

Simulating Quantum Drude Oscillators on a photonic quantum computer

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I. INTRODUCTION

Dispersion forces, also known as van der Waals (vdW) forces originate from the electromagnetic interaction between electrically neutral atoms or molecules which do not have permanent electric moments [1–4]. They are ever-present long-range forces between atoms or molecules arising from the zero-point fluctuations of the quantum electromagnetic field [5–9]. Their importance can be appreciated on both macro- and micro-scales; for instance, the first macroscopic signature of dispersion forces is the well-known correction to the equation of state of an ideal gas that led to the van der Waals equation [10]. Moreover, dispersion forces also influence the structure of liquids and solids such as the anomalies of water [11] as well as the macroscopic properties of macromolecules such as their structure [12], stability [13, 14], dynamics [15–17], and electric [18] and optical [19] responses. The most natural framework for the investigation of the response of matter subjected to such forces is quantum electrodynamics [20–24]. However, as widely shown in the literature, the inclusion of vdW dispersion interactions can be done by means of many-body methods [25–32]. Dispersion vdW interactions are often represented within the Lennard-Jones approach, namely, through a pairwise two-body interatomic potential [33–38] of the form C_6/R^6 (where R is the interatomic distance and C_6 a system-dependent constant). Among all the existing models in the literature, the many-body dispersion (MBD) framework has been undoubtedly proved to be an accurate approach [37, 39]. In the MBD framework, the drudonic response of valence drudons in atoms and molecules is supposed to be linear and this can be formally done through the introduction of the quantum Drude oscillator (QDO). A single QDO is coarse-grained quantum-mechanical model in which the properties of an atom are encompassed in a small number of parameters. The model consists in assimilating the atom to a point particle of mass m and electric charge $-q$ attached to a fixed (infinite mass) center of charge $+q$ by a harmonic spring characterized by a frequency ω . Molecules are then defined as a collection of QDOs in dipole-dipole in-

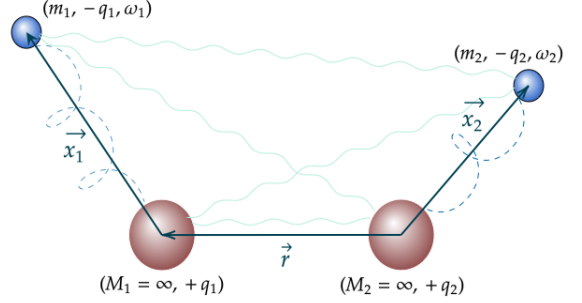


FIG. 1. a

teraction. For some specific choices of the matter system geometry, the quantum Hamiltonian can be exactly diagonalized. For instance, in a closed linear chain of molecules, one can analytically solve for the spectrum of the Hamiltonian [40]. In the case of a general geometry, instead, one can solve for the London-van der Waals interaction energy through a perturbative approach [41]. This simple model has been extensively used in various contexts, for instance in order to tackle the drudonic structure problem for isolated molecules, in particular long range interactions, as well as to study the impact of an ambient bath or of an external electric field on molecular properties [42, 43]. Though simple, through a numerical treatment this system was shown to capture long-range phenomena in large biomolecular systems [44]. By construction, the MBD framework relies on the dipole approximation of the drudon–drudon Coulomb interaction leaving aside any contribution coming from high-order terms. In the literature, dispersion forces have been addressed mostly for atomic dimers and small systems, via multipolar generalizations of the pairwise second-order perturbative approaches [33, 34, 38, 45]. In this Letter ... They did full coulomb FCI in [46]

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II. DEFINITION OF THE MODEL

A. Three-dimensional model

The Hamiltonian describing a system of N Quantum Drude Oscillators in three dimensions is given by:

$$H = \sum_{i=1}^N \left[\frac{\mathbf{p}_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 \mathbf{x}_i^2 \right] + \sum_{i<j} V_{\text{Coul}}(\mathbf{x}_i, \mathbf{x}_j), \quad (1)$$

with the Coulomb interaction receiving contributions from interacting drudon-drudon and drudon-nucleus pairs:

$$\frac{V_{\text{Coul}}(\mathbf{x}_i, \mathbf{x}_j)}{q_i q_j} = \frac{1}{r_{ij}} - \frac{1}{|\mathbf{r}_{ij} + \mathbf{x}_i|} - \frac{1}{|\mathbf{r}_{ij} - \mathbf{x}_j|} + \frac{1}{|\mathbf{r}_{ij} - \mathbf{x}_j + \mathbf{x}_i|}. \quad (2)$$

The usual approach consists in solving the theory in the multipolar expansion framework, in which the potential can be expressed as a power series in the inverse distance separating the two centers:

$$V_{\text{Coul}}(\mathbf{x}_i, \mathbf{x}_j) = \sum_{n \geq 0} V_n(\mathbf{x}_i, \mathbf{x}_j), \quad (3)$$

with the following scaling behavior in terms of the distance between the centers:

$$V_n(\mathbf{x}_i, \mathbf{x}_j) \propto r_{ij}^{-n-3}. \quad (4)$$

The potential V_0 corresponds then to the dipole-dipole interaction, and is at the core of the Many Body Dispersion (MBD) model. V_1 corresponds to the dipole-quadrupole interaction, and V_2 to the quadrupole-quadrupole and dipole-octupole interaction.

One obvious limitation of the multipolar expansion is the lower bound it imposes on the interatomic distance. One can easily see that within the MBD (dipole-dipole) model by direct diagonalization of the quadratic Hamiltonian in terms of normal modes. In that case, one of the normal modes (the center of mass mode) develops a purely imaginary frequency at short range. Higher order physical effects are also neglected in the MBD model, motivating the study of the QDO model with full Coulomb interaction potential between its constituents.

Let us define the following dimensionless position and momentum operators associated to QDO i :

$$\mathbf{X}_i := \sqrt{\frac{m_i \omega_i}{\hbar}} \mathbf{x}_i, \quad \mathbf{P}_i := \frac{\mathbf{p}_i}{\sqrt{2\hbar m_i \omega_i}}, \quad (5)$$

in terms of which the Hamiltonian reads

$$H = \sum_{i=1}^N \frac{\hbar \omega_i}{2} (\mathbf{X}_i^2 + \mathbf{P}_i^2) + \sum_{i<j} V_{\text{Coul}} \left(\sqrt{\frac{\hbar}{m_i \omega_i}} \mathbf{X}_i, \sqrt{\frac{\hbar}{m_j \omega_j}} \mathbf{X}_j \right) \quad (6)$$

One can define the $3N$ creation and annihilation operators

$$\mathbf{a}_i = \frac{\mathbf{X}_i + i\mathbf{P}_i}{\sqrt{2}}, \quad \mathbf{a}_i^\dagger = \frac{\mathbf{X}_i - i\mathbf{P}_i}{\sqrt{2}}, \quad (7)$$

in terms of which the Hamiltonian reads

$$H = \sum_{i=1}^N \frac{\hbar \omega_i}{2} \left(\mathbf{a}_i^\dagger \cdot \mathbf{a}_i + \frac{3}{2} \right) + \sum_{i<j} V_{\text{Coul}} \left(\sqrt{\frac{\hbar}{m_i \omega_i}} \frac{\mathbf{a}_i + \mathbf{a}_i^\dagger}{\sqrt{2}}, \sqrt{\frac{\hbar}{m_j \omega_j}} \frac{\mathbf{a}_j + \mathbf{a}_j^\dagger}{\sqrt{2}} \right). \quad (8)$$

Let us from now on restrict the problem to a pair of QDOs ($N = 2$) for concreteness, though the following developments carry to bigger systems. We denote by d the interatomic distance.

B. One-dimensional case

In order to reduce the complexity of the problem, let us define one-dimensional instances of the QDO model as follows: we restrict the movement of the two drudons to be along a common axis (directed by a unit vector $\hat{\mathbf{e}}_\theta$) which form an angle $\theta \in [0, \pi/2]$ with respect to the vector \mathbf{r}_{12} connecting the two nuclei. We therefore have a family of one-dimensional models which can be obtained from the full-fledged 3d model simply by setting to zero the contribution from the oscillator modes belonging to the plane perpendicular to $\hat{\mathbf{e}}_\theta$. Let us denote by (X, P) the remaining position and momentum degree of freedoms. As limiting cases, we obtain models in which the drudons are constrained to move either in the direction parallel to the axis separating the two nuclei ($\theta = 0$), or perpendicular to the latter ($\theta = \pi/2$). Those two models were studied in [47] in the dipole-dipole approximation. In that paper, the authors encode the states in the truncated Fock space of the system into the state of a set of qubits, and run VQE-type algorithms on IBQ quantum processors. However as we will see, the angle θ captures the competition between *existence of binding* (for small θ) and *smoothness* (for large θ), and interesting one-dimensional model actually sit at values of θ in the open segment $(0, \pi/2)$.

In the case of a generic angle θ and interatomic distance, the one-dimensional Coulomb potential reads:

$$\begin{aligned} \frac{V_{\text{Coul}}^{\theta,d}(x_1, x_2)}{q_1 q_2} &= \frac{1}{d} - \frac{1}{\sqrt{d^2 + 2d(\cos \theta)x_1 + x_1^2}} \\ &- \frac{1}{\sqrt{d^2 - 2d(\cos \theta)x_2 + x_2^2}} \\ &+ \frac{1}{\sqrt{d^2 - 2d(\cos \theta)(x_2 - x_1) + (x_2 - x_1)^2}}. \end{aligned} \quad (9)$$

The model we consider is therefore given by

$$H = \frac{\hbar\omega_1}{2} (P_1^2 + X_1^2) + \frac{\hbar\omega_2}{2} (P_1^2 + X_1^2) + V_{\text{Coul}}^{\theta,d} \left(\sqrt{\frac{\hbar}{m_1\omega_1}} X_1, \sqrt{\frac{\hbar}{m_2\omega_2}} X_2 \right). \quad (10)$$

Alternatively, in terms of creation and annihilation operators, one has

$$H = \frac{\hbar\omega_1}{2} \left(a_1^\dagger \cdot a_1 + \frac{3}{2} \right) + \frac{\hbar\omega_2}{2} \left(a_2^\dagger \cdot a_2 + \frac{3}{2} \right) + V_{\text{Coul}}^{\theta,d} \left(\sqrt{\frac{\hbar}{m_1\omega_1}} \frac{a_1 + a_1^\dagger}{\sqrt{2}}, \sqrt{\frac{\hbar}{m_2\omega_2}} \frac{a_2 + a_2^\dagger}{\sqrt{2}} \right). \quad (11)$$

For the simulation, we set $\hbar = 1$ as well as $m_i = q_i = \omega_i = 1$ for both QDOs.

III. PHOTONIC CIRCUIT

The main point of this letter is to probe the very natural idea that photonic-based continuous variable quantum hardware should be perfectly suited to simulate bosonic degrees of freedom. The simplest example one can come up with is a finite collection of interacting bosonic harmonic oscillators, which is precisely the case for the Quantum Drude Oscillator model. Another application is the computation of correlation functions of bosonic quantum fields on a lattice, for which an exponential speedup can be achieved with respect to standard methods [48].

The circuit implements a unitary $U(\theta)$ acting on an input reference state (the Fock vacuum for instance) that we simply take to be the vacuum state $|0\rangle$. The state prepared by the circuit is therefore given by

$$|\psi(\theta)\rangle = U(\theta)|0\rangle. \quad (12)$$

The circuit is therefore composed of at most quadratic optical components (squeezing operations for our ansatz). However for the other models, non-Gaussian operation should be added in the end of each layer in the ansatz circuit.

Once the ansatz state $|\psi(\theta)\rangle$ has been produced, one should extract the value of the energy in that state, namely compute the value of

$$\langle\psi(\theta)|H|\psi(\theta)\rangle \quad (13)$$

To be specific, let us take the model H , and let us denote by angular brackets the expectation in state $|\psi(\theta)\rangle$. One has

$$\begin{aligned} \langle H \rangle &= \sum_{i=1}^N \frac{\hbar\omega_i}{2} \left(\langle n_{i,x} \rangle + \langle n_{i,y} \rangle + \langle n_{i,z} \rangle + \frac{3}{2} \right) \\ &+ \sum_{i < j} \left\langle V_{\text{Coul}} \left(\sqrt{\frac{\hbar}{m_i\omega_i}} \mathbf{X}_i, \sqrt{\frac{\hbar}{m_j\omega_j}} \mathbf{X}_j \right) \right\rangle \end{aligned} \quad (14)$$

by linearity of the expectation. On the second line one has to compute something of the form $\langle f(X_{i,\alpha}) \rangle$, where $(X_{i,\alpha})$ denotes collectively the position quadrature of all the photon channels (including QDO and spatial index). One therefore needs to extract the statistics of quadratures by preparing and measuring the state $|\psi(\theta)\rangle$ in the quadrature basis. Once the joint density ρ of $(X_{i,\alpha})$ in the state $|\psi(\theta)\rangle$ is known, one can compute

$$\langle f(X_{i,\alpha}) \rangle = \int_{\mathbb{R}^{3N}} f(x_{i,\alpha}) \rho(x_{i,\alpha}) \prod_{i,\alpha} dx_{i,\alpha}. \quad (15)$$

There is a complication related to the fact that strawberryfields does not allow to access the joint law of the position quadratures. We therefore have to extract it by ourselves. Let us first suppose that there are K photon modes. The statevector is represented in the Fock basis as follows:

$$|\psi\rangle = \sum_{n_1, \dots, n_{2K}=0}^{\infty} \alpha_{n_1 \dots n_{2K}} |n_1\rangle \otimes \dots \otimes |n_{2K}\rangle. \quad (16)$$

The modes labeled by (n_1, \dots, n_K) correspond to QDO₁, while the modes (n_{K+1}, \dots, n_{2K}) are attached to QDO₂. The amplitude of a specific tuple of the quadratures (X_1, \dots, X_K) is therefore given by:

$$\langle X_1, \dots, X_{2K} | \psi \rangle = \sum_{n_1, \dots, n_{2K}=0}^{\infty} \alpha_{n_1 \dots n_{2K}} \prod_{i=1}^{2K} \frac{e^{-\sum_{i=1}^{2K} \frac{X_i^2}{2}} H_{n_i}(X_i)}{\sqrt{\pi^{1/2} 2^{n_i} n_i!}},$$

in terms of the Hermite polynomials. The joint law of the quadratures in the state $|\psi\rangle$ is therefore given by

$$\begin{aligned} \rho(X_1, \dots, X_{2K}) &= \sum_{\substack{n_1, \dots, n_{2K} \\ m_1, \dots, m_{2K}}} \alpha_{n_1 \dots n_{2K}} \alpha_{m_1 \dots m_{2K}}^* \\ &\times \prod_{i=1}^{2K} \frac{e^{-X_i^2} H_{n_i}(X_i) H_{m_i}(X_i)}{\sqrt{\pi^{1/2} 2^{n_i} n_i!} \sqrt{\pi^{1/2} 2^{m_i} m_i!}} \end{aligned} \quad (17)$$

After extracting as well the mean photon numbers $\langle n_{i,\alpha} \rangle$, one obtains $\langle H \rangle$.

IV. VARIATIONAL ALGORITHM

We define the following loss function:

$$\mathcal{C}(\theta) := \langle\psi(\theta)|H|\psi(\theta)\rangle \quad (18)$$

In order to compute this loss, one therefore has to measure both the photon number operator on each channel, as well as the position quadrature operator on each channel, as described in the previous section.

Algorithm 1: Extract distribution of observable

Parameters: statevector $|\psi\rangle$, observable \mathcal{O} , shots $M \in \mathbb{N}$

Result: Probability distribution of \mathcal{O} in state $|\psi\rangle$

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for  $m = 1$  to  $\dim(\text{spec}(\mathcal{O}))$  do
  | Initialize  $N_m \leftarrow 0$ ;
for  $j = 1$  to  $M$  do
  | Measure the state  $|\psi\rangle$  in the basis  $\mathcal{O}$ , obtain the
  | eigenvalue  $o_m$ , set  $N_m \leftarrow N_m + 1$ ;
for  $m = 1$  to  $\dim(\text{spec}(\mathcal{O}))$  do
  | Normalize  $N_m \leftarrow N_m/M$ ;
return  $(N_m)_m$ .

```

Algorithm 2: Computation of the loss

Parameters: $M \in \mathbb{N}$

Result: Value of the loss \mathcal{C}

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Initialize  $\mathcal{C} \leftarrow 0$ ;
Get the position quadratures distribution with alg.
(1);
Get the photon numbers distribution with alg. (1);
Compute the loss  $\mathcal{C}$  using eq. (14);
return  $\mathcal{C}$ .

```

Algorithm 3: Training of the parameterized photonic circuit

Parameters: $N_{\text{steps}} \in \mathbb{N}$, initial parameters $\theta_0 \in \mathbb{R}^K$, learning rate $\eta \in \mathbb{R}_+$

Result: Optimized hyperparameters $\theta \in \mathbb{R}^K$

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Initialize hyperparameters  $\theta \leftarrow \theta_0$ ;
for  $i = 1$  to  $N_{\text{steps}}$  do
  | Compute the loss  $\mathcal{C}$  with alg. (2);
  | Compute the gradient  $\nabla_{\theta}\mathcal{C}$  with shift rule and alg.
  | (2);
  | Update the parameters  $\theta \leftarrow \theta - \eta\nabla_{\theta}\mathcal{C}$ ;
end for
return  $\theta$ .

```

V. RESULTS

We gather here the results of the simulations. We focus on the case of 2 QDOs. In particular we plot the profile of the binding energy as a function of the distance between the two nuclei. The binding energy is simply defined as the ground state energy of the interacting system to which one subtract the ground state energy of the uninteracting system, namely that of a pair of free harmonic oscillators in this Drude model.

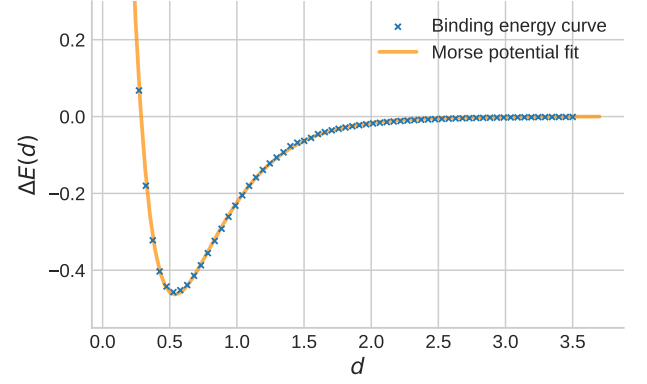


FIG. 2. a

VI. QUANTUM MUTUAL INFORMATION

Let us give the expression of the partial density matrix associated to QDO₁. The total state of the system, we recall, is expressed in the Fock basis as

$$|\psi\rangle = \sum_{n_1, \dots, n_{2K}=0}^{\infty} \alpha_{n_1 \dots n_{2K}} |n_1\rangle \otimes \dots \otimes |n_{2K}\rangle, \quad (19)$$

leading to the following expression for the density matrix:

$$\rho = \sum_{\substack{n_1, \dots, n_{2K} \\ m_1, \dots, m_{2K}}} \alpha_{m_1 \dots m_{2K}}^* \alpha_{n_1 \dots n_{2K}} |n_1\rangle \langle m_1| \otimes \dots \otimes |n_{2K}\rangle \langle m_{2K}|.$$

The partial trace associated to QDO₁ is therefore given by:

$$\rho_1 = \sum_{\substack{n_1, \dots, n_K \\ m_1, \dots, m_K \\ l_1, \dots, l_K}} \alpha_{m_1 \dots m_K l_1, \dots, l_K}^* \alpha_{n_1 \dots n_K l_1, \dots, l_K} |n_1\rangle \langle m_1| \otimes \dots \otimes |n_K\rangle \langle m_K|. \quad (20)$$

Since the state of the total system is pure, the von Neumann entropy of the total density matrix is zero. QDO number 2 can then be interpreted as purifying the system composed solely of QDO number 1. The two QDO therefore have identical von Neumann entropy $S(\rho_1)$, the entanglement entropy. The quantum mutual information of the system is therefore given by

$$I(1 : 2) = S(\rho_1) + S(\rho_2) - S(\rho) = 2S(\rho_1), \quad (21)$$

with the von Neumann entropy being defined as

$$S(\rho) = -\text{Tr} \rho \log \rho. \quad (22)$$

The profile of the quantum mutual information as a function of the interatomic distance is provided in fig. (...)

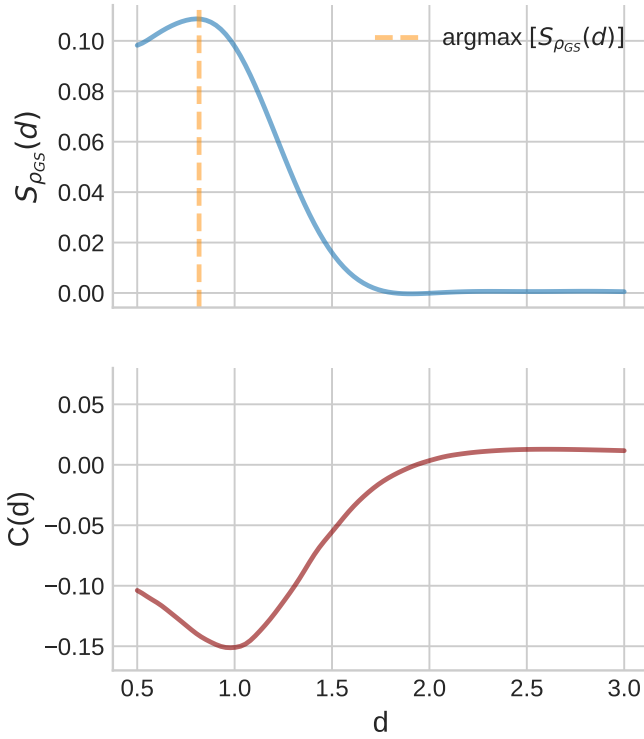


FIG. 3. a

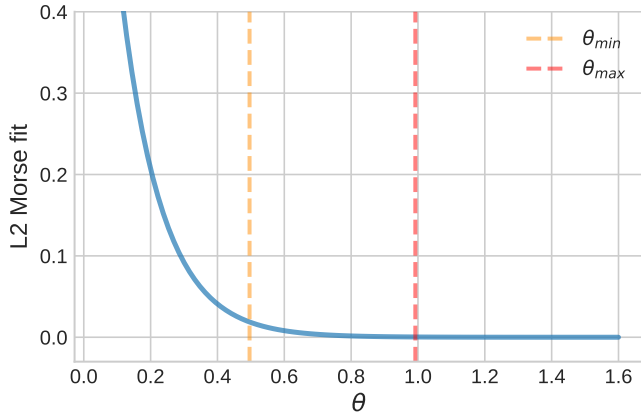


FIG. 4. a

VII. CONCLUSION

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CODE AVAILABILITY

The reader will find an open source python code accompanying this paper at github repository.

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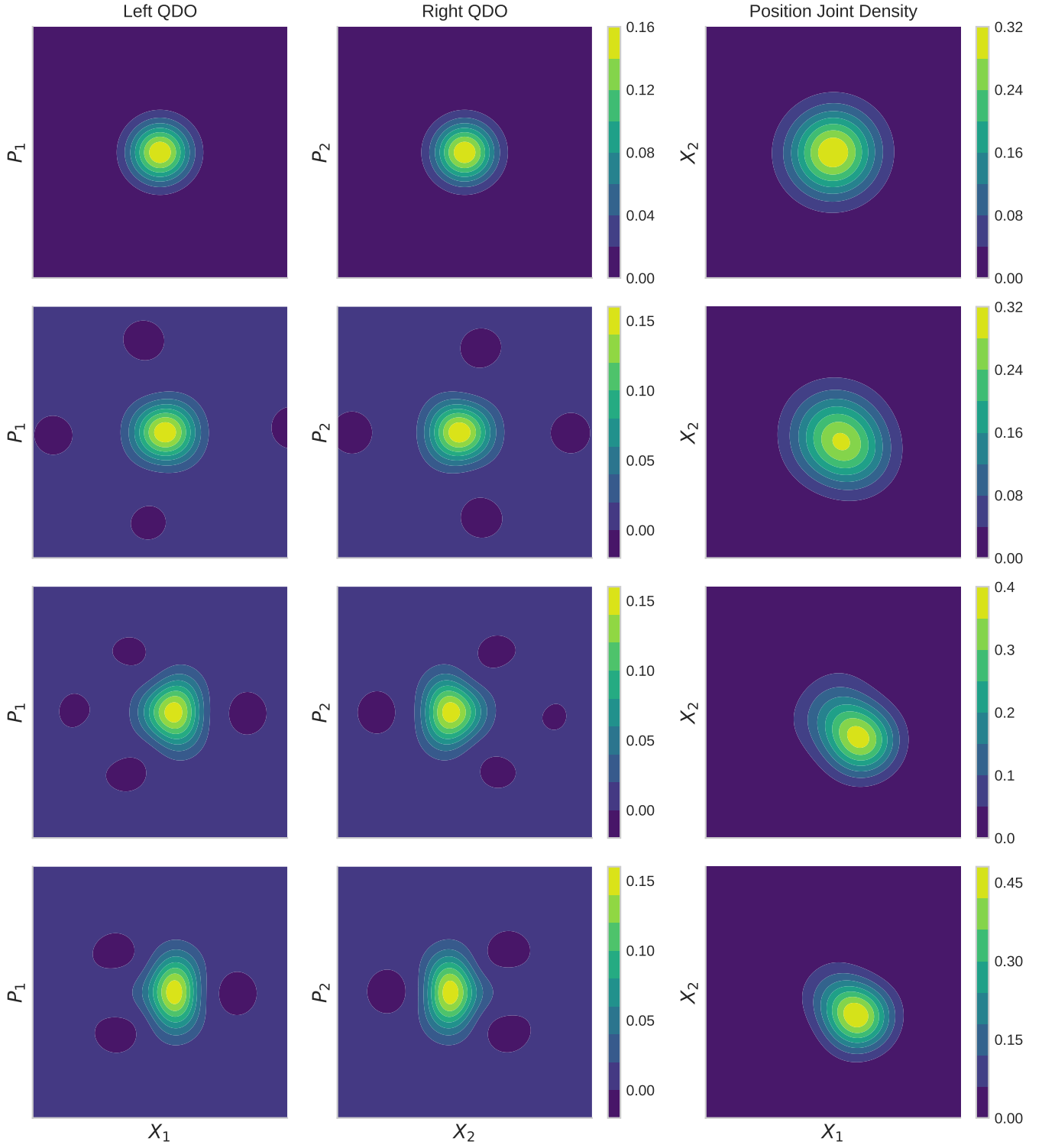


FIG. 5. a

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