3 h_3 f_1 f_2, ..., r_n h_n f_1 f_2 .. f_n, \frac{s_1\left(1 - h_1\right)}{f_1}, \frac{s_2\left(1 - h_2\right)}{f_2}, ..., \frac{s_{n-1}\left(1 - h_{n-1}\right)}{f_{n-1}}$

> $pp := 4 \cdot n - 2$

$$pp := 4\, n - 2 \tag{22}$$

> $rr := 2 \cdot n - 1$

$$rr := 2\, n - 1 \tag{23}$$

> $dd := pp - rr;$

$$dd := 2\, n - 1 \tag{24}$$

> # Apply extension Theomrem:

> # Need to first reparameterise in terms of estimable parameter combinations - then apply extension theorem

>

> # Hunter Survey data

> $kappa2 := \langle r[1]\cdot a[1], r[2]\cdot a[2], r[3]\cdot a[3]\rangle;$

$$\kappa 2 := \begin{bmatrix} r_1 a_1 \\ r_2 a_2 \\ r_3 a_3 \end{bmatrix} \quad (25)$$

> $pars2 := \langle r[1], r[2], r[3]\rangle;$

$$pars2 := \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad (26)$$

> $D2 := Dmat(kappa2, pars2);$

$$D2 := \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \quad (27)$$

> $nopars := Dimension(pars2); rr := Rank(D2); d := Dimension(pars2) - rr;$

$$nopars := 3$$
$$rr := 3$$
$$d := 0 \quad (28)$$

>

>

> # *Radio Tracking data*

> $kappa3 := \langle h[1]\cdot v[1], (1-h[1])\cdot(1-s[1])\cdot v[1], h[2]\cdot v[2], (1-h[2])\cdot(1-s[2])\cdot v[2], h[3]\cdot v[3], (1-h[3])\cdot(1-s[3])\cdot v[3]\rangle;$

$$\kappa 3 := \begin{bmatrix} h_1 v_1 \\ (1-h_1)(1-s_1) v_1 \\ h_2 v_2 \\ (1-h_2)(1-s_2) v_2 \\ h_3 v_3 \\ (1-h_3)(1-s_3) v_3 \end{bmatrix} \quad (29)$$

> $pars3 := \langle h_1, h_2, h_3, s_1, s_2, s_3\rangle;$

(30)

$$pars3 := \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \quad (30)$$

```
> D3 := Dmat(kappa3, pars3);
```

$$D3 := \begin{bmatrix} v_1 & -(1-s_1)\,v_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 & -(1-s_2)\,v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_3 & -(1-s_3)\,v_3 \\ 0 & -(1-h_1)\,v_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-h_2)\,v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1-h_3)\,v_3 \end{bmatrix} \quad (31)$$

```
> nopars := Dimension(pars3); rr := Rank(D3); d := Dimension(pars3) − rr;
```

$$nopars := 6$$
$$rr := 6$$
$$d := 0 \quad (32)$$

```
>
> # Considering different combinations of data sets:
> kappajoin := convert(⟨kappa1, kappa2⟩, Vector) :
> indets(kappajoin)
```

$$\{N_{1,1}, N_{1,2}, a_1, a_2, a_3, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2\} \quad (33)$$

```
> pars := ⟨f1, f2, h1, h2, h3, r1, r2, r3, s1, s2⟩ :
> Djoin := Dmat(kappajoin, pars) :
> nopars := Dimension(pars); rr := Rank(Djoin); d := Dimension(pars) − rr;
```

$$nopars := 10$$
$$rr := 8$$
$$d := 2 \quad (34)$$

```
> Estpar(Djoin, pars, 0);
```

$$\left\{ f(f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2) = \_F1\left(h_1, f_1\,h_2, h_3 f_2 f_1, r_1, r_2, r_3, \frac{s_1}{f_1}, -\frac{s_2\,(-1+h_2)}{f_2}\right) \right\} \quad (35)$$

```
> pp := 4·n − 2
```

$$pp := 4\,n - 2 \quad (36)$$

```
> rr := 3·n − 1
```

$$rr := 3\,n - 1 \quad (37)$$

```
> dd := pp − rr;
```

$$dd := n - 1 \tag{38}$$

```
> kappajoin := convert(⟨kappa1, kappa3⟩, Vector) :
> indets(kappajoin)
```

$$\{N_{1,1}, N_{1,2}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3, v_1, v_2, v_3\} \tag{39}$$

```
> pars := ⟨f[1], f[2], h[1], h[2], h[3], r[1], r[2], r[3], s[1], s[2], s[3]⟩ :
> Djoin := Dmat(kappajoin, pars) :
> nopars := Dimension(pars); rr := Rank(Djoin); d := Dimension(pars) - rr;
```

$$nopars := 11$$
$$rr := 11$$
$$d := 0 \tag{40}$$

```
> kappajoin := convert(⟨kappa2, kappa3⟩, Vector) :
> indets(kappajoin)
```

$$\{a_1, a_2, a_3, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3, v_1, v_2, v_3\} \tag{41}$$

```
> pars := ⟨h[1], h[2], h[3], r[1], r[2], r[3], s[1], s[2], s[3]⟩ :
> Djoin := Dmat(kappajoin, pars) :
> nopars := Dimension(pars); rr := Rank(Djoin); d := Dimension(pars) - rr;
```

$$nopars := 9$$
$$rr := 9$$
$$d := 0 \tag{42}$$

```
> kappajoin := convert(⟨kappa1, kappa2, kappa3⟩, Vector) :
> indets(kappajoin)
```

$$\{N_{1,1}, N_{1,2}, a_1, a_2, a_3, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3, v_1, v_2, v_3\} \tag{43}$$

```
> pars := ⟨f[1], f[2], h[1], h[2], h[3], r[1], r[2], r[3], s[1], s[2], s[3]⟩ :
> Djoin := Dmat(kappajoin, pars) :
> nopars := Dimension(pars); rr := Rank(Djoin); d := Dimension(pars) - rr;
```

$$nopars := 11$$
$$rr := 11$$
$$d := 0 \tag{44}$$

```
>
> # Identifiability of Elk using 2 state model - assuming starting values unknown
>
> C := ⟨⟨ h[t]·r[t]|0⟩, ⟨ 0| h[t]·r[t]⟩⟩;
  A := ⟨⟨0 | f[t-1]⟩, ⟨(1 - h[t-1])·s[t-1]|(1 - h[t-1])·s[t-1]⟩⟩;
  x1 := ⟨N[1, 1], N[1, 2]⟩;
```

$$C := \begin{bmatrix} h_t r_t & 0 \\ 0 & h_t r_t \end{bmatrix}$$

$$A := \begin{bmatrix} 0 & f_{t-1} \\ \left(1 - h_{t-1}\right) s_{t-1} & \left(1 - h_{t-1}\right) s_{t-1} \end{bmatrix}$$

$$\tag{45}$$

$$x1 := \begin{bmatrix} N_{1,1} \\ N_{1,2} \end{bmatrix} \tag{45}$$

```
> 
> #` Components of the exhaustive summary
```

> # $C_1 x_1$

> $kappa11 := MatrixMatrixMultiply(eval(C, t=1), x1);$

$$\kappa 11 := \begin{bmatrix} h_1\, r_1\, N_{1,1} \\ h_1\, r_1\, N_{1,2} \end{bmatrix} \tag{46}$$

> #` `$C_2 A_2 x_1$

> $kappa12 := MatrixMatrixMultiply(eval(C, t=2), MatrixMatrixMultiply(eval(A, t=2), x1));$

$$\kappa 12 := \begin{bmatrix} h_2\, r_2\, f_1\, N_{1,2} \\ h_2\, r_2\, \left(\left(1-h_1\right) s_1\, N_{1,1} + \left(1-h_1\right) s_1\, N_{1,2}\right) \end{bmatrix} \tag{47}$$

> #$C_3 A_3 A_2 x_1$

> $kappa13 := MatrixMatrixMultiply(eval(C, t=3), MatrixMatrixMultiply(eval(A, t=3), MatrixMatrixMultiply(eval(A, t=2), x1)));$

$$\kappa 13 := \begin{bmatrix} h_3\, r_3\, f_2\, \left(\left(1-h_1\right) s_1\, N_{1,1} + \left(1-h_1\right) s_1\, N_{1,2}\right) \\ h_3\, r_3\, \left(\left(1-h_2\right) s_2\, f_1\, N_{1,2} + \left(1-h_2\right) s_2\, \left(\left(1-h_1\right) s_1\, N_{1,1} + \left(1-h_1\right) s_1\, N_{1,2}\right)\right) \end{bmatrix} \tag{48}$$

> #$C_3 A_4 A_3 A_2 x_1$

> $kappa14 := MatrixMatrixMultiply(eval(C, t=4), MatrixMatrixMultiply(eval(A, t=4), MatrixMatrixMultiply(eval(A, t=3), MatrixMatrixMultiply(eval(A, t=2), x1))));$

$$\kappa 14 := \left[\left[h_4\, r_4\, f_3\, \left(\left(1-h_2\right) s_2\, f_1\, N_{1,2} + \left(1-h_2\right) s_2\, \left(\left(1-h_1\right) s_1\, N_{1,1} + \left(1-h_1\right) s_1\, N_{1,2}\right)\right)\right], \left[h_4\, r_4\, \left(\left(1-h_3\right) s_3\, f_2\, \left(\left(1-h_1\right) s_1\, N_{1,1} + \left(1-h_1\right) s_1\, N_{1,2}\right) + \left(1-h_3\right) s_3\, \left(\left(1-h_2\right) s_2\, f_1\, N_{1,2} + \left(1-h_2\right) s_2\, \left(\left(1-h_1\right) s_1\, N_{1,1} + \left(1-h_1\right) s_1\, N_{1,2}\right)\right)\right)\right]\right] \tag{49}$$

```
> #` Building exhaustive summary up one component at a time
```

> $kappa1 := \langle kappa11 \rangle;$

$$\kappa 1 := \begin{bmatrix} h_1\, r_1\, N_{1,1} \\ h_1\, r_1\, N_{1,2} \end{bmatrix} \tag{50}$$

> $indets(kappa1)$

$$\left\{N_{1,1}, N_{1,2}, h_1, r_1\right\} \tag{51}$$

> $pars1 := \langle N_{1,1}, N_{1,2}, h_1, r_1 \rangle;$

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$$pars1 := \begin{bmatrix} N_{1,1} \\ N_{1,2} \\ h_1 \\ r_1 \end{bmatrix} \tag{52}$$

> $D1 := Dmat(convert(kappa1, Vector), pars1);$

$$D1 := \begin{bmatrix} r_1\,h_1 & 0 \\ 0 & r_1\,h_1 \\ r_1\,N_{1,1} & r_1\,N_{1,2} \\ h_1\,N_{1,1} & h_1\,N_{1,2} \end{bmatrix} \tag{53}$$

> $nopars := Dimension(pars1); rr := Rank(D1); d := Dimension(pars1) - rr;$

$$nopars := 4$$
$$rr := 2$$
$$d := 2 \tag{54}$$

> $Estpar(D1, pars1, 0);$

$$\left\{ f\left(N_{1,1}, N_{1,2}, h_1, r_1\right) = \_F1\left(\frac{N_{1,2}}{N_{1,1}}, h_1\,r_1\,N_{1,1}\right) \right\} \tag{55}$$

> $kappa1 := \langle kappa11, kappa12 \rangle;$

$$\kappa 1 := \begin{bmatrix} h_1\,r_1\,N_{1,1} \\ h_1\,r_1\,N_{1,2} \\ h_2\,r_2\,f_1\,N_{1,2} \\ h_2\,r_2\,\left(\left(1-h_1\right)\,s_1\,N_{1,1} + \left(1-h_1\right)\,s_1\,N_{1,2}\right) \end{bmatrix} \tag{56}$$

> $indets(kappa1)$

$$\left\{N_{1,1}, N_{1,2}, f_1, h_1, h_2, r_1, r_2, s_1\right\} \tag{57}$$

> $pars1 := \left\langle N_{1,1}, N_{1,2}, f_1, h_1, h_2, r_1, r_2, s_1 \right\rangle;$

(58)

$$pars1 := \begin{bmatrix} N_{1,1} \\ N_{1,2} \\ f_1 \\ h_1 \\ h_2 \\ r_1 \\ r_2 \\ s_1 \end{bmatrix} \tag{58}$$

> $D1 := Dmat(convert(kappa1, Vector), pars1);$

$$D1 := \begin{bmatrix} r_1 h_1 & 0 & 0 & h_2 r_2 (1 - h_1) s_1 \\ 0 & r_1 h_1 & h_2 r_2 f_1 & h_2 r_2 (1 - h_1) s_1 \\ 0 & 0 & h_2 r_2 N_{1,2} & 0 \\ r_1 N_{1,1} & r_1 N_{1,2} & 0 & h_2 r_2 (-N_{1,1} s_1 - N_{1,2} s_1) \\ 0 & 0 & r_2 f_1 N_{1,2} & r_2 ((1 - h_1) s_1 N_{1,1} + (1 - h_1) s_1 N_{1,2}) \\ h_1 N_{1,1} & h_1 N_{1,2} & 0 & 0 \\ 0 & 0 & h_2 f_1 N_{1,2} & h_2 ((1 - h_1) s_1 N_{1,1} + (1 - h_1) s_1 N_{1,2}) \\ 0 & 0 & 0 & h_2 r_2 ((1 - h_1) N_{1,1} + (1 - h_1) N_{1,2}) \end{bmatrix} \tag{59}$$

> $nopars := Dimension(pars1); rr := Rank(D1); d := Dimension(pars1) - rr;$

$$nopars := 8$$
$$rr := 4$$
$$d := 4 \tag{60}$$

> $Estpar(D1, pars1, 0);$

$$\left\{ f(N_{1,1}, N_{1,2}, f_1, h_1, h_2, r_1, r_2, s_1) = \_F1\left(\frac{N_{1,2}}{N_{1,1}}, h_1 r_1 N_{1,1}, h_2 r_2 f_1 N_{1,1}, \frac{s_1 (h_1 - 1)}{f_1} \right) \right\} \tag{61}$$

> $kappa1 := \langle kappa11, kappa12, kappa13 \rangle;$

$$\kappa 1 := \begin{bmatrix} h_1 r_1 N_{1,1} \\ h_1 r_1 N_{1,2} \\ h_2 r_2 f_1 N_{1,2} \\ h_2 r_2 ((1 - h_1) s_1 N_{1,1} + (1 - h_1) s_1 N_{1,2}) \\ h_3 r_3 f_2 ((1 - h_1) s_1 N_{1,1} + (1 - h_1) s_1 N_{1,2}) \\ h_3 r_3 ((1 - h_2) s_2 f_1 N_{1,2} + (1 - h_2) s_2 ((1 - h_1) s_1 N_{1,1} + (1 - h_1) s_1 N_{1,2})) \end{bmatrix} \tag{62}$$

> indets(kappa1)

$$\{N_{1,1}, N_{1,2}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2\} \quad (63)$$

> pars1 := ⟨N[1, 1], N[1, 2], f[1], f[2], h[1], h[2], h[3], r[1], r[2], r[3], s[1], s[2]⟩ :

> D1 := Dmat(convert(kappa1, Vector), pars1) :

> nopars := Dimension(pars1); rr := Rank(D1); d := Dimension(pars1) − rr;

$$nopars := 12$$
$$rr := 6$$
$$d := 6 \quad (64)$$

> Estpar(D1, pars1, 0);

$$\left\{f\left(N_{1,1}, N_{1,2}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2\right) = \_F1\left(\frac{N_{1,2}}{N_{1,1}}, h_1\, r_1\, N_{1,1}, h_2\, r_2 f_1\, N_{1,1}, N_{1,1}\, r_3\, h_3 f_2 f_1, \frac{s_1\,(h_1-1)}{f_1}, \frac{s_2\,(-1+h_2)}{f_2}\right)\right\} \quad (65)$$

> # Clear pattern

> #Estimable parameter combinations are $\frac{N_{1,2}}{N_{1,1}}$, $r_1\, h_1 N_{1,1}$, $r_2 h_2 f_1 N_{1,1}$, $r_3 h_3 f_1 f_2 N_{1,1}$, ..., $r_n h_n f_1 f_2 .. f_n N_{1,1}$, $\frac{s_1\,(1-h_1)}{f_1}$, $\frac{s_2\,(1-h_2)}{f_2}$,...., $\frac{s_{n-1}\,(1-h_{n-1})}{f_{n-1}}$

> pp := 4·n;

$$pp := 4\, n \quad (66)$$

> rr := 2·n;

$$rr := 2\, n \quad (67)$$

> dd := pp − rr;

$$dd := 2\, n \quad (68)$$

> # Apply extension Theomrem:

> # Need to first reparameterise in terms of estimable parameter combinations - then apply extension theorem

>

> # Hunter Survey data

> kappa2 := ⟨r[1]·a[1], r[2]·a[2], r[3]·a[3]⟩;

$$\kappa 2 := \begin{bmatrix} r_1\, a_1 \\ r_2\, a_2 \\ r_3\, a_3 \end{bmatrix} \quad (69)$$

> pars2 := ⟨r[1], r[2], r[3]⟩;

$$pars2 := \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \quad (70)$$

> D2 := Dmat(kappa2, pars2);

$$D2 := \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \quad \textbf{(71)}$$

```
> nopars := Dimension(pars2); rr := Rank(D2); d := Dimension(pars2) - rr;
```

$$nopars := 3$$

$$rr := 3$$

$$d := 0 \quad \textbf{(72)}$$

```
>
>
> # Radio Tracking data
> kappa3 := ⟨h[1]·v[1], (1 - h[1])·(1 - s[1])·v[1], h[2]·v[2], (1 - h[2])·(1 - s[2])·v[2],
      h[3]·v[3], (1 - h[3])·(1 - s[3])·v[3]⟩;
```

$$\kappa 3 := \begin{bmatrix} h_1 v_1 \\ (1 - h_1)\,(1 - s_1)\,v_1 \\ h_2 v_2 \\ (1 - h_2)\,(1 - s_2)\,v_2 \\ h_3 v_3 \\ (1 - h_3)\,(1 - s_3)\,v_3 \end{bmatrix} \quad \textbf{(73)}$$

```
> pars3 := ⟨h_1, h_2, h_3, s_1, s_2, s_3⟩;
```

$$pars3 := \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} \quad \textbf{(74)}$$

```
> D3 := Dmat(kappa3, pars3);
```

(75)

$$D3 := \begin{bmatrix} v_1 & -(1-s_1)\,v_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 & -(1-s_2)\,v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_3 & -(1-s_3)\,v_3 \\ 0 & -(1-h_1)\,v_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1-h_2)\,v_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1-h_3)\,v_3 \end{bmatrix} \quad \textbf{(75)}$$

> $nopars := Dimension(pars3);\ rr := Rank(D3);\ d := Dimension(pars3) - rr;$

$$nopars := 6$$
$$rr := 6$$
$$d := 0 \quad \textbf{(76)}$$

>

> # Considering different combinations of data sets:

> $kappajoin := convert(\langle kappa1, kappa2\rangle, Vector) :$

> $indets(kappajoin)$

$$\{N_{1,1}, N_{1,2}, a_1, a_2, a_3, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2\} \quad \textbf{(77)}$$

> $pars := \langle N_{1,1}, N_{1,2}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2\rangle :$

> $Djoin := Dmat(kappajoin, pars) :$

> $nopars := Dimension(pars);\ rr := Rank(Djoin);\ d := Dimension(pars) - rr;$

$$nopars := 12$$
$$rr := 9$$
$$d := 3 \quad \textbf{(78)}$$

> $Estpar(Djoin, pars, 0);$

$$\left\{f\left(N_{1,1}, N_{1,2}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2\right) = \_F1\left(h_1\,N_{1,1}, \frac{N_{1,2}}{N_{1,1}}, f_1\,h_2\,N_{1,1}, N_{1,1}\,h_3\,f_2\,f_1, r_1, r_2, r_3, -\frac{s_1\,(h_1-1)}{f_1}, -\frac{s_2\,(-1+h_2)}{f_2}\right)\right\} \quad \textbf{(79)}$$

> $pp := 4\cdot n$

$$pp := 4\,n \quad \textbf{(80)}$$

> $rr := 3\cdot n$

$$rr := 3\,n \quad \textbf{(81)}$$

> $dd := pp - rr;$

$$dd := n \quad \textbf{(82)}$$

> $kappajoin := convert(\langle kappa1, kappa3\rangle, Vector) :$

> $indets(kappajoin)$

$$\{N_{1,1}, N_{1,2}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3, v_1, v_2, v_3\} \quad \textbf{(83)}$$

> $pars := \langle N_{1,1}, N_{1,2}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3\rangle :$

> $Djoin := Dmat(kappajoin, pars) :$

> $nopars := Dimension(pars);\ rr := Rank(Djoin);\ d := Dimension(pars) - rr;$

$$nopars := 13$$
$$rr := 12$$
$$d := 1 \quad \textbf{(84)}$$

> $kappajoin := convert(\langle kappa2, kappa3\rangle, Vector):$

> $indets(kappajoin)$

$$\{a_1, a_2, a_3, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3, v_1, v_2, v_3\} \quad \textbf{(85)}$$

> $pars := \langle h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3\rangle :$

> $Djoin := Dmat(kappajoin, pars):$

> $nopars := Dimension(pars); rr := Rank(Djoin); d := Dimension(pars) - rr;$

$$nopars := 9$$
$$rr := 9$$
$$d := 0 \quad \textbf{(86)}$$

> $kappajoin := convert(\langle kappa1, kappa2, kappa3\rangle, Vector):$

> $indets(kappajoin)$

$$\{N_{1,1}, N_{1,2}, a_1, a_2, a_3, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3, v_1, v_2, v_3\} \quad \textbf{(87)}$$

> $pars := \langle N_{1,1}, N_{1,2}, f_1, f_2, h_1, h_2, h_3, r_1, r_2, r_3, s_1, s_2, s_3\rangle :$

> $Djoin := Dmat(kappajoin, pars):$

> $nopars := Dimension(pars); rr := Rank(Djoin); d := Dimension(pars) - rr;$

$$nopars := 13$$
$$rr := 13$$
$$d := 0 \quad \textbf{(88)}$$

>

>

>

>

>