Constrained acyclic orientations of grids

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Abstract

Define a T**-odd orientation** of a graph $G = (V, E)$ as an orientation where all vertices in a given set $T \subseteq V$ have odd in-degree and all others have even in-degree. Finding T-odd orientations when one exists is well understood [1], but the problem becomes significantly harder with additional constraints. We consider the following problem:

Problem (Acyclic orientation with parity constraints) [2] Characterize graphs $G = (V, E)$ and sets $T \subseteq V$ that admit an acyclic T-odd orientation.

This problem has been well studied but remains open in general and is not even known to be in coNP. Szegedy and Szegedy [3] gave a randomized polynomial algorithm to decide if a graph has an acyclic T-odd orientation for a given T. A key step is reducing to the case $|V \setminus T| = 1$. Indeed, G admits an acyclic T-odd orientation if and only if $G' = (V \cup \{v\}, E \cup \{(v, u), u \in V \setminus T\})$ admits an acyclic $(V \setminus \{v\})$-odd orientation. Also, when $T = V \setminus \{v\}$, good characterizations exist for planar and cubic graphs [4]. In particular, a planar graph admits an acyclic $(V \setminus \{v\})$-odd orientation for any v if and only if its geometric dual is factor critical, i.e., has a perfect matching after removing any vertex.

In a more general setting however, when $|V \setminus T| > 1$, the construction of G' may not preserve properties like planarity or cubicity, and the problem remains open. As a step towards resolving this problem, we give a complete characterization of grid graphs $G = (V, E)$ and sets $T \subseteq V$ admitting an acyclic T-odd orientation. For most T, our main technique is to cut the grid into smaller parts to build a global acyclic T-odd orientation. For the remaining cases, we show that no acyclic T-odd orientation exists by constructing G', which happens to be planar, and proving its dual is not factor critical.

References

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