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Graph reconstruction from queries on triples

(Extended abstract)

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Abstract

Graph reconstruction has been studied for a long time. The famous Kelly-Ulam conjecture [3][6] states that any graph of order $n \geqslant 3$ is uniquely reconstructible (up to isomorphism) from the multiset of its subgraphs of order $n-1$. One way to approach this conjecture is to study reconstruction from other partial information. Let $k \leqslant n$ be an integer, and let $\mathbf{\Pi}$ be a partition of the set of graphs of order k. We are given a graph G with known vertices but unknown edges. We consider labelled graphs: the vertices are labelled, and we are looking for exact reconstruction as opposed to reconstruction up to isomorphism. For each induced subgraph of G of order k, we are given the element of $\mathbf{\Pi}$ to which it belongs, up to isomorphism. As such, $\mathbf{\Pi}$ defines a query on the graph G. For instance, in the case $k = 3$ and $\mathbf{\Pi} = \{\{\therefore, \dot{-}\}, \{\wedge, \triangle\}\}$, we are given a binary information on each subgraph H of order 3 of G: either H is isomorphic to $\therefore$ or $\dot{-}$, or it is isomorphic to $\wedge$ or $\triangle$. This example corresponds to graph reconstruction from the list of all connected triples of vertices in G. This problem has recently been studied by Bastide et al. [1], who provided a polynomial-delay algorithm to enumerate all graphs that are consistent with a given list of connected triples, and by Qi [5], who gave a structural characterization of graphs that can be uniquely reconstructed from their connected triples.

In this paper, we investigate all other possible partitions $\mathbf{\Pi}$ for $k = 3$. For each partition, we provide a structural characterization of graphs that are uniquely reconstructible from the information provided by the query, as well as a polynomial-delay algorithm to enumerate all graphs that are consistent with given query answers.

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1 Introduction

The most famous conjecture on graph reconstruction has been stated independently by Kelly [3] in 1942 and Ulam [6] in 1960.

Conjecture 1 (Kelly-Ulam)**.** *Every graph of order $n \geqslant 3$ is uniquely reconstructible (up to isomorphism) from the multiset of its subgraphs of order $n-1$.*

We introduce a new framework to consider reconstruction from partial information. A graph on the vertex set $V = \{1, \ldots, n\}$ is called a labelled n-graph. Consider an integer $k \leqslant n$ and a partition $\mathbf{\Pi}$ of the set of graphs of order k. A query $Q_{\mathbf{\Pi}}$ is a function which, for each labelled n-graph G and each k-subset $S \subseteq V$, provides the set of graphs $Q_{\mathbf{\Pi}}(G)(S) \in \mathbf{\Pi}$ to which $G[S]$ belongs, up to isomorphism.

Problem: Labelled Graph Reconstruction
Given an integer n and a partition $\mathbf{\Pi}$ of the set of graphs of order $k \leqslant n$:

- **Uniqueness:** Structural characterization of the labelled n-graphs G that are uniquely reconstructible from $\mathbf{\Pi}$, meaning that, for every labelled n-graph $H \neq G$, there exists a k-subset $S \subseteq V$ such that $Q_{\mathbf{\Pi}}(H)(S) \neq Q_{\mathbf{\Pi}}(G)(S)$.
- **Enumeration:** Polynomial-delay enumeration of all graphs that are consistent with some input information, i.e., given some function $f : \binom{V}{k} \to \mathbf{\Pi}$, enumerate all labelled n-graphs G such that, for all k-subset $S \subseteq V$, $Q_{\mathbf{\Pi}}(G)(S) = f(S)$.

The partition $\mathbf{C}_3 = \{\{\therefore, \overset{\cdot}{\_}\}, \{\wedge, \triangle\}\}$ corresponds to graph reconstruction from connected triples, for which the enumeration problem has been addressed by Bastide et al. [1] and the uniqueness problem has been solved by Qi [5]. In this paper, we solve these problems for all other possible queries on triples.

2 Main results

Our results are summed up in Table 1. Some of them extend to larger values of k (for partitions that admit a natural generalization to any k, such as $\mathbf{E}$ which generalizes to the query on the number of edges, or $\mathbf{K}$ which generalizes to the clique query).
The **complement** of a partition $\mathbf{\Pi}$ is the partition $\bar{\bar{\mathbf{\Pi}}}$ defined as $\bar{\bar{\mathbf{\Pi}}} = \{\{\bar{H} \mid H \in \pi\} \mid \pi \in \mathbf{\Pi}\}$. Whenever two partitions are complementary, we need only address one of the two. Indeed, G is uniquely reconstructible from $\mathbf{\Pi}$ if and only if $\overline{G}$ is uniquely reconstructible from $\bar{\bar{\mathbf{\Pi}}}$.

3 Uniqueness theorems

We now state the uniqueness theorem associated to each partition. For reasons of space, we only give a hint of the main idea used for some of the results. For some partitions, we also make an observation regarding the enumeration result.

Theorem 2 ($\mathbf{E} = \{\{\triangle\}, \{\wedge\}, \{\overset{\cdot}{\_}\}, \{\therefore\}\}$: query on the number of edges)**.**
Let n be an integer such that $n \geqslant 3$. A labelled n-graph G is uniquely reconstructible from $\mathbf{E}$ *if and only if G is not isomorphic to: P_3, $\overline{P_3}$, P_4, C_4, or $\overline{C_4}$.*

Notation	Query	Uniqueness	Enumeration	$k \geqslant 4$
-	$\{\{\triangle, \wedge, \dot{-}, \therefore\}\}$	-	-	-
$\mathbf{E}$	$\{\{\triangle\}, \{\wedge\}, \{\dot{-}\}, \{\therefore\}\}$	✓[3]	✓[3]	✓[3]
$\mathbf{K}$	$\{\{\triangle\}, \{\therefore, \dot{-}, \wedge\}\}$	✓[3]	✓[3]	✓[3]
$\bar{\mathbf{K}}$	$\{\{\therefore\}, \{\triangle, \wedge, \dot{-}\}\}$	✓[3]	✓[3]	✓[3]
$\mathbf{L}$	$\{\{\wedge\}, \{\therefore, \dot{-}, \triangle\}\}$	✓[3]	✓[3]	
$\bar{\mathbf{L}}$	$\{\{\dot{-}\}, \{\triangle, \wedge, \therefore\}\}$	✓[3]	✓[3]	
$\mathbf{C}$	$\{\{\triangle, \wedge\}, \{\dot{-}, \therefore\}\}$	✓[1]	✓[2]	
$\mathbf{J}$	$\{\{\triangle, \dot{-}\}, \{\wedge, \therefore\}\}$	✓[3]	✓[3]	
$\mathbf{A}$	$\{\{\triangle, \therefore\}, \{\wedge, \dot{-}\}\}$	✓[3]	✓[3]	
$\mathbf{B}$	$\{\{\triangle\}, \{\therefore\}, \{\wedge, \dot{-}\}\}$	✓[3]	✓[3]	
$\mathbf{F}$	$\{\{\wedge\}, \{\dot{-}\}, \{\triangle, \therefore\}\}$	✓[3]	✓[3]	
$\mathbf{D}$	$\{\{\triangle\}, \{\wedge\}, \{\therefore, \dot{-}\}\}$	✓[3]	✓[3]	
$\bar{\mathbf{D}}$	$\{\{\therefore\}, \{\dot{-}\}, \{\triangle, \wedge\}\}$	✓[3]	✓[3]	
$\mathbf{M}$	$\{\{\therefore\}, \{\wedge\}, \{\triangle, \dot{-}\}\}$	✓[3]	✓[3]	
$\bar{\mathbf{M}}$	$\{\{\triangle\}, \{\dot{-}\}, \{\therefore, \wedge\}\}$	✓[3]	✓[3]	

[1]Qi (2023) [2]Bastide et al. (2023) [3]GLMPW (2025+)

Table 1: All 15 possible queries on triples of vertices. A check in the column "$k \geqslant 4$" means the results extend to the generalization of the partition to larger values of k.

Sketch of proof. More generally, consider the query for general $k \leqslant n-1$ which returns the number of edges in the induced subgraph. For $k \leqslant n-2$, we can query all the tuples to compute the number of edges in every induced subgraph of G of order at least $n-2$, and then retrieve the edge set. For $k = n-1$, G is uniquely reconstructible if and only if it is uniquely characterized by its degree sequence. □

Theorem 3 ($\mathbf{F} = \{\{\wedge\}, \{\dot{-}\}, \{\triangle, \therefore\}\}$)**.**
Let n be an integer such that $n \geqslant 3$. A labelled n-graph G is uniquely reconstructible from $\mathbf{F}$ *if and only if G is neither a complete graph or a null graph and G is uniquely reconstructible from* $\mathbf{E}$.

Theorem 4 ($\mathbf{J} = \{\{\triangle, \dot{-}\}, \{\wedge, \therefore\}\}$: query on the parity of the number of edges)**.**
Let n be an integer such that $n \geqslant 3$. No graph is uniquely reconstructible from $\mathbf{J}$. *More precisely, for every labelled n-graph G, there exist exactly 2^{n-1} graphs consistent with $Q_{\mathbf{J}}(G)$ including G itself.*

Sketch of proof. The main argument is that $Q_{\mathbf{J}}(G) = Q_{\mathbf{J}}(H) \iff Q_{\mathbf{J}}(G \triangle H) = Q_{\mathbf{J}}(I_n)$ where $G \triangle H$ is the symmetric difference operator on edge sets and I_n is the null graph of order n. Note that $Q_{\mathbf{J}}(G) = Q_{\mathbf{J}}(I_n)$ if and only if G is a complete bipartite graph. □

Theorem 5 ($\mathbf{K} = \{\{\triangle\}, \{\therefore, \dot{-}, \wedge\}\}$: clique query)**.**
Let n be integers such that $n \geqslant 3$. A labelled n-graph G is uniquely reconstructible from $\mathbf{K}$ *if and only if:*

- *For every edge $uv \in E(G)$, there is a triangle in G containing uv.*

- *For every non-edge $uv \notin E(G)$, there is a triangle in $G + uv$ containing uv.*

Observation. The enumeration with polynomial delay is obtained using a branching tree where each node marks a choice between keeping the edge or not. Using the answers to the queries, we can cut a large number of branches and achieve polynomial delay.

Definition 6 (NEP graph)**.**
A graph $G = (V, E)$ is a ***non-edge partition*** *graph (NEP for short) if and only if there exists a proper partition $V = V_1 \uplus V_2$ such that for every pair $uv \notin E$ of vertices in V_1 (respectively V_2), $N(u) \cap V_2$ and $N(v) \cap V_2$ form a partition of V_2 (respectively $N(u) \cap V_1$ and $N(v) \cap V_1$ form a partition of V_1).*

Theorem 7 ($\mathbf{L} = \{\{\wedge\}, \{\therefore, \dot{-}, \triangle\}\}$)**.**
A labelled n-graph G is uniquely reconstructible from $\mathbf{L}$ if and only if all connected components of G are not NEP *and at most one of them has order* 1*.*

Sketch of proof. We prove that any connected labelled n-graph G containing a uniquely reconstructible induced subgraph is uniquely reconstructible. From that, we can prove that any connected labelled n-graph containing an independent set of size 3 is uniquely reconstructible. Note that such graphs cannot be NEP. We then prove that any connected graph with no such independent set is uniquely reconstructible if and only if it is not NEP. □

Theorem 8 ($\mathbf{M} = \{\{\wedge\}, \{\therefore\}, \{\triangle, \dot{-}\}\}$)**.**
Let n be an integer such that $n \geqslant 3$. A labelled n-graph G is uniquely reconstructible from $\mathbf{M}$ if and only if it is uniquely reconstructible from $\mathbf{L}$.

Theorem 9 ($\mathbf{A} = \{\{\triangle, \therefore\}, \{\wedge, \dot{-}\}\}$)**.**
Let n be an integer such that $n \geqslant 3$. No graph is uniquely reconstructible from $\mathbf{A}$.

Proof. $\overline{G}$ is consistent with $Q_{\mathbf{A}}(G)$ for every labelled n-graph G. □

Definition 10. *A (p, q)***-perfect-dominating pair** *in a graph G is a pair (S, T) of disjoint subsets of $V(G)$ such that $|S| = p$, $|T| = q$, and every $x \in V(G) \backslash (S \cup T)$ satisfies $N(x) \cap (S \cup T) \in \{S, T\}$.*

Theorem 11 ($\mathbf{B} = \{\{\triangle\}, \{\therefore\}, \{\wedge, \dot{-}\}\}$)**.**
Let n be an integer such that $n \geqslant 3$. A labelled n-graph G is uniquely reconstructible from $\mathbf{B}$ if and only if it contains no (p, q)-perfect-dominating pair (S, T) with $p, q \in \{1, 2\}$ such that $G[S \cup T]$ is K_3-free and I_3-free.

Observation. The enumeration with polynomial delay is achieved by reducing the partition to a 2-CNF formula, where the variables correspond to pairs of vertices and $True$ means that the pair is an edge. We then refer to Feder [2] algorithm to enumerate all satisfying assignments of the 2-CNF formula.

Theorem 12 ($\mathbf{C} = \{\{\triangle, \wedge\}, \{\dot{-}, \therefore\}\}$)**.** Refer to the result of Qi [5, Theorem 4.12].

Observation. This partition corresponds to the reconstruction problem from connected triples addressed by Bastide et al. [1] for the enumeration and Qi [5] for the uniqueness. The case $k \geqslant 4$ has also been explored in [1] and by Kluk et al. in [4].

Theorem 13 ($\mathbf{D} = \{\{\triangle\}, \{\wedge\}, \{\therefore, \overset{\circ}{\_\_}\}\}$).
Let n be an integer such that $n \geqslant 3$. A labelled n-graph G is uniquely reconstructible from $\mathbf{D}$ if and only if G has at most one isolated vertex and has no connected component inducing a P_2, P_3, P_4, or C_4.

Sketch of proof. We first prove that every connected graph of order 5 is uniquely reconstructible. As $\mathbf{D}$ is a refinement of $\mathbf{L}$, we also have the property that any connected graph G containing a uniquely reconstructible induced subgraph is uniquely reconstructible. □

4 Perspectives

From these results, several questions arise, suggesting multiple directions for future research. We employed various techniques to address both uniqueness and enumeration problems for each partition. It remains unknown whether a more unified approach could be developed. Additionally, we are interested in extending our results to larger values of k and exploring whether these results can be generalized to other types of queries.

Furthermore, the concept of partition raises several questions regarding graph reconstruction. Some partitions seem to provide more information than others, and it would be valuable to develop a method to quantify the amount of information conveyed by each partition. This could lead to a deeper understanding of the relationship between different partitions and their effectiveness in graph reconstruction. Specifically, uniqueness is rare (at times impossible) for some partitions, while for others, uniqueness is achieved without even needing to query every k-subset. This also asks the question of the minimum number of k-subsets that must be queried to achieve uniqueness, for various partitions.

Given a partition $\mathbf{\Pi}$, we are also interested in understanding which properties of the input function $f : \binom{V}{k} \to \mathbf{\Pi}$ allow for the existence of graphs that are consistent with f. If no graph is consistent with f, can we determine which graphs are closest for some adequate notion of distance, e.g. which graphs maximize the number of satisfied k-subsets ? If f has been obtained by adding noise to some f_0 such that there exists a unique graph consistent with f_0, when is it possible to recover this graph from f depending on the nature of the noise ?

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