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Faculty of Science

Fibred Lie groupoids and multiplicative connections

Masterthesis Mathematics

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Contents

Introduction

A prevalent object in differential geometry is that of a surjective submersion, may it be as a vector bundle, principal G-bundle, covering space, associated bundle or symplectic fibration. A point of view on these subjects is through the foliation of the total space into the fibres of the map. However, the nature of such foliations is rather tame as the only interesting geometry lies transversal to the leaves. Therefore, we are interested in the interplay between the geometry of the domain and codomain. In many geometrical theories using surjective submersion, like the ones mentioned before, there are additional homogeneity conditions imposed on the surjective submersion such that it locally resembles a product space. Such structures are known as fibre bundles, and they have been studied extensively as they give a strong relation between the domain —or total space —and the codomain —or base space —of the surjective submersion. For example, unlike general surjective submersions, a fibre bundle is a Serre fibration, cf. [?Husemoeller1994].

In many of the aforementioned examples —in particular, vector bundles, principal G-bundles, covering spaces and symplectic fibrations —an important aspect of the theories deals with the lifting of paths from the base space to the total space or parallel transport of points along them. Integral to these types of problems is a notion of parallelness or horizontality, which is introduced through the concept of a connection. While this notion may differ between these fields, from affine connections to connection 1-forms, they are all manifestations of (Ehresmann) connections on a surjective submersion, satisfying some compatibility conditions. An Ehresmann connection corresponds to a specific subbundle $\mathrm{E} \subset TM$, for $\pi\colon M \to B$, which is a complement to $\ker T\pi$, and they were introduced by Charles Ehresmann [?Ehresmann1959]. From the basic theory of vector bundles, it follows that such a connection always exists; therefore, they can be used as a standard tool in the theory of surjective submersions.

Given an Ehresmann connection E on a surjecrtive submersion $\pi\colon M \to B$, some curve $\gamma(0,1) \to B$ and a lift $x \in \pi^{-1}(\gamma(0))$, we can parallel transport x along γ by solving the following initial value problem:

$$\begin{cases} \widetilde{\gamma}(0) = x, \\ \dot{\widetilde{\gamma}}(t) \in E_{\gamma(t)}. \end{cases}$$

Generally, a solution is only local; when it always extends to the whole of $[0,1]$, a connection is called complete. In some cases —e.g. vector bundles, principal G-bundles and covering spaces —any connection with the correct compatibility conditions is complete. While the completeness of a connection is an analytical condition, these previous examples already show that geometric properties of a surjective submersion can ensure its existence.

In this thesis, we are interested in investigating such relations between the geometry imposed by the surjective submersion and the analytical properties of the connection. One of the main results of this thesis is the following:

Theorem. *A surjective submersion admits a complete connection if and only if it is a fibre bundle.*

The idea of our proof is based on [?delHoyo2016]; however, we have reworked and generalised many constructions in the proof to give a better overview of the objects in the construction. The preceding theory to this result lets us generalise to a multiplicative version as well.

For the multiplicative version, we are interested in another recurring topic within differential geometry: that of Lie groupoids. They were first introduced to study generalised symmetries in the 1950s by Charles Ehresmann [?Ehresmann1959] and were thoroughly investigated by his PhD students. However, they became mainstream mathematical objects due to two significant applications. Firstly, Alain Connes stressed their importance in his theory of noncommutative geometry, e.g. [?Connes1990]. Secondly, they are used to "integrate" Poisson structures, as introduced by Alan Weinstein in [?Weinstein1987].

Recent developments surrounding different normal form theorems have sparked particular interest in types of surjective submersions by Lie groupoids morphisms. For example, in the deformation theory of Lie groupoids and related structures, like symplectic Lie groupoids, one considers Lie groupoid morphisms mapping onto an identity Lie groupoid which are surjective submersions, cf. [?Crainic2018, ?Cardenas2021]. Alternatively, one can consider the groupoid generalisation of a group extension, which is a short exact sequence of groups, as discussed in [?LaurentGengoux2009]. In the current literature on this topic, for example [?Fernandes2023], a theory of Lie groupoid extensions using multiplicative Ehresmann connections has been developed in the case where the Lie groupoids are all over the same base space and the morphism covers the identity.

To provide a unifying framework for both these situations, we consider Lie groupoid morphisms, which may not cover the identity and which may not map to the identity groupoid, but which are surjective submersions. Such structures, we will call a *fibred Lie groupoid*.

While fibred Lie groupoids are a generalisation of the notions above, one of them still plays an integral part in the theory: Families of Lie groupoids. Inspired by the approach in [?Fernandes2023], part of the geometry of a fibred Lie groupoid can be reduced to an internal family of Lie groupoids. In a traditional Lie groupoid extension, the kernel of the morphisms defines a bundle of Lie groups; however, in our generalised setting, we obtain a family of Lie groupoids instead. This family of Lie groupoids will also play an important role in the main results of the thesis, as they admit a notion of local triviality where the multiplicative structure of the fibres is incorporated.

Much like for many theories involving surjective submersions, e.g., vector bundles, principal G-bundles, it is fruitful to consider connections with certain compatibility conditions. For a fibred Lie groupoid, this compatibility comes from the multiplicative structure on the tangent bundle of a Lie groupoid. This compatibility can be presented in terms of the lifting of multiplicable curves, which gives a geometric interpretation akin to other classical theories of connections. Connections satisfying these conditions are called *multiplicative*.

We would like to emulate the above theorem on surjective submersion in the case of fibred Lie groupoids, as this has already been done for Lie groupoid extensions, see [?Fernandes2023]. However, due to problems with local triviality, we can only formulate this for families of Lie groupoids. One of the directions of the previous theorem translates directly, namely, complete connections giving local triviality. For the other direction, we have a problem of glueing multiplicative connections, and in particular, the problem of the existence of connections. Again, in the special case of Lie groupoid extensions, the existence of multiplicative connections is well-known and controlled by a class in cohomology [?Grad2025, ?LaurentGengoux2009]. Under additional compactness and local triviality assumptions, we can ensure the existence of a multiplicative connection and its completeness as well.

Theorem. *Let $p\colon \mathcal{K} \to B$ be a locally trivial family of Lie groupoids with typical fibre $\mathcal{G}$. Suppose that $\mathcal{G}$ is a Lie groupoid whose source map is proper, then p admits a complete multiplicative connection.*

Additionally, we can reduce the completeness of multiplicative connections on arbitrary fibred Lie groupoids to the underlying kernel. A complete connection E on a fibred Lie groupoid $\phi\colon \mathcal{G} \to \mathcal{H}$ immediately defines a complete connection on its kernel given by $\mathrm{E}^{\mathcal{K}} = \mathrm{E} \cap T \ker \phi$. The converse of this can be shown to hold in the case where our morphism admits lifts to arbitrary sources, something we will call *arrow complete*.

Theorem. *If $\phi\colon \mathcal{G} \to \mathcal{H}$ is a fibred Lie groupoid that is arrow complete and a connection E such that $\mathrm{E}^{\mathcal{K}}$ is complete, then E is complete.*

Besides this application of arrow completeness, we will show that it automatically induces some equivalence between fibres of a fibred Lie groupoid, namely, Morita equivalence, even without the presence of a connection.

Lastly, this thesis discusses the notion of a symplectic Lie groupoid fibration, which incorporates the multiplicative structure of a fibred Lie groupoid such that it naturally combines with the fibred structure of a symplectic fibration. First, we give a digression on the uses of connections in symplectic fibrations, and in particular, we discuss why our proofs relating completeness to local triviality fail for a symplectic setting. We then show that symplectic Lie groupoid fibrations give a natural setting to translate classical results on symplectic fibrations to a multiplicative setting. Additionally, these types of structures seem to play a role in the theory of normal forms around Poisson submanifolds [?Fernandes2024].

Organisation

This thesis is organised as follows:

- Chapter 1 concerns itself with the classical case of surjective submersions and connections. While many proofs in this chapter are omitted, the last section gives a full and new proof of the main theorem, namely, the equivalence between surjective submersions with complete connections and fibre bundles.
- Chapter 2 describes the notion of a Lie groupoid, alongside some of the basic theory and constructions surrounding them. Secondly, we describe the notion of a Morita equivalence between Lie groupoids using principal bibundles.
- Chapter 3 gives a short overview of the theory of $\mathcal{VB}$-groupoids and multiplicative differential forms. Additionally, we show some new results relating to short exact sequences of $\mathcal{VB}$-groupoids.
- Chapter 4 defines the notion of fibred Lie groupoids and multiplicative connections on them, and in particular also families of Lie groupoids. We then prove some results regarding the completeness of multiplicative connections, relating them to local triviality conditions.
- Chapter 5 is a digression on the application of connections in the field of symplectic fibrations. Additionally, we provide a brief introduction to a possible multiplicative point of view on this topic, which incorporates the theory of multiplicative connections.

Chapter 1

Fibre bundles and connections

Fibre bundles provide a unifying framework that generalises many useful objects in differential geometry, including vector bundles and principal G-bundles. At their core, these objects consist of a space which is fibred over a base space through a surjective submersion in a locally trivial or homogeneous manner. Historically, they arose in questions posed in topology and geometry of manifolds. In this chapter, we adopt such a perspective, where we examine how the geometry of the base space imposes structure on the total space through the surjective submersion. A central concept in this analysis is that of a connection, which directly describes the relation between dynamics on the base space and the total space in a manner coherent with the surjective submersion.

We begin the chapter with a brief review of the theory of surjective submersions, focusing on their relationship with foliations, which will be essential for understanding their geometric structure. We then proceed to fibre bundles and local triviality. As a final piece of preliminary material, we discuss the notion of a connection and its associated parallel transport, which provides a geometrical way to interpret horizontal lifts. The final section contains the main result of this chapter, namely the equivalence of fibre bundles and surjective submersions with complete connections. Our proof is a new contribution based on the ideas of del Hoyo in [?delHoyo2016], but is based on a refinement of his main analytical lemma, [?delHoyo2016, Lem. 2], where we have changed it to measure the completeness of connections on trivial bundles and then apply this to fibre bundles locally.

Except for the last section, the contents of this chapter are primarily preliminaries for the rest of this thesis, and thus, most proofs have been omitted. Details of the constructions and some proofs can be found in many great books on differential geometry, like [?Candel2000, ?Husemoeller1994, ?Kolar1993, ?Lee2013, ?Moerdijk2003, ?Warner1983]. We will utilise many results from these first couple of sections throughout the rest of the thesis without further mention.

1.1 Surjective submersions

Recall that for a map $\pi\colon M \to B$ between manifolds the *rank at* $p \in M$ is the rank of its tangent map at p, which we denote by $T_p\pi\colon T_pM \to T_{\pi(p)}B$, i.e. it is the dimension of $\operatorname{im} T_p\pi \subset T_{\pi(p)}B$. In general, the rank is not a continuous map from $M \to \mathbb{Z}$, but only lower-semicontinuous. In particular, this means that the rank of a map is not necessarily constant, not even locally. We will say that a map has *constant rank* if it has the same rank at every point. According to the dimension theorem from linear algebra, the rank of $\pi\colon M \to B$ at p is bounded by the dimension of the domain or codomain, depending on which is smaller. This implies that the rank at p being maximal means that its tangent map is either surjective or injective. From this dichotomy, we will call a constant rank map a *submersion* if the tangent map is surjective and an *immersion* if it is injective. As the rank is lower-semicontinuous, the rank being maximal is an open condition: If it holds at p, then it holds in a neighbourhood of p.

By the rank theorem, a constant rank map admits charts in which it is linear. A particular corollary of the rank theorem is that constant rank maps are particularly well-behaved under taking level sets. By which we mean that the level set of a constant rank map is automatically a properly embedded submanifold. If π has constant rank, we refer to $M_b = \pi^{-1}(b)$ as the *fibre* of $\pi\colon M \to B$ at $b \in B$, and if all fibres are diffeomorphic to a fixed F, we will say that π has *typical fibre* F. We will use the shorthand $F \hookrightarrow M \xrightarrow{\pi} B$ to denote a surjective submersion $\pi\colon M \to B$ with typical fibre F.

We can describe a submersion in terms of its local sections, where a *local section* of $\pi\colon M \to B$ is a smooth map $\sigma\colon U \subset B \to M$ such that $\pi \circ \sigma = \mathrm{id}_U$. The existence of enough local sections is then precisely equivalent to the map being a submersion. From this alternate description of a submersion, we deduce that surjective submersions act like the quotient maps of the smooth category, with them being quotient maps in particular, as they are automatically open. Moreover, a surjective submersion lets us derive certain global properties from local properties (read: properties on fibres).

Proposition 1.1.1. *Let $\pi\colon M \to B$ be a proper surjective submersion; then its fibres are compact. Conversely, if $\pi\colon M \to B$ is a surjective submersion with compact and connected fibres, then it is proper.*

Proof. The first implication follows from the fact that points are compact, and by the assumption of properness, their inverse images as well. The second implication will follow as a result of our main theorem, see Proposition **??**. □

In the above proposition, we cannot drop the connectedness assumption, as any finite smooth covering with a closed subset removed gives a counterexample. We will refer to the domain of a surjective submersion as the *total space*, and its codomain as the *base space*.

Many examples of surjective submersions come from vector bundles, principal G-bundles, covering spaces and associated bundles. Outside of this context, the most basic example can be constructed for any base manifold B and typical fibre F by considering $\mathrm{pr}_1\colon B \times F \to B$. This we will refer to as the *trivial bundle* or *trivial bundle over B with fibre F*. From a surjective submersion $\pi\colon M \to B$, we can construct a more surjective submersion by restricting the base space to an open $U \subset B$, resulting in $\pi|_{\pi^{-1}(U)}\colon M|_U = \pi^{-1}(U) \to U$.

1.1.1 Foliations

Let us also consider a more geometric interpretation of surjective submersions in terms of foliations. In this context, it will also be clearer what a good notion of morphisms between surjective submersions might be. There are many different definitions, all with their merits; here we choose the one which is closest related to the notion of a surjective submersion via the rank theorem.

Definition 1.1.2. A *codimension q foliated atlas* of a n-manifold M is an atlas of the form

$$\left\{\psi_\alpha\colon U_\alpha \to \mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q\right\}_{\alpha\in\Lambda}$$

such that the transition functions are given by

$$\psi_{\beta\alpha}(x,y) = (g_{\beta\alpha}(x,y), h_{\beta\alpha}(y)), \quad \forall (x,y) \in \psi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^{n-q} \times \mathbb{R}^q.$$

The charts in a foliated atlas are called *foliated charts*. A *foliation of codimension q* of M is a maximal foliated atlas of M, and we will call such a pair $(M, \{U_\alpha, \psi_\alpha\}_{\alpha\in\Lambda})$ a *foliated manifold*.

A foliated manifold has a particularly nice geometric description: If (U, ψ) is a foliated chart of M, we obtain a partition of U by the connected components of

$$U_y = \psi^{-1}(\mathbb{R}^{n-q} \times \{y\}),$$

for any $y \in \mathbb{R}^q$, which we call the plaques of U. The conditions on a foliated atlas imply that transition functions map plaques of (U_α, ψ_α) to plaques of (U_β, ψ_β). Globally, we can glue such plaques into *leaves* by defining an equivalence relation on M, such that $x \sim y$ if there exists a sequence of foliated charts $\{(U_i, \psi_i)\}_{i=1}^n$ and points $p_0 = x, p_1, \ldots, p_n = y$, where p_{i-1} and p_i lie in the same plaque of U_i. In general, we denote $\mathcal{F}$ as the set of leaves of the foliated atlas on M, and call $(M, \mathcal{F})$ or $\mathcal{F}$ a *foliation*. The quotient $M/\mathcal{F}$, where we view $\mathcal{F}$ as the equivalence relation, is called the *leaf space* of the foliation. The leaves of a foliation are automatically immersed submanifolds. While they are not always embedded, they do satisfy slightly stronger conditions.

Proposition 1.1.3 ([?Warner1983, Thm. 1.62])**.** *The leaves of a foliated manifold are initial submanifolds.*

An alternative description of a foliation comes from distributions. Pointwise a foliation $\mathcal{F}$ on M defines a vector subspace of TM as $T_x\mathcal{F} = T_xL$, where $x \in L \in \mathcal{F}$. As a leaf is locally given by the level sets of a submersion, say π, the local sections of $T\mathcal{F}$ are π-related to the zero vector field, and it locally coincides with $\ker T\pi$. This implies that the Lie bracket of vector fields tangent to the leaves is also π-related to the zero vector field, implying that $\Gamma(T\mathcal{F}) \subset \mathfrak{X}(M)$ is a Lie subalgebra. This idea results in an equivalent description of foliations, given by considering subbundles of the tangent bundle of a manifold, which we call *distributions*. Given a distribution $D \subset TM$, we consider the restriction of the Lie bracket on $\mathfrak{X}(M)$ to $\Gamma(D)$ as the map

$$[\,\cdot\,,\,\cdot\,]\colon \Gamma(D) \times \Gamma(D) \to \mathfrak{X}(M)\colon (X, Y) \mapsto [X, Y],$$

If the image of this map lies in $\Gamma(D)$ again, we call the distribution *involutive*. The following theorem, called the Frobenius theorem, characterises foliations in terms of distributions.

Theorem 1.1.4 ([?Warner1983, Thm. 1.60])**.** *There is a one-to-one correspondence between involutive distributions and foliations.*

Given the geometric nature of foliations, if we understand them as their partition of a manifold into initial submanifolds, we can build a geometrically intuitive notion of maps. This notion of a map of foliation should preserve the partitioning and the smooth structure of the total space.

Definition 1.1.5. A *map of foliations* from $(M, \mathcal{F})$ to $(M', \mathcal{F}')$ is a smooth map $\psi\colon M \to M'$ such that ψ maps leaves of $\mathcal{F}$ to leaves of $\mathcal{F}'$.

Let us relate this to the notion of a surjective submersion, where we first focus on the case where the fibres are connected. If $\pi\colon M \to B$ is a surjective submersion with connected fibres, then the rank theorem gives us foliated charts on our total space, whose associated distribution is given by the *vertical bundle* $\mathrm{Ver} = \ker T\pi$. The leaves of the foliation are then the fibres of π, and the leaf space is exactly the base space B. Hence, π

acts like a particularly nice foliation where the leaves are embedded and it is parametrised by some smooth space. Therefore, they are called simple foliations. A map of foliations between two surjective submersions $\pi\colon M \to B$ and $\pi'\colon M' \to B'$ then is a map $\psi\colon M \to M'$ such that $\pi' \circ \psi$ is constant on the fibres of π. As π is a surjective submersion, this implies that there exists some smooth $\psi_0\colon B \to B'$ such that $\pi' \circ \psi = \psi_0 \circ \pi$.

We can generalise these ideas to the nonconnected case, where the foliated chart and distribution are the same, but the leaves now consist of the connected components of the fibres. The leaf space now is not simply the base space, as a point may have some multiplicity depending on the number of connected components of the fibres. As we want maps of our surjective submersions to still preserve some of the geometry of the base space, we cannot use maps of the associated foliations in general, but we need to specify them.

Definition 1.1.6. A *fibred map* between surjective submersions $\pi_i\colon M_i \to B_i$, for $i = 1, 2$, is a map $\psi\colon M_1 \to M_2$ such that $\pi_2 \circ \psi$ is constant on the fibres of π_1.

As above, a fibred map automatically induces a map on the base spaces, denoted with a subscript 0. Notice that indeed a fibred map induces a map of the associated foliations, but the converse is not true in general.

Example 1.1.7. Take the surjective submersion $\pi\colon M = \{0,1\} \times \mathbb{R} \to \mathbb{R}\colon (i, x) \mapsto x$, and consider the map $\psi\colon M \to M\colon (i, x) \mapsto (i, x + i)$. As the leaves of the associated foliation to π are just points, this clearly defines a map of foliations. However, this is not a fibred map as $\pi(1, x) = \pi(0, x)$, but $\pi \circ \psi(1, x) = \pi(1, x + 1) = x + 1$ and $\pi(1, x) = x$. //

Given a surjective submersion $\pi\colon M \to B$, we will denote the associated foliation by $\mathcal{F}_\pi$.

1.2 Fibre bundles

The interesting geometry of a surjective submersion, which differs from a general foliation, is the fact that the leaf space is smooth. To probe the transversal geometry of the total space, we need a stronger connection between the base space and the foliation. As the transversal geometry is a local phenomenon, we want to embed the base space into the total space locally, and not globally like a product manifold.

Definition 1.2.1. Let $\pi\colon M \to B$ be a surjective submersion. A pair (U, ψ) is called a *local trivialisation* if $U \subset B$ is open and $\psi\colon M|_U \to U \times F$ is a fibred isomorphism. This means they fit into the following commutative diagram:

$$\begin{tikzcd} M|_U \arrow[rr, "\psi", "\sim"'] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

A *fibre bundle* is a surjective submersion $\pi\colon M \to B$ with a collection of local trivialisations $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$ such that $\{U_\alpha\}_{\alpha \in \Lambda}$ is an open cover of B. Such a collection is called a *trivialising cover*.

Of course, the trivial example of a trivial bundle is always a fibre bundle, but there are many cases of surjective submersions which are not fibre bundles, for example, by taking out singular points of a fibre bundle. Under certain compactness conditions, fibre bundles and surjective submersions actually coincide.

Theorem 1.2.2 ([?Kolar1993, Lem. 9.2]). *Let $\pi\colon M \to B$ be a proper surjective submersion, then it is a fibre bundle.*

Additionally, we see that the local triviality of a fibre bundle gives a much stronger connection between the geometry of the base space and total space, as it lets us drop the assumption of connectedness in Proposition **??** and makes the proof purely topological instead of the analytic prove given in Proposition **??**.

Proposition 1.2.3. *If $\pi\colon M \to B$ is a fibre bundle, then π is proper if and only if its fibres are compact.*

Proof. The first implication follows from Proposition **??**. The converse goes as follows: Let $\pi\colon M \to B$ be a fibre bundle with compact fibres and $K \subset B$ compact. Pick a locally finite trivialising cover $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ such that there exists a precompact open cover $\{V_\alpha \subset U_\alpha\}_{\alpha\in\Lambda}$. In particular, $\{V_\alpha\}_{\alpha\in\Lambda}$ is an open cover of K, and therefore, there exists a finite subcover indexed by $\Lambda' \subset \Lambda$. This implies that

$$\pi^{-1}(K) \subset \pi^{-1}\left(\bigcup_{\alpha\in\Lambda'} V_\alpha\right) = \bigcup_{\alpha\in\Lambda'} \pi^{-1}(V_\alpha) \subset \bigcup_{\alpha\in\Lambda'} \pi^{-1}(\overline{V_\alpha}).$$

As $\psi_\alpha\colon \pi^{-1}(\overline{V_\alpha}) \to \overline{V_\alpha} \times F$ is a diffeomorphism, and $\overline{V_\alpha}$ and F are compact, it follows that $\pi^{-1}(\overline{V_\alpha})$ is compact. This implies that $\pi^{-1}(K)$ is compact as it is a closed set in a finite union of compact sets. Therefore, we conclude that π is proper. □

These propositions indicate that local triviality adds a significant amount of robustness to a surjective submersion. To explore this structure more in-depth, let us fix a fibre bundle $\pi\colon M \to B$. In a local trivialisation (U, ψ), with $\psi\colon M|_U \to U \times F$, any $b \in U$ defines a diffeomorphism

$$\psi_b\colon M_b \to F\colon x \mapsto \mathrm{pr}_2 \circ \psi(x).$$

Its inverse defines an embedding of F in M, we will denote this map by $\iota_b\colon F \to M_b \subset M$. This is sometimes called the *(fibre) inclusion*. In particular, we find that the restriction π to $M|_U$ has a typical fibre F.

Given a trivialising cover $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ of π, we deduce, by considering sequences of local trivialisations, that the fibre is constant on connected components over the base space. In other words, the restriction to a connected component of the base space has a typical fibre. Next, each $\alpha \in \Lambda$ and $b \in U_\alpha$ admits a diffeomorphisms $\psi_{\alpha,b}\colon M_b \to F$ and an inclusion $\iota_{\alpha,b}\colon F \to M$ as defined above. However, there is not a single canonical diffeomorphism, as for $b \in U_\alpha \cap U_\beta$ the maps $\psi_{\alpha,b}$ and $\psi_{\beta,b}$ may differ, and similarly for the inclusion.

These maps are canonically related through the so-called transition data. For simplicity, we will restrict ourselves to the case where B is connected, such that π has a typical fibre F. At any $b \in U_{\alpha\beta}$[1], we can define a diffeomorphism of F:

$$\psi_{\beta\alpha,b}\colon F \to F\colon f \mapsto \psi_{\beta,b} \circ \iota_{\alpha,b}(f).$$

By assembling these diffeomorphisms over $U_{\alpha\beta}$, we derive the *transition function* from (U_α, ψ_α) to (U_β, ψ_β) as the function

$$\psi_{\beta\alpha}\colon U_{\alpha\beta} \to \mathrm{Diff}(F)\colon b \mapsto \psi_{\beta\alpha,b}.$$

The full set of these transition functions, $\{\psi_{\beta\alpha}\colon U_{\alpha\beta} \to \mathrm{Diff}(F)\}$, constitutes the *transition data*. This collection forms a well-structured family of functions that encapsulate the geometry of a fibre bundle. Its signifi-

[1]From here on, we denote $U_{\alpha\beta} = U_\alpha \cap U_\beta$. For a finite family of sets $\{U_{\alpha_i} \subset B\}_{i=1}^n$, we denote $U_{\alpha_1\alpha_2\ldots\alpha_n} = \bigcap_{i=1}^n U_{\alpha_i}$.

cance is most effectively conveyed using the language of Čech 1-cocycles.

Definition 1.2.4. Given manifolds B and F, and a subgroup $G \subset \mathrm{Diff}(F)$, a *smooth Čech* 1*-cocycle* on B with values in G consists of an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ of B and a collection of maps $g_{\beta\alpha}\colon U_{\alpha\beta} \to G$ satisfying the *cocycle conditions*:

$$g_{\alpha\alpha,b} = \mathrm{id}, \quad g_{\gamma\alpha,b} = g_{\gamma\beta,b}g_{\beta\alpha,b}, \quad \forall\alpha,\beta,\gamma\in\Lambda,\ b\in U_{\alpha\beta\gamma}.$$

Additionally, there is a smooth criterion requiring the following map to be smooth:

$$\overline{g_{\beta\alpha}}\colon U_{\alpha\beta}\times F \to F\colon (b,f)\mapsto g_{\beta\alpha}(b)(f).$$

The naturality of the cocycle conditions comes from a more sheaf-theoretic perspective on bundles. Here, it makes sense as it is exactly the data needed to glue local geometry to a global structure. Therefore, fibre bundles $F \hookrightarrow M \overset{\pi}{\to} B$ are in 1-1 correspondence with smooth Čech 1-cocyles on B with values in $\mathrm{Diff}(F)$ (cf. [?Husemoeller1994, Thm. 5.3.2] for the topological case).

1.3 Connections

In Section ??, we saw that a surjective submersion has an intrinsically defined vertical direction, whether it be geometrically through the foliation or algebraically as its vertical bundle. This notion was intrinsic as they can be directly deduced from the map π, one as the connected components of the fibres and the other as $\ker T\pi$. However, we want to relate the horizontal geometry of the base to the total geometry. We can see that this is not a canonical relation as there is no canonical embedding of B into M. Algebraically, we can interpret this as the existence of the following short exact sequence of vector bundles over M:

$$0 \longrightarrow \mathrm{Ver} \longrightarrow TM \longrightarrow \pi^*TB \longrightarrow 0.$$

The horizontal direction of $\pi\colon M \to B$ is then identified with the vector bundle π^*TB, which is the pullback bundle of $\mathrm{pr}\colon TB \to B$ along π, given by $\pi^*TB = \{(x,v)\in M\times TB\colon \pi(x) = \mathrm{pr}(v)\}$, such that it is a vector bundle over M.

To properly study transversal geometry in this manner, we shortly recall some generalities of short exact sequences of vector bundles. Here, by a short exact sequence, we mean a pair of constant rank vector bundle maps $\iota\colon V \to V'$ and $\pi\colon V' \to V''$ covering the identity, such that ι is injective, $\mathrm{im}\,\iota = \ker\pi$ and π is surjective. Short exact sequences are useful in general as they let us describe bigger objects, namely the middle one, in terms of smaller ones, the outer objects, even though this is not canonical. Such a choice of description is called a splitting of the short exact sequence. Concretely, for a short exact sequence

$$0 \longrightarrow V \xrightarrow{\ \iota\ } V' \xrightarrow{\ \pi\ } V'' \longrightarrow 0,$$

a splitting is given by an isomorphism $\phi\colon V' \to V\oplus V''$ such that $\iota = \phi\circ\mathrm{incl}_1$ and $\pi = \mathrm{pr}_2\circ\phi$. If a short exact sequence admits such a splitting, it is called a split short exact sequence. In the case of a vector bundle, we remark that any short exact sequence is split, which will become clearer after the following result. Namely, given a splitting, we obtain a natural way to identify V'' inside of V', which will complement $\iota(V)$, and a way to project V' to V. In the category of vector bundles, we can show that any such choice would constitute a splitting.

Lemma 1.3.1 ([?Tu2017, Prp 27.20]). *Consider a short exact sequence of vector bundles over a manifold B:*

$$0 \longrightarrow V \xrightarrow{\iota} V' \xrightarrow{\pi} V'' \longrightarrow 0$$

There is a 1-1 *correspondence between the following:*

$$\{\textit{Right inverses to } \pi\} \longleftrightarrow \{\textit{Left inverses to } \iota\} \longleftrightarrow \left\{\begin{matrix}\textit{Splittings}\\ \phi\colon V' \to V \oplus V''\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{Complements to}\\ \iota(V) \textit{ in } V\end{matrix}\right\}$$

These correspondences are determined uniquely by $h \circ \pi + i \circ \theta = \mathrm{id}_\Omega$ and $C = \ker\theta = \operatorname{im} h$. Moreover, if C is a complement of Γ in Ω, then $\pi|_C\colon C \to \Gamma'$ is an isomorphism.

As we saw, a surjective submersion $\pi\colon M \to B$ induces a short exact sequence of vector bundles containing the vertical and horizontal directions. Identifying the horizontal direction is then a choice of a splitting, which, by the above lemma, can be defined in a multitude of ways. We will call any such identification a *connection*, but we will refer to them specifically as follows:

Definition 1.3.2. Let $\pi\colon M \to B$ be a surjective submersion. We define the following objects:

- *Horizontal lift*: A vector bundle morphism $h\colon \pi^*TB \to TM$ such that $\pi \circ h = \mathrm{id}_{\pi^*TB}$.
- *Vertical projection*: A vector bundle morphism $\mathrm{pr}\colon TM \to \mathrm{Ver}$ such that $\mathrm{pr} \circ \mathrm{incl} = \mathrm{id}_{\mathrm{Ver}}$, where $\mathrm{incl}\colon \mathrm{Ver} \to TM$ is the inclusion map.
- *Connection idempotent*: A vector bundle morphism $p\colon TM \to TM$ such that $\operatorname{im} p = \mathrm{Ver}$ and $p^2 = p$.
- *Ehresmann connection*: Vector subbundle $\mathrm{E} \subset TM$ such that $\mathrm{Ver} \oplus \mathrm{E} = TM$.
- *Connection 1-form*: Ver-valued 1-form $\alpha \in \Omega^1(M; \mathrm{Ver})$ such that $\alpha|_{\mathrm{Ver}} = \mathrm{id}_{\mathrm{Ver}}$. We will denote $\Omega_{\mathrm{conn}}(M; \mathrm{Ver})$ for the set of connection forms.

Example 1.3.3. Let $\pi\colon M \to B$ be a smooth covering space, then there exists a unique connection as the vertical bundle is trivial. //

Example 1.3.4. Let $G \circlearrowright P \xrightarrow{\pi} Q$ be a principal G-bundle and ω a connection 1-form. This defines an Ehresmann connection by setting $\mathrm{E} = \ker\omega$, with the property that $TL_g\,\mathrm{E}_p = \mathrm{E}_{gp}$. Conversely, any Ehresmann connection E on $\pi\colon P \to B$ such that $TL_g\,\mathrm{E}_p = \mathrm{E}_{gp}$ defines a connection 1-form, where Ver is canonically isomorphic to $P \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Composing the connection 1-form with this isomorphism defines a 1-form on the principal G-bundle. //

Example 1.3.5. Let $\pi\colon V \to B$ be a vector bundle with an affine connection ∇. As π is a submersion, any $v \in V_b$ can then be obtained as $v = \sigma(b)$ for a local section $\sigma\colon U \subset B \to V$. One can check that under the additional requirement that $\nabla\sigma(b) = 0$, we can still always find such a section. We then define the horizontal bundle as $\mathrm{E}_v = \operatorname{im} T_b\sigma$. As sections are right inverses to π, the images of the tangent maps intersect Ver trivially, and by counting dimensions, we see that E_v is a complement to Ver_v in T_vV.

The induced horizontal bundle will incorporate some of the linear structure of a vector bundle. Namely, let us denote its fibrewise scalar multiplication by $S_\lambda\colon V \to V\colon v \mapsto \lambda v$ for any $\lambda \in \mathbb{R}$. Let us fix some $v \in V$ and $\lambda \in \mathbb{R}$, then for a local section σ such that $\sigma(b) = v$ we have that $T_b(S_\lambda \circ \sigma) = T_vS_\lambda T_b\sigma$. If we assume that σ is flat, then $S_\lambda \circ \sigma$ is also flat and we find that

$$\mathrm{E}_{\lambda v} = \operatorname{im} T_b(S_\lambda \circ \sigma) = \operatorname{im}(T_v S_\lambda \circ T_b \sigma) = T_v S_\lambda \, \mathrm{E}_v \, .$$

Conversely, any Ehresmann connection on π with this property will define an affine connection. //

In the above examples, we always know that a connection exists. In the general case of a surjective submersion, this follows from the theory of short exact sequences of vector bundles, again, as any short exact sequence of vector bundles admits a splitting. Let us show this in the particular case of a surjective submersion.

Proposition 1.3.6. *Any surjective submersion admits a connection.*

Proof. Let $\pi \colon M \to B$ be a surjective submersion and take an atlas for M, say $\{(U_\alpha, \chi_\alpha = (x^i_\alpha))\}_{\alpha\in\Lambda}$. On each open, U_α, we define the canonical Riemannian metric $g_\alpha = \chi^*_\alpha g = g_{ij} dx^i_\alpha dx^j_\alpha$. For some partition of unity, $\{\phi_\alpha\}_{\alpha\in\Lambda}$, subordinate to $\{U_\alpha\}_{\alpha\in\Lambda}$, the gluing $g = \sum_\alpha \phi_\alpha g_\alpha$ defines a Riemannian metric on M. The fibrewise orthogonal complement of Ver with respect to g then defines a connection on M. □

1.3.1 Connections in terms of modules

Recall that a vector bundle $\pi \colon V \to M$ defines a $C^\infty(M)$ module by taking global sections, denoted $\Gamma(V)$. In the case of a tangent bundle, this gives the vector fields. Additionally, if we pullback $\pi \colon V \to M$ along $f \colon N \to M$, then $\Gamma(f^*V) = C^\infty(N) \otimes_{C^\infty(M)} \Gamma(V)$, where $C^\infty(M)$ acts on $C^\infty(N)$ by first precomposing with f.

As a horizontal lift is a vector bundle morphism, it will define a module morphism on the associated modules of global sections. Hence, if $h \colon \pi^*TB \to TM$ is a horizontal lift on $\pi \colon M \to B$, then it defines a morphism of $C^\infty(M)$-modules $h \colon C^\infty(M) \otimes_{C^\infty(B)} \mathfrak{X}(B) \to \mathfrak{X}(M)$. By precomposing with the natural inclusion $\mathfrak{X}(B) \to C^\infty(M) \otimes_{C^\infty(B)} \mathfrak{X}(B) \colon X \mapsto 1 \otimes X$, we obtain a horizontal lifting map $h \colon \mathfrak{X}(B) \to \mathfrak{X}(M)$. If E is the associated horizontal distribution, then h maps $\mathfrak{X}(B)$ injectively into $\Gamma(\mathrm{E})$, but never surjectively unless $B = M$. Concretely, for some $X \in \mathfrak{X}(B)$ this map is defines as $h(X)_x = h(x, X_{\pi(x)})$. Moreover, as any tangent vector extends to a vector field, this contains all the information of the connection.

In the case where $M = B \times F$, we can naturally identify the sections of TM using the fact that taking sections and direct sums "commute". Therefore, the vector fields on M can be written as

$$\mathfrak{X}(M) = (C^\infty(M) \otimes_{C^\infty(B)} \mathfrak{X}(B)) \oplus (C^\infty(M) \otimes_{C^\infty(F)} \mathfrak{X}(F)),$$

Which has a canonical projection to its first and second components. As h is a horizontal lift, and this is a right inverse to pr_1, it is a right inverse to the first projection on the level of modules as well. This implies that all information on the connection is contained in the second component, such that we associate h with the second projection composed with h. Hence, we consider the map

$$h \colon C^\infty(M) \otimes_{C^\infty(B)} \mathfrak{X}(B) \to C^\infty(M) \otimes_{C^\infty(F)} \mathfrak{X}(F) \colon f \otimes X \mapsto \mathrm{pr}_2 \circ h(f \otimes X).$$

Furthermore, we can restrict the domain and codomain to the fibre M_b by taking the tensor product with $C^\infty(M_b)$ over $C^\infty(M)$. One can verify that the following map defines an isomorphism of $C^\infty(M_b)$-modules:

$$C^\infty(M_b) \otimes_{C^\infty(B)} \mathfrak{X}(B) \to C^\infty(M_b) \otimes_{\mathbb{R}} T_b B \colon f \otimes X \mapsto f \otimes X_b.$$

The surjectivity of this map follows by extending a tangent vector to a local vector field. For the injectivity, we remark that it shows that the left tensor product is really only dependent on the value of X at b. First, remark that elements of $C^\infty(M_b) \otimes_{C^\infty(B)} \mathfrak{X}(B)$ depend on $\mathfrak{X}(B)$ only locally. If $f \in C^\infty(M_b)$,$X \in \mathfrak{X}(B)$ and ψ is a bump function such that $\psi|_U = 1$, for a neighbourhood U of b.

$$f \otimes \psi X = f \cdot \psi \otimes X = f \otimes X,$$

where we used that $\psi \in C^\infty(B)$ acts on $C^\infty(M_b)$ as $(f \cdot \psi)(x) = f(x)\psi(b) = f(x)$. Hence, suppose that ψ is supported in some coordinate chart $(U, (x^i))$, it follows that

$$f \otimes X = f \otimes \psi X = f \otimes \psi X^i \partial_i = f \cdot X^i \otimes \psi \partial = f \cdot X^i(b) \otimes \psi \partial_i = f \otimes \psi X^i(b)\partial_i.$$

We conclude that $f \otimes X_b$ being zero implies that $f \otimes X$ vanishes as well. Therefore, the mapping is an isomorphism of $C^\infty(M_b)$-modules. We obtain a module morphism

$$h\colon C^\infty(M_b) \otimes_{\mathbb{R}} T_bB \to \mathfrak{X}(F),$$

covering the map $C^\infty(M_b) \to C^\infty(F)$ induced by the diffeomorphism $F \to M_b$.

Again, we can precompose this map with the inclusion $T_bB \to C^\infty(M_b) \otimes_{\mathbb{R}} T_bB\colon v \mapsto 1 \otimes v$, to obtain an $\mathbb{R}$-linear map $h_b\colon T_bB \to \mathfrak{X}(F)$. The collection $\{h_b\colon T_bB \to \mathfrak{X}(F)\}_{b\in B}$ contains all the information of the connection as we can explicitly recover it as $h((b, f), v) = (v, h_b(v)_f)$.

On a fibre bundle, we obtain similar expressions which are dependent on the choice of local trivialisation. Let $F \hookrightarrow M \xrightarrow{\pi} B$ be a fibre bundle with trivialising cover $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$. In each trivialisation (U_α, ψ_α) we can consider the following sequence:

$$\pi^*TU_\alpha \xrightarrow{h} TM_{U_\alpha} \xrightarrow{T\psi_\alpha} \psi_\alpha^* T(U_\alpha \times F) \xrightarrow{T\,\mathrm{pr}_2} \psi_\alpha^* \,\mathrm{pr}_2^* TF = (\mathrm{pr}_2 \circ \psi_\alpha)^* TF$$

By taking sections and pulling back to M_b like before, we obtain a map $h_{\alpha,b}\colon T_bB \to \mathfrak{X}(F)$, which is explicitly given by

$$h_{\alpha,b}(v)(f) = T\,\mathrm{pr}_2 \circ T\psi_\alpha \circ h(\iota_{\alpha,b}(f), v).$$

To transition to a different local trivialisation, we obtain the following expression:

$$\begin{aligned}(v, h_{\alpha,b}(v)(f)) &= T\psi_\alpha \circ h(\iota_{\alpha,b}(f), v) = T\psi_{\alpha\beta} \circ T\psi_\beta \circ h(\iota_{\beta,b}(\psi_{\beta\alpha,b}(f)), v)\\ &= (v, T\psi_{\alpha\beta,b} h_{\beta,b}(v)(\psi_{\alpha\beta,b}(f))).\end{aligned}$$

This implies that $h_{\alpha,b}(v) = T\psi_{\alpha\beta,b} h_{\beta,b}(v)$, such that the transition data exactly dictate the transition between the local horizontal lifts.

1.3.2 The space of connections

We remark that $\Omega_{\mathrm{conn}}(M; \mathrm{Ver}) \subset \Omega^1(M; \mathrm{Ver})$ is not closed under scalar multiplication, as a connection form must be the identity on Ver, and it does not contain the zero form. Therefore, it cannot be viewed as a vector subspace. However, it does have some affinelike structure, in the sense that it is a vector space up to a choice of basis, such that it can be written as $\alpha + V$ for some vector subspace $V \subset \Omega^1(M; \mathrm{Ver})$ and $\alpha \in \Omega_{\mathrm{conn}}(M; \mathrm{Ver})$. However, in this case we will be interested in a slightly more nuanced structure, namely, affine $C^\infty(M)$-modules. By this, we mean that it is of the form $\alpha + M$, where $M \subset \Omega^1(M; \mathrm{Ver})$ is a

$C^\infty(M)$-submodule.

Proposition 1.3.7. *For any connection form α on $\pi\colon M \to B$, the set of connection forms is given by*

$$\Omega_{\text{conn}}(M; \text{Ver}) = \alpha + \mathscr{A}(M; \text{Ver}),$$

where $\mathscr{A}(M; \text{Ver}) = \{\tau \in \Omega^1(M; \text{Ver})\colon \tau|_{\text{Ver}} = 0\}$ is a $C^\infty(M)$-module. Hence, it is an affine $C^\infty(M)$-module.

Proof. Let $\pi\colon M \to B$ be a surjective submersion, fix an arbitrary $\alpha \in \Omega_{\text{conn}}(M; \text{Ver})$, and define

$$\mathscr{A}(M; \text{Ver}) = \{\tau \in \Omega^1(M; \text{Ver})\colon \tau|_{\text{Ver}} = 0\}$$

Given any $\alpha' \in \Omega_{\text{conn}}(M; \text{Ver})$ and $v \in \text{Ver}$ it follows that

$$(\alpha' - \alpha)(v) = \alpha'(v) - \alpha(v) = v - v = 0.$$

Hence, $\alpha' - \alpha \in \mathscr{A}(M; \text{Ver})$ such that $\Omega_{\text{conn}}(M; \text{Ver}) - \alpha \subset \mathscr{A}(M; \text{Ver})$. Moreover, if $\tau \in \mathscr{A}(M; \text{Ver})$ and $v \in \text{Ver}$, then

$$(\alpha + \tau)(v) = \alpha(v) + \tau(v) = \operatorname{id}|_{\text{Ver}}(v) = v.$$

This implies that $\alpha + \mathscr{A}(M; \text{Ver}) \subset \Omega_{\text{conn}}(M; \text{Ver})$ and thus they indeed coincide.

Lastly, if we denote $\iota\colon\ \text{Ver} \to TM$ for the inclusion, then $\mathscr{A}(M; \text{Ver}) = \ker \iota^*$, where we denote

$$\iota^*\colon \Omega^1(M; \text{Ver}) \to \operatorname{End}(\text{Ver})\colon \alpha \mapsto \alpha \circ \iota.$$

This is a $C^\infty(M)$-module morphism and therefore the kernel is a $C^\infty(M)$-module. Therefore, we can conclude that the connection forms, $\Omega_{\text{conn}}(M; \text{Ver})$, define an affine $C^\infty(M)$-module. $\square$

Corollary 1.3.8. *The connection forms $\Omega_{\text{conn}}(M; \text{Ver})$ of a surjective submersion $\pi\colon M \to B$ are a convex subset of $\Omega^1(M; \text{Ver})$ as a $C^\infty(M)$-module.*

We observe that this convex structure naturally translates to the notions of horizontal lifts, vertical projections, and connection idempotents by taking the convex combination of the vector bundle morphisms. For the Ehresmann connection counterpart, this description is not as neat and is better written as the kernel of the associated connection forms. However, if E_i are Ehresmann connections and $t_i \in (0, 1)$, both for $i = 1, 2$, such that $t_1 + t_2 = 1$ then the "convex combination", which we will denote by $\sum_i t_i \mathrm{E}_i$, satisfies the following inclusions:

$$\mathrm{E}_1 \cap \mathrm{E}_2 \subset \sum_i t_i \mathrm{E}_i \subset \mathrm{E}_1 + \mathrm{E}_2\,.$$

This notation, of course, extends to arbitrary convex combinations as well.

1.3.3 Induced splitting of forms

An Ehresmann connection E on a surjective submersion $\pi\colon M \to B$ is a splitting of TM. This decomposition extends to higher exterior powers of TM, and thus a decomposition of differential forms on M. The decomposition of $\bigwedge^k TM$ is by the following canonical isomorphisms

$$\bigwedge^k TM \cong \bigwedge^k (\mathrm{Ver} \oplus \mathrm{E}) \cong \bigoplus_{l_1+l_2=k} \bigwedge^{l_1} \mathrm{Ver} \otimes \bigwedge^{l_2} \mathrm{E}\,.$$

By viewing a k-form τ on M as a vector bundle morphism $\tau\colon \bigwedge^k TM \to \mathbb{R}$ covering the map $M \to \{*\}$, we obtain a decomposition of ω into a collection $\left\{\omega_{(l_1,l_2)}\right\}_{l_1+l_2=k}$. A component in this decomposition is given by the restriction to $\bigwedge^{l_1} \mathrm{Ver} \otimes \bigwedge^{l_2} \mathrm{E}$ as a subset of $\bigwedge^k TM$ via the above isomorphism.

Through this identification, the space of k-forms on M decomposes as

$$\Omega^k(M) \cong \bigoplus_{l_1+l_2=k} \Gamma\left(\bigwedge^{l_1} \mathrm{Ver}^*\right) \otimes \Gamma\left(\bigwedge^{l_2} \mathrm{E}^*\right).$$

Example 1.3.9. Let E be a connection on $\pi\colon M \to B$ and $\omega \in \Omega^2(M)$. On a pair of tangent vectors $u = u^\perp + u^\top$, $v = v^\perp + v^\top$, with $u^\perp, v^\perp \in \mathrm{Ver}$ and $u^\top, v^\top \in \mathrm{E}$, the above splitting yields:

$$\omega(u,v) = \omega_{(2,0)}(u^\perp, v^\perp) + \omega_{(1,1)}(u^\perp, v^\top) - \omega_{(1,1)}(v^\perp, u^\top) + \omega_{(0,2)}(u^\top, v^\top).$$

For example, if we are given a form $\tau\colon \bigwedge^2 \mathrm{Ver} \to \mathbb{R}$, we can easily define $\omega \in \Omega^2(M)$ extending it by setting $\omega_{(2,0)} = \tau$, $\omega_{(1,1)} = 0$ and $\omega_{(0,2)} = 0$.

In the case of 2-forms, we will have a particular interest in those that split with respect to our connection into a vertical part and a horizontal part. Therefore, we will call a 2-form ω for which $\omega_{(1,1)}$ vanishes E-*compatible*, or we will say it is compatible with the connection. //

1.4 Parallel transport

To make full analytic use of a connection on a surjective submersion and use it to probe at the transverse geometry of the total space with respect to the base space, we want to relate the curve spaces of the base and the total space. In the trivial example of a product manifold, $M = B \times F$, a curve $\gamma\colon I \to B$ and some choice $f \in F$ canonically defines a curve $\widetilde{\gamma}_f\colon I \to M\colon t \mapsto (\gamma(t), f)$ such that $\mathrm{pr}_1 \circ \widetilde{\gamma} = \gamma$ and $\mathrm{pr}_2 \circ \widetilde{\gamma} = \mathrm{const}_f$. Additionally, in the canonical splitting $T(B \times F) = \mathrm{pr}_1^* TB \oplus \mathrm{pr}_2^* TF$, its tangent vector has the form $\dot{\widetilde{\gamma}}_f(t) = (\dot{\gamma}(t), 0)$. In other words, the curve $\widetilde{\gamma}_f$ lifts γ to the total space, such that it moves solely in the horizontal direction.

In the general case, i.e. a surjective submersion $\pi\colon M \to B$, we can always lift a curve $\gamma\colon I \to B$ locally around some $t_0 \in I$ to any $x \in M_{\gamma(t_0)}$ by picking a local section $\sigma\colon U \subset B \to M$, where U is an open neighbourhood of $\gamma(t_0)$, with $\sigma(\gamma(t_0)) = x$ and setting $\widetilde{\gamma} = \sigma \circ \gamma$. However, this is only defined on $\gamma^{-1}(U)$, which might be a proper subset of I and is dependent on the choice of local section, which is not unique. To obtain a unique lift of γ, we need to fix a direction at each point which is horizontal to B, which is exactly the problem solved in the previous section by picking a connection.

Definition 1.4.1. Let $\pi\colon M \to B$ be a surjective submersion and E an Ehresmann connection. We call a curve $\gamma\colon I \to M$ *horizontal* if $\dot{\gamma}(t) \in \mathrm{E}_{\gamma(t)}$. If $\gamma\colon I \to B$ is a curve, then $\widetilde{\gamma}\colon J \subset I \to M$ is a *horizontal lift of* γ if $\pi \circ \widetilde{\gamma} = \gamma$ and it is horizontal.

This notion of horizontal paths coincides with the usual notion of being parallel along a curve on a vector bundle or a principal bundle after identifying the Ehresmann connection to the affine connection and connection 1-form, respectively, as in Examples **??** and **??**. Additionally, in the unique connection on a smooth covering space, the notion of a horizontal lift corresponds to a path lifting.

Finding a horizontal lift of a curve with a fixed starting point now has become an initial value problem for an ordinary differential equation, namely: Given $\pi\colon M \to B$, with a horizontal lift h and $\gamma\colon I \to B$ with $t_0 \in I$ and $x \in M_{\gamma(t_0)}$, a horizontal lift $\widetilde{\gamma}$ with $\widetilde{\gamma}(t_0) = x$ is a solution to the following system:

$$\begin{cases} \widetilde{\gamma}(t_0) = x, \\ \dot{\widetilde{\gamma}}(t) = h(\dot{\gamma}(t)). \end{cases}$$

In the cases of vector bundles and principal bundles, we know that the horizontal lift is always defined on the total domain of definition of the curve. In the general case, we cannot ensure such global existence, but as lifting horizontally is a solution to a differential equation, we can ensure local existence.

Proposition 1.4.2. *Let $\pi\colon M \to B$ be a surjective submersion with a connection and $\gamma\colon I \to B$ a curve. For any choice $t_0 \in I$, there exists*

- *a neighbourhood $U \subset M_{\gamma(t_0)} \times I$ of $M_{\gamma(t_0)} \times \{t_0\}$,*
- *a map $\widetilde{\gamma}\colon U \to M$,*

such that for any choice $f \in M_{\gamma(t_0)}$ we obtain a curve $\widetilde{\gamma}_f\colon \operatorname{pr}_2(U \cap (\{f\} \times I)) \to M\colon t \mapsto \widetilde{\gamma}(f,t)$ which is a horizontal lift of γ such that $\widetilde{\gamma}_f(t_0) = f$.
Additionally, U can be chosen maximally such that this lift is unique, i.e. for any horizontal lift of γ, say $\overline{\gamma}\colon J \to M$ with $f = \overline{\gamma}(t_0)$, we have $\overline{\gamma} = \widetilde{\gamma}_f|_J$.

Proof. Suppose that $\pi\colon M \to B$ is a surjective submersion with a horizontal lift $h\colon \pi^*TB \to TM$, let $\gamma\colon I \to B$ be a curve and fix some $t_0 \in I$. We want to determine a vector field on M which is the lift of $\dot{\gamma}$ through h. Consider the following vector field:

$$X\colon \gamma^*M = I\ {}_{\gamma}{\times}_{\pi} M \to T(\gamma^*M)\colon (s,x) \mapsto \left(\left.\frac{\partial}{\partial t}\right|_s, h(x, \dot{\gamma}(s)) \right)$$

Notice that $T(\gamma^*M) = TI\ {}_{T\gamma}{\times}_{T\pi} TM$ and that the image of X lies in $TI\ {}_{T\gamma}{\times}_{T\pi}$ E per construction. After picking a $t_0 \in I$, we can define $\widetilde{\gamma}$ and its domain as follows:

$$U = \left\{ (f,t) \in M_{\gamma(t_0)} \times I\colon (t - t_0, t_0, f) \in \mathscr{D}(X) \right\}, \quad \widetilde{\gamma}\colon U \to M\colon (f,t) \mapsto \operatorname{pr}_2 \circ \phi_X(t - t_0, t_0, f),$$

where ϕ_X denotes the flow of X and $\mathscr{D}(X)$ its domain of definition.

Fix an $f \in M_{\gamma(t_0)}$ and consider the curve $\widetilde{\gamma}_f\colon U \cap (\{f\} \times I) \to M\colon t \mapsto \widetilde{\gamma}(f,t)$. We remark that on γ^*M the maps $\gamma \circ \operatorname{pr}_1$ and $\pi \circ \operatorname{pr}_2$ coincide, such that

$$\pi \circ \widetilde{\gamma}(f,t) = \pi \circ \operatorname{pr}_2 \circ \phi_X(t - t_0, f) = \gamma \circ \operatorname{pr}_1 \circ \phi_X(t - t_0, f)$$

As ϕ_X is the flow of X and $\mathrm{pr}_1(X) = \frac{\partial}{\partial t}$, it follows that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_s (\mathrm{pr}_1 \circ \phi_X(t-t_0, t_0, f)) = \mathrm{pr}_1\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_s \phi_X(t-t_0,t_0,f)\right) = \mathrm{pr}_1(X_{\phi_X(s-t_0,t_0,f)})$$
$$= \left.\frac{\partial}{\partial t}\right|_{\mathrm{pr}_1(\phi_X(s-t_0,t_0,f))}$$

This implies that $\mathrm{pr}_1 \circ\phi_X(t-t_0,t_0,f) = t$, and thus $\pi\circ\widetilde{\gamma} = \gamma$. Additionally, we already remarked that the image of $\mathrm{im}\,\mathrm{pr}_2 \circ X$ lies in E, such that $\dot{\widetilde{\gamma}}_f(t) = \mathrm{pr}_2(X_{\phi_X(t-t_0,t_0,,f)}) \in \mathrm{E}_{\widetilde{\gamma}_f}$. Hence, we conclude that it is indeed a horizontal lift of γ and $\widetilde{\gamma}_f(t_0) = \mathrm{pr}_2 \circ\phi_X(t_0-t_0,t_0,f) = \mathrm{pr}_2(t_0,f) = f$.

The maximality and unicity then follow from the maximality of the flow domain and the unicity of the flow. □

Definition 1.4.3. Given a curve $\gamma\colon I \to B$ on a surjective submersion $\pi\colon M \to B$ with a connection, we will call $\widetilde{\gamma}\colon U \to M$ the *parallel transport map* and $\widetilde{\gamma}_f\colon U_f \to M$ *the horizontal lift to* f, where $U_f = U \cap (\{f\}\times I)$ is the maximal interval on which $\widetilde{\gamma}_f$ can be defined.

We can remark that the solution of the differential equation is dependent only on the image of the curve, and therefore only on the geometry, and not on the specific parametrisation of the curve.

Proposition 1.4.4. *Let $\pi\colon M \to B$ be a surjective submersion with a connection, $\gamma\colon I\to B$ a curve, and $\phi\colon J \to I$ is a diffeomorphism. Then $\widetilde{\gamma\circ\phi} = \widetilde{\gamma}\circ(\mathrm{id}\times\phi)$.*

Proof. Suppose $\pi\colon M\to B$ is a surjective submersion with an Ehresmann connection E and $\gamma\colon I\to B$ a curve with $f \in M_{\gamma(t_0)}$. Let $\phi\colon J\to I$ be some diffeomorphism and define $\eta = \gamma\circ\phi\colon J \to B$. Next define $\widetilde{\eta}_f = \widetilde{\gamma}_f\circ\phi$, then $\pi\circ\widetilde{\eta}_f = \pi\circ\widetilde{\gamma}_f\circ\phi = \gamma\circ\phi = \eta$ and

$$\dot{\widetilde{\eta}}_f = T\widetilde{\eta}_f\left(\frac{\partial}{\partial t}\right) = T(\widetilde{\gamma}\circ\phi)\left(\frac{\partial}{\partial t}\right) = T\widetilde{\gamma}\left(\frac{\partial\phi}{\partial t}\frac{\partial}{\partial t}\right) = \frac{\partial\phi}{\partial t}T\widetilde{\gamma}\left(\frac{\partial}{\partial t}\right) = \frac{\partial\phi}{\partial t}\dot{\widetilde{\gamma}}_f \in \mathrm{E}\,.$$

Therefore, $\widetilde{\eta}_f$ is the horizontal lift of η to f as it exists uniquely and we can conclude that $\widetilde{\eta} = \widetilde{\gamma}\circ(\mathrm{id}\times\phi)$. □

In the case where $\dot{\gamma}$ extends to a vector field X on B, in the sense that $\dot{\gamma}(t) = X_{\gamma(t)}$, we obtain a more direct expression of the horizontal lift using h as a map $\mathfrak{X}(B)\to\mathfrak{X}(M)$. Set $\widetilde{X} = h(X)$ and let $\phi_{\widetilde{X}}$ denote the flow and $\mathscr{D}(\widetilde{X})$ its flow domain. The parallel transport can then be defined as

$$U = \left\{(f,t)\in M_{\gamma(t_0)}\times I\colon (t-t_0,f)\in\mathscr{D}(\widetilde{X})\right\},\quad \widetilde{\gamma}\colon U\to M\colon (f,t)\mapsto\phi_{\widetilde{X}}(t-t_0,f).$$

Clearly $\widetilde{\gamma}_f$ is horizontal for any f as $\dot{\widetilde{\gamma}}_f(t) = \widetilde{X}(\phi_{\widetilde{X}}(t-t_0,f)) \in \mathrm{E}$. Additionally,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_s \pi\circ\widetilde{\gamma}_f = T\pi\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_s \phi_{\widetilde{X}}(t-t_0,f)\right) = T\pi\left(\widetilde{X}(\widetilde{\gamma}_f(s))\right) = X(\pi(\widetilde{\gamma}_f(s))).$$

This implies that $\pi\circ\widetilde{\gamma}$ is an integral curve of X, which also starts at $\gamma(t_0)$. We conclude that $\widetilde{\gamma}_f$ is the horizontal lift of γ to f.

Locally on a fibre bundle $F\hookrightarrow M\xrightarrow{\pi} B$, we can also obtain a more concrete description of parallel transport as a map on the fibre in terms of the local description of a connection. Let $\{(U_\alpha,\psi_\alpha)\}_{\alpha\in\Lambda}$ be a trivialising cover, take a curve $\gamma\colon I \to U_\alpha \subset B$ starting at b_0 and some $x \in M_{b_0}$, which corresponds to $\psi_\alpha(x) = (b_0, f_0)$. The horizontal lift of γ to x then has a local representation as $\psi_\alpha(\widetilde{\gamma}_x(t)) = (\gamma(t), \gamma^F(t))$, where $\gamma^F = \mathrm{pr}_2 \circ \psi_\alpha \circ \widetilde{\gamma}_x$. The construction implies that γ^F is a solution to the following initial value problem:

$$\begin{cases} \gamma^F(0) = f_0, \\ \dot{\gamma}^F(t) = h_{\alpha,\gamma(t)}(\dot{\gamma}(t))(\gamma^F(t)). \end{cases}$$

If we consider the time-dependent vector field $X_t = h_{\alpha,\gamma(t)}(\dot{\gamma}(t))$, the curve γ^F then becomes an integral curve of this vector field flowing from f at $t = 0$.

Still, the horizontal lift may not always be defined; consider a smooth finite covering space with a single point removed. Under some compactness conditions, however, we can resolve this analytically.

Proposition 1.4.5. *Let $\pi\colon M \to B$ be a surjective submersion with a connection and $\gamma\colon [0,1] \to B$ a curve. If $f \in M_{\gamma(0)}$ and $\widetilde{\gamma}_f : U_f \to M$ maps into a compact set, then $U_f = [0,1]$.*

Proof. Suppose that $\pi\colon M \to B$ is a surjective submersion with a connection and let $\gamma\colon [0,1] \to B$ be a curve. Fix an $f \in M_{\gamma(0)}$, and suppose that $\operatorname{im} \widetilde{\gamma}_f$ lies inside a compact set. Assume for contradiction that $U_f \neq [0,1]$, and recall that U_f is maximal. Without loss of generality, suppose that $\sup U_f < 1$ and set $a = \sup U_f$. Since U_f is an interval and open subset of $[0,1]$ whose supremum is strictly smaller than 1, we have $U_f = [0,a)$.

By assumption, the image of the lift $\widetilde{\gamma}_f$ is contained in a compact set and therefore the limit $\lim_{t\to a} \widetilde{\gamma}_f(t)$ must exist, such that $\widetilde{\gamma}_f$ admits a continuous extension to a. Furthermore, as $\widetilde{\gamma}_f$ is a solution to an ordinary differential equation determined by the horizontal lift condition, the Picard-Lindelöf theorem implies that there exists $\epsilon > 0$ such that the lift extends to an open interval $(a-\epsilon, a+\epsilon)$, contradicting the maximality of U_f. Hence, we conclude that $U_f = [0,1]$. □

1.4.1 Holonomy

The parallel transport and horizontal lifts of curves contain a lot of information on the connection. But while the restriction of the parallel transport to horizontal lifts is the natural lifts of curves, we can also consider the restriction of parallel transport to a map between fibres.

Definition 1.4.6. Let $\pi\colon M \to B$ be a surjective submersion with a connection and $\gamma\colon I \to B$ a curve. For $t_0, t_1 \in I$, the *holonomy map from t_0 to t_1 along γ* is the map

$$\tau_\gamma^{t_1,t_0} : U_{t_1,t_0} \subset M_{\gamma(t_0)} \to M_{\gamma(t_1)} : f \mapsto \widetilde{\gamma}(t_1, f),$$

where $U_{t_1,t_0} = \mathrm{pr}_1(U_{t_0} \cap (M_{\gamma(t_0)} \times \{t_1\}))$.

Hence, given a connection and a curve, we obtain a collection of maps $\left\{\tau_\gamma^{t_1,t_0}\right\}_{t_0,t_1\in I}$ which contain all the information on the parallel transport of the curve γ. This set of data again contains all the data of the connection as given some $v \in T_bB$ and $x \in M_b$, if $\gamma\colon (-\epsilon,\epsilon)$ integrates v, then we recover h, by using the fact that $\widetilde{\gamma}_x(t) = \tau_\gamma^{t,0}(x)$, as

$$h(x,v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \tau_\gamma^{t,0}(x).$$

Moreover, the set of holonomies has a nice structure with respect to some composition laws.

Proposition 1.4.7. *The holonomy maps are smooth embeddings, and for* $t_0 \leq t_1 \leq t_2$ *they satisfy*

$$\tau_\gamma^{t_2,t_1} \circ \tau_\gamma^{t_1,t_0}|_{\widetilde{U}} = \tau_\gamma^{t_0,t_2}|_{\widetilde{U}}, \quad \textit{where } \widetilde{U} = U_{t_2,t_0} \cap {\tau_\gamma^{t_1,t_0}}^{-1}(U_{t_2,t_1}).$$

Proof. The fact that the holonomy maps are smooth follows from the fact that $\widetilde{\gamma}$ is smooth, and thus, after restricting to an embedded submanifold and fixing t_1, it will still be smooth. The second property follows from the fact that the flow of a vector field is a one-parameter group action. □

1.5 Complete connections

The domains $U_f \subset U$ capture the extendibility of the horizontal lifts of $\gamma\colon I \to B$. As mentioned before, this domain of definition may be strictly smaller than I. However, in the trivial case $B \times F$ with the canonical connection, this is always maximal. In this section, we will show that it is exactly the existence of such maximal horizontal lifts which differentiates a fibre bundle from a surjective submersion.

Definition 1.5.1. Let $\pi\colon M \to B$ be a surjective submersion with a connection and $\gamma\colon I \to B$ a curve. The parallel transport of γ, for some $t_0 \in I$, is called *complete* if $\widetilde{\gamma}\colon U \to M$ is defined on $M_{\gamma(t_0)} \times I$. A connection on $\pi\colon M \to B$ is *complete* all parallel transports of all curves are complete.

To check the completeness of a connection, one has to check many paths over varying domains. However, as we saw in Proposition **??**, horizontal lifts are geometrically independent of the domain. Therefore, we can greatly reduce the collection of paths on which we need to check the completeness.

Proposition 1.5.2. *A connection is complete if and only if all the parallel transport of curves defined on* $[0, 1]$ *are complete.*

Proof. Let $\pi\colon M \to B$ be a surjective submersion and $h\colon \pi^*TB \to TM$ a horizontal lift. Clearly, if h is complete, then all parallel transports of curves $\gamma\colon\ [0,1] \to B$ are complete.

Conversely, suppose that all horizontal lifts of curves defined on $[0, 1]$ exist and are defined on $[0, 1]$. Let $\gamma\colon I \to B$ be an arbitrary curve, with I not necessarily $[0, 1]$, and take some arbitrary $t_0 \in I$ and $f \in M_{t_0}$. For any $t_1 \in I$, where we assume that $t_1 > t_0$, can consider a reparametrization $\phi\colon\ [0,1] \to [t_0, t_1]\colon$ and remark that $\eta = \gamma \circ \phi$ lifts to the whole of $[0, 1]$. By Proposition **??**, the lift of $\gamma|_{[t_0,t_1]}$ can be obtained from the lift of η. In particular, it follows that $\widetilde{\gamma}_f(t_1) = \widetilde{\eta}_f(1)$ and thus as t_1 was arbitrary $\widetilde{\gamma}_f$ is defined on the whole of I. □

The above proposition lets us consider only the curve space of the form $C^\infty([0,1]\,, B)$. In particular, a complete connection defines a nice characterisation in terms of these curve spaces.

Corollary 1.5.3. *Let* $\pi\colon M \to B$ *be a surjective submersion with a connection. Then the connection is complete if and only if the following map is well-defined:*

$$C^\infty([0,1]\,, B)\,{}_{\mathrm{ev}_0}{\times}_\pi M \to C^\infty([0,1]\,, M)\colon (\gamma, x) \mapsto \widetilde{\gamma}_x,$$

where $\mathrm{ev}_0 \colon C^\infty([0,1], B) \to B \colon \gamma \mapsto \gamma(0)$.

While completeness of connections is somewhat of an analytical condition, as it relates to the existence of solutions to differential equations, Proposition **??** already suggests that there are geometrical elements to it as well. Under some additional compactness conditions on a surjective submersion, or a fibre bundle, we can show that any connection is complete.

Proposition 1.5.4. *Let $\pi \colon M \to B$ be a surjective submersion, whose fibres are compact and connected; then all connections are complete.*

Proof. Let $\pi \colon M \to B$ be a surjective submersion whose fibres are compact and connected, and take an arbitrary connection. Consider an arbitrary curve $\gamma \colon [0,1] \to B$ and let us denote $\widetilde{\gamma} \colon U \to M$ for its parallel transport. By the tube lemma, there must exist some $\epsilon > 0$ such that $\pi^{-1}(\gamma(0)) \times [0, \epsilon) \subset U$, and suppose that this ϵ is maximal. For any $t \in [0, \epsilon)$, the associated holonomy $\tau_\gamma^{0,t} \colon \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(t))$ is injective as it has a left inverse given by $\tau^{t,0}$, see Proposition **??**. Additionally, this implies that it is an immersion, and thus a local diffeomorphism, as the fibres are of the same dimension. As $\pi^{-1}(\gamma(0))$ and $\pi^{-1}(\gamma(t))$ are connected, it will automatically define a diffeomorphism as it maps connected component to connected component.

Consider the parallel transport of γ at $t = \epsilon$, which we will denote with $\overline{\gamma} \colon U_\epsilon \to M$. By the tube lemma, we find that a $\delta > 0$ such that $\pi^{-1}(\gamma(\epsilon)) \times (\epsilon - \delta, \epsilon + \delta) \subset U_\epsilon$. By a similar argument as before, we can conclude that $\tau_\gamma^{t,\epsilon} \colon \pi^{-1}(\gamma(\epsilon)) \to \pi^{-1}(\gamma(t))$, for $t \in (\epsilon - \delta, \epsilon + \delta)$, defines a diffeomorphism. In particular, we find that the holonomy $\tau_\gamma^{\epsilon,0} = \tau_\gamma^{\epsilon,t} \circ \tau_\gamma^{t,0}$ is a diffeomorphism, by considering some $t \in (\epsilon - \delta, \epsilon)$. Hence, we can conclude that $\pi^{-1}(\gamma(0)) \times [0, \epsilon + \delta) \subset U$, which is a contradiction with the maximality of ϵ.

We conclude that the parallel transport along γ is complete, and as our connection was arbitrary, it follows that all connections on π are complete. □

We can drop the connectedness assumptions if we instead assume that our surjective submersion is a fibre bundle. This result was already known to Ehresmann in his seminal paper, [?Ehresmann1952], where he introduced the notion of fibre bundles and connections.

Proposition 1.5.5 ([?Ehresmann1952, Prp. 3.1])**.** *Let $\pi \colon M \to B$ be a fibre bundle with compact fibres; then all connections are complete.*

As we have mentioned before, a fibre bundle has a lot more geometrical relations between the base space and the total space. Therefore, we will see that the local triviality is exactly the condition one needs to obtain a complete connection, giving us a geometric condition for their existence instead.

Let us note some historical remarks made in [?delHoyo2016]. The statement, relating completeness and local triviality, was first made in [?Wolf1964, Cor. 2.5], accompanied by an incomplete proof. Later, it reappeared in books like [?Kolar1993, ?Michor2008] with a proof relying on the convex glueing of fibred metrics, which was not a sound argument. This was later resolved in [?delHoyo2016, Thm. 5] using a glueing of connections which varied over the fibres. The statement is as follows:

Theorem 1.5.6. *A surjective submersion admits a complete connection if and only if it is a fibre bundle.*

Before we move to the proof, which involves some preliminary steps, let us give a result which is long overdue.

Proposition 1.5.7. *Let $\pi\colon M \to B$ be a surjective submersion whose fibres are compact and connected; then it is a proper map.*

Proof. Let $\pi\colon M \to B$ be a surjective submersion whose fibres are compact and connected, and pick an arbitrary connection. It follows from Proposition **??** that this is a complete connection, and thus by Theorem **??**, it is a fibre bundle. Notice that we can now apply Proposition **??** to conclude that π is a proper map. ☐

Let us turn to the proof of Theorem **??**. Our proof will be based on some of the material from [?delHoyo2016], but changed in a manner such that we can later apply it to the multiplicative cases as well. The first implication is a standard exercise for fibre bundles and consists of taking a contractible cover of the base space and letting the contractions define paths along which one can parallel transport. The converse is the more intricate argument and will be based on the following lemma and corollary, which is based on Proposition **??**, which lets us measure the completeness of connection in the trivial case and the locally trivial case by extension.

Lemma 1.5.8. *Let* E *be an Ehremann connection on* $\mathrm{pr}_1\colon B \times F \to B$ *and suppose there exists* $S \subset F$ *such that:*

- ⋄ $E|_{B\times S} = TB \times 0_S$*, where* 0_S *is the zero section* $F \to TF$ *restricted to* S*.*
- ⋄ *The connected components of* $F \backslash S$ *are relatively compact.*

Then E *is a complete connection.*

Proof. Suppose that E is an Ehresmann connection on $\mathrm{pr}_1\colon B \times F \to B$ and $S \subset F$ as above. Let $\gamma\colon [0,1] \to B$ be a curve starting at b, and take an arbitrary $f \in F$. We are in either of two cases: $f \in S$ or $f \notin S$.

i) Suppose that $f \in S$, we then define $\widetilde{\gamma}_{(b,f)}(t) = (\gamma(t), f)$ and note this is indeed a horizontal lift to (b, f) as

$$\mathrm{pr}_1 \circ \widetilde{\gamma}_{(b,f)}(t) = \mathrm{pr}_1(\gamma(t), f) = \gamma(t). \qquad \dot{\widetilde{\gamma}}_{(b,f)}(t) = (\dot\gamma(t), 0) \in T_{\gamma(t)}B = \mathrm{E}_{(\gamma(t),f)}\,.$$

Additionally, we find that $\widetilde{\gamma}_{(b,f)}(0) = (\gamma(0), f) = (b, f)$. Therefore, this lift is defined on $[0, 1]$.

ii) Suppose that $f \notin S$, and let $\widetilde{\gamma}_{(f,s)}$ denote for the horizontal lift. We remark that $\mathrm{pr}_2 \circ \widetilde{\gamma}_{(b,f)}$ must stay within a connected component of $F\backslash S$ as horizontal lifts are unique, and if $\mathrm{pr}_2 \circ \widetilde{\gamma}_{(b,f)}$ maps into S, the horizontal lift is given by the first case. Therefore, $\widetilde{\gamma}_{(b,f)}$ must stay within a compact set, namely $\mathrm{im}\,\gamma \times \overline{\mathrm{Conn}(F\backslash S, f)}$, where $\mathrm{Conn}(F\backslash S, f)$ denotes the connected component of $F\backslash S$ containing f. By Proposition **??** it follows that $\widetilde{\gamma}_{(b,f)}$ is defines on the whole of $[0, 1]$.

We can conclude that the horizontal lift of γ is always defined on the whole of $[0, 1]$ and thus the parallel transport is always complete. By Proposition **??**, this shows that E is a complete connection. ☐

Using an argument similar to the one used in the proof of the path-lifting property on covering spaces, we can extend this argument to fibre bundles.

Proposition 1.5.9. *Let* E *be an Ehresmann connection on a fibre bundle* $F \hookrightarrow M \xrightarrow{\pi} B$*. If there exists a trivialising cover* $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ *and for all* $\alpha \in \Lambda$ *there exists* $S_\alpha \subset F$ *such that:*

i) $T\psi_\alpha(\mathrm{E}|_{\psi_\alpha^{-1}(U_\alpha \times S_\alpha)}) = TU_\alpha \times 0_{S_\alpha}$, *where* 0_{S_α} *denotes the zero section of* $F \to TF$ *restricted to* S_α.

ii) *The connected components of* $F \backslash S_\alpha$ *are precompact.*

Then E *is a complete connection.*

Proof. Take a fibre bundle $F \hookrightarrow M \xrightarrow{\pi} B$ with an Ehresmann connection E and fix some trivialising cover $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ with S_α as above. Let $\gamma\colon [0,1] \to B$ be a curve, and notice that $\operatorname{im}\gamma$ is compact, so that there exists a finite subset $\{(U_i, \psi_i)\}_{i=1}^n$ covering $\operatorname{im}\gamma$. Additionally, we can assume, after possibly reordering the cover, that there exists a partition $0 = a_0 < a_1 < \cdots < a_n = 1$ such that $\operatorname{im}\gamma|_{[a_{i-1},a_i]} \subset U_i$. Let us denote the restriction γ to each subinterval as $\gamma_i = \gamma|_{[a_{i-1},a_i]}$. We can then remark that $\psi_i \circ \gamma|_{[a_{i-1},a_i]}$ has complete parallel transport with respect to the connection $T\psi_i(\mathrm{E}|_{M|_{U_i}})$ on $U_i \times F$. Therefore γ_i has complete parallel transport by conjugating with ψ_i.

We can then define $x_0 = x$, and define x_i recursively as $x_i = \widetilde{\gamma}_{i x_{i-1}}(a_i)$, the horizontal lift of γ_i to x_{i-1}, and set

$$\widetilde{\gamma}_x(t) = \widetilde{\gamma}_{i x_{i-1}}(t), \text{ when } t \in [a_{i-1}, a_i]\,.$$

This will clearly define a horizontal lift of γ to x, which is defined on the whole of $[0, 1]$. This implies that E is a complete connection. □

This lemma gives us a more geometric picture to check the completeness of connections on a fibre bundle. However, this still starts with a connection. In Theorem **??**, we want to construct a connection and then show that it satisfies the requirements of Proposition **??**. Under specific conditions on $S_\alpha \subset F$, we can indeed define compatible connections.

Proposition 1.5.10. *Let* $F \hookrightarrow M \xrightarrow{\pi} B$ *be a fibre bundle with trivialising cover* $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ *which is locally finite, and* $\{U_\alpha\}_{\alpha\in\Lambda}$ *an open cover of* B *with* $\overline{U_\alpha} \subset V_\alpha$. *Suppose that for each* $\alpha \in \Lambda$ *we have a closed subset* $S_\alpha \subset F$, *such that they satisfy*

$$S_\alpha \cap \psi_{\alpha\beta,b}(S_\beta) = \emptyset, \quad \forall \alpha \neq \beta \in \Lambda,\ b \in \overline{U_{\alpha\beta}}.$$

Then there exists an Ehresmann connection E *such that* $T\psi_\alpha(\mathrm{E}|_{\psi_\alpha^{-1}(U_\alpha\times S_\alpha)}) = TU_\alpha \times 0_{S_\alpha}$ *for all* $\alpha \in \Lambda$.

Proof. Suppose that $F \hookrightarrow M \xrightarrow{\pi} B$ is a fibre bundle with a trivialising cover $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ and let $\{U_\alpha\}_{\alpha\in\Lambda}$ and $\{S_\alpha\}_{\alpha\in\Lambda}$ be as in the statement. Remark that for each $\alpha \in \Lambda$ we take on $M|_{U_\alpha}$ the canonical connection induced by the trivialisation, namely

$$h_\alpha\colon \pi^*TU_\alpha \to TM|_{U_\alpha}\colon (x, v) \mapsto T_{\psi_\alpha(x)}\psi_\alpha^{-1}(v, 0).$$

Next, for each $\alpha \in \Lambda$ we can define an open subset W_α by

$$W_\alpha = \pi^{-1}(U_\alpha) \backslash \bigcup_{\beta\neq\alpha} \psi_\beta^{-1}(\overline{U_\beta} \times S_\beta).$$

Notice that the collection $\{\psi_\beta^{-1}(\overline{U_\beta} \times S_\beta)\}_{\beta\neq\alpha}$ is locally finite as the cover $\{V_\alpha\}_{\alpha\in\Lambda}$ is, moreover, they are all closed as all S_β are. Their union is then also closed, see [?Munkres2000, Lemma 39.1 (c)], and thus W_α is open. Moreover, one can verify that $\{W_\alpha\}_{\alpha\in\Lambda}$ defines a cover of M:

- If $x \in M\setminus \bigcup_\beta \psi_\beta^{-1}(\overline{U_\beta} \times S_\beta)$, then there must exists a $\alpha \in \Lambda$ such that $x \in \pi^{-1}(U_\alpha)$. This implies that it is an element of W_α.

- If $x \in \bigcup_\beta \psi_\beta^{-1}(\overline{U_\beta} \times S_\beta)$, we remark that it in particular lies in $\psi_\alpha^{-1}(\overline{U_\alpha} \times S_\alpha)$ for some α, as it is an element of some $\pi^{-1}(U_\alpha)$. Let us show that it cannot be an element of $\psi_\beta^{-1}(\overline{U_\beta} \times S_\beta)$ for $\beta \neq \alpha$. Assume the converse, i.e. $x \in \psi_\beta^{-1}(\overline{U_\beta} \times S_\beta)$. Remark that we can write them in the local trivialisations as $\psi_\beta(x) = (b, s_\beta)$ for $b \in \overline{U_{\alpha\beta}}$ and $s_\beta \in S_\beta$. Similarly, we find $s_\alpha \in S_\alpha$ such that $\psi_\alpha(x) = (b, s_\alpha)$. By definition, we have
$$\psi_{\alpha\beta,b}(s_\beta) = \mathrm{pr}_2 \circ\psi_\alpha \circ \psi_\beta^{-1}(b, s_\beta) = s_\alpha.$$
This implies that $s_\alpha \in \psi_{\alpha\beta,b}(S_\beta)$, which is a contradiction with our assumption on $\{S_\alpha\}_{\alpha\in\Lambda}$. It follows that $\psi_\alpha^{-1}(\overline{U_\alpha} \times S_\alpha) \cap \psi_\beta^{-1}(\overline{U_\beta} \times S_\beta) = \emptyset$ and thus $x \in W_\alpha$.

Let $\{\phi_\alpha\}_{\alpha\in\Lambda}$ denote the partition of unity subordinate to the open cover $\{W_\alpha\}$ and define $h = \sum_\alpha \phi_\alpha h_\alpha$. By Corollary ??, this defines a connection, and let E denote the associated Ehresmann connection. By our construction of $\{W_\alpha\}$, we find that $x \in \psi_\alpha^{-1}(U_\alpha \times S_\alpha)$ implies that $x \in W_\alpha$ and $x \notin W_\beta$ for $\beta \neq \alpha$. Therefore, if $x = \psi_\alpha^{-1}(b, s)$, for some $s \in S_\alpha$ and $v \in T_{\pi(x)}U_\alpha$, then
$$h(x, v) = \sum_\gamma \phi_\gamma(x)h_\gamma(x, v) = h_\alpha(x, v) = T_{\psi_\alpha(x)}\psi_\alpha^{-1}(v, 0)$$

From this, we can conclude that $T\psi_\alpha\left(\mathrm{E}\,|_{\psi_\alpha^{-1}(U_\alpha\times S_\alpha)}\right) = TU_\alpha \times 0_{S_\alpha}$. □

We will now end this section with a proof of Theorem **??** based on the previous corollaries and proposition to construct a connection and check its completeness.

Proof of Theorem **??**. Let $\pi\colon M \to B$ be a surjective submersion.

$\Longrightarrow$: Suppose E is a complete Ehresmann connection and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of B where each U_α is contractible. Let $c_\alpha\colon U_\alpha \times [0, 1] \to U_\alpha$ be contractions to some $b_\alpha \in U_\alpha$, i.e. $c_\alpha(b, 0) = b,\ c_\alpha(b, 1) = b_\alpha$. Define the path $\gamma^{\alpha,b}\colon [0, 1] \to U_\alpha\colon t \mapsto c_\alpha(b, t)$. Using these paths, we obtain local trivialisations via the parallel transport along these paths:
$$\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \pi^{-1}(b_\alpha)\colon x \mapsto (\pi(x), \widetilde{\gamma^{\alpha,\pi(x)}}_x(1)).$$

This parallel transport is then well-defined at 1 as the connection is complete. One can readily verify that these maps are diffeomorphisms preserving the fibred structure.

$\Longleftarrow$: Suppose that $F \hookrightarrow M \xrightarrow{\pi} B$ is a locally trivial fibre bundle. To prove that π admits a complete connection, we will construct an Ehresmann connection using Proposition **??** such that it satisfies the conditions of Proposition **??**.

Pick a trivialising cover $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ by relatively compact sets which is locally finite and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a cover of B such that $\overline{U_\alpha} \subset V_\alpha$. Remark that we can assume that $\Lambda = \mathbb{N}$ or $\Lambda = \{0, \ldots, n\}$ without loss of generality. Let $f\colon F \to \mathbb{R}_{\geq 0}$ be a proper function, which exists by [?Lee2013, Prp. 2.28]. We can let S_α be the preimage of some infinite discrete subsets, N_α, of $\mathbb{R}_{\geq 0}$ under this map.

To construct the sets $N_\alpha \subset \mathbb{N}$, we will define a map $n\colon \mathbb{N}\times\Lambda \to \mathbb{N}$ inductively with respect to the lexicographical ordering[2]. We first define $n(0,0) = 0$. Next, suppose that n is defined on all (j,β), with $(j,\beta) < (i,\alpha)$, and consider the set

$$\widetilde{C}_{i,\alpha} = \bigcup_{(j,\beta)<(i,\alpha)} \psi_\beta^{-1}(\overline{U_{\alpha\beta}} \times f^{-1}(n(j,\beta))) \subset \pi^{-1}(\overline{U_\alpha}).$$

Notice that the precompactness of U_β and properness of f implies that $\psi_\beta^{-1}(\overline{U_\beta}\times f^{-1}(n(j,\beta)))$ is compact. Next, we notice that the collection $\{\psi_\beta^{-1}(\overline{U_\beta}\times f^{-1}(n(j,\beta)))\}_{(\beta,j)\leq(i,\alpha)}$ is a locally finite collection of compact sets. Remark that this implies that $\psi_\beta^{-1}(\overline{U_\beta}\times f^{-1}(n(i,\alpha)))$ can only have nonempty intersection with finitely many $\psi_\beta^{-1}(\overline{U_\beta}\times f^{-1}(n(j,\beta)))$[3]. We can conclude that $\widetilde{C}_{i,\alpha}$ is compact as the union consists of finitely many nonempty compacts. Therefore, the set $C_{i,\alpha} = \mathrm{pr}_2 \circ\psi_\alpha(\widetilde{C}_{i,\alpha}) \subset F$, is compact as well and we can set $n(i,\alpha)$ to be the smallest integer strictly larger than $\sup\{f(x) \mid x \in C_{i,\alpha}\}$.

Having constructed the full map $n\colon \mathbb{N}\times\Lambda\to\mathbb{N}$, we define $N_\alpha = \{n(i,\alpha) \mid i\in\mathbb{N}\}$ and $S_\alpha = f^{-1}(N_\alpha)$.

Next, we show that $S_\alpha \cap \psi_{\alpha\beta,b}(S_\beta) = \emptyset$ for all $\alpha\neq\beta\in\Lambda$ and $b\in\overline{U_{\alpha\beta}}$. Suppose $s\in S_\alpha\cap\psi_{\alpha\beta,b}(S_\beta)$; then there exists $i,j\in\mathbb{N}$ with $f(s) = n(i,\alpha)$ and $f(\psi_{\beta\alpha,b}(s)) = n(j,\beta)$, such that $s\in\psi_{\alpha\beta,b}(f^{-1}(n(j,\beta)))$. Without loss of generality, we may assume that $(j,\beta) < (i,\alpha)$. By definition, (i,α) must satisfy $n(i,\alpha) > \sup\{f(x)\mid x\in C_{i,\alpha}\}$. Remark that $C_{i,\alpha}$ can be rewritten as

$$\begin{aligned}C_{i,\alpha} &= \mathrm{pr}_2\circ\psi_\alpha(\widetilde{C}_{i,\alpha}) = \mathrm{pr}_2\circ\psi_\alpha\left(\bigcup_{(k,\gamma)<(i,\alpha)} \psi_\gamma^{-1}(\overline{U_{\alpha\gamma}}\times f^{-1}(n(k,\gamma)))\right)\\ &= \bigcup_{(k,\gamma)<(i,\alpha)} \mathrm{pr}_2\circ\psi_\alpha\left(\bigcup_{b\in\overline{U_{\alpha\gamma}}}\psi_\gamma^{-1}(\{x\}\times f^{-1}(n(k,\gamma)))\right) = \bigcup_{(k,\gamma)<(i,\alpha)}\bigcup_{b\in\overline{U_{\alpha\gamma}}}\psi_{\alpha\gamma,b}(f^{-1}(n(k,\gamma))).\end{aligned}$$

Remark that in particular, $s\in C_{i,\alpha}$ as $(j,\beta)<(i,\alpha)$ and $s\in\psi_{\alpha\beta,b}(f^{-1}(n(j,\beta)))$ by assumption. Therefore, by our construction, we have the following inequalities:

$$f(s) = n(i,\alpha) > \sup_{x\in C_{i,\alpha}} f(x) \geq f(s),$$

which leads to a contradiction. We conclude that $S_\alpha\cap\psi_{\alpha\beta,b}(S_\beta) = \emptyset$ for all $\alpha\neq\beta\in\Lambda$ and $b\in\overline{U_{\alpha\beta}}$.

It follows from Proposition **??**, that there exists an Ehresmann connection E on $\pi\colon M\to B$ such that

$$T\psi_\alpha\left(\mathrm{E}\,|_{\psi_\alpha^{-1}(U_\alpha\times S_\alpha)}\right) = TU_\alpha\times 0_{S_\alpha}.$$

Combining this with the fact that the connected components of $F\backslash S_\alpha$ are all precompact, as they are subsets of $f^{-1}\left[n(\alpha,i), n(\alpha,i+1)\right]$, it follows from Proposition **??** that E is a complete connection. □

[2]Given two partial orders $(P,\leq_P)$ and $(Q,\leq_Q)$, the lexicographical ordering on $P\times Q$ is defines as $(p,q)\leq(p',q')$ if and only if $p<p'$, or $p=p'$ and $q\leq q'$. In particular, we first consider the first coordinate in this ordering.

[3]Suppose $\mathscr{A}$ is a locally finite collection of compact sets. For a $K\in\mathscr{A}$, any $x\in K$ admits a neighbourhood U_x such that it has finitely many nonempty intersections with elements in $\mathscr{A}$. As K is compact, pick a finite subcover $\{U_{x_i}\}_{i=1}^n$. For each $i=1,\dots,n$, the neighbourhood U_{x_i} has finitely many nonempty intersections with elements of $\mathscr{A}$ and there are finitely many i, thus K will also have finitely many nonempty intersection with elements of $\mathscr{A}$.

Chapter 2

Lie groupoids

While Lie groups give a great description of classical symmetries of manifolds, there are more general symmetries we might want to capture. One way of doing this is by considering groupoids, which are like symmetry groups but possibly between manifolds. This flexible structure lets us capture a multitude of different concepts besides groups. For example, they can also encode the data of a group action, contain manifolds, and also different generalisations of groups, like pseudo-groups.

Groupoids were first studied, or at least their smooth version, by Charles Ehresmann [?Ehresmann1959], who used them to describe the general symmetries of systems. Now they are indispensable tools in different aspects of mathematics, like Poisson geometry, where they are used to integrate the Lie algebroid associated to a Poisson manifold, and noncommutative geometry, where groupoid C^*-algebras generate a range of objects with interesting index and K-theory.

In this chapter, we will give a short introduction to groupoids and describe the smooth version of these objects, which can be viewed as a groupoid object inside Diff, the category of smooth manifolds. The main tools one can use to work with them are similar to the ones used in Lie group theory, like translations, but also ones which make use of the categorical nature of groupoids, such as bisections, isotropy groups, and properties on source fibres. We will then go over some examples and constructions of Lie groupoids, which will prove useful later. The chapter finishes with a discussion of Morita equivalences based on bibundles. The main references are [?delHoyo2013, ?Mackenzie2005, ?Mackenzie2007, ?Meinrenken2017].

2.1 Categories of Lie groupoids

As the name suggests, Lie groupoids are the smooth version of a groupoid, just like a Lie group is the smooth version of a group. Hence, to introduce Lie groupoids, we first need to define groupoids themselves. The study of groupoids is not limited to Lie groupoids, and has been studied for far longer, cf. [?Brown1968]. We will start with a short, yet effective definition of a groupoid.

Definition 2.1.1. A *groupoid* is a small category whose morphisms are all invertible. Additionally, a *subgroupoid* is a subcategory which contains all inverses.

This definition of a groupoid is efficient, yet not very insightful into the internal structure. As we want to impose additional geometric requirements on groupoids later, we will unravel the intrinsic structure of such objects. Hence, we will give a second equivalent definition where we view a groupoid as an algebraic object similar to a group, where we only have a partial multiplication. Alternatively, one can view them as groups which have multiple identity elements.

Definition 2.1.2. A *groupoid* consists of a pair of sets $\mathcal{G}$ and $\mathcal{G}_0$, called the set of *arrows* and set of *objects*, respectively. Associated with these sets, we have *structure maps*:

- The *source map* $\boldsymbol{s}\colon \mathcal{G} \to \mathcal{G}_0$ and *target map* map $\boldsymbol{t}\colon \mathcal{G} \to \mathcal{G}_0$. If $g \in \mathcal{G}$ has $\boldsymbol{s}(g) = x$ and $\boldsymbol{t}(g) = y$, then we sometimes denote this as $y \xleftarrow{g} x \in \mathcal{G}$. Additionally, we define the following sets[a] :
$$\mathcal{G}_x = \boldsymbol{s}^{-1}(x), \quad \mathcal{G}^y = \boldsymbol{t}^{-1}(y), \quad \mathcal{G}_x^y = \mathcal{G}_x \cap \mathcal{G}^y,$$
$$\mathcal{G}^{(n)} = \{(g_1, \ldots, g_n) \in \mathcal{G}^n |\, \boldsymbol{s}(g_i) = \boldsymbol{t}(g_{i+1})\}\,.$$
The set $\mathcal{G}_x$ is called the $\boldsymbol{s}$-fibre at x (read: source-fibre), and $\mathcal{G}^y$ the $\boldsymbol{t}$-fibre at y (read: target-fibre). We call $\mathcal{G}^{(n)}$ the *n-composable arrows*, or if $n = 2$ just the *composable arrows*, and if $(g, h) \in \mathcal{G}^{(2)}$ we call g and h *composable*.
- The *multiplication* $\boldsymbol{m}\colon \mathcal{G}^{(2)} \to \mathcal{G}\colon (g, h) \mapsto gh$.
- The *unit map* or *object inclusion* $\boldsymbol{u}\colon \mathcal{G}_0 \to \mathcal{G}\colon x \mapsto 1_x$.
- The *inversion* $\boldsymbol{i}\colon \mathcal{G} \to \mathcal{G}\colon g \mapsto g^{-1}$.

The source and target maps interact with the other structure maps as follows:

- $\boldsymbol{s}(gh) = \boldsymbol{s}(h)$ and $\boldsymbol{t}(gh) = \boldsymbol{t}(g)$ for all $(g, h) \in \mathcal{G}^{(2)}$.
- $\boldsymbol{s}(1_x) = \boldsymbol{t}(1_x) = x$ for all $x \in \mathcal{G}_0$.
- $\boldsymbol{s}(g^{-1}) = \boldsymbol{t}(g)$ and $\boldsymbol{t}(g^{-1}) = \boldsymbol{s}(g)$ for all $g \in \mathcal{G}$.

The other structure maps then abide by grouplike axioms.

- $g(hk) = (gh)k$ for all $g, h, k \in \mathcal{G}$ such than $(g, h), (h, k) \in \mathcal{G}^{(2)}$.
- $g1_x = 1_y g = g$ for all $y \xleftarrow{g} x \in \mathcal{G}$.
- $g^{-1}g = 1_x$ and $gg^{-1} = 1_y$ for all $y \xleftarrow{g} x \in \mathcal{G}$.

A *subgroupoid* in this sense is a pair $\mathcal{H} \subset \mathcal{G}$ and $\mathcal{H}_0 \subset \mathcal{G}_0$ such that $\boldsymbol{s}(\mathcal{H}) = \boldsymbol{t}(\mathcal{H}) = \mathcal{H}_0$, and it is closed under mulltplication and inversion, i.e. $\boldsymbol{m}(\mathcal{H} \times \mathcal{H} \cap \mathcal{G}^{(2)}) \subset \mathcal{H}$ and $\boldsymbol{i}(\mathcal{H}) \subset \mathcal{H}$.

[a] We can define similar notions where x, y are subsets $U, V \subset M$, which are then denoted as $\mathcal{G}_U, \mathcal{G}^V, \mathcal{G}_U^V$.

Notation. While a groupoid consists of a pair $\mathcal{G}$ and $\mathcal{G}_0$, we will often view $\mathcal{G}_0$ as internal to $\mathcal{G}$, by considering its inclusion via $\boldsymbol{u}$. Therefore, we will associate a groupoid with its set of arrows $\mathcal{G}$. In this case, the object set is always denoted with a subscript 0.
If we want to specify a set of objects, say M, we will write $\mathcal{G} \rightrightarrows M$.

Notice that a groupoid still abides by some group-like structure. For example, the units and inverses are unique, in the sense that if $y \xleftarrow{g} x \in \mathcal{G}$ and $(h, g) \in \mathcal{G}^{(2)}$ then $hg = g$ implies that $h = 1_x$, and if $hg = 1_y$, then $h = g^{-1}$.

Let us consider some examples, which showcase how groupoids can describe generalisations of symmetries.

Example 2.1.3. Consider a vector bundle $\pi\colon V \to M$ and define $\mathrm{Gl}(E)$ as the set

$$\mathrm{Gl}(E) = \{(x, A, y) | x, y \in M \text{ and } A\colon E_x \to E_y \text{ a linear isomorphism}\}.$$

This defines a groupoid over M where $y \overset{(x,A,y)}{\longleftarrow} x$ and the multiplication is $(y, A, z)(x, B, y) = (x, AB, z)$. //

Example 2.1.4. Consider the fibre bundle $\mathrm{pr}_1\colon M \times M \to M$ and let C^∞_M denote the sheaf of local sections of this surjective submersion. A section of this sheaf, say $f \in C^\infty_M(U)$, can be viewed as a map $f\colon U \to M$ by considering its second projection. We can then consider the subpresheaf Diff_M defined by

$$\mathrm{Diff}_M = \{f \in C^\infty_M | \, f\colon \mathrm{dom}\, f \to \mathrm{im}\, f \text{ is a diffeomorphism}\}$$

This defines only a presheaf, as glueing diffeomorphisms may make them noninjective. Yet this set admits a composition

$$\circ\colon \ \mathrm{Diff}_M \times \mathrm{Diff}_M \to \mathrm{Diff}_M\colon \ (f, g) \mapsto f \circ g|_{g^{-1}(\mathrm{dom}\, f)}$$

This composition is associative and has a unit, namely id_M, and it admits inverses.

A pseudogroup (on M) is a subset $G \subset \mathrm{Diff}_M$ such that

- $G \circ G \subset G, G^{-1} \subset G$ and $\mathrm{id}_M \in G$,
- if $f \in \mathrm{Diff}_M(U)$ and $U = \bigcup_{i\in I} U_i$ then $f \in G(U)$ if and only if $f|_{U_i} \in G(U_i)$ for all $i \in I$.

This means that G has grouplike properties and it is defined by local data.

A pseudogroup in particular defines a groupoid, by restricting the multiplication to a fibred product over the domain and image map, i.e. $\mathrm{Diff}_M \,{}_{\mathrm{dom}}{\times}_{\mathrm{im}}\, \mathrm{Diff}_M$, where $\mathrm{dom}\colon \ \mathrm{Diff}_M \to \mathcal{O}_M$ and $\mathrm{im}\colon \ \mathrm{Diff}_M \to \mathcal{O}_M$ send a diffeomorphism to its domain and image, respectively, where $\mathcal{O}_M$ denotes the collection of open subsets of M. The source and target are then exactly given by dom and im, while the multiplication is the restriction of $\circ$. The unit at an open is simply the identity map of that open, and the inverse is by taking the inverse.

Clearly, this manner of taking a groupoid loses a lot of information. We can retain more information about a pseudogroup on M by considering germs of the map. The objects of our groupoid are given by M and the arrows by

$$\mathcal{G} = \{\mathrm{germ}_x f | \, f \in G \text{ and } x \in \mathrm{dom}\, f\}$$

The structure maps are defined as follows:

$$\boldsymbol{s}(\mathrm{germ}_x f) = x, \quad \boldsymbol{t}(\mathrm{germ}_x f) = f(x),$$

$$\mathrm{germ}_y f \cdot \mathrm{germ}_x g = \mathrm{germ}_x(f \circ g), \quad 1_x = \mathrm{germ}_x \mathrm{id}_M, \quad (\mathrm{germ}_x f)^{-1} = \mathrm{germ}_{f(x)}(f^{-1})$$

One can verify that this indeed defines a groupoid structure on $\mathcal{G} \rightrightarrows M$.

Notice that some $f \in \mathrm{Diff}_M$ is fully determined its collection of germs, $\{\mathrm{germ}_x f\}_{x\in \mathrm{dom}\, f}$, as a subset of $\mathcal{G}$. Namely, we recover f via $f(x) = \boldsymbol{t}(\mathrm{germ}_x f)$. This implies that it is a so-called separated sheaf. //

To fully describe the structure of groupoids, we want to describe an appropriate category into which they fit. As we view a groupoid as a category, there is an obvious choice of maps, namely functors. Given the remarks after Definition **??**, we will describe this in more detail related to the internal structure of a groupoid.

Definition 2.1.5. A *groupoid morphism* from $\mathcal{G}$ to $\mathcal{H}$ consists of a pair of maps $\phi\colon \mathcal{G} \to \mathcal{H}$ and $\phi_0\colon \mathcal{G}_0 \to \mathcal{H}_0$ such that

$$\phi_0 \circ \boldsymbol{s} = \boldsymbol{s} \circ \phi, \quad \phi_0 \circ \boldsymbol{t} = \boldsymbol{t} \circ \phi, \quad \phi(gh) = \phi(g)\phi(h) \text{ for all } (g,h) \in \mathcal{G}^{(2)}.$$

We call ϕ the *map of arrows* and ϕ_0 the *map of objects* or *base map*.

Notation. Similar to how a groupoid is determined by the set of arrows and the structure maps, where we can view the objects as a structure internal to it, a groupoid morphism is fully determined by the map of arrows. The object map of a groupoid morphisms (ϕ, ϕ_0) can be recovered as $\phi_0 = \boldsymbol{s} \circ \phi \circ \boldsymbol{u}$. Therefore, we will identify a groupoid map with its map of arrows. We will always implicitly assume ϕ_0 is the object map of a groupoid map ϕ if there is no possible confusion.

By using the above notation, the composition of groupoid morphisms is simply given by the composition of the maps of arrows, and in particular, this defines a category.

Definition 2.1.6. The category of groupoids with groupoid morphisms is denoted by **Grpd**.

Let us now extend these ideas to the smooth category. This definition will very much be in the same spirit as a Lie group; however, we need some more conditions for it to make sense. We will also generalise the concept of a groupoid morphism and introduce the concept of a Lie subgroupoid.

Definition 2.1.7. A *Lie groupoid* is a groupoid $\mathcal{G}$, where both the spaces of arrows and the base space are manifolds, the structure maps are smooth, and $\boldsymbol{s}$ and $\boldsymbol{t}$ are submersions.
A *Lie groupoid morphism* between Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ is a groupoid morphism which is also a smooth map.
We denote the associated category of Lie groupoids with Lie groupoid morphisms by **LieGrpd**.

Remark. In this thesis, we will assume our Lie groupoids to be Hausdorff. In general, one often assumes only the source and target fibres to be Hausdorff, but not the whole space in total, as there are natural constructions of groupoids with a nonHausdorff topology. //

Let us delay giving examples of Lie groupoids until we have defined some more properties for us to discuss in tandem.

The translation from groupoid to Lie groupoid is akin to that from group to Lie group, except for the additional requirement that $\boldsymbol{s}$ and $\boldsymbol{t}$ are submersions. This assumption is needed to make sure that $\mathcal{G}^{(2)}$ is an embedded submanifold of $\mathcal{G} \times \mathcal{G}$, such that the multiplication can be smooth in a canonical manner. Of course, one could go with a weaker transversality condition like clean intersections. However, the source and target map being submersions will prove useful in the future, for example, it automatically implies that their fibres are embedded submanifolds. The structure maps have some additional nice geometric properties.

Proposition 2.1.8. *Let $\mathcal{G}$ be a Lie groupoid; then the inversion is a diffeomorphism; moreover, it restricts to a diffeomorphism $\boldsymbol{i}\colon \mathcal{G}_x \to \mathcal{G}^x$ for any $x \in \mathcal{G}_0$. Additionally, the unit map is a closed embedding.*

Proof. As the inversion is its own inverse, it is a diffeomorphism. As $\boldsymbol{s}$ and $\boldsymbol{t}$ are submersions, the $\boldsymbol{s}$- and $\boldsymbol{t}$-fibres are automatically embedded submanifolds and thus $\boldsymbol{i}\colon \mathcal{G}_x \to \mathcal{G}$ is a smooth map whose image lies in $\mathcal{G}^x$, and similarly for $\boldsymbol{i}\colon \mathcal{G}^x \to \mathcal{G}$. These restrictions are then each other's inverses, and therefore $\boldsymbol{i}\colon \mathcal{G}_x \to \mathcal{G}^x$ is a diffeomorphism.

To see that the inclusion is an embedding, we remark that it has a left inverse given by the source (or target). Moreover, it is closed as the map $\delta\colon \mathcal{G} \to \mathcal{G} \times \mathcal{G}\colon g \mapsto (g, gg^{-1})$ is continuous and the units can be identified with $\delta^{-1}(\Delta)$, where Δ denotes the diagonal. This diagonal is closed as $\mathcal{G}$ is assumed to be Hausdorff. □

Remark. If we only assume that $\boldsymbol{s}$ is a submersion, then the fact that $\boldsymbol{i}$ is a diffeomorphism and $\boldsymbol{t} = \boldsymbol{s} \circ \boldsymbol{i}$ implies that $\boldsymbol{t}$ is automatically a submersion. Remark that $\boldsymbol{i}$ being a diffeomorphism only depends on itself being smooth. We will use this without mentioning it going forward, as it reduces a lot of arguments. //

Proposition 2.1.9. *A Lie groupoid morphism which is a diffeomorphism is a Lie groupoid isomorphism.*

Proof. Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a Lie groupoid morphism with a smooth inverse ϕ^{-1}. Suppose $(h_1, h_2) \in \mathcal{H}^{(2)}$, then there exist $g_i \in \mathcal{G}$, with $i = 1, 2$, such that $\phi(g_i) = h_i$. Remark that $\phi_0 \circ \boldsymbol{s}(g_1) = \boldsymbol{s}(h_1) = \boldsymbol{t}(h_2) = \phi_0 \circ \boldsymbol{t}(g_2)$. As ϕ_0 is the restriction of ϕ to the units, it is also a diffeomorphism. This implies that $(g_1, g_2) \in \mathcal{G}^{(2)}$ and we can conclude that:

$$\phi^{-1}(h_1h_2) = \phi^{-1}(\phi(g_1)\phi(g_2)) = \phi^{-1}(\phi(g_1g_2)) = g_1g_2 = \phi^{-1}(h_1)\phi^{-1}(g_2).$$

This shows that ϕ^{-1} is indeed a Lie groupoid morphism and ϕ is thus a Lie groupoid isomorphism. □

Lastly, we want to define the notion of a subgroupoid. Where normally, one takes an internal object of the groupoid, we will give a slightly more general definition, as this is the norm in the theory of Lie groupoids.

Definition 2.1.10. A *Lie subgroupoid* of a Lie groupoid $\mathcal{G}$ is a pair $(\mathcal{H}, \iota)$, where $\mathcal{H}$ is Lie groupoid and $\iota\colon \mathcal{H} \to \mathcal{G}$ is a Lie groupoid morphism which is additionally an injective immersion.

In the case that a Lie groupoid is also an embedded submanifold, we automatically see that $\iota_0(\mathcal{H}_0) \subset \mathcal{G}_0$ is also an embedded submanifold. Therefore, the restriction of the structure maps of $\mathcal{G}$ to $\iota(\mathcal{H})$ is smooth. Additionally, using that $\mathcal{H}$ is a Lie groupoid and ι a Lie groupoid morphism, we deduce that the source map is a submersion. Therefore $\iota(\mathcal{H}) \subset \mathcal{G}$ is a Lie groupoid in itself, which is then isomorphic as Lie groupoids to $\mathcal{H}$. Hence, we will identify Lie subgroupoids which are embedded with their images in the total groupoid.

2.1.1 Translations, bisections, isotropy groups and orbits

For Lie groups, much of the rich geometric structure follows from their homogeneous nature, in the sense that the left and right translations define diffeomorphisms of the whole group. In the case of Lie groupoids, given an arrow in a groupoid, say $y \xleftarrow{g} x \in \mathcal{G}$, we do not obtain a diffeomorphism on the Lie groupoid, but only on the source and target fibres:

$$l_g\colon \mathcal{G}^x \to \mathcal{G}^y\colon h \mapsto gh, \quad r_g\colon \mathcal{G}_y \to \mathcal{G}_x\colon h \mapsto hg,$$

which we call the *left* and *right translation maps by g*, respectively. Additionally, we define the *conjugation by g* as $c_g\colon \mathcal{G}_x^x \to \mathcal{G}_y^y\colon h \mapsto ghg^{-1}$.

Similar to Lie groups, cf. [?Duistermaat2000], the first-order approximation of the multiplication is given by the left and right translations.

Proposition 2.1.11. *For $(g, h) \in \mathcal{G}^{(2)}$ with $\boldsymbol{s}(g) = \boldsymbol{t}(h) = x$, $X \in T_g\mathcal{G}_x$ and $Y \in T_h\mathcal{G}^x$ we have*

$$T_{(g,h)}\boldsymbol{m}(X, Y) = T_g r_h(X) + T_h l_g(Y).$$

Proof. Take some $(g, h) \in \mathcal{G}^{(2)}$ with $\boldsymbol{s}(g) = \boldsymbol{t}(h) = x$, $X \in T_g\mathcal{G}_x$ and $Y \in T_h\mathcal{G}^x$. The tangent spaces of the $\boldsymbol{s}$- and $\boldsymbol{t}$-fibres are given by $T_g\mathcal{G}_x = \ker T_g\boldsymbol{s}$ and $T_h\mathcal{G}^x = \ker T_h\boldsymbol{t}$, such that $T_g\boldsymbol{s}(X) = 0 = T_h\boldsymbol{t}(Y)$. Notice that the tangent space of a fibre product is isomorphic to the fibre product of the tangent space, and thus we have the following:

$$(X, Y), (0, Y), (X, 0) \in T_g\mathcal{G}\ {}_{T_g\boldsymbol{s}}\!\times_{T_h\boldsymbol{t}} T_h\mathcal{G} = T_{(g,h)}\mathcal{G}^{(2)}$$

We can conclude that $\in T_{(g,h)}\mathcal{G}^{(2)}$. The following then holds by linearity:

$$T_{(g,h)}\boldsymbol{m}(X, Y) = T_{(g,h)}\boldsymbol{m}(X, 0) + T_{(g,h)}\boldsymbol{m}(0, Y)$$

We will only prove that $T_{(g,h)}\boldsymbol{m}(X, 0) = T_g r_h(X)$, as an analogous proof will give $T_{(g,h)}\boldsymbol{m}(0, Y) = T_h l_g(Y)$, such that the result indeed holds.

Let us pick a path $\phi\colon (-\epsilon, \epsilon) \to \mathcal{G}_x$ such that $\phi(0) = g$ and $\dot{\phi}(0) = X$. We can then extend this path to $\gamma\colon (-\epsilon, \epsilon) \to \mathcal{G}^{(2)}\colon t \mapsto (\phi(t), h)$. Notice that $\dot{\gamma}(0) = (X, 0)$ and:

$$\boldsymbol{m} \circ \gamma\colon (-\epsilon, \epsilon) \to \mathcal{G}\colon t \mapsto \boldsymbol{m}(\gamma(t)) = \boldsymbol{m}(\phi(t), h) = \phi(t)h = r_h(\phi(t)).$$

This implies that $T_{(g,h)}\boldsymbol{m}(X, 0) = T_g r_h(X)$. As remarked before, this proves our result. $\square$

The above proposition implies that at a unit 1_x, the first order differential of the multiplication is simply the restriction of the addition. Explicitly, for some $X \in T_{1_x}\mathcal{G}_x$ and $Y \in T_{1_x}\mathcal{G}^x$, it satisfies

$$T_{(1_x,1_x)}\boldsymbol{m}(X, Y) = X + Y.$$

For a Lie group, this equality implies that the first-order derivative of the multiplication cannot capture any nonabelianess. As for a Lie groupoid, this is slightly more complicated as the converse multiplication of some composable g and h might not even be defined, as $\boldsymbol{s}(h)$ does not need to equal $\boldsymbol{t}(h)$. However, we can restrict to subsets where these expressions do make sense.

Definition 2.1.12. Let $\mathcal{G}$ be a Lie groupoid and $x \in \mathcal{G}_0$. The *isotropy groups* at x is given by $\mathcal{G}_x^x$.

In particular, the isotropy groups define a bundle of groups $p\colon G \to M$, where $G = \bigcup_{x \in M} \mathcal{G}_x^x$ and $p(g) = \boldsymbol{s}(g) = \boldsymbol{t}(g)$. Remark that the fibres of p are indeed groups, where the multiplication is simply the restrictions of gr. However, we can even show that they are Lie groups by proving that they are embedded submanifolds of $\mathcal{G}$. To show this, we will need a little more machinery. The motivation of which comes from the fact that the left and right translations by an arrow are not defined on the whole groupoid. We can try to extend this definition as follows:

A *left-translation* is a pair of maps $L\colon \mathcal{G} \to \mathcal{G}$ and $L_0\colon \mathcal{G}_0 \to \mathcal{G}_0$ such that $\boldsymbol{t} \circ L = L_0 \circ \boldsymbol{t}$, $\boldsymbol{s} \circ L = \boldsymbol{s}$ and on each $\mathcal{G}^x$ there exists an $y \xleftarrow{g} x \in \mathcal{G}$ such that $L|_{\mathcal{G}^x} = l_g$. Hence, this map is not characterised by a single arrow, but by a family of arrows indexed by object. This leads us to the following definition:

Definition 2.1.13. A *local bisection* of a Lie groupoid $\mathcal{G}$ is a map $\sigma\colon U \subset \mathcal{G}_0 \to \mathcal{G}$ such that $\boldsymbol{s}\circ\sigma = \mathrm{id}_U$ and $\boldsymbol{t}\circ\sigma$ is a diffeomorphism. We will call a local bisection defined on the whole of $\mathcal{G}_0$ simply a bisection.

Given a bisection $\sigma\colon \mathcal{G}_0 \to \mathcal{G}$ we can define a left-translation by:

$$L_\sigma\colon \mathcal{G} \to \mathcal{G}\colon g \mapsto \sigma \circ \boldsymbol{t}(g)g, \quad L_{0,\sigma}\colon \mathcal{G}_0 \to \mathcal{G}_0\colon x \mapsto \boldsymbol{t}(\sigma(x))$$

This is an injective mapping, and as composition gives a group structure to the set of left translations, this induces a group structure on the set of bisections.

One may wonder about the existence of such bisections, and much like for submersions, we can always ensure the existence of a local bisection through any arrow.

Proposition 2.1.14. *Let $g \in \mathcal{G}$, then there exists a local bisection σ such that $g \in \operatorname{im}\sigma$.*

Proof. Notice that $\boldsymbol{s}$ and $\boldsymbol{t}$ are submersions and thus $\dim\ker T_g\boldsymbol{s} = \dim\ker T_g\boldsymbol{t} = \dim\mathcal{G} - \dim\mathcal{G}_0$. We now remark that we can find some linear subspace $C \subset T_g\mathcal{G}$ such that

$$T_g\mathcal{G} = \ker T_g\boldsymbol{s} \oplus C = \ker T_g\boldsymbol{t} \oplus C.$$

This follows from some simple linear algebra: Pick a basis $\{u_i\}$ for $\ker T_g\boldsymbol{s} \cap \ker T_g\boldsymbol{t}$ and extend these to a basis $\{u_i, v_j\}$ of $\ker T_g\boldsymbol{s}$ and $\left\{u_i, v_j'\right\}$ of $\ker T_g\boldsymbol{t}$, remark that the dimensions are equal and therefore their bases have the same size. Moreover, the union $\left\{v_j, v_j'\right\}$ is linearly independent. We then remark that we can extend the basis $\left\{u_i, v_j, v_j'\right\}$ to a basis of V, say $\left\{u_i, v_j, v_j', w_k\right\}$. Then consider $C = \left\langle v_j + v_j', w_k\right\rangle_{\mathbb{R}}$.

We can then consider an embedded submanifold $S \ni g$ such that $T_gS = C$. Remark that if we restrict $\boldsymbol{s}$ and $\boldsymbol{t}$ to S, they are a submersion at g. Therefore, we can pick an open neighbourhood of $g \in S$ such that they are of full rank everywhere. However, by counting dimensions, we see that they must be local diffeomorphisms. We can then restrict to a neighbourhood U such that $\boldsymbol{s}\colon U \to \boldsymbol{s}(U)$ is a diffeomorphism. The inverse of $\boldsymbol{s}|_U$ will then be a local section attaining the value g. $\square$

From the existence of local sections, we can find the following interesting results on the restrictions of the target map to source fibres.

Proposition 2.1.15. *Let $x \in \mathcal{G}_0$, then $\boldsymbol{t}|_{\mathcal{G}_x}\colon \mathcal{G}_x \to \mathcal{G}_0$ has constant rank.*

Proof. To show that $\boldsymbol{t}|_{\mathcal{G}_x}$ has constant rank, we will relate its differential at some $g, h \in \mathcal{G}_x$ by using a translation via a bisection. By Proposition **??**, we can find a local bisection $\sigma\colon U \to \mathcal{G}$ attaining gh^{-1}, and let us denote $V = (\boldsymbol{t}\circ\sigma)(U)$. Notice that V is diffeomorphic to U as $\boldsymbol{t}\circ\sigma$ is a diffeomorphism. Consider the left-translation induced by this local bisection, defined as

$$L_\sigma\colon \mathcal{G}^U \to \mathcal{G}^V\colon g' \mapsto \sigma \circ \boldsymbol{t}(g')g'.$$

Notice that this left translation satisfies $L_\sigma(h) = g$. We want to show that this is a diffeomorphism and then let it translate the tangent map of the target map from g to h. One can verify that the left translation with the following local bisection defines an inverse.

$$\sigma' \colon V \to \mathcal{G} \colon x \mapsto (\boldsymbol{i} \circ \sigma(\boldsymbol{t} \circ \sigma)^{-1})(x) = \sigma((\boldsymbol{t} \circ \sigma)^{-1}(x))^{-1}$$

By a calculation, one can verify that indeed $L_{\sigma'}$ is the inverse of L_σ. Moreover, we notice that L_σ induces the following commutative diagram:

$$\begin{tikzcd} \mathcal{G}_x^U \arrow[r, "L_\sigma"] \arrow[d, "\boldsymbol{t}|_{\mathcal{G}_x^U}"'] & \mathcal{G}_x^V \arrow[d, "\boldsymbol{t}|_{\mathcal{G}_x^V}"] \\ U \arrow[r, "\boldsymbol{t}\circ\sigma"'] & V \end{tikzcd}$$

As $\mathcal{G}_x^U$ and $\mathcal{G}_x^V$ are open neighbourhoods of g and h in $\mathcal{G}_x$ respectively, we can conclude that

$$T_h \boldsymbol{t}|_{\mathcal{G}_x} \circ T_g L_\sigma|_{\mathcal{G}_x} = T_{\boldsymbol{t}(g)}(\boldsymbol{t} \circ \sigma) \circ T_g \boldsymbol{t}|_{\mathcal{G}_x}$$

As L_σ and $\boldsymbol{t} \circ \sigma$ are diffeomorphisms, we can conclude that $\boldsymbol{t}$ has constant rank. □

Corollary 2.1.16. *For any $x, y \in \mathcal{G}_0$, the set $\mathcal{G}_x^y \subset \mathcal{G}$ is embedded, and the isotropy groups are Lie groups.*

Corollary 2.1.17. *The subset $\mathscr{O}_x = \boldsymbol{t}(\mathcal{G}_x)$ is an immersed submanifold of $\mathcal{G}_0$.*

For $x \in \mathcal{G}_0$, we call the set $\mathscr{O}_x = \boldsymbol{t}(\mathcal{G}_x)$ the *orbit of x*, and we will denote $\mathcal{G}_0/\mathcal{G}$ for the set of orbits. Notice that the partition into orbits defines an equivalence relation on $\mathcal{G}_0$, which is exactly given by the image of $(\boldsymbol{t}, \boldsymbol{s}) \colon \mathcal{G} \to \mathcal{G}_0 \times \mathcal{G}_0$.

This partition of $\mathcal{G}_0$ into the orbits carries some important information on $\mathcal{G}$, which will, in particular, be invariant under Lie groupoid isomorphisms. Therefore, we let $(\boldsymbol{t}, \boldsymbol{s})$ dictate some properties of $\mathcal{G}$.

Definition 2.1.18. A Lie groupoid $\mathcal{G}$ is called *proper/transitive*, if $(\boldsymbol{t}, \boldsymbol{s})$ is a proper/ surjective map, respectively.

Proposition 2.1.19. *Let $\mathcal{G}$ be a Lie groupoid and denote $f = (\boldsymbol{t}, \boldsymbol{s}) \colon \mathcal{G} \to \mathcal{G}_0 \times \mathcal{G}_0$. For any $x \in \mathcal{G}_0$ and $g \in f^{-1}(x)$ we have $T_g f^{-1}(x) = \ker T_g f$. In particular, if f is injective, then it is an immersion.*

Proof. Let $\mathcal{G}$ be a Lie groupoid, and denote $f = (\boldsymbol{t}, \boldsymbol{s})$. Pick some $y \xleftarrow{g} x \in \mathcal{G}$ and $v \in \ker T_g f$, which implies that $v \in \ker T_g \boldsymbol{t} \cap \ker T_g \boldsymbol{s}$. As $\boldsymbol{t}|_{\mathcal{G}_x}$ as constant rank, it follows that

$$T_g \mathcal{G}_x^y = T_g \boldsymbol{t}^{-1}(y) = \ker T_g \boldsymbol{t}|_{\mathcal{G}_x} = \ker T_g \boldsymbol{t} \cap \ker T_g \boldsymbol{s}.$$

Thus $v \in T_g \mathcal{G}_x^y$. Hence, if f is injective, it follows that $T_g \mathcal{G}_x^y = T_g \{g\} = \{0\}$. Therefore $T_g f$ is injective, and thus it is an immersion. □

2.1.2 Properties on fibres

Lastly, before we move to examples and constructions of Lie groupoids, we will go over a localised version of imposing topological restrictions on our groupoids by only imposing them on the source and, thus, target fibres.

Definition 2.1.20. Given a diffeomorphism invariant property P of manifolds, we will say that a Lie groupoid $\mathcal{G}$ is $\boldsymbol{s}$-P if each $\boldsymbol{s}$ fibre has property P. A similar definition holds for $\boldsymbol{t}$-P.

As inversion is a diffeomorphism between $\boldsymbol{s}$- and $\boldsymbol{t}$-fibres, a Lie groupoid is $\boldsymbol{s}$-P if and only if it is $\boldsymbol{t}$-P. Notice that the Lie groupoid itself having property P, does not necessarily imply that it is $\boldsymbol{s}$-P or vice versa. These notions do coincide for Lie groups, as there the source fibre is the whole group. The idea is that many statements holding for Lie groups with property P should hold for $\boldsymbol{s}$-P groupoids. An important example of this is the following proposition.

Proposition 2.1.21. *Let $\mathcal{G}$ be a $\boldsymbol{s}$-connected Lie groupoid, then any open neighbourhood U of $\mathcal{G}_0 \subset \mathcal{G}$ generates the whole groupoid in the following sense: For any $g \in \mathcal{G}$, there exist a finite collection $\{u_i\}_{i=1}^n \subset U$ such that $g = u_1 u_2 \cdots u_n$.*

Proof. Let $\mathcal{G}$ be a $\boldsymbol{s}$-connected Lie groupoid and $U \subset \mathcal{G}$ and open neighbourhood of $\mathcal{G}_0$. Remark that we can assume that $\boldsymbol{i}(U) = U$, as we can always consider $\boldsymbol{i}(U) \cap U$, which will contain $\mathcal{G}_0 = \boldsymbol{i}(\mathcal{G}_0)$. Let us denote $\langle U \rangle$ for the following set

$$\langle U \rangle = \left\{ u_1 \cdots u_n \in \mathcal{G} | \, n \in \mathbb{N}, \; u_i \in U, \; (u_1, \dots, u_n) \in \mathcal{G}^{(n)} \right\}.$$

We will show that $\langle U \rangle_x = \langle U \rangle \cap \mathcal{G}_x$ is clopen for any $x \in \mathcal{G}_0$, which, combined with the fact that the $\boldsymbol{s}$-fibres are connected, implies that $\langle U \rangle_x = \mathcal{G}_x$ and thus $\langle U \rangle = \mathcal{G}$.

To see that it is open, define $U_x^n = \left\{ u_1 \cdots u_n | \, u_i \in U, \; (u_1, \dots, u_n) \in \mathcal{G}^{(n)}, \; \boldsymbol{s}(u_1) = x \right\}$ and notice that $\langle U \rangle_x = \bigcup_{n=1}^{\infty} U_x^n$. We can rewrite the set U_x^n inductively using the right translation of the Lie groupoid as

$$U_x^n = \bigcup_{g \in U_x^1} r_g \left(U_{\boldsymbol{t}(g)}^{n-1} \right) \subset \mathcal{G}_x$$

As for the case $n = 1$, we recover $U_x^1 = U \cap \mathcal{G}_x$, which is open in the subspace topology of $\mathcal{G}_x$, we can conclude that U_x^n is open for any n and thus $\langle U \rangle_x$ is open.

Next, remark that if $g \in \mathcal{G}_x \setminus \langle U \rangle_x$, then $r_g(U \cap \mathcal{G}_{\boldsymbol{t}(g)})$ is open in $\mathcal{G}_x$ and it is disjoint of $\langle U \rangle_x$. This shows that $\langle U \rangle_x$ is a closed subset of $\mathcal{G}_x$.

We conclude that $\langle U \rangle_x$ is clopen in $\mathcal{G}_x$, and using the $\boldsymbol{s}$-connectedness it follows that $\langle U \rangle_x = \mathcal{G}_x$. This implies that $\langle U \rangle = \mathcal{G}$ and thus any g can be written as the product of elements in U. □

Remark. Notice that this proof only relies on the topological properties of the Lie groupoid. //

Proposition 2.1.22. *Let $\mathcal{G}$ be a Lie groupoid, and $\mathcal{H}$ a subgroupoid (notice this is a priori not a Lie groupoid). If $\mathcal{H}$ is an embedded submanifold of $\mathcal{G}$, and it is $\boldsymbol{s}$-connected, then $\mathcal{H}$ is a Lie subgroupoid.*

The proof of this proposition uses the following somewhat unusual lemma.

Lemma 2.1.23 ([?Kolar1993, Thm. 1.13])**.** *If $f \colon M \to M$ is a smooth map such that $f \circ f = f$, then $\operatorname{im} f \subset M$ is embedded. Moreover, there exists an open neighbourhood $U \subset M$ of $f(M)$ such that $f \colon U \to f(M)$ is a submersion.*

Proof of Proposition **??**. Remark that $\boldsymbol{u}\circ\boldsymbol{s}\colon \mathcal{G}\to\mathcal{G}$ restricts to a smooth map $\mathcal{H}\to\mathcal{H}$ such that $\boldsymbol{u}\circ\boldsymbol{s}\circ\boldsymbol{u}\circ\boldsymbol{s} = \boldsymbol{u}\circ\boldsymbol{s}$. By Lemma **??**, it follows that $\operatorname{im}\boldsymbol{u}\circ\boldsymbol{s}$ is an embedded submanifold of $\mathcal{H}$, however, this is exactly the set of units $\mathcal{H}_0\subset\mathcal{H}$. The structure maps of $\mathcal{H}$, except for the multiplication, will then be automatically smooth as they are the restriction of smooth maps. Let us denote the structure maps of $\mathcal{H}$ with a tilde.

Lemma **??** additionally provides an open neighbourhood $U\subset\mathcal{H}$ of $\mathcal{H}_0$ such that $\widetilde{\boldsymbol{u}}\circ\widetilde{\boldsymbol{s}}|_U$ is a submersion. As $\boldsymbol{u}$ is an embedding, it follows that $\widetilde{\boldsymbol{s}}|_U$ is a submersion as well. Remark that by possibly shrinking U, we find that $\widetilde{\boldsymbol{t}}|_U$ is a submersion and $\boldsymbol{i}(U)=U$. It follows that the restriction of the multiplication map of $\mathcal{H}$ to U in the right components i.e. the following map:

$$\mathcal{H}\,{}_{\widetilde{\boldsymbol{s}}}\!\times_{\widetilde{\boldsymbol{t}}} U\to\mathcal{H}\colon (g,k)\mapsto gk,$$

is still smooth. Therefore, the restriction of the right translation for some $k\in U$ is also smooth and by the choice of U it is a diffeomorphism. This results in an isomorphism of tangent spaces for any $(g,k)\in\mathcal{H}\,{}_{\widetilde{\boldsymbol{s}}}\!\times_{\widetilde{\boldsymbol{t}}} U$:

$$T_{gk}r_{k^{-1}}\colon\ \ker T_{gk}\widetilde{\boldsymbol{s}}\to\ker T_g\widetilde{\boldsymbol{s}},$$

where we identify $T_{gk}\mathcal{H}_{\boldsymbol{s}(k)} = \ker T_{gk}\widetilde{\boldsymbol{s}}$ and $T_g\mathcal{H}_{\boldsymbol{t}(k)} = \ker T_g\widetilde{\boldsymbol{s}}$. Using Proposition **??**, we find that $\mathcal{H}$ is generated by U. From this, we can conclude that $\widetilde{\boldsymbol{s}}$ is a submersion at all points of $\mathcal{H}$. As mentioned before, this automatically implies that $\boldsymbol{t}$ is a submersion as well. This then implies that $\widetilde{\boldsymbol{m}}\colon \mathcal{H}\,{}_{\widetilde{\boldsymbol{s}}}\!\times_{\widetilde{\boldsymbol{t}}}\mathcal{H}\to\mathcal{H}$ is smooth as well, such that we can conclude that $\mathcal{H}$ is a Lie subgroupoid. $\square$

Without the assumption that $\mathcal{H}$ is $\boldsymbol{t}$-connected, this result may fail to hold as seen from the following example.

Example 2.1.24. Consider $\mathbb{R}\times\mathbb{R}\rightrightarrows\mathbb{R}$, where the source and target maps are $\boldsymbol{s}(x,y)=\boldsymbol{t}(x,y)=x$ and the multiplication map is the addition on the second component, such that

$$(x,y)(x,z)=(x,y+z),\quad 1_x=(x,0),\quad (x,y)^{-1}=(x,-y).$$

In other words, it is a bundle of groups over $\mathbb{R}$, where the group is $(\mathbb{R},+)$. Consider a map $f\colon\mathbb{R}\to\mathbb{R}\colon\ \mapsto \psi(x)x^{1/3}+1$, where $\psi\colon\mathbb{R}\to\mathbb{R}$ is some bump function with support in $[-1,1]$ such that $\psi|_{(-\epsilon,\epsilon)}\equiv 1$ for some $\epsilon>0$. Remark that the graph of this map is a submanifold of $\mathbb{R}^2$ and that the tangent space at $(0,f(0))$ is given by $\left\langle\frac{\partial}{\partial y}\right\rangle$. We can define a set-theoretical subgroupoid of $\mathcal{G}$ by

$$\mathcal{H}=\{(x,kf(x))\colon\ x\in\mathbb{R},\ k\in\mathbb{Z}\}.$$

However, this does not define a Lie groupoid as the source and target maps are not submersions at $(0,0)$. //

In particular, this lets us show that any Lie groupoid contains some $\boldsymbol{s}$-connected subgroupoid.

Corollary 2.1.25. *The set $\mathcal{G}^0=\bigcup_{x\in\mathcal{G}_0}C(\mathcal{G}_x,1_x)$, where $C(\mathcal{G}_x,1_x)$ denotes the connected component of 1_x in $\mathcal{G}_x$, is an $\boldsymbol{s}$-connected Lie subgroupoid of $\mathcal{G}$.*

Proof. Let $\mathcal{G}$ be a Lie groupoid, we remark that $\mathcal{G}^0$ can be determined using the source-foliation of $\mathcal{G}$, denoted $\mathcal{F}_{\boldsymbol{s}}$, as

$$\mathcal{G}^0=\cup_{L\in\mathcal{F}_{\boldsymbol{s}},\,L\cap\mathcal{G}_0\neq\emptyset}L.$$

Notice that $\mathcal{G}_0$ is a transversal to $\mathcal{F}_{\boldsymbol{s}}$ and thus its saturation, given by $\mathcal{G}^0$, is open. Therefore, it is an embedded submanifold of $\mathcal{G}$ and by construction it is $\boldsymbol{s}$-connected.

Lastly, we need to check that it is indeed a subgroupoid, such that it is closed under multiplication and inversion. For the multiplication, we remark that for $y \xleftarrow{g} x \in \mathcal{G}^0$, the right translation defines a diffeomorphism $r_g\colon \mathcal{G}_y \to \mathcal{G}_x$ and it in particular maps connected components to connected components. As $r_g(1_y) = g$, it will indeed map $C(\mathcal{G}_y, 1_y)$ to $C(\mathcal{G}_x, 1_x)$ and thus the multiplication restricts to a map on $\mathcal{G}^0$. As for the inversion, we notice that $r_g(g^{-1}) = 1_x$ such that g^{-1} is in $C(\mathcal{G}_y, 1_y)$ by a similar argument. Clearly, it also contains all the units.

It now follows from Proposition **??** that $\mathcal{G}^0$ is indeed a Lie subgroupoid of $\mathcal{G}$. □

2.2 Examples of (Lie) groupoids

Let us now discuss a series of examples of (Lie) groupoids. Notice that many of the following constructions could be performed in the nonsmooth or topological case, leading to weaker versions of groupoids like topological groupoids.

Example 2.2.1. Given a Lie group G, we can interpret it as a Lie groupoid

$$\begin{array}{c} G \\ \boldsymbol{t}\Big\downarrow\Big\downarrow\boldsymbol{s} \\ \{*\} \end{array} \qquad \text{(arrows: } * \xleftarrow{g} *\text{)}$$

Conversely, any Lie groupoid where the base manifold is a point is a Lie group. In this setting, where the object space is a point, Lie group homomorphisms are exactly the same as morphisms of Lie groupoids. //

Example 2.2.2. Given a manifold M, we can consider the *pair groupoid*

$$\begin{array}{c} M \times M \\ \mathrm{pr}_1\Big\downarrow\Big\downarrow\mathrm{pr}_2 \\ M \end{array} \qquad \text{(arrows: } y \xleftarrow{(x,y)} x\text{)}$$

where the source map is $\boldsymbol{s} = \mathrm{pr}_2$ and the target map $\boldsymbol{t} = \mathrm{pr}_1$. As this is the Lie groupoid with a single arrow between any two objects, the multiplication, units and inverses are uniquely determined by the source and target relations they satisfy. A map of manifolds induces a Lie groupoid morphism on the pair groupoids, and Pair is therefore a functor. Remark that any Lie groupoid $\mathcal{G} \rightrightarrows M$ admits a Lie groupoids morphism to the pair groupoid of its object set, given by $(\boldsymbol{t}, \boldsymbol{s})\colon \mathcal{G} \to \mathrm{Pair}(M)$.

If we additionally have a submersion $\mu\colon M \to N$, we can define the *submersion groupoid* as fibre product $M\, {}_\mu\!\times_\mu M$:

$$\begin{array}{c} M\, {}_\mu\!\times_\mu M \\ \mathrm{pr}_1\Big\downarrow\Big\downarrow\mathrm{pr}_2 \\ M \end{array} \qquad \text{(arrows: } y \xleftarrow{(x,y)} x \text{ if } \mu(y) = \mu(x)\text{)}$$

It is not hard to see that this is a Lie subgroupoid of the pair groupoid. In particular, if $\mu = \mathrm{id}_M$, then $M\, {}_\mu\!\times_\mu M = M$ and we will call it the *identity groupoid*. //

Example 2.2.3. If G is a Lie group with a left action on M, denoted by $\alpha\colon G \times M \to M$, then define $G \ltimes M$, called the *action groupoid*:

$$\begin{tikzcd} G \times M \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\mathrm{pr}_1"] \\ M \end{tikzcd} \qquad \text{(arrows: } gx \xleftarrow{(g,x)} x\text{)}$$

where the structure maps are as follows:

$$\boldsymbol{s}(g,m) = m, \quad \boldsymbol{t}(g,m) = gm, \quad \mathrm{gr}((g,m),(h,n)) = (gh,n).$$

This groupoid encapsulates various properties of the action. Specifically, the action is free, transitive, or proper if and only if $(\boldsymbol{t}, \boldsymbol{s})$ is injective, surjective, or proper, respectively. Furthermore, the isotropy groups of the action groupoid correspond precisely to the stabilisers of the action, while its orbits are exactly the orbits of the action.

Remark that for right actions a similar construction exists, which we denote by $M \rtimes G$. //

Example 2.2.4. Let $\pi \colon P \to M$ be a principal G-bundle, then consider the diagonal action G on $P \times P$, explicitly given by $(p,q)g = (pg, qg)$. As the action on P is free and proper, so is this action. The quotient of the product by the diagonal action is then a well-defined manifold, and $\overline{\pi} \colon P \times P \to (P \times P)/G$ is a G-invariant surjective submersion. Remark that $\pi \circ \mathrm{pr}_1$ and $\pi \circ \mathrm{pr}_2$ are constant on the fibres of $\overline{\pi}$ and thus they descend to the quotient, denoted by $\overline{\boldsymbol{t}}$ and $\overline{\boldsymbol{s}}$ respectively. Moreover, as they are surjective submersions, so is the map on the quotient. We then obtain the *Gauge groupoid*, denoted by P:

$$\begin{tikzcd} (P \times P)/G \arrow[d, shift right, "\overline{\boldsymbol{t}}"'] \arrow[d, shift left, "\overline{\boldsymbol{s}}"] \\ M \end{tikzcd} \qquad \text{(arrows: } \pi(p) \xleftarrow{[p,q]} \pi(q)\text{)},$$

The source and target are explicitly given by $\overline{\boldsymbol{s}}\,[p,q] = \pi(q)$ and $\overline{\boldsymbol{t}}\,[p,q] = \pi(p)$. For the multiplication, we remark that a principal G-bundle comes with a diffeomorphism:

$$P \times G \to P \times P \colon (p,g) \mapsto (p, pg),$$

whose inverse is written as

$$P\,{}_{\pi}\!\times_{\pi} P \to P \times G \colon (p,q) \mapsto (p, [p:q]).$$

Where $[p:q] \in G$ is the unique element such that $[p\colon q]\,q = p$. Hence, if $([p,q], [p',q']) \in P^{(2)}$, then $p' = [p'\colon q]\,q$ and therefore we can define the multiplication as:

$$\boldsymbol{m} \colon P^{(2)} \to P \colon ([p,q], [p',q']) \to \left[[p'\colon q]\,p, q'\right].$$

It is an easy check that this is indeed independent of the choice of representative.

An important property of gauge groupoids is that they are transitive, and they classify all transitive groupoids up to isomorphism. //

Example 2.2.5. Let $\mathcal{G}$ be a Lie groupoid, we define $\mathcal{G}^{\mathrm{op}}$ for the groupoid whose objects are given by $\mathcal{G}_0$ and arrows by $\mathcal{G}$. The structure maps are given by:

$$\boldsymbol{s}_{\mathrm{op}}(g) = \boldsymbol{t}(g), \quad \boldsymbol{t}_{\mathrm{op}}(g) = \boldsymbol{s}(g),$$

$$\boldsymbol{m}_{\mathrm{op}} \colon \mathcal{G}\,{}_{\boldsymbol{s}_{\mathrm{op}}}\!\times_{\boldsymbol{t}_{\mathrm{op}}} \mathcal{G} = \mathcal{G}\,{}_{\boldsymbol{t}}\!\times_{\boldsymbol{s}} \mathcal{G} \to \mathcal{G} \colon (h,g) \mapsto \boldsymbol{m}(g,h), \quad \boldsymbol{u}_{\mathrm{op}} = \boldsymbol{u}, \quad \boldsymbol{i}_{\mathrm{op}} = \boldsymbol{i}.$$

Heuristically, this corresponds to reversing all the arrows in de groupoid; therefore, it is called the *opposite groupoid* of $\mathcal{G}$. Notice that $\boldsymbol{i}\colon \mathcal{G} \to \mathcal{G}^{\text{op}}$ defines a Lie groupoid isomorphism. //

Example 2.2.6. Take a Lie groupoid $\mathcal{G}$ and a manifold M, then $C^\infty(M, \mathcal{G})$ is a groupoid over $C^\infty(M, \mathcal{G}_0)$, where the structure maps are pointwise, i.e. for $F, G \in C^\infty(M, \mathcal{G})$:

$$(\boldsymbol{s}(F))(x) = \boldsymbol{s}(F(x)), \quad (\boldsymbol{t}(F))(x) = \boldsymbol{t}(F(x)), \quad \boldsymbol{m}(F, G)(x) = F(x)G(x).$$

Similarly, the unit and the inversion are pointwise. Notice that we can identify the composable arrows of $(C^\infty(M, \mathcal{G}))$ with $C^\infty(M, \mathcal{G}^{(2)})$. This identification is given by sending a pair $(F, G) \in (C^\infty(M, \mathcal{G}))^{(2)}$ to the map $(F, G)\colon M \to \mathcal{G} \times \mathcal{G}\colon x \mapsto (F(x), G(x))$. Notice that

$$\boldsymbol{s}(F(x)) = (\boldsymbol{s}(F))(x) = (\boldsymbol{t}(G))(x) = \boldsymbol{t}(G(x)),$$

such that $\operatorname{im}(F, G) \subset \mathcal{G}^{(2)}$.

Given $\phi\colon \mathcal{G} \to \mathcal{H}$ a Lie groupoid morphism, we obtain an induced groupoid morphism, called the push-forward, given by:

$$\phi_*\colon C^\infty(M, \mathcal{G}) \to C^\infty(M, \mathcal{H})\colon F \mapsto \phi \circ F.$$

Dually, if we start with a map $f\colon M \to N$, then we obtain the pullback:

$$f^*\colon C^\infty(M, \mathcal{G}) \to C^\infty(N, \mathcal{G})\colon F \mapsto F \circ f.$$

This also defines a groupoid morphism. //

Example 2.2.7. Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids, then $\mathcal{G} \times \mathcal{H}$ has the structure of a Lie groupoid over $\mathcal{G}_0 \times \mathcal{H}_0$ with the component-wise structure maps. This groupoid is called the *product groupoid*.

The product of two groupoids comes with the usual universal property of products and thus also with canonical projection maps $\pi_{\mathcal{G}}\colon \mathcal{G} \times \mathcal{H} \to \mathcal{G}\colon (g, h) \mapsto g$ and $\pi_{\mathcal{H}}\colon \mathcal{G} \times \mathcal{H} \to \mathcal{H}\colon (g, h) \mapsto h$, which are Lie groupoid morphism. Moreover, for any choice $x \in \mathcal{H}_0$ we get a Lie groupoid morphism which is the inclusion at x, defined by $\iota_x\colon \mathcal{G} \to \mathcal{G} \times \mathcal{H}\colon g \mapsto (g, 1_x)$. A similar inclusion exists for $x \in \mathcal{G}_0$. //

One useful application of the product groupoid is that it measures whether a smooth map is a Lie groupoid morphism in the following sense.

Lemma 2.2.8. *For Lie groupoids $\mathcal{G}$ and $\mathcal{H}$ and a smooth map $\phi\colon \mathcal{G} \to \mathcal{H}$, the following are equivalent:*

i) *ϕ is a Lie groupoid morphism,*

ii) $\operatorname{gr}(\phi) \rightrightarrows \operatorname{gr}(\phi_0)$ *is a Lie subgroupoid of $\mathcal{G} \times \mathcal{H}$.*

Proof. Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids, and suppose that $\phi\colon \mathcal{G} \to \mathcal{H}$ is a map between them.

i) $\Longrightarrow$ ii): Suppose ϕ is a Lie groupoid morphism. As it is a smooth map, its graph is a closed embedded submanifold of $\mathcal{G} \times \mathcal{H}$. Moreover, if $((g, \phi(g)), (h, \phi(h))) \in (\mathcal{G} \times \mathcal{H})^{(2)}$, then

$$(g, \phi(g))(h, \phi(h)) = (gh, \phi(g)\phi(h)) = (gh, \phi(gh)) \in \operatorname{gr} \phi,$$

and

$$(g, \phi(g))^{-1} = (g^{-1}, \phi(g)^{-1}) = (g^{-1}, \phi(g^{-1})) \in \operatorname{gr} \phi.$$

This implies that it is closed under multiplication and inversion. Additionally, if $(x, \phi_0(x)) \in \mathrm{gr}(\phi_0)$, then the unit satisfies $(1_x, 1_{\phi_0(x)}) = (1_x, \phi(1_x)) \in \mathrm{gr}(\phi)$. Moreover, we can check that $\boldsymbol{s}$ and $\boldsymbol{t}$ map into $\mathrm{gr}(\phi_0)$. We conclude that $\mathrm{gr}(\phi) \rightrightarrows \mathrm{gr}(\phi_0)$ is a subgroupoid of $\mathcal{G} \times \mathcal{H}$.

Next, we remark that if $(u, v) \in T_{(x,y)} \operatorname{gr} \phi_0$, then $v = T\phi_0(u)$ and we can find a path $\gamma\colon (-\epsilon, \epsilon) \to \mathcal{G}_0$ such that $\dot{\gamma}(0) = u$. Additionally, suppose that $(g, h) \in \operatorname{gr} \phi$ such that $\boldsymbol{s}(g, h) = (x, y)$, we can then pick a section $\sigma\colon U \subset \mathcal{G}_0 \to \mathcal{G}$ of $\boldsymbol{s}$ such that $\sigma(x) = g$. Set $\widetilde{\gamma}(t) = \sigma(\gamma(t))$, then it follows that $T\boldsymbol{s}(\dot{\widetilde{\gamma}}(0)) = u$ and $T\boldsymbol{s}(T\psi(\dot{\widetilde{\gamma}}(0))) = T\psi_0(T\boldsymbol{s}(\dot{\widetilde{\gamma}}(0))) = T\psi_0(u) = v$. Therefore, $\boldsymbol{s}$ is a submersion and $\operatorname{gr} \phi$ is a Lie subgroupoid of $\mathcal{G} \times \mathcal{H}$.

ii) $\implies$ i) Suppose that $\operatorname{gr} \phi$ is a Lie subgroupoid, then for any $g, h \in \mathcal{G}$ we remark that

$$(gh, \phi(g)\phi(h)) = (g, \phi(g))(h, \phi(h)) \in \operatorname{gr} \phi$$

This implies that $\phi(gh) = \phi(g)\phi(h)$ and thus ϕ is a Lie groupoid morphism. □

Example 2.2.9. Let $\mathcal{G}$ and $\mathcal{H}$ be two Lie groupoids such that $\dim \mathcal{G} = \dim \mathcal{H}$ and $\dim \mathcal{G}_0 = \dim \mathcal{H}_0$. Define $\mathcal{G} \coprod \mathcal{H}$ as the Lie groupoid over $\mathcal{G}_0 \coprod \mathcal{H}_0$ where the structure maps are induced by the disjoint union. //

Example 2.2.10. Let $\mathcal{G}$ be a Lie groupoid, then $T\mathcal{G}$ defines a Lie groupoid over $T\mathcal{G}_0$ where the structure maps are the tangent maps of the original structure maps. This groupoid is called the *tangent groupoid* //

2.2.1 Constructions via clean intersections

Besides some general examples and basic constructions of Lie groupoids and groupoids, we now want to work on some of the more involved constructions using more of the geometrical data. In particular, the main goal of this section is to describe the fibre product and pullback construction of Lie groupoids. However, as we are working in the smooth setting, these will not always exist. To fix this, we will introduce the concept of a clean intersection and show that this is enough to ensure the smoothness of the structure maps and the submersiveness of the source and target. A great summary of the application of these results for Lie groupoids can be found in [?Meinrenken2017, Section 4.9], but the core results, that on fibred products, stem from [?Bursztyn2016].

Before discussing the geometrical situation, we go over the constructions for set-theoretical groupoids. As groupoids are algebraic objects, the category **Grpd** is rather well-behaved, in the sense that it is closed under many constructions.

Example 2.2.11. Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a groupoid morphism which covers an injective map; then $\operatorname{im} \phi$ defines a groupoid. Remark that if ϕ does not cover an injective map, then this may not be the case, as $\operatorname{im} \phi$ might not be closed under multiplication. For example, we can consider the groupoid $I = \{1_x, 1_y, g, g^{-1}\}$ over x, y where $y \xleftarrow{g} x$. Then, consider the map $\phi\colon I \to \mathbb{Z}$ which maps $\phi(g) = 1$ and $\phi(g^{-1}) = -1$. This defines a Lie groupoid morphism whose image is $\{0, \pm 1\}'$, which is not a subgroup of $\mathbb{Z}$. //

Example 2.2.12. Let $\mathcal{H}, \mathcal{H}' \subset \mathcal{G}$ be subgroupoids, then $\mathcal{H} \cap \mathcal{H}' \rightrightarrows \mathcal{H}_0 \cap \mathcal{H}'_0$ is a subgroupoid of $\mathcal{G}$. //

Example 2.2.13. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids, $\phi\colon \mathcal{G} \to \mathcal{H}$ a groupoid morphism, and take some subgroupoid $\mathcal{H}' \subset \mathcal{H}$. The inverse image $\phi^{-1}(\mathcal{H}')$ defines a subgroupoid of $\mathcal{G}$. In particular, the *kernel* of a groupoid morphism is defined as $\ker \phi = \phi^{-1}(\boldsymbol{u}(\mathcal{H}_0))$. //

Example 2.2.14. For a groupoid $\mathcal{G}$ and a function $f\colon X \to \mathcal{G}_0$ we define the *pullback groupoid*, denoted $f^!\mathcal{G}$ as the groupoid:

$$\begin{array}{c} X\ {}_f{\times}_{\boldsymbol{t}}\mathcal{G}\ {}_{\boldsymbol{s}}{\times}_f X \\ {\scriptstyle \mathrm{pr}_1}\downdownarrows{\scriptstyle \mathrm{pr}_2} \\ X \end{array} \qquad \text{(arrows: } y \xleftarrow{(y,g,x)} x \text{ where } \boldsymbol{t}(g) = f(y), \quad \boldsymbol{s}(g) = f(x))$$

Hence, the source and target maps are the projections. Meanwhile, the multiplication is simply the multiplication of the arrows, while adjoining the correct source and target:

$$(z, g, y)(y, h, x) = (z, gh, x)$$

The units are given by $1_x = (x, 1_{f(x)}, x)$ and the inversion becomes $(y, g, x)^{-1} = (x, g^{-1}, y)$. The pullback groupoid lets us make a base change to a different object set. //

Example 2.2.15. If $\phi\colon \mathcal{G} \to \mathcal{K}$ and $\psi\colon \mathcal{H} \to \mathcal{K}$ are groupoid morphisms, then we can define the *fibred product groupoid*, denoted by $\mathcal{G}\ {}_\phi{\times}_\psi\mathcal{H}$ as

$$\begin{array}{c} \mathcal{G}\ {}_\phi{\times}_\psi\mathcal{H} \\ {\scriptstyle \boldsymbol{t}\times\boldsymbol{t}}\downdownarrows{\scriptstyle \boldsymbol{s}\times\boldsymbol{s}} \\ \mathcal{G}_0\ {}_{\phi_0}{\times}_{\psi_0}\mathcal{H}_0 \end{array} \qquad \text{(arrows: } (y, n) \xleftarrow{(g,h)} (x, m) \text{ where } y \xleftarrow{g} x,\ n \xleftarrow{h} m)$$

Here, the structure maps are induced by the inclusion $\mathcal{G}\ {}_\phi{\times}_\psi\mathcal{H} \subset \mathcal{G} \times \mathcal{H}$, such that it becomes a subgroupoid.

We can remark that the fibred product is a generalisation of all three previous constructions:

- Let $\mathcal{H}, \mathcal{H}' \subset \mathcal{G}$ be subgroupoids and denote $\iota\colon \mathcal{H} \to \mathcal{G}$ and $\iota'\colon \mathcal{H}' \to \mathcal{G}$ be their inclusions, then $\mathcal{H} \cap \mathcal{H}' \cong \mathcal{H}\ {}_\iota{\times}_{\iota'}\mathcal{H}'$.
- Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids, $\phi\colon \mathcal{G} \to \mathcal{H}$ a groupoid morphism, and take some subgroupoid $\mathcal{H}' \subset \mathcal{H}$. The inverse of $\mathcal{H}'$ under ϕ is isomorphic to $\mathcal{G}\ {}_\phi{\times}_\iota\mathcal{H}'$ as groupoids, where $\iota\colon \mathcal{H}' \to \mathcal{H}$ is the inclusion.
- Let $\mathcal{G}$ be a groupoid and $f\colon X \to \mathcal{G}_0$ a function, then the pullback groupoid along f is isomorphic to $(X \times X)\ {}_{f\times f}{\times}_{(\boldsymbol{t},\boldsymbol{s})}\mathcal{G}$ as groupoids.

Categorically, fibred products are therefore the only important construction. //

If we directly translate these constructions to Lie groupoids, we run into the problem that the obtained spaces may not carry a smooth structure any longer. Many examples of this can be found by considering the identity manifolds and remark that the category of manifolds is not closed under taking inverse images, intersections and fibred products. We argue that solving these differential obstructions allows us to perform these constructions in the category of Lie groupoids, without having to solve the submersiveness of the source and target. The "correct" notion to fix these obstructions is that of clean intersections, which are a generalisation of transversality of maps and are often neater to work with.

Definition 2.2.16. Let M be a manifold.

- Two embedded submanifolds $S_1, S_2 \subset M$ are said to *intersect cleanly* if $S_1 \cap S_2$ is an embedded submanifold and $T(S_1 \cap S_2) = TS_1 \cap TS_2$.

- A map $f\colon N \to M$ and an embedded submanifold $S \subset M$ *intersect cleanly* if $f^{-1}(S) \subset N$ is embedded and $(T_x f)^{-1}(T_{f(x)}S) = T_x f^{-1}(S)$ for all $x \in f^{-1}(S)$.
- Two smooth maps $f_i\colon N_i \to M$ *intersect cleanly* if $f_1 \times f_2\colon N_1 \times N_2 \to M \times M$ intersects cleanly with $\Delta \subset M \times M$.

Proposition 2.2.17. *If $f_i\colon N_i \to M$ for $i = 1,2$ have clean intersection, then their fibre product, $N_1\,{}_{f_1}\!\times_{f_2} N_2$, is a an embedded manifold of $N_1 \times N_2$ such that*

$$T(N_1\,{}_{f_1}\!\times_{f_2} N_2) = TN_1\,{}_{Tf_1}\!\times_{Tf_2} TN_2.$$

Proof. Suppose that $f_i\colon N_i \to M$ for $i = 1,2$ have clean intersection,then remark that

$$N_1\,{}_{f_1}\!\times_{f_2} N_2 = (f_1 \times f_2)^{-1}(\Delta).$$

It then follows from the definition that it is embedded with the appropriate tangent bundle. □

While categorically, the fibred product is the most important construction, in the geometrical case, we will translate all the constructions to intersections instead. Hence, we first show that the intersections of cleanly intersecting Lie subgroupoids are Lie subgroupoids.

Theorem 2.2.18. *Let $\mathcal{G}$ be a Lie groupoid and $\mathcal{H}, \mathcal{H}' \subset \mathcal{G}$ embedded Lie subgroupoids which intersect cleanly, then $\mathcal{H} \cap \mathcal{H}'$ is a Lie subgroupoid.*

Proof. Let $\mathcal{G}$ be a Lie groupoid, and let $\mathcal{H}, \mathcal{H}' \subset \mathcal{G}$ be embedded Lie subgroupoids that intersect cleanly. In particular, their intersection as manifolds $\mathcal{K} = \mathcal{H} \cap \mathcal{H}'$ is an embedded submanifold of $\mathcal{G}$. This implies that the map $\boldsymbol{u}_{\mathcal{G}} \circ \boldsymbol{s}_{\mathcal{G}}\colon \mathcal{G} \to \mathcal{G}$, when restricted to $\mathcal{K}$, remains smooth. The image of this restriction is exactly the image of the intersection of the object sets $\mathcal{K}_0 = \mathcal{H}_0 \cap \mathcal{H}'_0$ under $\boldsymbol{u}_{\mathcal{G}}$, and hence by Lemma **??**, the unit inclusion is an embedding. It follows that all structure maps of the groupoid, except for the multiplication, restrict to smooth maps on $\mathcal{K}$. The remaining task is to verify that the source and target maps restrict to submersions, which would then imply the smoothness of the multiplication.

We restrict our focus to the source map, as this will automatically imply that the target map is a submersion. From Lemma **??**, there exists a neighbourhood U of $\boldsymbol{u}_{\mathcal{K}}(\mathcal{K}_0)$ in $\mathcal{K}$ such that $\boldsymbol{u}_{\mathcal{K}} \circ \boldsymbol{s}_{\mathcal{K}}|_U$ is a submersion. Since $\boldsymbol{u}_{\mathcal{K}}$ is an embedding, it follows that $\boldsymbol{s}_{\mathcal{K}}|_U$ is a submersion as well. To extend this to all of $\mathcal{K}$, we use the right translation in $\mathcal{G}$, which induces an isomorphism of tangent spaces:

$$T_g r_{g^{-1}}\colon\ \ker T_g \boldsymbol{s}_{\mathcal{G}} \to \ker T_{1_{\boldsymbol{t}(g)}} \boldsymbol{s}_{\mathcal{G}},$$

where we identify the source fiber tangent space $T_g \mathcal{G}_{\boldsymbol{s}(g)}$ with $\ker T_g \boldsymbol{s}_{\mathcal{G}}$.

Since $grH \subset \mathcal{G}$ is an embedded subgroupoid, it intersects cleanly with the source fibres of $\mathcal{G}$. Indeed, for $x \in \mathcal{G}_0$, we have $\mathcal{H} \cap \mathcal{G}_x = \mathcal{H}_x$, which is an embedded submanifold. Moreover, as $\boldsymbol{s}_{\mathcal{H}} = \boldsymbol{s}_{\mathcal{G}}|_{\mathcal{H}}$, we have $T_g \boldsymbol{s}_{\mathcal{H}} = T_g \boldsymbol{s}_{\mathcal{G}} \circ T_g \iota$, where $\iota\colon \mathcal{H} \to \mathcal{G}$ is the inclusion. Since ι is an immersion, it follows that at $g \in \mathcal{H}$

$$\ker T_g \boldsymbol{s}_{\mathcal{H}} = T_g \mathcal{H} \cap \ker T_g \boldsymbol{s}_{\mathcal{G}}.$$

A similar expression holds for $\mathcal{H}'$. Hence, for any $g \in \mathcal{K}$, we obtain the following isomorphisms:

$$T_g r_{g^{-1}} \colon T_g \mathcal{H} \cap \ker T_g \boldsymbol{s}_{\mathcal{G}} \to T 1_{\boldsymbol{t}(g)} \mathcal{H} \cap \ker T 1_{\boldsymbol{t}(g)} \boldsymbol{s}_{\mathcal{G}},$$
$$T_g r_{g^{-1}} \colon T_g \mathcal{H}' \cap \ker T_g \boldsymbol{s}_{\mathcal{G}} \to T 1_{\boldsymbol{t}(g)} \mathcal{H}' \cap \ker T 1_{\boldsymbol{t}(g)} \boldsymbol{s}_{\mathcal{G}}.$$

Since $\mathcal{H}$ and $\mathcal{H}'$ intersect cleanly, we can take the intersection of the respective tangent spaces, yielding:

$$T_g r_{g^{-1}} \colon T_g \mathcal{K} \cap \ker T_g \boldsymbol{s}_{\mathcal{G}} \to T 1_{\boldsymbol{t}(g)} \mathcal{K} \cap \ker T 1_{\boldsymbol{t}(g)} \boldsymbol{s}.$$

Now observe that $\ker T_g \boldsymbol{s}_{\mathcal{K}} = T_g \mathcal{K} \cap \ker T_g \boldsymbol{s}$, since $\boldsymbol{s}_{\mathcal{K}} = \boldsymbol{s} \circ \iota$. Therefore, this defines an isomorphism between $\ker T_g \boldsymbol{s}_{\mathcal{K}}$ and $\ker T 1_{\boldsymbol{s}(G)} \boldsymbol{s}_{\mathcal{K}}$, which we know is minimal. Hence, we conclude that $\boldsymbol{s}$ and $\boldsymbol{t}$ are submersions, such that $\mathcal{K}$ is a Lie groupoid. $\square$

Corollary 2.2.19. *Let $\phi \colon \mathcal{G} \to \mathcal{H}$ be a Lie groupoid morphism and $\mathcal{H}'$ is a Lie subgroupoid of $\mathcal{H}$ such that ϕ intersects cleanly with $\mathcal{H}'$, then the inverse image groupoid $\phi^{-1}(\mathcal{H}')$ is a Lie subgroupoid of $\mathcal{G}$.*

Proof. Let $\phi \colon \mathcal{G} \to \mathcal{H}$ and $\mathcal{H}' \subset \mathcal{H}$ be as in the lemma. Remark that the inverse image of $\mathcal{H}'$ under ϕ can be rewritten as $\mathrm{pr}_1(\mathrm{gr}(\phi) \cap \mathcal{G} \times \mathcal{H}')$. Remark that the restriction of pr_1 to the graph of ϕ is a Lie groupoid isomorphism. Therefore, if $\mathrm{gr}(\phi)$ and $\mathcal{G} \times \mathcal{H}'$ have a clean intersection, then $\phi^{-1}(\mathcal{H}')$ has an induced Lie groupoid structure through this isomorphism.

Clearly, $\phi^{-1}(\mathcal{H}')$ is an embedded submanifold and as pr_1 is a diffeomorphism it follows that $\mathrm{gr}(\phi) \cap \mathcal{G} \times \mathcal{H}'$ is an embedded submanifold. For the tangent condition, we remark the following:

$$T_{(g,\phi(g))}(\mathrm{gr}\,\phi \cap \mathcal{G} \times \mathcal{H}') = T_g(\mathrm{pr}_1^{-1}) T_g \phi^{-1}(\mathcal{H}') = T_g(\mathrm{pr}_1^{-1})(T_g\phi)^{-1}(T_{\phi(g)}\mathcal{H}').$$

Using the fact that $\mathrm{pr}_1^{-1} = (\mathrm{id}, \phi)$, we can remark that the following are equivalent:

$$\begin{aligned} v \in (T_g\phi)^{-1}(T_{\phi(g)}\mathcal{H}') &\iff T_g\phi(v) \in T_{\phi(g)}\mathcal{H}', \\ &\iff (v, T_g\phi(v)) \in T_g\mathcal{G} \times T_{\phi(g)}\mathcal{H}' = T_{(g,\phi(g))}(\mathcal{G} \times \mathcal{H}'), \\ &\iff (v, w) \in T_{(g,\phi(g))}\mathrm{gr}\,\phi \cap T_{(g,\phi(g))}(\mathcal{G} \times \mathcal{H}'). \end{aligned}$$

Combining these two lines, we conclude that $T_{(g,\phi(g))}(\mathrm{gr}\,\phi \cap \mathcal{G} \times \mathcal{H}') = T_{(g,\phi(g))}\mathrm{gr}\,\phi \cap T_{(g,\phi(g))}(\mathcal{G} \times \mathcal{H}')$ such that the intersection of $\mathrm{gr}\,\phi$ and $\mathcal{G} \times \mathcal{H}'$ is clean. From Theorem **??** we can conclude that $\mathrm{gr}\,\phi \cap \mathcal{G} \times \mathcal{H}'$ is a Lie subgroupoid of $\mathcal{G} \times \mathcal{H}$ and thus $\phi^{-1}(\mathcal{H}') \subset \mathcal{G}$ is as well. $\square$

Corollary 2.2.20. *If $\phi \colon \mathcal{G} \to \mathcal{K}$ and $\psi \colon \mathcal{G} \to \mathcal{K}$ are Lie groupoid morphisms such that ϕ and ψ intersect cleanly, then $\mathcal{G} \,{}_{\phi}{\times}_{\psi} \mathcal{H}$ is a Lie subgroupoid of the product groupoid.*

Proof. We remark that the fibred product of ϕ and ψ is the subgroupoid of $\mathcal{G} \times \mathcal{H}$ given by $(\phi, \psi)^{-1}(\Delta)$. By the previous corollary, we find that this is a Lie subgroupoid as ϕ and ψ intersect cleanly. $\square$

Corollary 2.2.21. *For a Lie groupoid $\mathcal{G}$ and $f \colon M \to \mathcal{G}_0$ smooth such that $f \times f$ and $(\boldsymbol{t}, \boldsymbol{s})$ have clean intersection, then $f^!\mathcal{G} \colon = (M \,{}_{f}{\times}_{\boldsymbol{t}} \mathcal{G} \,{}_{\boldsymbol{s}}{\times}_{f} M \rightrightarrows M)$ is a Lie groupoid.*

Proof. As remarked in Example **??**, the pullback groupoid is a special case of a fibred product groupoid. The assumptions made above are exactly such that we can apply the previous corollary. □

2.3 Morita equivalences

In the simplest sense, we think of two Lie groupoids being the same if there exists a Lie groupoid isomorphism between them. However, we can also construct a more general notion of equivalence between Lie groupoids, where we do not focus on the internal structure but on the way they act on sets, called Morita equivalence. To introduce this notion, we first have to discuss groupoid actions and bundles, before we can define so-called bibundles.

2.3.1 Groupoid actions

A groupoid can be seen as a generalised symmetry of a system: whereas a group describes symmetries of a single object, a groupoid captures symmetries between multiple objects. For example, $\mathrm{Gl}(E)$ for a vector bundle E or the groupoid of germs associated to a pseudogroup, see Examples **??** and **??**. internally, a groupoid $\mathcal{G}$ acts on its object set $\mathcal{G}_0$, by moving along the arrows. A core ingredient in these symmetries, is the fact that some $g \in \mathcal{G}$ does not act on all elements of the associated set, but this is mediated by some map: For example, $A \in \mathrm{Gl}(E)$ "acts" on $E_x = \pi^{-1}(x)$, where $x = \boldsymbol{s}(A)$, but not on the whole of E. Let us generalise this type of symmetry to arbitrary manifolds M.

Definition 2.3.1. For a Lie groupoid $\mathcal{G}$ and map $\mu\colon M \to \mathcal{G}_0$, a *(left) action* of $\mathcal{G}$ on μ is a smooth map:

$$\alpha\colon \mathcal{G}\,{}_{\boldsymbol{t}}{\times}_{\mu} M\colon (g,x) \mapsto gx$$

which satisfies the following:

$$\mu(gx) = \boldsymbol{t}(g), \quad h(gx) = (hg)x, \quad 1_{\mu(x)}x = x.$$

We denote a left action by $\mathcal{G} \circlearrowright_{\mu} M$, and call (M,μ) or M a *(left) $\mathcal{G}$-space* and μ the *moment map*. Denote $\mathscr{O}_x = \{gx \in M\colon g \in \mathcal{G}_{\mu(x)}\}$ for the *orbit* of x[a], and $\mathcal{G}_x^y = \{g \in \mathcal{G}\colon gx = y\}$. If $x = y$, we call $\mathcal{G}_x^x$ the *stabilizer of* x.

[a] Similar definition exist for subset $U \subset M$, which we denote by $\mathscr{O}_U$

Remark. A *right action* and other associated notions are defined similarly by interchanging the roles of $\boldsymbol{s}$ and $\boldsymbol{t}$ and are denoted by $M\,{}_{\mu}\!\circlearrowleft \mathcal{G}$. Yet, when it is clear on which side the groupoid acts, we will simply refer to it by an action. //

Remark. Notice that there is a 1-1 correspondence between right $\mathcal{G}$-spaces and left $\mathcal{G}^{\mathrm{op}}$-spaces ,where $\mathcal{G}^{\mathrm{op}}$ is as in Example **??**. //

We already saw that a Lie group action can be encapsulated in an action groupoid. A similar construction can be done for groupoid action as follows: Let $\mathcal{G} \circlearrowright_{\mu} M$ be a $\mathcal{G}$-space, and define the following groupoid:

$$\begin{array}{c} \mathcal{G}\,{}_{\boldsymbol{s}}{\times}_{\mu} M \\ {\scriptstyle\alpha}\Big\downarrow\Big\downarrow{\scriptstyle\mathrm{id}} \\ M \end{array} \qquad \text{(arrows: } gx \xleftarrow{(g,x)} x \text{ where } \boldsymbol{s}(g) = \mu(x)\text{)}$$

where the structure maps are defined similarly to Example **??**. This is called the *action groupoid* and is denoted by $\mathcal{G} \ltimes_\mu M$. Remark that a similar construction exists for right actions, which we will denote by $M_\mu \rtimes \mathcal{G}$. The properties of groupoid actions are then defined through this action groupoid, inspired by the way action groupoids encapsulate a Lie group action.

Definition 2.3.2. Let (M, μ) be a $\mathcal{G}$-space, and $(\boldsymbol{t}, \boldsymbol{s})\colon \mathcal{G} \ltimes_\mu M \to M$ the source and target map of the associated action groupoid. The action is called *free*, *transitive*, or *proper* if $(\boldsymbol{t}, \boldsymbol{s})$ is injective, surjective or proper, respectively.

Example 2.3.3. Let $\mathcal{G}$ be a Lie groupoid, then $\mathcal{G}$ acts on $\mathcal{G}_0$ over id as

$$\mathcal{G}\,{}_{\boldsymbol{s}}\!\times_{\mathrm{id}}\mathcal{G}_0 \to \mathcal{G}_0\colon (g, x) \mapsto \boldsymbol{t}(g).$$

In particular, a Lie group acts on its object space, which is a singleton.

Notice that the orbits of this action are exactly the orbits of the groupoid, as described after Corollary **??**. The stabilisers of this action coincide with the isotropy groups of the Lie groupoid.

There is also a right action of $\mathcal{G}$ on $\mathcal{G}_0$ over id where we map to the source of g, but remark that these give the exact same orbits and stabilisers. //

Example 2.3.4. A Lie groupoid $\mathcal{G}$ acts on itself from both the left (over $\boldsymbol{t}$) and right (over $\boldsymbol{s}$) by left or right multiplication, respectively. The associated action groupoids are isomorphic to submersion groupoids via the following isomorphisms:

$$\mathcal{G} \ltimes_{\boldsymbol{t}} \mathcal{G} \to \mathcal{G}\,{}_{\boldsymbol{s}}\!\times_{\boldsymbol{s}}\mathcal{G}\colon (g, h) \mapsto (gh, h),$$
$$\mathcal{G}_{\boldsymbol{s}} \rtimes \mathcal{G} \to \mathcal{G}\,{}_{\boldsymbol{t}}\!\times_{\boldsymbol{t}}\mathcal{G}\colon (g, h) \mapsto (g, gh).$$

From these isomorphisms, we can deduce that these actions are free and proper. Additionally, the action is transitive if and only if the object set is a singleton, i.e. $\mathcal{G}$ is a Lie group. //

To extend our theory of $\mathcal{G}$-spaces, we also need a notion of a map with respect to the $\mathcal{G}$-space structure, which we can do either invariantly or equivariantly, depending on the context. These definitions are again similar to that of G-spaces, for a group G.

Definition 2.3.5. Let (M, μ) be a $\mathcal{G}$-space and N a manifold, a smooth map $f\colon M \to N$ is $\mathcal{G}$-*invariant* if for all $(g, x) \in \mathcal{G}\,{}_{\boldsymbol{s}}\!\times_\mu M$ it satisfies $f(gx) = f(x)$.

Besides a good notion of an invariant map, we can also consider an invariant notion of a subset of a $\mathcal{G}$-space.

Definition 2.3.6. Let (M, μ) be a $\mathcal{G}$-space, an embedded submanifold $X \subset M$ is called $\mathcal{G}$-*invariant* if for all $(g, x) \in \mathcal{G}\,{}_{\boldsymbol{s}}\!\times_\mu X$ it satisfies $gx \in X$.

Proposition 2.3.7. *Let (M, ν) be a $\mathcal{G}$-space and $X \subset M$ is a $\mathcal{G}$-invariant subspace if and only if $\mathcal{O}_x \subset X$ for all $x \in X$. In particular, the $\mathcal{G}$ action over μ restricts to X, and X coincides with its orbit in the $\mathcal{G}$ action on M, i.e $X = \mathcal{O}_X$.*

Lastly, let us give a notion of a map which intertwines two actions, i.e. which is equivariant for two actions.

Definition 2.3.8. Let (M, μ) be a $\mathcal{G}$-space and (N, ν) be a $\mathcal{H}$-space, and $\phi\colon \mathcal{G} \to \mathcal{H}$ a Lie groupoid morphism. A map $f\colon M \to N$ is a *$\mathcal{G}$-$\mathcal{H}$-equivariant map over ϕ* if $f(gx) = \phi(g)f(x)$ for any choice $(g, x) \in \mathcal{G}\ {}_{\boldsymbol{s}}\!\times_{\mu} M$. In particular, the following diagram must commute:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\mu"] & N \arrow[d, "\nu"] \\ \mathcal{G}_0 \arrow[r, "\phi_0"] & \mathcal{H}_0 \end{tikzcd}$$

A particular example of an equivariant map is the moment map of a $\mathcal{G}$-space with respect to the canonical action of $\mathcal{G}$ on $\mathcal{G}_0$.

2.3.2 Quotients by proper free actions

Given a $\mathcal{G}$-space (M, μ) we obtain an equivalence relation induced by the image of $(\boldsymbol{t}, \boldsymbol{s})$ associated to the action groupoid, i.e. the image of the map

$$\mathcal{G}\ {}_{\boldsymbol{s}}\!\times_{\mu} M \to M \times M(g, x) \mapsto (gx, x).$$

Similar to group actions, we can consider the orbit space of the action, denoted by $M/\mathcal{G}$, or for left actions sometimes as $\mathcal{G}\backslash M$. This space is exactly the quotient by the induced equivalent relation. However, again, similar to the case of Lie groups, this does not have an induced manifold structure. The geometric structure of the quotient is controlled by the geometry of the induced equivalence relation through Godement's criterion.

Proposition 2.3.9 ([?Serre2006, Thm. 12.2])**.** *Let M be a manifold and $R \subset M \times M$ an equivalence relation on M, then the following are equivalent:*

i) *$R \subset M \times M$ is a properly embedded submanifold and $\mathrm{pr}_2 : R \to M$ is a submersion.*

ii) *M/R is a manifold and $q : M \to M/R$ is a surjective submersion.*

Proof. Let R be an equivalence relation on a manifold M and let $q\colon M \to M/R$ denote its quotient map.

i) $\Longrightarrow$ ii): Suppose that R is properly embedded in $M \times M$ and $\mathrm{pr}_2 : R \to M$ is a submersion. To show that M/R admits a smooth structure such that $q\colon M \to M/R$ is a submersion, we will first show that it can be reduced to a local statement by showing it need only hold on an open cover, and then we will construct such an open cover.

1) Suppose that $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of M by saturated sets, i.e. they satisfy $U_\alpha = q^{-1}(q(U_\alpha))$ for each $\alpha \in \Lambda$, such that the quotient U_α/R_α, where $R_\alpha = R \cap (U_\alpha \times U_\alpha)$ is the restricted equivalence relation, has a manifold structure. Additionally, assume that $q\colon U_\alpha \to U_\alpha/R_\alpha$ is a surjective submersion.

   As the quotient map is open and $q(U_\alpha) = U_\alpha/R_\alpha$ it follows that $\{U_\alpha/R_\alpha\}_{\alpha\in\Lambda}$ defines an open cover of M/R. Due to U_α and U_β being saturated, their intersection, $U_{\alpha\beta} = U_\alpha \cap U_\beta$, is as well. Let us denote the induces equivalence relation as $R_{\alpha\beta} = R \cap (U_{\alpha\beta} \times U_{\alpha\beta})$. It follows that $U_{\alpha\beta}/R_{\alpha\beta} \subset$

$U_\alpha/R_\alpha \cap U_\beta/R_\beta$, as $U_{\alpha\beta}$ being saturated implies that any orbit of an element in the relation R is completely contained in $U_{\alpha\beta}$.

By assumption U_α/R_α and U_β/R_β carry a manifold structure, and as $U_{\alpha\beta}/R_{\alpha\beta}$ is open it inherits one from both, in both of which the quotient map $q\colon U_{\alpha\beta} \to U_{\alpha\beta}/R_{\alpha\beta}$ is a surjective submersion as it is simply the restriction to open subsets. However, this implies that the induced smooth structures are diffeomorphic and thus the transition maps between the smooth structures of U_α/R_α and U_β/R_β are smooth. This implies that they glue together to a smooth structure on M/R as well.

2) Next, we will show that we can drop the assumption of the cover being by saturated sets. Suppose that $U \subset M$ is open, such that U/R_U, where $R_U = R \cap (U \times U)$, is a manifold and $q\colon U \to U/R_U$ is a surjective submersion. We will show that this translates to its saturations, given by $\mathrm{Sat}(U) = q^{-1}(q(U))$ or $\mathrm{Sat}(U) = \mathrm{pr}_2(R \cap (U \times M))$. Firstly, we remark that it will still be open as q is a continuous open map.

Secondly, we need to show that $\mathrm{Sat}(U)/R_{\mathrm{Sat}(U)}$ admits a manifold structure for which the quotient map, i.e. $q\colon\ \mathrm{Sat}(U) \to \mathrm{Sat}(U)/R_{\mathrm{Sat}(U)}$, is a surjective submersion. Remark that we have the following canonically induced map:

$$\alpha\colon U/R_U \to \mathrm{Sat}(U)/R_{\mathrm{Sat}(U)}\colon\ [x]_{R_U} \mapsto [x]_{R_{\mathrm{Sat}(U)}}.$$

For the well-definedness, see that $R_U \subset R_{\mathrm{Sat}(U)}$. Additionally, it is surjective as for any $[y]_{R_{\mathrm{Sat}(U)}}$ there exists an $x \in U$ such that $q(y) = q(x)$, i.e. $\alpha([x]_{R_U}) = [x]_{R_{\mathrm{Sat}(U)}} = [y]_{R_{\mathrm{Sat}(U)}}$. Lastly, injectivity follows as for any $[x]_{R_{\mathrm{Sat}(U)}} = [y]_{R_{\mathrm{Sat}(U)}}$ with $x, y \in U$, then

$$(x, y) \in R_{\mathrm{Sat}(U)} \cap (U \times U) = R \cap (\mathrm{Sat}(U) \times \mathrm{Sat}(U)) \cap (U \times U) = R \cap (U \times U) = R_U.$$

We can conclude that $[x]_{R_U} = [y]_{R_U}$, such that α is injective as well.

We now consider the following commutative diagram

$$\begin{tikzcd}
 & R \cap (U \times \mathrm{Sat}(U)) \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
U \arrow[d, "q"'] & & \mathrm{Sat}(U) \arrow[d, "q"] \\
U/R_U \arrow[rr, "\alpha"] & & \mathrm{Sat}(U)/R_{\mathrm{U}}
\end{tikzcd}$$

As $q \circ \mathrm{pr}_1$ is a surjective submersion and pr_2 is surjective, it follows that $\alpha^{-1} \circ q$ is a surjective submersion. The restriction $\alpha^{-1} \circ q|_U$ is still a submersion as U is open. We can then define a manifold structure on $\mathrm{Sat}(U)/R_{\mathrm{Sat}(U)}$ such that α is a diffeomorphism and by composing, we see that the quotient map $q : \mathrm{Sat}(U) \to \mathrm{Sat}(U)/R_{\mathrm{Sat}(U)}$ is also a submersion and thus $\mathrm{Sat}(U)/R_{\mathrm{Sat}(U)}$ is a quotient manifold.

We can conclude that, given a cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of M where U_α/R_α is a manifold and $q\colon U_\alpha \to U_\alpha/R_\alpha$ is a submersion for each $\alpha \in \Lambda$, their saturations $\{\mathrm{Sat}(U)_\alpha\}_{\alpha \in \Lambda}$ satisfies these properties as well.

3) The last step then needs to construct a cover of such opens. For some $x_0 \in M$, define the following set

$$N = \left\{ v \in T_{x_0}M \,|\, (v, 0) \in T_{(x_0,x_0)}R \right\}, \quad \text{with } T_{(x_0,x_0)}R \subset T_{(x_0,x_0)}(M \times M) = T_{x_0}M \oplus T_{x_0}M.$$

Pick some embedded submanifold $W' \subset M$ which complements N, i.e. $T_{x_0}W' \oplus N = T_{x_0}M$, and we set $\Sigma = R \cap (W' \times M)$, and we claim the following:

a) $\Sigma \subset R$ is an embedded submanifold;

b) $\mathrm{pr}_2 \colon \Sigma \to M$ is a local diffeomorphism at (x_0, x_0).

Let us prove these statements.

a) We remark that Σ is recovered from W' as $\Sigma = \mathrm{pr}_1^{-1}(W')$, where $\mathrm{pr}_1 \colon R \to M$ is a submersion as pr_2 is, this implies that Σ is an embedded submanifold.

b) To show that $\mathrm{pr}_2 \colon \Sigma \to M$ is a local diffeomorphism at (x_0, x_0), we need to show that its differential at this point is an isomorphism.

For injectivity, we remark that a tangent vector in factors as

$$(v_1, v_2) \in T_{(x_0,x_0)}\Sigma \subset T_{(x_0,x_0)}R \cap (T_{x_0}W' \oplus T_{x_0}M).$$

Thus, if $(v_1, v_2) \in \ker T_{(x_0,x_0)} \mathrm{pr}_2$ it follows that $v_1 \in T_{x_0}W'$, but $0 = T_{(x_0,x_0)} \mathrm{pr}_2(v_1, v_2) = v_2$, such that $v_1 \in N$ as well. As $T_{x_0}W' \cap N = \{0\}$, it follows that $\ker T_{(x_0,x_0)} \mathrm{pr}_2$ is trivial and thus it is injective.

For surjectivity, if $u \in T_{x_0}M$ we can find some $v \in T_{x_0}M$ such that $(v, u) \in T_{(x_0,x_0)}R$. As $T_{x_0}W' \oplus N$, we can find $v_1 \in T_{x_0}W'$ and $v_2 \in N$ such that $v = v_1 + v_2$. However, as $(v_2, 0) \in T_{(x_0,x_0)}R$ it follows that $(v_1, u) \in T_{(x_0,x_0)}R$ as well. This elements maps to u under $T_{(x_0,x_0)} \mathrm{pr}_2$ and thus pr_2 has surjective differential at (x_0, x_0).

We can conclude that $\mathrm{pr}_2 \colon \Sigma \to M$ is a local diffeomorphism at (x_0, x_0).

From these two properties of Σ, we can conclude that there exist open neighbourhoods $U_1 \subset W' \times M$ of (x_0, x_0) and $U_2 \subset M$ of x_0 such that $\mathrm{pr}_2 \colon \Sigma \cap U_1 \to U_2$ is a diffeomorphism. We can then find some smooth map $r \colon U_2 \to \mathrm{pr}_1(U_1)$ such that $(r(x), x) \in \Sigma \cap U_1$. We remark that on $W' \cap U_2$ this map takes the form $r(x) = x$ as $\mathrm{pr}_2(x, x) = x = \mathrm{pr}_2(r(x), x)$. We now define the following sets

$$U = \{x \in U_2 | \, r(x) \in W'\} = r^{-1}(W' \cap U_2) \quad \text{and} \quad W = U \cap W'$$

These then define open subsets of W' which therefore automatically carry a manifold structure. Additionally, we remark that $r(U) \subset W$ as $r(r(x)) = r(x)$ due to $r|_{W' \cap U_2} = \mathrm{id}$, and for all $x \in U$ the element $r(x) \in W$ is the only one equivalent to x as pr_2 is injective on this set. In particular, this implies that there exists a bijection $\phi \colon U/R_U \to W$ which makes the following diagram commute:

$$\begin{tikzcd} & U \arrow[dl, "q"'] \arrow[dr, "r"] & \\ U/R_U \arrow[rr, "\phi", "\sim"'] & & W \end{tikzcd}$$

Through this bijection, we can induce a manifold structure on U/R_U which is compatible with the assumptions made in part **??** of this proof.

The construction in step **??** can be done around any point M, and thus this defines a cover of opens which satisfy the conditions of step **??**. Therefore, M/R obtains a manifold structure.

ii) $\implies$ i): Suppose that M/R is a manifold and $q : M \to M/R$ is a surjective submersion. Notice that we can obtain R as the following pullback:

$$\begin{tikzcd} R \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & M \arrow[d, "q"] \\ M \arrow[r, "q"] & M/R \end{tikzcd}$$

It follows that R is a closed embedded manifold as q is a surjective submersion. Moreover, we can remark that for any $(x, y) \in R$ such that $z = q(x) = q(y)$ and we have that

$$T_{(x,y)}R = T_xM\ {}_{T_xq}\!\times_{T_yq} T_yM$$

Next, pick some $v \in T_yM$, and consider $T_yq(v) \in T_{[y]}M/R$. As $T_xq\colon T_xM \to T_{[x]}M/R = T_{[y]}M/R$ is surjective, we can find some $u \in T_xM$ such that $T_xq(u) = T_yq(v)$. In particular, it follows that $(u, v) \in T_{(x,y)}R$ and $T_{(x,y)}\,\mathrm{pr}_2(u, v) = v$, such that it is indeed a surjective submersion. □

In particular, this has the following corollary.

Corollary 2.3.10. *Let $\mathcal{G}$ act freely and properly on M over μ, then the quotient $M/\mathcal{G}$ is a manifold.*

Proof. Suppose that $\mathcal{G}$ acts propertly and freely over $\mu\colon M \to \mathcal{G}_0$, in other words

$$\alpha\colon \mathcal{G}\ {}_{\boldsymbol{s}}\!\times_{\mu} M \to M \times M\colon (g, x) \mapsto (gx, x),$$

is a proper injective map and an immersion by Proposition **??**. This implies that the induced equivalence relation is an embedded submanifold.

Remark that the second projection $\mathrm{pr}_2\colon\ \mathrm{im}\,\alpha \to M$ is a submersion, as for any $(y, x) \in \mathrm{im}\,\alpha$ we can find a $g \in \mathcal{G}$ such that $xg = y$. Suppose that $U \subset M$ is an open neighbourhood of x, such that $\sigma_1 : U \to \mathcal{G}$ local bisection such that $\sigma_1(x) = g^{-1}$. The map $\sigma\colon U \to M \times M : x \mapsto (x\sigma_1(x)^{-1}, x)$ is a section of pr_2 whose image lies in $\mathrm{im}\,\alpha$.

We can now apply Proposition **??**, and we find that the quotient is a manifold. □

Remark. Given that Corollary **??** holds, we also obtain Proposition **??**. Namely, given an equivalence relation R which is closed embedded in $M \times M$ such that pr_2 is a submersion, then it is a Lie subgroupoid of the pair groupoid of M. The quotient of M by this groupoid is then the quotient by the equivalence relation. //

We finish this section with a last definition, defining principal $\mathcal{G}$-bundles, which are the groupoid equivalent of the classical principal bundles.

Definition 2.3.11. Let $\mathcal{G}$ be a Lie groupoid. A *principal $\mathcal{G}$-bundle* is given by a $\mathcal{G}$-space (P, μ) with a $\mathcal{G}$-invariant surjective submersion $\pi\colon P \to M$ such that the map

$$\mathcal{G} \ltimes P \to P\ {}_{\pi}\!\times_{\pi} P\colon (g, p) \mapsto (gp, p)$$

is a diffeomorphism.

Notice that, just like in the classical case, there is a correspondence between principal $\mathcal{G}$-bundles and free and proper actions of $\mathcal{G}$.

Proposition 2.3.12. *A $\mathcal{G}$-space (P, μ) with a $\mathcal{G}$-invariant surjective submersion $\pi\colon P \to M$ is a principal $\mathcal{G}$-bundle if and only if the action is free and proper, and $\overline{\pi}\colon \mathcal{G}\backslash P \to M$ is a diffeomorphism.*

2.3.3 Products with $\mathcal{G}$-spaces

Given a $\mathcal{G}$-space M and some manifold N, we obtain a canonical action on $M \times N$ over $\mu \circ \mathrm{pr}_1$, where μ is the moment map of the $\mathcal{G}$ action on M. This action is then defined by

$$(M \times N)\,{}_{\mu\circ\mathrm{pr}_1}\!\times_{\boldsymbol{t}}\mathcal{G} \to M \times N\colon (x, y, g) \mapsto (xg, y).$$

This induced action acts nicely within the collection of $\mathcal{G}$-spaces.

Proposition 2.3.13. *Let (M, μ) be a $\mathcal{G}$-space and N some manifold, and let $M \times N$ carry the action as above. Then:*

i) *The first projection* $\mathrm{pr}_1\colon M \times N \to M$ *is $\mathcal{G}$-equivariant.*

ii) *The second projection* $\mathrm{pr}_2\colon M \times N \to N$ *is $\mathcal{G}$-invariant.*

iii) *If the action on M is free (resp. proper), then the action on $M \times N$ is free (resp. proper).*

Proof. The fact that the induced map defines an action, such that the projections are equivariant and invariant, should be clear. Additionally, if the action on M is free, then $g(x, y) = (x, y)$ if and only if $gx = x$ if and only if g is a unit.

Lastly, suppose that $\mathcal{G}$ acts properly on M, we need to show that the following map is proper:

$$\alpha_\times\colon (M \times N)\,{}_{\mu\circ\mathrm{pr}_1}\!\times_{\boldsymbol{t}}\mathcal{G} \to M \times N \times M \times N\colon (x, y, g) \mapsto (xg, y, x, y),$$

Let us denote $\alpha_M\colon M\,{}_{\mu}\!\times_{\boldsymbol{t}}\mathcal{G} \to M \times M\colon (x, g) \mapsto (xg, g)$, which is proper by assumption, and pr_i for the projection of $(M \times N) \times (M \times N)$ onto the i-component. Suppose that $K \subset M \times N \times M \times N$ is compact, and remark that $K_1 = \mathrm{pr}_1 \times \mathrm{pr}_3(K) \subset M \times M$ and $K_2 = \mathrm{pr}_4(K) \subset N$ are compact as well. One can readily verify that $\alpha_\times^{-1}(K) \subset \alpha_M^{-1}(K_1) \times K_2$, and therefore it is a closed subset of a compact subset, which implies it is compact itself. $\square$

We can also take the product of two $\mathcal{G}$-spaces, (M, μ) and (N, ν), by taking a sort of diagonal action. However, the naive approach of trying to define an action on $M \times N$ fails, as there is no canonical moment map on this set such that an $g \in \mathcal{G}$ can act on both components of $M \times N$. To solve this, we instead consider their fibred $M\,{}_{\mu}\!\times_{\nu} N$, which has a canonically induced map $\mu \times \nu = \mu \circ \mathrm{pr}_1 = \nu \circ \mathrm{pr}_2\colon M\,{}_{\mu}\!\times_{\nu} N \to \mathcal{G}_0$.

Proposition 2.3.14. *Let (M, μ) and (N, ν) be a $\mathcal{G}$-space, such that μ and ν have a clean intersection, then $M\,{}_{\mu}\!\times_{\nu} N$ is a $\mathcal{G}$-space with the action:*

$$\mathcal{G}\,{}_{\boldsymbol{s}}\!\times_{\mu\times\nu}(M\,{}_{\mu}\!\times_{\nu} N) \to M\,{}_{\mu}\!\times_{\nu} N\colon (g, x, y) = (gx, gy).$$

Moreover, with respect to this action:

i) *The projections* $\mathrm{pr}_1 \colon M \times N \to M$ *and* $\mathrm{pr}_2 \colon M \times N \to N$ *are* $\mathcal{G}$*-equivariant.*

ii) *If the action on* M *or* N *is free (resp. proper), then the action on* $M\,{}_\mu\!\times_\nu N$ *is free (resp. proper).*

Proof. Here, we need the condition of a clean intersection to make sure that $M\,{}_\mu\!\times_\nu N$ is an embedded manifold, such that the action is automatically smooth. The rest of this proof is analogous to the proof of Proposition **??**. □

2.3.4 Bibundles

To relate the spaces of actions by different Lie groupoids, we work with bibundles. These are manifold which have an action of two Lie groupoids, which commute. We will see that in certain cases, these let us translate between the spaces of Lie groupoid actions of different Lie groupoids.

Definition 2.3.15. For two Lie groupoids $\mathcal{G}$ and $\mathcal{H}$, a $\mathcal{G}$-$\mathcal{H}$ *bibundle* is a triple (P, μ, ν) such that

◇ $\mathcal{G} \circlearrowright_\mu P$ is a left action,

◇ $P\,{}_\nu\!\circlearrowleft \mathcal{H}$ is a right action,

◇ μ is $\mathcal{H}$-invariant, and ν is $\mathcal{G}$-invariant,

◇ for some $(g, x, h) \in \mathcal{G}\,{}_{\boldsymbol{s}}\!\times_\mu P\,{}_\nu\!\times_{\boldsymbol{t}} \mathcal{H}$ we have $(gx)h = g(xh)$.

Example 2.3.16. Let M be a left $\mathcal{G}$-space, with moment map μ. Consider the right action of the identity groupoid $M \rightrightarrows M$ on M, over $\mathrm{id}\colon M \to M$. This defines a $\mathcal{G}$-M bibundle. //

Remark. Notice that the collection of $\mathcal{G} - \mathcal{H}$ bibundles corresponds with $\mathcal{G} \times \mathcal{H}^{\mathrm{op}}$-spaces. From this correspondence, the notions of orbits and stabilisers immediately translate to bibundles. //

A bibundle comes with two intrinsic invariant maps, namely μ and ν. By imposing additional conditions on these maps, we obtain a stronger type of bibundle.

Definition 2.3.17. If (P, μ, ν) is a $\mathcal{G}$-$\mathcal{H}$ bibundle we call $\mathcal{G} \circlearrowright_\mu P \xrightarrow{\nu} \mathcal{H}_0$ and $\mathcal{G}_0 \xleftarrow{\mu} P\,{}_\nu\!\circlearrowleft \mathcal{H}$ the *left and right underlying bundle*, respectively.
A bibundle is then called *left or right principal* if the left or right underlying bundle is principal, respectively. In the case where it is both left and right principal, it is simply called *principal*.

Example 2.3.18. A Lie groupoid $\mathcal{G}$ is a principal $\mathcal{G}$-$\mathcal{G}$ bibundle with left and right multiplication as action. Clearly, this defines a bibundle, and it is free and principal as the left and right multiplication are automatically free and proper, see Example **??**. //

Notation. Much like actions, we may omit writing the moment maps. Moreover, we will denote the collection of $\mathcal{G}$-$\mathcal{H}$ bibundles that are right principal by $\mathrm{Pbb}_{\mathrm{rgt}}(\mathcal{G}, \mathcal{H})$, and $\mathrm{Pbb}(\mathcal{G}, \mathcal{H})$ denotes the $\mathcal{G}$-$\mathcal{H}$ bibundles that are principal.

Much like a normal groupoid action, we can capture the geometric behaviour of a bibundle in a groupoid structure as follows: Given a bibundle $\mathcal{G} \circlearrowright_{\mu} P \;{}_{\nu}\!\circlearrowleft \mathcal{H}$, we can define the action groupoid $\mathcal{G} \ltimes P \rtimes \mathcal{H}$ as

$$\begin{tikzcd} \mathcal{G}\;{}_{\boldsymbol{s}}\!\times_{\mu} P\;{}_{\nu}\!\times_{\boldsymbol{t}} \mathcal{H} \arrow[d, shift right, "\boldsymbol{t}"'] \arrow[d, shift left, "\boldsymbol{s}"] \\ P \end{tikzcd} \qquad \text{(arrows: } gph \xleftarrow{(g,p,h)} p\text{)}$$

One can verify that $(g', gph, h')(g, p, h) = (g'g, p, hh')$ defines a groupoid multiplication. Alternatively, we can view this as the action groupoid of the induced action of $\mathcal{G} \times \mathcal{H}^{\mathrm{op}}$ on P. This automatically implies that it defines a Lie groupoid. Moreover, we obtain canonical projection maps

$$\mathrm{pr}_{\mathcal{G}} \colon \mathcal{G}\;{}_{\boldsymbol{s}}\!\times_{\mu} P\;{}_{\nu}\!\times_{\boldsymbol{t}} \mathcal{H} \to \mathcal{G} \colon (g, p, h) \mapsto g \quad \text{and} \quad \mathrm{pr}_{\mathcal{H}} \colon \mathcal{G}\;{}_{\boldsymbol{s}}\!\times_{\mu} P\;{}_{\nu}\!\times_{\boldsymbol{t}} \mathcal{H} \to \mathcal{H} \colon (g, p, h) \mapsto h^{-1},$$

which are Lie groupoid morphisms.

The notion of a map of $\mathcal{G}$-spaces now extends to these double action structures, and so does the notion of an invariant subspace.

Definition 2.3.19. An *equivalence of $\mathcal{G}$-$\mathcal{H}$ bibundles*, say P and Q, is a diffeomorphism $f \colon P \to Q$ which is both $\mathcal{G}$- and $\mathcal{H}$-equivariant. Additionally, a subset $Q \subset P$ is called *biinvariant* if it is invariant for the $\mathcal{G}$ and $\mathcal{H}$ action.

Next, we want to use bibundles to describe a category of Lie groupoids up to their representation theory, or spaces of actions. In other words, we want to realise these bibundles as the morphisms in some category. For this, we in particular need a notion of a composition of bibundles.

Proposition 2.3.20. *Given $P \in \mathrm{Pbb}_{\mathrm{rgt}}(\mathcal{G}, \mathcal{H})$ and $Q \in \mathrm{Pbb}_{\mathrm{rgt}}(\mathcal{H}, \mathcal{K})$, then the space*

$$P \otimes Q = (P\;{}_{\mu}\!\times_{\alpha} Q)/\mathcal{H}$$

has the structure of a $\mathcal{G}$-$\mathcal{K}$ bibundle that it right principle. Moreover, this assignment satisfies the following:

- *It is associative up to isomorphism, i.e. $(P \otimes Q) \otimes R \cong P \otimes (Q \otimes R)$.*
- *It is well-defined on isomorphism classes, i.e. if $P \cong P' \in \mathrm{Pbb}_{\mathrm{rgt}}(\mathcal{G}, \mathcal{H})$ and $Q \cong Q' \in \mathrm{Pbb}_{\mathrm{rgt}}(\mathcal{H}, \mathcal{K})$, then $P \otimes Q \cong P' \otimes Q'$*

Proof. Take some $(P, \mu, \nu) \in \mathrm{Pbb}_{\mathrm{rgt}}(\mathcal{G}, \mathcal{H})$ and $(Q, \alpha, \beta) \in \mathrm{Pbb}_{\mathrm{rgt}}(\mathcal{H}, \mathcal{K})$, which reads as the following diagram:

$$\begin{tikzcd} \mathcal{G} \arrow[r, bend left] \arrow[d, "\boldsymbol{s}"] & P \arrow[dl, "\mu"] \arrow[dr, "\nu"'] & \mathcal{H} \arrow[l, bend right] \arrow[r, bend left] \arrow[d, shift right, "\boldsymbol{s}"'] \arrow[d, shift left, "\boldsymbol{t}"] & Q \arrow[dl, "\alpha"] \arrow[dr, "\beta"'] & \mathcal{K} \arrow[l, bend right] \arrow[d, "\boldsymbol{t}"] \\ \mathcal{G}_0 & & \mathcal{H}_0 & & \mathcal{K}_0 \end{tikzcd}$$

As α is a submersion, it has a clean intersection with ν and thus we obtain an induced action on $P\;{}_{\nu}\!\times_{\alpha} Q$, cf. Proposition **??**. Due to the action of $\mathcal{H}$ being free and proper on P, the diagonal action on $P\;{}_{\nu}\!\times_{\alpha} Q$ is also free and proper, see Proposition **??** as well. This implies that we can take its quotient to obtain a set $P \otimes Q = (P\,{}_{\mu}\!\times_{\alpha} Q)/\mathcal{H}$.

On this set, we have an induced principal $\mathcal{G} - \mathcal{K}$ bibundle structure by remarking the following: Consider the induced action of $\mathcal{G}$ on $P \times Q$ and remark that $P\,{}_{\mu}\!\times_{\alpha} Q$ is an invariant subset of this action as μ is $\mathcal{G}$ invariant. This implies that this has an induced $\mathcal{G}$ action and as the actions of $\mathcal{G}$ and $\mathcal{H}$ commute on P, the also commute on $P\,{}_{\mu}\!\times_{\alpha} Q$, such that it restricts to the quotient. Similarly, $\mathcal{K}$ induces an action on $P \otimes Q$ as well.

To verify that the action is free, suppose that $[x, y] = [x', y']$ and $(x, yk) = [x', y'l]$. The first equality implies that there exists some $h \in \mathcal{H}$ with $x' = xh$ and $y' = hy$ such that $(x, yk) = [x', y'l] = \left[xh, h^{-1}yl\right] = [x, yl]$, where we used that the action commute. Therefore, we find some $h' \in \mathcal{H}$ such that $xh' = x$ and $h'^{-1}yk = yl$. As the action of $\mathcal{H}$ on P is free, we find that h' is an identity and thus $yk = yl$. Now, because the $\mathcal{K}$ action on Q is free, we find that $k = l$ and our $\mathcal{K}$ action on $P \otimes Q$ is also free.

For properness, we remark that $\mathcal{K}$ acts properly on $P \times Q$, see Proposition **??**, and thus this descends to a proper action on $P\,{}_{\mu}\!\times_{\nu} Q$. Moreover, we have the following commuting diagram:

$$\begin{tikzcd}
(P\,{}_{\nu}\!\times_{\alpha} Q)\,{}_{\overline{\beta}}\!\times_{\boldsymbol{t}} \mathcal{K} \arrow[r, "{\alpha\colon (x,y,k)\mapsto(xk,yk,k)}"] \arrow[d, "q\times \mathrm{id}"] & (P\,{}_{\nu}\!\times_{\alpha} Q) \times (P\,{}_{\nu}\!\times_{\alpha} Q) \arrow[d, "q\times q"] \\
P \otimes Q\,{}_{\overline{\beta}}\!\times_{\boldsymbol{t}} \mathcal{K} \arrow[r, "{\beta\colon ([x,y],k)\mapsto([xk,yk],[x,y])}"] & P \otimes Q \times P \otimes Q
\end{tikzcd}$$

Consider some compact $K \subset P \otimes Q \times P \otimes Q$ and let $\{U_i\}$ be a cover of K such that for each i the closure $\overline{U_i}$ is compact and we can find sections $\sigma_i \colon \overline{U_i} \to (P\,{}_{\nu}\!\times_{\alpha} Q) \times (P\,{}_{\nu}\!\times_{\alpha} Q)$ of $q \times q$. Remark that we can find such a section as $q \times q$ is a submersion and a manifold is locally compact. As K is compact, we can assume this is a finite cover.

We can conclude that $\widetilde{U}_i = \alpha^{-1}(\sigma_i(\overline{U_i}))$ is compact as well and thus $\widetilde{U} = \bigcup_i \widetilde{U}_i$ is compact as it is a finite union. Per construction, we know that $\beta^{-1}(K) \subset q \times \mathrm{id}(\widetilde{U})$, which is a closed subset of a compact and thus it is itself compact. Therefore, $\mathcal{K}$ acts properly on $P \otimes Q$.

We conclude that $P \otimes Q \in \mathrm{Pbb}_{\mathrm{rgt}}(\mathcal{G}, \mathcal{K})$ and the associativity up to isomorphism follows from the map:

$$(P \otimes Q) \otimes R \to P \otimes (Q \otimes R)\colon \ [[p, q]\,, r] \mapsto [p, [q, r]]$$

Moreover, this construction is well-defined on isomorphism classes of bibundles. Namely, given equivalences $\psi\colon P \to P'$ and $\phi\colon Q \to Q'$, we obtain an equivalence

$$P \otimes Q \to P' \otimes Q'\colon \ [p, q] \mapsto [\psi(p), \phi(q)]\,.$$

□

The above proposition implies that bibundles that are right principal define a type of morphism on Lie groupoids, when considered up to equivalence. We will denote the category by $\mathbf{LieGrpd}_{\mathrm{weak}}$, and remark that this is a much bigger category compared to taking only Lie groupoid morphisms. In particular, given a Lie groupoid morphism $\phi\colon \mathcal{H} \to \mathcal{G}$, we obtain the bibundle. $\mathcal{G} \circlearrowright_{\phi_0 \circ \boldsymbol{t}} \mathcal{H}\ {}_{\boldsymbol{s}}\!\circlearrowleft \mathcal{H}$, where the right action is the normal multiplication, thus it is free and proper, and the left action is the action by applying ϕ. This bibundle is right principal, and thus defines a morphism in $\mathbf{LieGrpd}_{\mathrm{weak}}$. As we allow for more morphisms in the category $\mathbf{LieGrpd}_{\mathrm{weak}}$, we obtain a weaker notion of equivalence, which we will call *Morita equivalence*. In other words, two Lie groupoids are *Morita equivalent* if they are isomorphic in $\mathbf{LieGrpd}_{\mathrm{weak}}$.

Moreover, under the identification of a $\mathcal{G}$-space with the right principal $\mathcal{G} - M$ bibundle, we can see that $P \in \mathrm{Pbb}_{\mathrm{rgt}}(\mathcal{G}, \mathcal{H})$ defines a map

$$P\colon \mathcal{G}\text{-spaces} \to \mathcal{H}\text{-spaces}\colon\; M \mapsto P \otimes M.$$

Therefore, they define maps on the collections of $\mathcal{G}$-spaces.

We will finish this chapter with a slight reformulation of the notion of a Morita equivalence.

Proposition 2.3.21 ([?delHoyo2013, Thm. 4.6.3])**.** *Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids, then the following are equivalent:*

- *$\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent.*
- *There exist a principal $\mathcal{G} - \mathcal{H}$ bibundle.*

Remark that there is an equivalent description in terms of generalised maps, which are constructed using localisation with respect to so-called weak equivalences. For details on this construction, refer to [?Moerdijk2003, ?delHoyo2013].

Chapter 3

$\mathcal{VB}$-groupoids

As we saw in the first chapter, vector bundles play a critical role in the theory of surjective submersions and fibre bundles, through the use of connections. With an eye on the goal of describing such objects in the multiplicative setting of Lie groupoids, we need to translate these ideas to involve the multiplicative setting as well. The primal example of the structure we want to emulate, it that of the tangent groupoid of a Lie groupoid, which is a pair of vector bundles $T\mathcal{G} \to \mathcal{G}$ and $TM \to M$, which also fit into a pair of Lie groupoids $T\mathcal{G} \rightrightarrows TM$ and $\mathcal{G} \rightrightarrows M$. Moreover, these have some compatible structures. We will capture this in the notion of a $\mathcal{VB}$-groupoid. The definitions and different notions of $\mathcal{VB}$-groupoids are taken from [?GraciaSaz2017] and [?Mackenzie2005], while the algebraic constructions like the direct sum and kernel are based on [?LiBland2011]. Then we discuss a new result on the splitting of short exact sequences in this category. We will finish the chapter with a description of multiplicative forms on Lie groupoids and describe them with values in $\mathcal{VB}$-groupoids as in [?Drummond2019].

3.1 Different notions of $\mathcal{VB}$-groupoids

There are many equivalent ways of defining $\mathcal{VB}$-groupoids, each with its own merits. We will start with a more classical notion and then turn to some more categorical definitions, each highlighting a different part of the structures. They all start out with a quadruple $(\Gamma, V, \mathcal{G}, M)$ which forms a *diagram of Lie groupoids and vector bundles*, i.e. it fits into the following diagram:

$$\begin{tikzcd} \Gamma \arrow[r, shift left, "\widetilde{t}"] \arrow[r, shift right, "\widetilde{s}"'] \arrow[d, "\widetilde{q}"] & V \arrow[d, "q"] \\ \mathcal{G} \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & M \end{tikzcd}$$

where $\Gamma \rightrightarrows V$ and $\mathcal{G} \rightrightarrows M$ are Lie groupoids, and $\widetilde{q}\colon \Gamma \to \mathcal{G}$ and $q\colon V \to M$ are vector bundles. We will denote the structure maps of $\Gamma \rightrightarrows V$ with a tilde, and let $\widetilde{0}\colon \mathcal{G} \to \Gamma$ and $0\colon M \to V$ denote the zero-sections of the vector bundles. We call all these maps of the internal structures, the *structure maps* of the $\mathcal{VB}$-groupoid.

> *Terminology.* Let $(\Gamma, V, \mathcal{G}, M)$ be a diagram of Lie groupoids and vector bundles. We call $\mathcal{G} \rightrightarrows M$ and $V \to M$ the base groupoid and vector bundle, respectively, and $\Gamma \rightrightarrows V$ and $\Gamma \to \mathcal{G}$ the top groupoid and vector bundle, respectively.

Our guiding example, the tangent groupoid $T\mathcal{G} \rightrightarrows TM$ of a groupoid $\mathcal{G} \rightrightarrows M$ fits into such a diagram. The structure maps of the tangent groupoid admit a lot more compatibility conditions for the internal structures. For example, they are all vector bundle morphisms, while the projection $\widetilde{q}\colon T\mathcal{G} \to \mathcal{G}$ defines a Lie groupoid morphism. A $\mathcal{VB}$-groupoid incorporates all of this structure as well.

Definition 3.1.1. A *$\mathcal{VB}$-groupoid* is a diagram of Lie groupoids and vector bundles such that the following holds:

i) $(\widetilde{\boldsymbol{s}}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{t}}, \boldsymbol{t})$ are vector bundle morphisms,

ii) $(\widetilde{q}, q)$ is a Lie groupoid morphism,

iii) The interchange law holds:

$$\widetilde{\boldsymbol{m}}(\gamma_1 + \gamma_3, \gamma_2 + \gamma_4) = \widetilde{\boldsymbol{m}}(\gamma_1, \gamma_2) + \widetilde{\boldsymbol{m}}(\gamma_3, \gamma_4),$$

where $(\gamma_1, \gamma_2), (\gamma_3, \gamma_4) \in \Gamma^{(2)}$ with $\widetilde{q}(\gamma_1) = \widetilde{q}(\gamma_3)$ and $\widetilde{q}(\gamma_2) = \widetilde{q}(\gamma_4)$.

This definition slightly deviates from the definition in [?Mackenzie2005], where there is the technical condition that the map

$$\rho\colon \Gamma \to V\,{}_q\!\times_{\boldsymbol{s}} G\colon \gamma \mapsto (\widetilde{\boldsymbol{s}}(\gamma), \widetilde{q}(\gamma))$$

is a surjective submersion, which is also called the "double source condition". However, it was shown that this is actually redundant in [?LiBland2010, Lem. A.3]. Here, they showed that the assumption that $(\widetilde{\boldsymbol{s}}, \boldsymbol{s})$ is a vector bundle morphism implies that ρ is a surjective submersion.

Example 3.1.2. Of course the tangent groupoid of a Lie groupoid is a $\mathcal{VB}$-groupoid, but even if we start with some $\mathcal{VB}$-groupoid $(\Gamma, V, \mathcal{G}, M)$ we can consider the tangent $\mathcal{VB}$-groupoid $(T\Gamma, TV, T\mathcal{G}, TM)$ with all the associated tangent maps of the structure maps. //

While this definition of a $\mathcal{VB}$-groupoid is all good and well, we could have also imagined them as being some objects inside categories, in a similar fashion to how we first described Lie groupoids.

Definition 3.1.3. A *Lie groupoid object in the category of vector bundles* is a diagram of Lie groupoids and vector bundles such that it satisfies the following conditions:

i) $\widetilde{q} \times \widetilde{q}\colon \Gamma^{(2)} \to \mathcal{G}^{(2)}$ is a vector bundle with the obvious structure maps.

ii) $(\widetilde{\boldsymbol{s}}, \boldsymbol{s})$, $(\widetilde{\boldsymbol{t}}, \boldsymbol{t})$ and $(\widetilde{\boldsymbol{m}}, \boldsymbol{m})$ are vector bundle morphisms.

A *vector bundle object in the category of Lie groupoids* is a diagram of Lie groupoids and vector bundles such that it satisfies the following:

i) $(\widetilde{q}, q)$ is a Lie groupoid morphism

ii) $\Gamma\,{}_{\widetilde{q}}\!\times_{\widetilde{q}} \Gamma \rightrightarrows V\,{}_q\!\times_q V$ is a Lie groupoid with the obvious structure maps.

iii) The addition $+\colon \Gamma\,{}_{\widetilde{q}}\!\times_{\widetilde{q}} \Gamma \to \Gamma$ is a Lie groupoid morphism over $+\colon V\,{}_q\!\times_q V \to V$.

The different descriptions of $\mathcal{VB}$-groupoids luckily coincide, and therefore any of these views is equally valid. We will not give a proof of this and only give the statement.

Proposition 3.1.4 ([?GraciaSaz2017, Prp. 3.5])**.** *Let $(\Gamma, V, \mathcal{G}, M)$ be a diagram of Lie groupoids and vector bundles. The following are equivalent:*

i) *It is a $\mathcal{VB}$-groupoid.*

ii) *It is a Lie groupoid object in the category of vector bundles.*

iii) *It is a vector bundle object in the category of Lie groupoids.*

Using all these different interpretations of $\mathcal{VB}$-groupoids, we can more easily deduce some of the algebraic properties of a $\mathcal{VB}$-groupoid when compared to just diagrams of Lie groupoids and vector bundles. In particular, we obtain the following:

Corollary 3.1.5. *Let $(\Gamma, V, \mathcal{G}, M)$ be a $\mathcal{VB}$-groupoid, then unit and inverse pairs are also vector bundle morphisms, such that for $x \in M$ we have*

$$\widetilde{1}_{0_x} = \widetilde{0}_{1_x}.$$

Notation. Given a $\mathcal{VB}$-groupoid $(\Gamma, V, \mathcal{G}, M)$, we will often refer to it simply by Γ. The other structures can then all be found internally via the natural embeddings along the unit map or zero section. Additionally, we will refer to both Lie groupoid objects in the category of vector bundles and vector bundle objects in the category of Lie groupoids as a $\mathcal{VB}$-groupoid, due to their equivalence.
Additionally, we will say that Γ is a $\mathcal{VB}$-groupoid over $\mathcal{G}$, when $\mathcal{G}$ is its base groupoid, and sometimes we will denote $(\Gamma, V, \mathcal{G}, M)$ as the $\mathcal{VB}$-groupoid (Γ, V) over $\mathcal{G} \rightrightarrows M$.

Again, we need to complete the category of $\mathcal{VB}$-groupoids by defining their morphisms. These morphisms should preserve all the internal structure, both of the vector bundles and the Lie groupoids.

Definition 3.1.6. A *$\mathcal{VB}$-groupoids morphism* from $(\Gamma, V, \mathcal{G}, M)$ to $(\Omega, W, \mathcal{H}, N)$ is a map $\Phi\colon \Gamma \to \Omega$ such that there exist maps

$$\Phi_0\colon V \to W, \quad \phi\colon \mathcal{G} \to \mathcal{H}, \quad \phi_0\colon M \to N$$

such that they fit into the following diagrams of $\mathcal{VB}$-groupoids:

$$\begin{tikzcd}
\Gamma \arrow[r, shift left] \arrow[r, shift right] \arrow[d] \arrow[rr, bend left, "\Phi"] & V \arrow[d] \arrow[rr, bend left, "\Phi_0"] & \Omega \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & W \arrow[d] \\
\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[rr, bend right, "\phi"'] & M \arrow[rr, bend right, "\phi_0"'] & \mathcal{H} \arrow[r, shift left] \arrow[r, shift right] & N
\end{tikzcd}$$

By which we mean the following:

- (Φ, ϕ) and (Φ_0, ϕ_0) are vector bundle morphisms.

- (Φ, Φ_0) and (ϕ, ϕ_0) are Lie groupoid morphisms.

If both of the $\mathcal{VB}$-groupoids are over the same groupoid, we will assume that (ϕ, ϕ_0) is the identity morphism unless explicitly stated.
The composition of $\mathcal{VB}$-groupoid morphisms is simply given by the composition of all the associated maps.

Clearly, if Φ is a $\mathcal{VB}$-groupoid morphism, we can recover Φ_0 as it fits into a Lie groupoid map and ϕ can be recovered by restricting to the zero section of $\Gamma \to \mathcal{G}$, the map ϕ_0 is then recovered as the restriction of Φ_0 to the zero section or ϕ to the units. Hence, like in the definition and the notation remark after Corollary **??**, we often only denote the map on the top left space of the $\mathcal{VB}$-groupoid.

Using the definition of a $\mathcal{VB}$-groupoid morphism, we can also easily define a $\mathcal{VB}$-subgroupoid based on the notion of Lie subgroupoids defined in Definition **??**.

Definition 3.1.7. A *$\mathcal{VB}$-subgroupoid* of Γ is a $\mathcal{VB}$-groupoid Ω with an embedding $\Phi\colon \Gamma \to \Omega$ which fits into a $\mathcal{VB}$-groupoid morphism.

In particular, when we restrict a $\mathcal{VB}$-subgroupoid to any of its internal Lie groupoids or vector bundles, it will define a subobject of the associated internal structure of the original groupoid.

Lastly, we recall that a vector bundle morphism which is an isomorphism on each fibre, covering a diffeomorphism, will automatically be a vector bundle isomorphism, as its fibrewise inverse is then automatically smooth. For Lie groupoids, we saw that being a diffeomorphism automatically makes it a Lie groupoid isomorphism. With this in mind, we obtain the following statement.

Proposition 3.1.8. *Let $\Phi\colon \Gamma \to \Omega$ be a $\mathcal{VB}$-groupoid morphism covering a Lie groupoid isomorphism. If on each fibre of Γ as a vector bundle it is an isomorphism of vector spaces, then it is a $\mathcal{VB}$-groupoid isomorphism.*

Proof. Let $\Phi\colon \Gamma \to \Omega$ be a $\mathcal{VB}$-groupoid morphism as above. We remark that $\Phi\colon \Gamma \to \Omega$ automatically defines an isomorphism of vector bundles, as it covers a diffeomorphism. Additionally, this implies that it is a diffeomorphism Lie groupoid morphism, and thus it is also an isomorphism of the top Lie groupoid structures.

We conclude that its inverse will define a $\mathcal{VB}$-groupoid morphism, where the associated maps are exactly the inverses of the associated maps. □

3.2 Constructions on $\mathcal{VB}$-groupoids

As a $\mathcal{VB}$-groupoid contains the algebraic structures of both Lie groupoids and vector bundles, we can translate some of the "algebraic" constructions to this setting as well. In this case, we are mostly interested in the constructions coming from vector bundle theory, which are associated to connections, see Section **??**, namely, direct sums, pullbacks, kernels and images. While the objects are often the canonically induced ones by considering the underlying vector bundle structures and remarking that they are functorial, there is often still some checking to do to make sure that the multiplicative structure translates as well.

Example 3.2.1. Let us fix a Lie groupoid $\mathcal{G} \rightrightarrows M$, and suppose that (Γ, V) and (Ω, W) are $\mathcal{VB}$-groupoids over $\mathcal{G}$. The *direct sum of $\mathcal{VB}$-groupoids over $\mathcal{G}$* of Γ and Ω is defined by the following diagram of Lie groupoids and vector bundles:

$$\begin{tikzcd}
\Gamma \oplus \Omega \arrow[r, shift left, "\widetilde{s}_\oplus"] \arrow[r, shift right, "\widetilde{t}_\oplus"'] \arrow[d, "\widetilde{q}"'] & V \oplus W \arrow[d, "q"] \\
\mathcal{G} \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & M
\end{tikzcd}$$

Here, we remark that $\Gamma \oplus \Omega = \Gamma \ {}_{\widetilde{q}_\Gamma}\!\times_{\widetilde{q}_\Omega} \Omega$ is a fibred product and thus we have a canonical map $\widetilde{q}\colon \Gamma \oplus \Omega \to \mathcal{G}$. Similarly, $V \oplus W$ obtains a canonical map to M, which we denoted by q. Moreover, as the source and target maps on Γ and Ω are vector bundle morphisms over the source and target of $\mathcal{G}$, by functoriality we obtain maps defined on the direct sums of the vector bundles as well, which we denote by $\boldsymbol{s}_\oplus$ and $\boldsymbol{t}_\oplus$. In particular, the pairs $(\boldsymbol{s}_\oplus, \boldsymbol{s})$ and $(\boldsymbol{t}_\oplus, \boldsymbol{t})$ define vector bundle morphism. Moreover, we automatically know that $\Gamma \oplus \Omega \rightrightarrows V \oplus W$ defines a Lie groupoid as it is the fibred product over surjective submersions. We can conclude that this indeed defines a $\mathcal{VB}$-groupoid.

Additionally, one readily verifies that this construction is functorial and thus pairs of $\mathcal{VB}$-groupoid morphisms define a $\mathcal{VB}$-groupoid morphism on their respective direct sums.

Remark that this categorically defines a direct sum as the inclusion into each component, and the projections define $\mathcal{VB}$-groupoid morphisms. //

Example 3.2.2. Let (Γ, V) be a $\mathcal{VB}$-groupoid over $\mathcal{G} \rightrightarrows M$, and $\phi\colon \mathcal{H} \to \mathcal{G}$ a Lie groupoid morphism, where $\mathcal{H}$ is a Lie groupoid over N. As vector bundles, we can then define $\phi^*\Gamma \to \mathcal{H}$ and $\phi_0^* V \to N$. Due to the functoriality of this assignment, we obtain induced structure maps on the top Lie groupoid. Moreover, this is indeed a Lie groupoid by Corollary **??**, as we can rewrite it as

$$(\phi^*\Gamma \rightrightarrows \phi_0^* V) \cong (\Gamma \ {}_{\widetilde{q}}\!\times_{\phi} \mathcal{H} \rightrightarrows V \ {}_{q}\!\times_{\phi_0} N) \cong (\Gamma \rightrightarrows V) \ {}_{\widetilde{q}}\!\times_{\phi} (\mathcal{H} \rightrightarrows N).$$

We remark that $\widetilde{q}$ is a surjective submersion, such that its intersection with ϕ is clean. We call the resulting $\mathcal{VB}$-groupoid the *pullback $\mathcal{VB}$-groupoid*. //

Example 3.2.3. To determine the kernel of a $\mathcal{VB}$-groupoid morphism, we remark that the inverse images of Lie groupoids were only defined under the assumption of clean intersection. However, for vector bundles, we need the additional assumption of constant rank, which automatically implies clean intersection.

Let $\Phi\colon \Gamma \to \Omega$ be a $\mathcal{VB}$-groupoid morphism, where (Γ, V) is a $\mathcal{VB}$-groupoid over $\mathcal{G} \rightrightarrows M$ and (Ω, W) over $\mathcal{H} \rightrightarrows N$, such that both Φ and Φ_0 have constant rank as a vector bundle morphism, then it will in particular have a clean intersection with the zero section of Ω. As the vector bundle kernel of Φ is given by the following fibred product:

$$\ker \Phi = \{\gamma \in \Gamma \mid \Phi(\gamma) = 0\} \cong \Gamma \ {}_{\Phi}\!\times_{\widetilde{0}} \mathcal{H}.$$

We can remark that it fits into a Lie groupoid over $\ker \Phi_0 \cong V \ {}_{\Phi_0}\!\times_0 N$ by Corollary **??**, where one has to imagine these as a fibred product of Lie groupoids. Again, one can readily verify that that $(\ker \Phi, \ker \Phi_0)$ defines a $\mathcal{VB}$-groupoid over $\mathcal{G} \rightrightarrows M$. In particular, this is a $\mathcal{VB}$-subgroupoid of Γ.

Remark that this construction will, in particular, hold if Φ is fibrewise surjective, as this will imply that ϕ_0 is surjective as well, and they are both of constant rank.

A similar restriction needs to be placed on the map when we want to determine the image. However, in this case, we run into additional problems as the images of Lie groupoid morphisms may not be a Lie groupoid, see Example **??**. In the case where Φ is of constant rank, Φ_0 is injective and ϕ is an embedding, we can quite easily check that $(\operatorname{im} \Phi, \operatorname{im} \Phi_0)$ defines a $\mathcal{VB}$-groupoid over $\operatorname{im} \phi$. //

Terminology. In the above example, the $\mathcal{VB}$-groupoid morphism Φ admits two kernels, namely as a Lie groupoid morphism and as a $\mathcal{VB}$-groupoid morphism. To differentiate between these two kernels, we will write $\ker_{\mathrm{LG}} \Phi$ for the Lie groupoid kernel and reserve $\ker \Phi$ for the full $\mathcal{VB}$-groupoid as defined in the above example.

3.3 Short exact sequences

Another indispensable element in describing connections was that of short exact sequences. As mentioned, in any general category with kernels and images, one can define a short exact sequence. However, as we saw in Example **??**, not every $\mathcal{VB}$-groupoid morphism defines an image or a kernel. However, as we will restrict ourselves to sequences of $\mathcal{VB}$-groupoids covering a single groupoid, $\mathcal{G} \rightrightarrows M$, we can drop some assumptions.

Definition 3.3.1. A *short exact sequence* of $\mathcal{VB}$-groupoids Γ, Γ' and Γ'' consists of a pair of $\mathcal{VB}$-groupoid morphisms $\iota\colon \Gamma \to \Gamma'$ and $\pi\colon \Gamma' \to \Gamma''$ covering the identity map on the base groupoid, denoted as

$$0 \longrightarrow \Gamma \xrightarrow{\ \iota\ } \Gamma' \xrightarrow{\ \pi\ } \Gamma'' \longrightarrow 0$$

such that they fit into the following short exact sequence of vector spaces at any $g \in \mathcal{G}$:

$$0 \longrightarrow \Gamma_g \xrightarrow{\ \iota_g\ } \Gamma'_g \xrightarrow{\ \pi_g\ } \Gamma''_g \longrightarrow 0$$

In the definition of a short exact sequence, we only assume that the sequence is short exact on the top vector bundle, as this will automatically imply that the sequence on the base vector bundles is also short exact.

Lemma 3.3.2. *Let (Γ, E), (Γ', V') and (Γ'', V'') be $\mathcal{VB}$-groupoids over $\mathcal{G}$ which fit into the following short exact sequence of $\mathcal{VB}$-groupoids:*

$$0 \longrightarrow \Gamma \xrightarrow{\ \iota\ } \Gamma' \xrightarrow{\ \pi\ } \Gamma'' \longrightarrow 0$$

Then we also have a short exact sequence of vector bundles over M of the form:

$$0 \longrightarrow V \xrightarrow{\ \iota_0\ } V' \xrightarrow{\ \pi_0\ } V'' \longrightarrow 0$$

Proof. Suppose that (Γ, V), (Γ', V') and (Γ'', V''), and ι and π are as in the Lemma. Remark that we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{1_x} \arrow[r, "\iota"] \arrow[d, shift left, "\widetilde{s}"] & \Gamma'_{1_x} \arrow[r, "\pi"] \arrow[d, shift left, "\widetilde{s}"] & \Gamma''_{1_x} \arrow[r] \arrow[d, shift left, "\widetilde{s}"] & 0 \\
0 \arrow[r] & V_x \arrow[r, "\iota_0"] \arrow[u, bend left, "\widetilde{u}"] & V'_x \arrow[r, "\pi_0"] \arrow[u, bend left, "\widetilde{u}"] & V''_x \arrow[r] \arrow[u, bend left, "\widetilde{u}"] & 0
\end{tikzcd}$$

We can then, by chasing this diagram and in particular, using the fact that $\widetilde{\boldsymbol{s}} \circ \widetilde{\boldsymbol{u}} = \mathrm{id}$, show that the exactness of the top row implies the exactness of the bottom.

For the exactness at V, take an arbitrary $v \in \ker \iota_0|_m$. As (ι, ι_0) is a Lie groupoid map we find that

$$\iota(\widetilde{\boldsymbol{u}}_\Gamma(v)) = \widetilde{\boldsymbol{u}}'_\Gamma(\iota_0(v)) = \widetilde{\boldsymbol{u}}'_\Gamma(0_m) = \widetilde{0}_{\boldsymbol{u}(m)},$$

where the last step follows that $(\widetilde{\boldsymbol{u}}'_\Gamma, \boldsymbol{u})$ is a vector bundle morphism. This implies that $\widetilde{\boldsymbol{u}}_\Gamma(v) \in \ker_{\text{Vect}} \iota$ and thus it vanishes, i.e. $\widetilde{\boldsymbol{u}}_\Gamma(v) = \widetilde{0}_{\boldsymbol{u}(m)}$. However, notice that $\widetilde{\boldsymbol{u}}_\Gamma(0_m) = \widetilde{0}_{\boldsymbol{u}(m)}$ as well, and as $\widetilde{\boldsymbol{u}}_\Gamma$ is injective, $0_m = v$.

Next, we remark that the following holds as the base maps of Lie groupoids are determined by their total maps:

$$\pi_0 \circ \iota_0 = \widetilde{\boldsymbol{s}}_{\Gamma''} \circ \pi \circ \widetilde{\boldsymbol{u}}'_\Gamma \circ \widetilde{\boldsymbol{s}}'_\Gamma \circ \iota \circ \widetilde{\boldsymbol{u}}_\Gamma = \widetilde{\boldsymbol{s}}_{\Gamma''} \circ \pi \circ \iota \circ \widetilde{\boldsymbol{u}}_\Gamma = \widetilde{\boldsymbol{s}}_{\Gamma''} \circ\ 0\widetilde{\boldsymbol{u}}_\Gamma = 0,$$

where the last step follows as $(\widetilde{\boldsymbol{s}}, \boldsymbol{s})$ is a vector bundle morphism. This immediately implies that $\operatorname{im} \iota_0 \subset \ker \pi_0$. Now, if $f \in \ker \pi_0|_m$, then remark that, using again the fact that our structure maps are vector bundle morphisms,

$$\widetilde{0}_{\boldsymbol{u}(m)} = \widetilde{\boldsymbol{u}}_\Gamma(0_m) = \widetilde{\boldsymbol{u}}_\Gamma(\pi_0(f)) = \pi(\boldsymbol{u}(f)) = \pi(\widetilde{\boldsymbol{u}}'_\Gamma(f)).$$

This implies that $\widetilde{\boldsymbol{u}}'_\Gamma(f) \in \ker_{\text{Vect}} \pi = \operatorname{im} \iota$. Take $\gamma \in \Gamma$ be such that $\iota(\gamma) = \widetilde{\boldsymbol{u}}'_\Gamma(f)$. Taking the source of γ then gives us our desired element.

$$\iota_0(\widetilde{\boldsymbol{s}}_\Gamma(\gamma)) = \widetilde{\boldsymbol{s}}'_\Gamma(\iota(\gamma)) = \widetilde{\boldsymbol{s}}'_\Gamma(\widetilde{\boldsymbol{u}}'_\Gamma(f)) = f.$$

Therefore, we also find that $\ker \pi_0 \subset \operatorname{im} \iota_0$ and hence the sequence is exact at V'.

For the last exactness, we remark that if $v \in V''$, then $\widetilde{\boldsymbol{u}}_{\Gamma''}(v) \in \operatorname{im} \pi$, and thus there exists some $\omega \in \Gamma'$ with $\pi(\omega) = \widetilde{\boldsymbol{u}}_{\Gamma''}(v)$. It then follows that

$$\pi_0(\widetilde{\boldsymbol{s}}'_\Gamma(\omega)) = \widetilde{\boldsymbol{s}}_{\Gamma''}(\pi(\omega)) = \widetilde{\boldsymbol{s}}_{\Gamma''}(\widetilde{\boldsymbol{u}}_{\Gamma''}(v)) = v.$$

Therefore π_0 is surjective at each fibre and the base sequence is therefore short exact. $\square$

Recall that for short exact sequences of vector bundles, a splitting always exists due to the existence of fibrewise inner products, and we obtain multiple equivalent definitions of splittings, see Lemma **??**. For a short exact sequence of $\mathcal{VB}$-groupoids, a splitting might not exist; however, we can show that the different ways of describing them still coincide for $\mathcal{VB}$-groupoids.

Lemma 3.3.3. *Let Γ, Γ' and Γ'', fit into a short exact sequence of $\mathcal{VB}$-groupoids over $\mathcal{G} \rightrightarrows M$.*

$$0 \longrightarrow \Gamma \xrightarrow{\ i\ } \Gamma' \xrightarrow{\ \pi\ } \Gamma'' \longrightarrow 0$$

Then, there is a 1-1 correspondence between then following objects:

- ◇ *$\mathcal{VB}$-groupoid morphisms $h\colon \Gamma'' \to \Gamma'$ such that $\pi \circ h = \mathrm{id}$.*
- ◇ *$\mathcal{VB}$-groupoid morphisms $p\colon \Gamma' \to \Gamma$ such that $p \circ \iota = \mathrm{id}$.*
- ◇ *Isomorphisms of $\mathcal{VB}$-groupoids $\phi\colon \Gamma' \to \Gamma \oplus \Gamma''$ which is a splitting.*
- ◇ *Complements to $\iota(\Gamma)$ in Γ' as which are $\mathcal{VB}$-subgroupoids.*

These correspondences are determined uniquely by $h \circ \pi + i \circ \theta = \mathrm{id}_\Omega$ and $C = \ker\theta = \operatorname{im} h$. Moreover, if C is a complement of Γ in Ω, then $\pi|_C\colon C \to \Gamma'$ is an isomorphism of $\mathcal{VB}$-groupoids.

Proof of Lemma **??**. Let Γ, Γ' and Γ'', be $\mathcal{VB}$-groupoids over $G \rightrightarrows M$ with two $\mathcal{VB}$-groupoid morphisms $\iota\colon \Gamma \to \Gamma'$ and $\pi\colon \Gamma' \to \Gamma''$, covering the identity, such that they fit into a short exact sequence of $\mathcal{VB}$-groupoids over G. We will prove that right splittings are 1-1 with complements, and that left splittings are 1-1 with complements. The properties are then a direct consequence of Lemma **??**. In this proof, we will often implicitly use Proposition **??**

Complements $\iff$ splittings: Clearly, a complement defines a splitting and vice versa by chasing some diagrams.

Right splitting $\iff$ splitting: Suppose $h\colon \Gamma'' \to \Gamma'$ is a right splitting of the short exact sequence, such that $\pi \circ h = \mathrm{id}_{\Gamma''}$. We will show that $\Gamma'' \oplus \Gamma$ defines a $\mathcal{VB}$-groupoid which is isomorphic to Γ' via the isomorphism $\phi\colon \Gamma'' \oplus \Gamma \to \Gamma'\colon (u, v) \mapsto h(u) + i(v)$. As h and i are $\mathcal{VB}$-groupoid maps, and $+$ is a Lie groupoid map, this will define a $\mathcal{VB}$-groupoid map as well. Lastly, we need to show that this is injective, as surjectivity will then follow by counting dimensions. If we suppose $\phi(u, v) = 0$, then it follows that

$$0 = \pi(0) = \pi(\phi(u, v)) = \phi(h(u)) + \pi(i(v)) = u.$$

We can conclude that $0 = \pi(u) + i(v) = i(v)$. As i is injective, $v = 0$ as well and thus ϕ is injective and an isomorphism.

Conversely, suppose that $\phi\colon \Gamma' \to \Gamma \oplus \Gamma''$ is a splitting of the short exact sequence. We can then define a right inverse to π as $h\colon \Gamma''\colon \Gamma'\colon u \mapsto \phi^{-1}(\mathrm{incl}_2(u))$. By the assumption that ϕ is a splitting it follows that

$$\pi(h(u)) = \mathrm{pr}_2 \circ \phi \circ \phi^{-1} \circ \mathrm{incl}_2(u) = \mathrm{pr}_2 \circ \mathrm{incl}_2(u) = u.$$

Therefore, it indeed defines a right splitting.

Left splitting $\iff$ complement: Suppose that $\theta\colon \Gamma \to \Gamma'$ is a left splitting, such that $\theta \circ i = \mathrm{id}_\Gamma$. We can then define $C = \ker_{\mathcal{VB}} \theta$, which is well-defined as θ is fibrewise surjective. Then consider the $\mathcal{VB}$-groupoid morphism $\psi\colon C \oplus \Gamma \to \Gamma'\colon (u, v) \mapsto u + i(v)$, as before this is a $\mathcal{VB}$-groupoid morphism as i is and $+$ is a Lie groupoid morphism. So see that this is an isomorphism, we again only need to check that it is injective. Suppose $\psi(u, v) = 0$, then

$$0 = \theta(\phi(u, v)) = \theta(u + i(v)) = \theta \circ i(v) = v.$$

It follows that $0 = \phi(u, v) = u$ and thus ϕ is indeed injective.

Conversely, given a complement C of Γ in Γ', then the projection onto the first component defines a left splitting.

□

3.4 Multiplicative differential forms

As a last application of $\mathcal{VB}$-groupoid, we will define multiplicative differential forms, and in particular with values in $\mathcal{VB}$-groupoids. To obtain a multiplicative structure on forms, we first discuss the simplest case: 0-forms.

Given a map $f\colon \mathcal{G} \to \mathbb{R}$, there is a canonical way of requiring multiplicativity:

$$f(gh) = f(g) + f(h), \quad \text{for all } (g, h) \in \mathcal{G}^{(2)}.$$

Hence, a 0-form is simply multiplicative if it defines a Lie groupoid map to $\mathbb{R}$. We can then rewrite the set of multiplicative functions as the kernel of the following map:

$$\partial\colon C^\infty(\mathcal{G}) \to C^\infty(\mathcal{G}^{(2)})\colon f \mapsto (\boldsymbol{m}^* - \mathrm{pr}_1^* - \mathrm{pr}_2^*)f.$$

With some effort, we can recognise it is obtained as the pullback along the differential of the cochain complex associated to the nerve of $\mathcal{G}$, see [?Cardenas2021]. In particular, this then has an easy generalisation to higher-degree differential forms

Definition 3.4.1. For a Lie groupoid $\mathcal{G}$ we call a differential form $\tau \in \Omega(\mathcal{G})$ *multiplicative* if

$$\boldsymbol{m}^*\tau = \mathrm{pr}_1^*\tau + \mathrm{pr}_2^*\tau$$

where $\boldsymbol{m}, \mathrm{pr}_1, \mathrm{pr}_2\colon \mathcal{G}^{(2)} \to \mathcal{G}$. We denote the multiplicative forms on $\mathcal{G}$ by $\Omega_{\mathrm{mult}}(\mathcal{G})$.

Here, we remark that this is, in particular, a single level in the Bott-Shulmann-Stasheff double complex, as laid out in [?Bott1976]. However, there is an alternative description of multiplicative forms by realising them as maps of vector bundles. For working with multiplicative forms, it is useful to realise them as morphisms.

Proposition 3.4.2. *A k-form $\tau \in \Omega^k(\mathcal{G})$ on a Lie groupoid $\mathcal{G}$ is multiplicative if and only if the map*

$$\begin{tikzcd} \bigoplus^k T\mathcal{G} \arrow[r, "\overline{\tau}"] \arrow[d, shift right, "T\boldsymbol{s}_\oplus"'] \arrow[d, shift left, "T\boldsymbol{t}_\oplus"] & (\mathbb{R}, +) \arrow[d] \\ \bigoplus^k T\mathcal{G}_0 \arrow[r] & \{*\} \end{tikzcd} \qquad \overline{\tau}\colon v = (v_1, \dots, v_k) \mapsto \tau(v_1, \dots, v_k)$$

is a Lie groupoid morphism.

Proof. Let $\mathcal{G}$ be a Lie groupoid and $\tau \in \Omega^k(\mathcal{G})$. By Example **??**, the k-fold direct sum of $T\mathcal{G}$ is a $\mathcal{VB}$-groupoid and thus in particular a Lie groupoid. Remark that for any $(u, v) \in \left(\bigoplus^k T\mathcal{G}\right)^{(2)}$, we have

$$\begin{aligned} \partial\tau(u, v) &= (\boldsymbol{m}^* - \mathrm{pr}_1^* - \mathrm{pr}_2^*)\tau(u, v), \\ &= \boldsymbol{m}^*\tau(u, v) - \mathrm{pr}_1^*\tau(u, v) - \mathrm{pr}_2^*\tau(u, v), \\ &= \overline{\tau}(T\boldsymbol{m}(u, v)) - \overline{\tau}(T\,\mathrm{pr}_1(u, v)) - \overline{\tau}(T\,\mathrm{pr}_2(u, v)), \\ &= \overline{\tau}(T\boldsymbol{m}(u, v)) - \overline{\tau}(u) - \overline{\tau}(v) \end{aligned}$$

This implies that $\partial\tau = 0$ if and only if $\overline{\tau}(T\boldsymbol{m}(u, v)) = \overline{\tau}(u) + \overline{\tau}(v)$, i.e. $\overline{\tau}$ is a Lie groupoid morphism. □

As this relates the multiplicativity of a k-form to its induced map on the underlying $\mathcal{VB}$-groupoids, we can more easily translate this notion to general settings as well.

The first of these is an extension of multiplicative forms to $\mathcal{VB}$-groupoids. Recall that one can define a k-form on a vector bundle V as a section of $\bigwedge^k V^*$, but that these are in 1-1 correspondence with alternating fibrewise multilinear maps $\bigoplus^k V \to \mathbb{R}$. This is exactly the setting in which Proposition **??** defines multiplicativity.

Definition 3.4.3. Let $\Gamma \rightrightarrows V$ be a $\mathcal{VB}$-groupoid, a k-form $\tau \in \Omega^k(\Gamma)$ is *multiplicative* if the map

$$\begin{tikzcd}
\bigoplus^k \Gamma \arrow[r, "\overline{\tau}"] \arrow[d, shift right, "T\boldsymbol{s}_\oplus"'] \arrow[d, shift left, "T\boldsymbol{t}_\oplus"] & (\mathbb{R}, +) \arrow[d] \\
\bigoplus^k V \arrow[r] & \{*\}
\end{tikzcd} \qquad \overline{\tau}\colon v = (v_1, \dots, v_k) \mapsto \tau(v_1, \dots, v_k)$$

is a Lie groupoid morphism. We denote the multiplicative forms on Γ by $\Omega_{\mathrm{mult}}(\Gamma)$.

Alternatively, they can we defined as the k-forms on Γ which lie in the kernel of the map

$$\partial\colon \Omega(\Gamma) \to \Omega(\Gamma^{(2)})\colon \tau \mapsto \tau \circ (\widetilde{\boldsymbol{m}} - \mathrm{pr}_1 - \mathrm{pr}_2).$$

This definition is again more similar to our original definition of multiplicative forms, but it makes it harder to work with.

The second generalisation we can make is not on the left-hand side of the diagram, but on the right-hand side. Recall that a manifold can admit forms with values in a vector bundle $V \to M$ by considering sections of $\bigwedge^k T^*M \otimes V$, or equivalently, we are interested in alternating fibrewise multilinear maps $\bigotimes^k TM \to V$. Again, using Proposition **??**, this easily translates to the multiplicative setting.

Definition 3.4.4. Let $\mathcal{G}$ be a Lie groupoid and $\Gamma \rightrightarrows V$ a $\mathcal{VB}$-groupoid over $\mathcal{G}$. A k-form τ on $\mathcal{G}$ with values in Γ is called *multiplicative* if the map $\overline{\tau}\colon \bigoplus^k T\mathcal{G} \to \Gamma$ defined as

$$\begin{tikzcd}
\bigoplus^k T\mathcal{G} \arrow[r, "\overline{\tau}"] \arrow[d, shift right, "T\boldsymbol{s}_\oplus"'] \arrow[d, shift left, "T\boldsymbol{t}_\oplus"] & \Gamma \arrow[d, shift right] \arrow[d, shift left] \\
\bigoplus^k T\mathcal{G}_0 \arrow[r] & V
\end{tikzcd} \qquad \overline{\tau}\colon v = (v_1, \dots, v_k) \mapsto \tau(v_1, \dots, v_k)$$

is a Lie groupoid morphism. We denote the multiplicative forms with values in Γ by $\Omega_{\mathrm{mult}}(\mathcal{G}; \Gamma)$.

Chapter 4

Fibred Lie Groupoids and Multiplicative connections

The theory of what we will call fibred Lie groupoids is a generalisation of the theory of Lie groupoid extensions as proposed in [?LaurentGengoux2009] and [?Fernandes2023], which are a generalisation of short exact sequences of groups and surjective submersions. While they focused on Lie groupoid morphisms, which are surjective submersions and cover the identity map, we will relax the second condition so that they may cover an arbitrary map. These types of objects can then be viewed as surjective submersions in the category of Lie groupoids, and we therefore will try to replicate the theory of Chapter **??**. Additionally, they will be a generalisation of the notion of a family of Lie groupoids, as in [?Cardenas2021], which we will discuss in more detail in this chapter as well.

The central focus of this chapter is the development of the basic notions of fibred Lie groupoids and the associated families of Lie groupoids, for which we introduce a concept of local triviality. We then define multiplicative Ehresmann connections using the language of $\mathcal{VB}$-groupoids, and in particular, demonstrate that these connections are also the algebraically correct formulation. Using this framework of multiplicative connections, we prove an analogue of Theorem **??** for families of Lie groupoids. We further examine multiplicative connections on arbitrary fibred Lie groupoids by relating their completeness to an internal family of Lie groupoids and the base surjective submersion. Finally, we show that a family of Lie groupoids admitting sufficiently many lifts of arrows, meaning it admits a cleavage, gives rise to internal equivalences between unit fibres, in the sense of Morita equivalence.

4.1 Basic Definitions

Let us start with the definition of the objects of interest.

Definition 4.1.1. A *fibred Lie groupoid* is a Lie groupoid morphism $\phi\colon \mathcal{G} \to \mathcal{H}$ such that ϕ is a surjective submersion.

Recall that the base map of a Lie groupoid morphism can be obtained as the unique map ϕ_0, such that $\phi_0 \circ \boldsymbol{s} = \boldsymbol{s} \circ \phi$. In particular, we see that the fibre of a fibred Lie groupoid $\phi\colon \mathcal{G} \to \mathcal{H}$ at some $y \xleftarrow{h} x \in \mathcal{H}$ is automatically an embedded submanifold of $\mathcal{G}$, and that the source and target maps restrict to

$$\boldsymbol{s}\colon \phi^{-1}(h) \mapsto \phi_0^{-1}(x), \quad \boldsymbol{t}\colon \phi^{-1}(h) \mapsto \phi_0^{-1}(y).$$

If $\boldsymbol{s}(h) \neq \boldsymbol{t}(h)$, then the restriction of $\boldsymbol{m}$ to $\phi^{-1}(h)$ is defined on an empty set. Even if $\boldsymbol{s}(h) = \boldsymbol{t}(h)$, then the fibre $\phi^{-1}(h)$ can only contain a unit if h is a unit. This implies that a lot of fibres, namely any fibre above $\mathcal{H}\backslash\mathcal{H}_0$, cannot contain any algebraic data on its own. In particular, this means that a local trivialisation of ϕ, which would look like $U \times F$, cannot contain any interesting data as $\phi^{-1}(U)$ is not necessarily a groupoid.

Luckily, a fibred Lie groupoid does contain an internal fibred structure whose fibres are automatically Lie groupoids, namely: $\ker\phi \subset \mathcal{G}$, which is smooth by Corollary **??**. We can see that these are exactly the fibres of ϕ which have a natural groupoid structure as they contain units and are closed under multiplication. As $\ker\phi$ and $\mathcal{H}_0$ are embedded submanifold, we obtain a fibred Lie groupoid $\phi\colon \ker\phi \to \mathcal{H}_0$ covering $\phi_0\colon \mathcal{G}_0 \to \mathcal{H}_0$, where we interpret $\mathcal{H}_0$ as the identity groupoid. Given a fibred Lie groupoid ϕ, we will refer to $\phi\colon \ker\phi \to \mathcal{H}_0$ as the kernel bundle.

Remark. If ϕ_0 is a diffeomorphism, then the kernel bundle is a bundle of Lie groups. //

Clearly, using Corollary **??** the fibres of $\phi\colon \ker\phi \to \mathcal{H}_0$ are Lie groupoids. The kernel, therefore, carries the natural structure of a collection of Lie groupoids, which is parametrised by some manifold.

4.1.1 Families of Lie groupoids

To highlight the structure of a kernel of a Lie groupoid morphism as a collection of Lie groupoids, we will discuss this structure separately. We will first come up with a slightly different definition of such families, but quickly see that they are equivalent.

Definition 4.1.2. A *family of Lie groupoids over B* consists of a Lie groupoid $\mathcal{K}$ and a surjective submersion $p\colon \mathcal{K}_0 \to B$ such that $p \circ \boldsymbol{s} = p \circ \boldsymbol{t}$.
An *equivalence of families of Lie groupoids* $\mathcal{K}$, with map p, and $\mathcal{H}$, with map p', over B is a Lie groupoid isomorphism $\phi\colon \mathcal{K} \to \mathcal{H}$ such that $p' \circ \phi = p$.

Remark. In the case that p is a diffeomorphism, this becomes a family of Lie groups. These are much more friendly, and this is where the theory of general fibred Lie groupoid diverges from Lie groupoid extensions. //

Notice that a family of Lie groupoids $\mathcal{K}$ with map $p\colon \mathcal{K}_0 \to B$ induces a map $\widetilde{p}\colon \mathcal{K} \to B\colon k \mapsto p \circ \boldsymbol{s}(k)$ which fits into the following commutative diagram:

$$\begin{tikzcd} \mathcal{K} \arrow[r, "\widetilde{p}"] \arrow[d, shift right, "\boldsymbol{t}"'] \arrow[d, shift left, "\boldsymbol{s}"] & B \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\mathrm{id}"] \\ \mathcal{K}_0 \arrow[r, "p"] & B \end{tikzcd}$$

Such that $\widetilde{p}\colon \mathcal{K} \to B$ defines a Lie groupoid morphism to the identity groupoid, and as p is a surjective submersion, so is $\widetilde{p}$. In this way, we obtain a 1-1 correspondence between families of Lie groupoids over B and fibred Lie groupoid over $B \rightrightarrows B$. Hence, we will simply denote $p\colon \mathcal{K} \to B$ for a family of Lie groupoids where we view p as a Lie groupoid map, and let $p_0\colon \mathcal{K}_0 \to B$ be its map on the base, which is recovered as $p_0 = p \circ \boldsymbol{u}$.

Another characterisation of families of Lie groupoids is in terms of the orbit space of the Lie groupoid $\mathcal{K}$. Let us denote this by $X = \mathcal{K}_0/\mathcal{K}$ and $q\colon \mathcal{K}_0 \to X$ for the quotient map. We can call a map $f\colon X \to B$, where B is a manifold, *smooth* if $f \circ q\colon \mathcal{K}_0 \to B$ is smooth and a *submersion* if $f \circ \pi$ is. Then there exists an equivalence between maps $p\colon \mathcal{K} \to B$ defining a family of Lie groupoids and smooth submersions $p'\colon X \to B$. Firstly, the base map of a family $p\colon \mathcal{K} \to B$ is constant on the fibres of q, and thus descends to the quotient. Going the other way, from a smooth submersion $f\colon X \to B$, we can define $p = f \circ q \circ \boldsymbol{s}$ to obtain a family of Lie groupoids.

In this general setting, we obtain a similar result to the remark on kernels, which justifies the name of a family of Lie groupoids.

Proposition 4.1.3. *Let $p\colon \mathcal{K} \to B$ and $p'\colon \mathcal{H} \to B$ be families of Lie groupoids. The fibres of p are Lie subgroupoids over the fibres of p_0, i.e. $p^{-1}(b) \rightrightarrows p_0^{-1}(b)$, and an equivalence of families of Lie groupoids $\phi\colon \mathcal{K} \to \mathcal{H}$ induces a Lie groupoid isomorphism between the fibres.*

Proof. Remark that $b \subset B$ is a subgroupoid, and as p is a surjective submersion, it will have a clean intersection. Therefore $p^{-1}(b)$ defines a Lie groupoid over $p_0^{-1}(b)$ by Corollary **??**.

To see that an equivalence of families of Lie groupoids ϕ induces Lie groupoid morphisms, we notice that the restrictions are well-defined. As ϕ is an isomorphism, there exists a Lie groupoid morphism ϕ^{-1} which is its inverse. If we restrict this inverse to the fibres, we obtain exactly the inverse of ϕ after restricting to the fibres. Therefore, it induces an isomorphism of the fibres as Lie groupoids. □

Our main example of families of Lie groupoids, at least the one we are most interested in, is the kernel of a fibred Lie groupoid. However, many examples come from deformation theory, see [`?Cardenas2021`], but also from working with bundles of Lie groups. Of course, there is also a trivial example.

Example 4.1.4. Given a Lie groupoid $\mathcal{F}$ and manifold B, consider the product groupoid $\mathcal{F} \times B \rightrightarrows \mathcal{F}_0 \times B$ where we view $B \rightrightarrows B$ as the identity groupoids. With the map $\mathrm{pr}_1\colon B \times \mathcal{F} \to B$, this is a family of Lie groupoids over B, called the *trivial family with fibre $\mathcal{F}$*. Moreover, we will call a family of Lie groupoids *trivial* if it is isomorphic to the trivial family. //

Proposition 4.1.5. *Let $p\colon \mathcal{K} \to B$ be a family of Lie groupoids and $U \subset B$ an open subset. The restriction $p\colon p^{-1}(U) \to U$ is a family of Lie groupoids.*

Proof. Remark that $U \subset B$ is an open subgroupoid and therefore by Corollary **??**, $p^{-1}(U)$ is a Lie groupoid. As $T_g p^{-1}(U) = T_g\mathcal{K}$, it follows that $p\colon p^{-1}(U) \to U$ is still a surjective submersion and thus it is a family of Lie groupoids. □

Using the fact that we have a trivial model and that a family of Lie groupoids descends to local data via restrictions, we can define local triviality in the sense of families of Lie groupoids.

Definition 4.1.6. A family of Lie groupoids $p\colon \mathcal{K} \to B$ is called *locally trivial* if there exists a trivialising cover $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ of p as a surjective submersion such that $\psi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathcal{F}$ is a Lie groupoid isomorphism, where $U_\alpha \times \mathcal{F}$ is the trivial family of Lie groupoids with as fibre some Lie groupoid $\mathcal{F}$.

Terminology. In the context of families of Lie groupoids, we will refer to a trivialising cover by equivalences of families of Lie groupoids, i.e. one in the above sense, simply by a trivialising cover.

Proposition 4.1.7. *If $p\colon \mathcal{K} \to B$ is a locally trivial family of Lie groupoids, then $p\colon \mathcal{K}_0 \to B$ is a fibre bundle.*

Proof. Suppose that $p\colon \mathcal{K} \to B$ is a locally trivial family of Lie groupoids and let $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ be a trivialising cover. As $\psi_\alpha\colon p^{-1}(U_\alpha) \to U_\alpha \times \mathcal{H}$ is a Lie groupoid morphism, it will cover a map on the base, denoted by $\phi_{\alpha,0}\colon p_0^{-1}(U_\alpha) \to U_\alpha \times \mathcal{H}_0$. This then fits into the following diagram

$$\begin{tikzcd}
p^{-1}(U) \arrow[r, "\psi_\alpha"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & U_\alpha \times \mathcal{H} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
p_0^{-1}(U) \arrow[r, "\psi_{\alpha,0}"] \arrow[d, "p_0"] & U_\alpha \times \mathcal{H}_0 \arrow[d, "\mathrm{pr}_1"] \\
U_\alpha \arrow[r, equal] & U_\alpha
\end{tikzcd}$$

As ψ_α is an isomorphism, it has an inverse, and the base map of the inverse will be an inverse to $\psi_{\alpha,0}$. Moreover, as $\psi_{\alpha,0}$ is a map which preserve the fibres of p_0 and pr_1, we obtain a local trivialisation.

Collecting all these trivialisations, we conclude that $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ is a trivialising cover for p_0. $\square$

Notice that the map $p\colon \mathcal{K} \to B$ must be a fibre bundle if we want it to be a locally trivial family of Lie groupoids; however, it is not sufficient. Even under mild compactness conditions, like properness of the fibres, the converse implication still does not hold, cf. Remark 7.2 in [?Crainic2018]. We can show that under certain compactness conditions, a family of Lie groupoids is automatically locally trivial.

Proposition 4.1.8 ([?Crainic2018, Thm. 7.8]). *Let $\colon \mathcal{K} \to B$ be a family of Lie groupoids with $\mathcal{K}$ compact, then it is a locally trivial family of Lie groupoids.*

4.2 Multiplicative connections

To describe multiplicative connections on fibred Lie groupoids, we take inspiration from the classical notion, where we defined it as a splitting of the short exact sequence induced by the tangent map. As a fibred Lie groupoid $\phi\colon \mathcal{G} \to \mathcal{H}$ consists of a pair of surjective submersions, we also obtain a pair of vertical bundles, namely $\mathrm{Ver} = \ker T\phi$ and $\mathrm{Ver}_0 = \ker T\phi_0$, where the kernel is a vector bundle morphism. Because ϕ is a Lie groupoid morphism, we can see that these combine to form the $\mathcal{VB}$-groupoid morphism kernel of $T\phi\colon T\mathcal{G} \to T\mathcal{H}$, as was shown in Example **??**. Notice that these then fit into a short exact sequence of $\mathcal{VB}$-groupoids covering $\mathcal{G}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ver} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & T\mathcal{G} \arrow[r, "T\phi"] \arrow[d, shift right] \arrow[d, shift left] & \phi^* T\mathcal{H} \arrow[r] \arrow[d, shift right] \arrow[d, shift left] & 0 \\
0 \arrow[r] & \mathrm{Ver}_0 \arrow[r] & T\mathcal{G}_0 \arrow[r, "T\phi_0"] & \phi_0^* T\mathcal{H}_0 \arrow[r] & 0
\end{tikzcd}$$

Recall that $\phi^*T\mathcal{H} \rightrightarrows \phi_0^*T\mathcal{H}_0$ is the pullback of a $\mathcal{VB}$-groupoid along a Lie groupoid morphism as defined in Example **??**. As we have an analogue to Lemma **??** in the multiplicative setting, namely Lemma **??**, we can easily translate the definition of a connection as a splitting of this short exact sequence to the multiplicative setting.

Definition 4.2.1. Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a fibred Lie groupoid, then define the following:

- *Multiplicative horizontal lift*: A right inverse $h\colon \phi^*T\mathcal{H} \to T\mathcal{G}$ as a $\mathcal{VB}$-groupoid map.
- *Multiplicative connection idempotent*: An idempotent $\mathcal{VB}$-groupoid map $p\colon T\mathcal{G} \to T\mathcal{G}$ such that $\operatorname{im} p = \mathrm{Ver}$.
- *Multiplicative Ehresmann connection*: A $\mathcal{VB}$-subgroupoid $E \subset TM$ complementary to Ver.
- *Multiplicative connection form*: A Ver-valued multiplicative 1-form $\alpha \in \Omega^1_{\text{mult}}(\mathcal{G}; \mathrm{Ver})$ such that $\alpha|_{\mathrm{Ver}} = \mathrm{id}\,|_{\mathrm{Ver}}$. Let us denote $\Omega_{\text{conn}}(\mathcal{G}; \mathrm{Ver})$ for the set of connection forms.

By the nature of $\mathcal{VB}$-groupoid morphisms and $\mathcal{VB}$-subgroupoids, a multiplicative connection canonically defines a connection on the base space as well. In particular, if E is a multiplicative Ehresmann connection on $\phi\colon \mathcal{G} \to \mathcal{H}$, then $\mathrm{E}_0 = \mathrm{E} \cap T\boldsymbol{u}(T\mathcal{G}_0)$ and thus $T\boldsymbol{s}, T\boldsymbol{t}\colon\ \mathrm{E} \to \mathrm{E}_0$ as surjective.

Let us come with a last, but useful interpretation of multiplicative connections in terms of parallel transport, which also shows that these connections correctly intertwine the algebraic and geometric structure of a fibred Lie groupoid.

Proposition 4.2.2. *Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a fibred Lie groupoid and* E *an Ehresmann connection on $\phi\colon \mathcal{G} \to \mathcal{H}$ as a surjective submersion. The following are equivalent:*

i) *E is a multiplicative Ehresmann connection,*

ii) *For $(\gamma_1, \gamma_2) \in C^\infty([0,1]\,, \mathcal{H}^{(2)})$ and $g_i \in \phi^{-1}(\gamma_i(0))$, such that $(g_1, g_2) \in \mathcal{G}^{(2)}$, the following hold:*
$$\widetilde{\gamma_1\gamma_2}_{g_1g_2} = \widetilde{\gamma}_{1g_1}\widetilde{\gamma}_{2g_2}, \quad (\widetilde{\gamma}_{1g_1})^{-1} = \widetilde{(\gamma_1^{-1})}_{g_1^{-1}},$$
wherever the horizontal lifts are defined.

Proof. Fix a $\phi\colon \mathcal{G} \to \mathcal{H}$ be a fibred Lie groupoid and E an Ehresmann connection on $\phi\colon \mathcal{G} \to \mathcal{H}$.

i) $\Longrightarrow$ ii): Suppose that E is a multiplicative connection and recall from Example **??** that $C^\infty([0,1]\,, \mathcal{H})$ carries a groupoid structure. Let us also fix some $(\gamma_1, \gamma_2) \in C^\infty([0,1]\,, \mathcal{H})^{(2)}$ and $g_i \in \phi^{-1}(\gamma_i(0))$, such that $(g_1, g_2) \in \mathcal{G}^{(2)}$.

This proof will rely on the fact that the horizontal lifts are defined locally as a solution to an ordinary differential equation with some initial values: The horizontal lift of a curve γ to x by a connection E is a solution to:
$$\dot{\widetilde{\gamma}}_x(t) = h(\dot\gamma(t)) \in \mathrm{E}_{\widetilde{\gamma}_x(t)}, \quad \widetilde{\gamma}_x(s) = x$$
Therefore, proving that these identities hold comes down to checking the initial values and showing that the derivatives are horizontal.

Let us now show that $\widetilde{\gamma_1\gamma_2}_{g_1g_2} = \widetilde{\gamma}_{1g_1}\widetilde{\gamma}_{2g_2}$. In particular, we need to show that $\boldsymbol{s}(\widetilde{\gamma}_{1g_1}) = \boldsymbol{t}(\widetilde{\gamma}_{2g_2})$. Let us show that $\boldsymbol{s}(\widetilde{\gamma}_{1g_1})$ is the horizontal lift of $\boldsymbol{s}(\gamma_1)$ to $\boldsymbol{s}(g_1)$ with respect to the induced connection E_0, and by symmetry of the problem and the proof this will also implies that $\boldsymbol{t}(\widetilde{\gamma}_2)_{g_2}$ is the horizontal lift of $\boldsymbol{t}(\gamma_2)$ to

$\boldsymbol{t}(g_2)$. Firstly, notice that $\boldsymbol{s}(\widetilde{\gamma_1}_{g_1})(0) = \boldsymbol{s}(g_1)$ and $\phi_0(\boldsymbol{s}(\widetilde{\gamma_1}_{g_1})) = \boldsymbol{s}(\phi(\widetilde{\gamma_1}_{g_1})) = \boldsymbol{s}(\gamma_1)$. Lastly, we need to check that $\boldsymbol{s}(\widetilde{\gamma_1}_{g_1})$ is horizontal. This also follows from a simple calculation and the fact that $\mathrm{E}_0 = T\boldsymbol{s}\,\mathrm{E}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{s}(\widetilde{\gamma}_g) = T\boldsymbol{s}(\dot{\widetilde{\gamma}}_g) \in T\boldsymbol{s}\,\mathrm{E} = \mathrm{E}_0\,.$$

Hence, we can conclude that $\boldsymbol{s}(\widetilde{\gamma}_g) = \widetilde{\boldsymbol{s}(\gamma)}_{\boldsymbol{s}(g)}$ and $\boldsymbol{t}(\widetilde{\gamma}_g) = \widetilde{\boldsymbol{t}(\gamma)}_{\boldsymbol{t}(g)}$. From our choice of γ_i and g_i we find that $\boldsymbol{s}(\gamma_1) = \boldsymbol{t}(\gamma_2)$ and $\boldsymbol{s}(g_1) = \boldsymbol{t}(g_2)$. Applying the argument of the previous paragraph to $\widetilde{\gamma_1}_{g_1}$ and $\widetilde{\gamma_2}_{g_2}$, we obtain $\boldsymbol{s}(\widetilde{\gamma_1}_{g_1}) = \boldsymbol{t}(\widetilde{\gamma_2}_{g_2})$ and thus the lifts can indeed be multiplied.

This implies that the multiplication of $\widetilde{\gamma_1}_{g_1}$ and $\widetilde{\gamma_2}_{g_2}$ is actually well-defined. To show that this multiplication is indeed the horizontal lift of the multiplication, we need to check the initial conditions and the fact that it is horizontal. This is followed by some easy calculations:

$$(\widetilde{\gamma_1}_{g_1}\widetilde{\gamma_2}_{g_2})(0) = \widetilde{\gamma_1}_{g_1}(0)\widetilde{\gamma_2}_{g_2}(0) = g_1g_2, \quad \phi(\widetilde{\gamma_1}_{g_1}\widetilde{\gamma_2}_{g_2}) = \phi(\widetilde{\gamma_1}_{g_1})\phi(\widetilde{\gamma_2}_{g_2}) = \gamma_1\gamma_2,$$

and the fact that E is closed under multiplication, given by $T\boldsymbol{m}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{\gamma_1}_{g_1}\widetilde{\gamma_2}_{g_2}) = T\boldsymbol{m}(\dot{\widetilde{\gamma_1}}_{g_1}, \dot{\widetilde{\gamma_2}}_{g_2}) \in T\boldsymbol{m}(\mathrm{E}^{(2)}) \subset \mathrm{E}\,.$$

Hence, we can conclude that $\widetilde{\gamma_1\gamma_2}_{g_1g_2} = \widetilde{\gamma_1}_{g_1}\widetilde{\gamma_2}_{g_2}$

Next, we want to check that $(\widetilde{\gamma_1}_{g_1})^{-1} = \widetilde{(\gamma_1^{-1})}_{g_1^{-1}}$. Notice that the following hold:

$$(\widetilde{\gamma_1}_{g_1})^{-1}(0) = (\widetilde{\gamma_1}_{g_1}(0))^{-1} = g_1^{-1}, \quad \phi((\widetilde{\gamma_1}_{g_1})^{-1}) = (\phi(\widetilde{\gamma_1}_{g_1}))^{-1} = \gamma_1^{-1},$$

and as E is closed under the inversion, $T\boldsymbol{i}$, we also get:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{\gamma_1}_{g_1})^{-1} = T\boldsymbol{i}(\dot{\widetilde{\gamma_1}}_{g_1}) \in T\boldsymbol{i}(\mathrm{E}) \subset \mathrm{E}\,.$$

We can conclude that $(\widetilde{\gamma_1}_{g_1})^{-1} = \widetilde{(\gamma_1^{-1})}_{g_1^{-1}}$.

ii) $\implies$ i): Suppose that condition ii) holds. We want to show that E is multiplicative; thus, we need to show that it is closed under multiplication and inversion, as the source and target maps will automatically be surjective, as they are fibrewise surjective and they cover a submersion.

For the multiplication, pick some $(u_1, u_2) \in \mathrm{E}\,{}_{T\boldsymbol{s}}\!\times_{T\boldsymbol{t}} \mathrm{E}$, and set $g_i \in \mathcal{G}$ such that $u_i \in \mathrm{E}_{g_i}$. As $T\boldsymbol{s}(u_1) = T\boldsymbol{t}(u_2)$ it follows in particular that $\boldsymbol{s}(g_1) = \boldsymbol{t}(g_2)$. From this we can conclude that $(\phi(g_1), \phi(g_2)) \in \mathcal{H}^{(2)}$ and $(T\phi(u_1), T\phi(u_2)) \in (T\mathcal{H})^{(2)} = T(\mathcal{H}^{(2)})$.

Let $\eta\colon (-\epsilon, \epsilon)\colon \mathcal{H}^{(2)}$ be such that $\eta(0) = (\phi(g_1), \phi(g_2))$ and $\dot{\eta}(0) = (T\phi(u_1), T\phi(u_2))$. If we let $\mathrm{pr}_i\colon \mathcal{H}^{(2)} \to \mathcal{H}$ Suppose that the second condition holds, we need to show that E is closed under multiplication and inversion.

For multiplication, take some $(u_1, u_2) \in \mathrm{E}\,{}_{d\boldsymbol{s}}\!\times_{d\boldsymbol{t}} \mathrm{E}$, which we assume is nonempty. We can then find $g_1, g_2 \in \mathcal{G}$ such that $u_1 \in E_{g_1}$ and $u_2 \in E_{g_2}$. Notice that $d\boldsymbol{s}(u_1) = d\boldsymbol{t}(u_2)$, while $d\boldsymbol{s}(u_1) \in \mathrm{E}_{0,\boldsymbol{s}(g_1)}$ and $d\boldsymbol{t}(u_2) \in \mathrm{E}_{0,\boldsymbol{t}(g_2)}$, such that $\boldsymbol{s}(g_1) = \boldsymbol{t}(g_2)$, i.e. $(g_1, g_2) \in \mathcal{G}^{(2)}$. Moreover, $(\phi(g_1), \phi(g_2)) \in \mathcal{H}^{(2)}$ and $(T\phi(u_1), T\phi(u_2)) \in T\mathcal{H}\,{}_{d\boldsymbol{s}}\!\times_{d\boldsymbol{t}} T\mathcal{H} = T\mathcal{H}^{(2)}$.

Let $\eta\colon (-\epsilon, \epsilon) \to \mathcal{H}^{(2)}$ be such that $\eta(0) = (\phi(g_1), \phi(g_2))$ and $\dot{\eta}(0) = (T\phi(u_1), T\phi(u_2))$. Then define $\gamma_i = \mathrm{pr}_i \circ \eta\colon (-\epsilon, \epsilon) \to \mathcal{H}$, such that $(\gamma_1, \gamma_2) \in C^\infty((-\epsilon, \epsilon), \mathcal{H})^{(2)}$.

From the second conditions, it follows that $\widetilde{\gamma_1}_{g_1}\widetilde{\gamma_2}_{g_2} = \widetilde{\gamma_1\gamma_2}_{g_1g_2}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t_0} i(g_i) = u_i$. Hence, it follows that:

$$d\boldsymbol{m}(u_1,u_2) = d\boldsymbol{m}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\widetilde{\gamma_1}_{g_1}, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\widetilde{\gamma_2}_{g_2}\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\left(\widetilde{\gamma_1}_{g_1}\widetilde{\gamma_2}_{g_2}\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\widetilde{\gamma_1\gamma_2}_{g_1g_2}.$$

We remark that $\widetilde{\gamma_1\gamma_2}_{g_1g_2}$ is horizontal and thus $d\boldsymbol{m}(u_1,u_2) \in \mathrm{E}$.
For invertibility, if $u \in \mathrm{E}_g$ consider $\gamma\colon (-\epsilon,\epsilon) \to \mathcal{H}$ such that $\gamma(0) = \phi(g)$ and $\dot\gamma(0) = T\phi(u)$. Then the assumptions of the second condition imply $(\widetilde{\gamma}_g)^{-1} = \widetilde{\gamma^{-1}}_{g^{-1}}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t_0}\widetilde{\gamma}_g$. We find that

$$d\boldsymbol{i}(u) = d\boldsymbol{i}\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\widetilde{\gamma}_g\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\left(\widetilde{\gamma}_g^{-1}\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}\widetilde{\gamma^{-1}}_{g^{-1}}.$$

As $\widetilde{\gamma^{-1}}_{g^{-1}}$ is horizontal, we have that $d\boldsymbol{i}(u) \in \mathrm{E}$.
We conclude that E is a multiplicative connection. □

In particular, it follows from the proof that the following must hold as well.

Corollary 4.2.3. *Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a fibred Lie groupoid with some multiplicative connection* E. *For a curve $\gamma\colon\ [0,1] \to \mathcal{H}$ and $g \in \phi^{-1}(\gamma(0))$ we have that $\boldsymbol{s}(\widetilde{\gamma}_g) = \widetilde{\boldsymbol{s}(\gamma)}_{\boldsymbol{s}(g)}$ and $\boldsymbol{t}(\widetilde{\gamma}_g) = \widetilde{\boldsymbol{t}(\gamma)}_{\boldsymbol{t}(g)}$*

4.2.1 Existence problem of multiplicative connections

While the existence of a connection on a surjective submersion is automatic, because any vector subbundle admits a complement, this construction is nontrivial in the case of $\mathcal{VB}$-groupoids.

Firstly, notice that even in the case of fibred Lie groupoids covering the identity, there does not always exist a multiplicative connection. The obstructions to the existence have been well researched and were already known in [?LaurentGengoux2009, Prp. 6.13]. Here, they construct a cohomology class controlling the existence of multiplicative connections. In [?Grad2025], in particular Proposition 5.2, they prove that this obstruction class is dependent on the data for the obstruction class which lies in $\mathcal{G}$ and a particular subbundle of its Lie algebroid, coming from the Lie groupoid morphism. In particular, if $\phi\colon \mathcal{G} \to \mathcal{H}$ is a fibred Lie groupoid covering the identity and $\mathcal{G}$ is proper, then it always admits a multiplicative connection [?Fernandes2023, Thm. 4.2].

In our more general case, the properness of $\mathcal{G}$ is not enough to ensure the existence of multiplicative connections. Without going into too much detail, this is related to the fact that a general fibred Lie groupoid does not induce a bundle of ideals, as defined in [?Fernandes2023] and used in [?Grad2025, Prp. 5.2]. In particular, we can remark the following class of examples.

Example 4.2.4. Consider an action $G \circlearrowright M$ of a connected Lie group on a compact manifold M and let $G \ltimes M$ denote the action groupoid, as in Example **??**. Remark that the projection on G defines a Lie groupoid morphism, i.e. the map

$$\mathrm{pr}_1\colon G \ltimes M \to G\colon (g,x) \mapsto g.$$

Moreover, as the space of arrows of $G \ltimes M$ is the product $G \times M$, the projection defines a fibre bundle with compact fibres. It therefore defines a fibred Lie groupoid covering the trivial map $M \to \{*\}$.

Suppose that a multiplicative connection on pr_1 exists, and let E be a multiplicative connection. Remark that the induced connection on the base map is the zero bundle as $\mathrm{Ver} = TM$. The lift of a curve $[0,1] \to \{*\}$

through this connection to some $x \in M$ is the constant curve $[0,1] \to M : t \mapsto x$. Notice that both E and E_0 are complete connections by Proposition **??**.

Take some curve $\gamma\colon [0,1] \to G$ with $\gamma(0) = g$ and notice that $\boldsymbol{s} \circ \gamma(t) = * = \boldsymbol{t} \circ \gamma(t)$. By Corollary **??**, we find that

$$\mathrm{pr}_2 \circ \widetilde{\gamma}_{(g,x)}(t) = \boldsymbol{s}(\widetilde{\gamma}_{(g,x)})(t) = \widetilde{\boldsymbol{s}(\gamma)}_x(t) = x.$$

As $\widetilde{\gamma}_{(g,x)}$ is a lift of γ through pr_1 it follows that $\widetilde{\gamma}_{(g,x)}(t) = (\gamma(t), x)$. However, by Corollary **??** and completeness of the lift, it follows that

$$\gamma(t)x = \boldsymbol{t}(\widetilde{\gamma}_{(g,x)}(t)) = \widetilde{\boldsymbol{t}(\gamma)}_{gx} = gx.$$

We can conclude that the action of $G \circlearrowright M$ is constant on the connected components of G. Specifically, the action of $G^\circ \subset G$, the connected component of the identity, is trivial. //

The above shows that there are fibred Lie groupoids $\phi\colon \mathcal{G} \to \mathcal{H}$ where $\mathcal{G}$ is proper, such that there does not exist a multiplicative connection —take the action groupoid of a nontrivial proper action of a connected Lie group on a compact manifold. The results from [`?Fernandes2023`, `?Grad2025`, `?LaurentGengoux2009`], therefore, do not directly translate to our generalised case as the obtained obstruction classes are more intricate and need to incorporate the geometry of the base map. In Section **??**, we will come back to this issue and give some geometric conditions for existence.

4.3 Completeness

The main result of Chapter **??** was the equivalence between the existence of a complete connection and local triviality. We would like to generalise this to the groupoid case as well, where a multiplicative connection is called complete if it is complete as a connection on the top surjective submersion. We have already run into an obstacle here, in the fact that a fibred Lie groupoid $\phi\colon \mathcal{G} \to \mathcal{H}$ does not admit any useful local trivialisations, and we will therefore have to focus on families of Lie groupoids for such a result. Let us first show that the canonical internal family of Lie groupoids, the kernel bundle, almost completely characterises completeness. Then, we will show how local triviality and completeness almost coincide in the case of families of Lie groupoids.

Remark that in the case of complete connections, there is a variant of Proposition **??** which incorporates the map

$$C^\infty([0,1], \mathcal{H})\, {}_{\mathrm{ev}_0}\!\times_\phi \mathcal{G} \to C^\infty([0,1], \mathcal{G}) : (\gamma, g) \mapsto \widetilde{\gamma}_g.$$

We remark that $C^\infty([0,1], \mathcal{H})\, {}_{\mathrm{ev}_0}\!\times_\phi \mathcal{G}$ defines a subgroupoid of $C^\infty([0,1], \mathcal{H}) \times \mathcal{G}$. Moreover, this is a subgroupoid over the fibred product $C^\infty([0,1], \mathcal{H}_0)\, {}_{\mathrm{ev}_0}\!\times_{\phi_0} \mathcal{G}_0$. We remark that the proof of Proposition **??** implies the following proposition.

Proposition 4.3.1. *Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a fibred Lie groupoid and* E *a complete connection, then the following are equivalent:*

i) E *is multiplicative*

ii) *The horizontal lift map, given by*

$$C^\infty([0,1], \mathcal{H})\, {}_{\mathrm{ev}_0}\!\times_\phi \mathcal{G} \to C^\infty([0,1], \mathcal{G}) : (\gamma, g) \mapsto \widetilde{\gamma}_g,$$

is a groupoid morphism covering the map

$$C^\infty([0,1], \mathcal{H}_0)\,{}_{\mathrm{ev}_0}\!\times_{\phi_0} \mathcal{G}_0 \to C^\infty([0,1], \mathcal{G}_0)\colon (\gamma, x) \mapsto \widetilde{\gamma}_x.$$

In particular, the connection on the base of a complete multiplicative connection is automatically complete.

Proof. This follows directly from Proposition **??** and its proof. □

4.3.1 Reduction to the kernel

Let us fix some fibred Lie groupoid $\phi\colon \mathcal{G} \to \mathcal{H}$, a multiplicative connection E, and denote $\mathcal{K} = \ker \phi \rightrightarrows \mathcal{G}_0$. Let us denote $p\colon \mathcal{K} \to \mathcal{H}_0$ for the induced map, i.e. $p = \phi|_{\mathcal{K}}$. We notice that there is an induced connection on $\mathcal{K}$, given by $\mathrm{E}^{\mathcal{K}} = \mathrm{E} \cap T\mathcal{K}$. Notice that this defines a vector bundle as it is the complement to $\ker Tp = \ker T\phi \cap T\mathcal{K}$ in $T\mathcal{K}$. Therefore, the intersection is clean and from Theorem **??** it follows that $\mathrm{E}^{\mathcal{K}}$ defines a $\mathcal{VB}$-subgroupoid of $T\mathcal{K}$ and thus a multiplicative connection on p.

Proposition 4.3.2. *Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a fibred Lie groupoids with a multiplicative connection E and suppose that $\gamma\colon [0,1] \to \mathcal{H}_0$. Then the horizontal lift to some $g \in \phi^{-1}(1_{\gamma(0)})$ by E and $\mathrm{E}^{\mathcal{K}}$ coincide.*

Proof. Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a fibred Lie groupoids with a multiplicative connection E, $\gamma\colon [0,1] \to \mathcal{H}_0 \subset \mathcal{H}$ a curve and set $x = \gamma(0)$. For any $g \in \phi^{-1}(1_x)$ remark that $\phi \circ \widetilde{\gamma}^{\mathrm{E}}_g(t) = \gamma(t) \in \mathcal{H}_0$. This implies that $\widetilde{\gamma}^{\mathrm{E}}_g\colon [0,1] \to \mathcal{K} \subset \mathcal{G}$ and $\dot{\widetilde{\gamma}}^{\mathrm{E}}_g(t) \in E \cap T\mathcal{K} = \mathrm{E}^{\mathcal{K}}$, such that $\widetilde{\gamma}^{\mathrm{E}}_g = \widetilde{\gamma}^{\mathrm{E}^{\mathcal{K}}}_g$. □

As mentioned before, the kernel of a fibred Lie groupoid defines a family of Lie groupoids, which are arguably much nicer structures to work with. Therefore, we want to reduce our completeness of E to a completeness of $\mathrm{E}^{\mathcal{K}}$. One of these implications follows directly from the above proposition.

Corollary 4.3.3. *If E is a complete, then $\mathrm{E}^{\mathcal{K}}$ is a complete.*

Ideally, this implication would have a direct converse; however, this is not true in general, as can be seen from the following example.

Example 4.3.4. Consider the groupoid $\mathcal{H}$ defined as the following bundle of groups:

$$\begin{array}{c} \mathbb{R} \times (\mathbb{Z}/2\mathbb{Z}) \\ {\scriptstyle \mathrm{pr}_1}\downarrow\downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathbb{R} \end{array} \qquad \text{(arrows: } x \xleftarrow{(x,n)} x\text{)}$$

We define the multiplication to be $(x,n)(x,m) = (x, n+m)$. Then consider the complement of $(0,\hat{1})$, $\mathcal{H} \setminus \{(0,\hat{1})\}$, and remark that it is closed under multiplication, inversion and contains all the units, while also being an open submanifold, such that the restriction of the source map is still a submersion. This implies that $\mathcal{H} \setminus \{(0,\hat{1})\}$ is still a Lie groupoid over $\mathbb{R}$, and in particular it is a Lie subgroupoid of $\mathcal{H}$.

Next, we consider the disjoint union groupoids $\mathcal{G} = \mathcal{H} \coprod (\mathcal{H} \setminus \{(0,\hat{1})\})$ as in Example **??**, which is a Lie groupoid over $\mathbb{R} \coprod \mathbb{R}$. Moreover, it obtains an induced map $\phi : \mathcal{G} \to \mathcal{H}$ as the disjoint union of the identity and inclusion map, which is a fibred Lie groupoid.

Notice that ϕ is a local diffeomorphism, but not a covering space, as any neighbourhood of $(0, \hat{1})$ is not evenly covered. In particular, this implies that the kernel of $T\phi$ is the trivial bundle and thus there is just a unique multiplicative connection on ϕ. However, this connection is not complete: Consider the curve $\gamma : [0,1] \to \mathcal{H} : t \mapsto (1-t, 1)$ and notice that this cannot be lifted on the whole of $[0,1]$ to the second component of $\mathcal{G}$, i.e. to $((0,\hat{1}),1) \in \mathcal{H} \setminus \{(0,\hat{1})\} \subset \mathcal{G}$, due to $((0,\hat{1}),1) \notin \mathcal{G}$ by construction.

The kernel bundle of ϕ is given by $\mathbb{R} \times \{0\} \coprod \mathbb{R} \times \{0\}$, on which the induced connection is trivial and a lift of a curve is simply given by the inclusion into a component. Remark that $\boldsymbol{s} : \phi^{-1}(h) \to \phi_0^{-1}(\boldsymbol{s}(h))$ is not surjective for $h = (0, \hat{1})$ as $\phi^{-1}(0,\hat{1}) = \{((0,\hat{1}),0)\}$ and $\phi_0^{-1}(0,\hat{1}) = \{(0,0),(0,\hat{1})\}$. //

Under some additional conditions, we do obtain a converse of Corollary **??**. In particular, the failure of the map $\boldsymbol{s}\colon \phi^{-1}(h) \to \phi_0^{-1}(\boldsymbol{s}(h))$ to be surjective in Example **??** was problematic remark however, the following does hold

Proposition 4.3.5. *Given a fibred Lie groupoid $\phi\colon \mathcal{G} \to \mathcal{H}$, for any $h \in \mathcal{H}$, the restrictions of the source and target map, $\boldsymbol{s}\colon \phi^{-1}(h) \to \phi_0^{-1}(\boldsymbol{s}(h))$ and $\boldsymbol{t}\colon \phi^{-1}(h) \to \phi_0^{-1}(\boldsymbol{t}(h))$, are submersions.*

Proof. Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a fibred Lie groupoid and take some $h \in \mathcal{H}$. Remark that it is enough to show it for $\boldsymbol{s}$, as it follows for $\boldsymbol{t}$ by applying the diffeomorphism $\boldsymbol{i}$, thus let us denote $\boldsymbol{s}(h) = x$. We rewrite the tangent spaces as follows:

$$T_g\phi^{-1}(h) = \ker T_g\phi = \mathrm{Ver}_g, \quad T_y\phi_0^{-1}(x) = \ker T_y\phi_0 = \mathrm{Ver}_{0,y}\,.$$

It follows from Example **??** that $T_g\boldsymbol{s}(\mathrm{Ver}_g) = \mathrm{Ver}_{0,y}$ when $\boldsymbol{s}(g) = y$. Therefore $\boldsymbol{s}\colon \phi^{-1}(h) \to \phi_0^{-1}(x)$ is a submersion. □

A solution to this is to consider only specific fibred Lie groupoids, the class of which is inspired by the definition of fibred categories, cf. [`?stacks-project`, Tag 02XJ].

Definition 4.3.6. A fibred Lie groupoid $\phi\colon \mathcal{G} \to \mathcal{H}$ is said to be *arrow complete* if the following map is surjective:

$$\boldsymbol{s} \times \phi\colon \mathcal{G} \to \mathcal{G}_0\,{}_{\phi_0}\!\times_{\boldsymbol{s}} \mathcal{H}\colon g \mapsto (\boldsymbol{s}(g), \phi(g)).$$

In particular, this definition is equivalent to the particular restrictions of the sources being surjective.

Proposition 4.3.7. *A fibred Lie groupoid $\phi\colon \mathcal{G} \to \mathcal{H}$ is arrow complete if and only if for all $h \in \mathcal{H}$ the restriction $\boldsymbol{s}\colon \phi^{-1}(h) \to \phi_0^{-1}(\boldsymbol{s}(h))$ is surjective.*

This class of Lie groupoid morphisms is related to a so-called cleavage of a Lie groupoid morphism, which is understood as a subset $C \subset \mathcal{G}$ such that $\boldsymbol{s} \times \phi\colon C \mapsto \mathcal{G}_0\,{}_{\phi_0}\!\times_{\boldsymbol{s}} \mathcal{H}$ is a bijection, cf. [`?delHoyo2025`]. These objects then coincide with right inverses of $\boldsymbol{s} \times \phi\colon \mathcal{G} \to \mathcal{G}_0\,{}_{\phi_0}\!\times_{\boldsymbol{s}} \mathcal{H}$ and thus their existence is equivalent to the surjectivity of this map, i.e the fibred Lie groupoid being arrow complete.

Under this additional assumption, we obtain a converse to Corollary **??**.

Theorem 4.3.8. *If $\phi\colon \mathcal{G} \to \mathcal{H}$ is a fibred Lie groupoid that is arrow complete and a connection* E *such that* $\mathrm{E}^{\mathcal{K}}$ *is complete, then* E *is complete.*

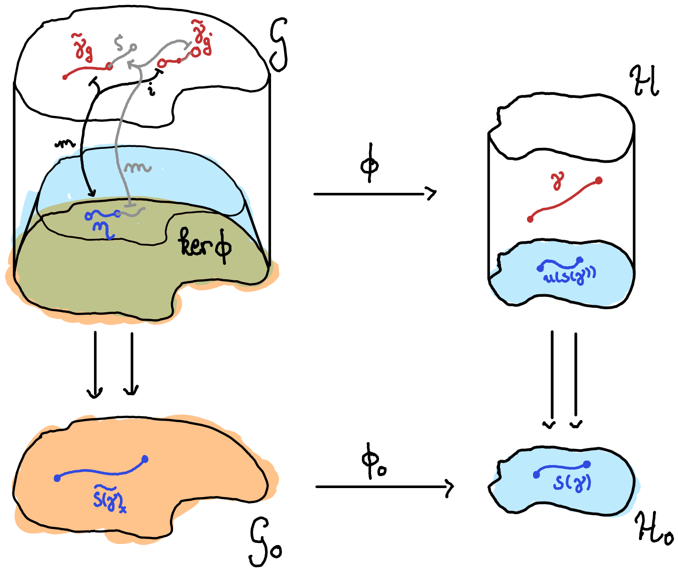

Figure 4.1: The map $\phi\colon \mathcal{G} \to \mathcal{H}$ is a fibred Lie groupoid and the red curves are all horizontal lifts of γ, which is a curve in $\mathcal{H}$. We can then notice that by suitable choice of g', the curves $\widetilde{\gamma}_g$ and $\widetilde{\gamma}_{g'}$ have the same source and therefore they are related by some curve η in the kernel. This curve is a lift of $\boldsymbol{s}(\gamma)$ and therefore it extends. An extension of $\widetilde{\gamma}_g$ is then obtain by the multiplication of η and $\widetilde{\gamma}_{g'}$.

The idea of the proof is as follows: Pick a curve and lift it maximally, and suppose that this is not defined on the whole domain. To extend it further, we remark that the source of our lift can be lifted to the kernel bundle instead, on which we know our connection is complete. Therefore, we can find a suitable extension of the source of the lift. Due to the surjectivity of the source map, we can then find a lift of this source to a suitable fibre, to which we can again find a horizontal lift. As this is redefined on some open neighbourhood, the original lift and this new lift must be defined on some common open interval. Here, they are related by a curve lying in the kernel, and therefore, this relation can be extended to the full interval. We can then extend the original lift using this new lift and this extended relation. A schematic drawing can be found in Figure **??**.

Proof. Suppose that ϕ is a fibred Lie groupoid which is arrow complete with a multiplicative connection E such that $\mathrm{E}^{\mathcal{K}}$ is complete. Let $\gamma\colon [0,1] \to \mathcal{H}$ be some curve starting at h_0 and take $g \in \phi^{-1}(h_0)$ with $\boldsymbol{s}(g) = x$. Then there exists some $\epsilon > 0$ such that $\widetilde{\gamma}^{\mathrm{E}}_g$ is defined on $[0,\epsilon)$. We remark that $\boldsymbol{s}(\widetilde{\gamma}_g) = \widetilde{\boldsymbol{s}(\gamma)}_x\colon [0,\epsilon) \to \mathcal{G}_0$, see Corollary **??**. As $\boldsymbol{s}(\gamma)$ maps into the units, we find that $\widetilde{\boldsymbol{s}(\gamma)}_x = \widetilde{\boldsymbol{s}(\gamma)}_x$, such that this lift is actually defined on $[0,1]$.

Set $1_y = \widetilde{\boldsymbol{s}(\gamma)}_x(\epsilon)$, and take some $g' \in \phi^{-1}(\gamma(\epsilon)) \cap \mathcal{G}_y$, which exists as we assume arrow completeness. The the lift $\widetilde{\gamma}_{g'}$ is defined on some $(\epsilon-\delta, \epsilon+\delta)$. We remark that $\boldsymbol{s}(\widetilde{\boldsymbol{s}(\gamma)}_x)$ defines a horizontal lift of $\boldsymbol{s}(\gamma)$ by the connection E_0 as $\mathrm{E}_0 = T\boldsymbol{s}(\mathrm{E})$ and the fact that $\phi_0 \circ \boldsymbol{s}|_{\mathcal{K}} = \phi|_{\mathcal{K}}$. By the uniqueness of lifts, it follows that on $(\epsilon-\delta,\epsilon)$ the lifts $\boldsymbol{s}(\widetilde{\gamma}_g)$ and $\boldsymbol{s}(\widetilde{\gamma}_{g'})$ coincide. Consider the following:

$$\eta\colon (\epsilon-\delta,\epsilon) \to \mathcal{K}\colon t \mapsto \widetilde{\gamma}_g(t)(\widetilde{\gamma}_{g'}(t))^{-1}.$$

This is a curve in the kernel and $\phi \circ \eta = \boldsymbol{u}(\boldsymbol{t}(\gamma))$, such that this curve lifts completely to $[0,1]$, let us also denote this extension by η. Notice that $\boldsymbol{s}(\eta) = \boldsymbol{t}(\widetilde{\gamma}_g)$ on $(\epsilon-\delta,\epsilon)$, which are both horizontal lifts of $\boldsymbol{t}(\gamma)$ by E_0 as the connection is multiplicative. Therefore, it extends to $(\epsilon-\delta,\epsilon+\delta)$ by Corollary **??**. An extension of

$\widetilde{\gamma}_g$ can then be defined as

$$\zeta\colon [0, \epsilon + \delta) \to \mathcal{G}\colon t \mapsto \begin{cases} \widetilde{\gamma}_g(t) & \text{if } t < \epsilon \\ \eta(t)\widetilde{\gamma}_{g'}(t) & \text{else} \end{cases}$$

This implies that the lift of γ can always be extended maximally to $[0, 1]$. ◻

Besides this condition being sufficient, we can also show that, in some cases, it is necessary.

Proposition 4.3.9. *Let $\phi\colon \mathcal{G} \to \operatorname{gr}\mathcal{H}$ be a fibred Lie groupoid with a complete multiplicative connection and suppose that $\mathcal{H}$ is $\boldsymbol{s}$-connected, then it is arrow complete.*

Proof. Suppose that E is complete and $\mathcal{H}$ is $\boldsymbol{s}$-connected and fix some $y \xleftarrow{h} x \in \mathcal{H}$, $z \in \phi_0^{-1}(x)$. By the $\boldsymbol{s}$-connectedness, take a curve $\gamma\colon\ [0, 1] \to \mathcal{H}_x$, where $\gamma(0) = 1_x$ and $\gamma_1(1) = h$. The lift $\widetilde{\gamma}_{1_z}$ then satisfies $\boldsymbol{s}(\widetilde{\gamma}_{1_z}) = z$ by Corollary **??** and the fact that $\boldsymbol{s}(\gamma)(t) = y$ is a constant curve. This implies that $g = \widetilde{\gamma}_{1_z} \in \mathcal{G}_z$ such that $\phi(g) = h$. Therefore, ϕ is arrow complete. ◻

4.3.2 Completeness for families of Lie groupoids

The next step in understanding completeness is getting a grasp of complete connections on families of Lie groupoids and their relation to local triviality. Similar to the classical case, one of these directions follows easily.

Proposition 4.3.10. *Let $p\colon \mathcal{K} \to B$ be a family of Lie groupoids. If it admits a complete multiplicative connection, then it is locally trivial as a family of Lie groupoids.*

Proof. This proof is completely analogous to the proof in Theorem **??**, but we will comment on some details.

Given a contraction $c\colon U \times [0, 1] \to U$, for $U \subset B$ open, we can construct some trivialisation, where we define $\gamma_b\colon\ [0, 1] \to \mathcal{K}\colon t \mapsto c(b, t)$ and set $\psi(g) = (p(g), \widetilde{\gamma_{p(g)}}_g(1))$. Fix some $(g, h) \in \mathcal{K}^{(2)}$, such that in particular $p(g) = b = p(h)$. By Proposition **??**, it follows that

$$\psi(gh) = (b, \widetilde{(\gamma_b)}_{gh}(1)) = (b, \widetilde{(\gamma_b)}_g(1)\widetilde{\gamma}_{bh}(1)) = (b, \widetilde{(\gamma_b)}_g(1))(b, \widetilde{(\gamma_b)}_h(1)) = \psi(g)\psi(h)$$

Therefore, the trivialisation is an equivalence of families of Lie groupoids. By covering B in contractible opens, we obtain a cover of trivialisations by equivalences of families of Lie groupoids. ◻

We do not have a general converse to this statement, as multiplicative connections are, in a sense, 'rarer' compared to general connections. However, we can find sufficient conditions similar to Proposition **??**, combined with some algebraic conditions, such that a multiplicative connection exists on a locally trivial family of Lie groupoids. Yet, we can find additional conditions similar to that of Proposition **??**, such that we can still find a connection with similar conditions.

Proposition 4.3.11. *Let $\mathcal{G} \hookrightarrow \mathcal{K} \xrightarrow{\phi} B$ be a locally trivial family of Lie groupoids with trivialising cover $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ which is locally finite, and $\{U_\alpha\}_{\alpha\in\Lambda}$ an open cover of B with $\overline{U}_\alpha \subset V_\alpha$, and suppose that $\mathcal{K}$ is proper. Suppose that for each $\alpha \in \Lambda$ we have a closed subset $S_\alpha \subset \mathcal{G}$ which are $\mathcal{G}$-biinvariant,* *such that*

$$S_\alpha \cap \psi_{\alpha\beta,b}(S_\beta) = \emptyset, \quad \forall \alpha \neq \beta \in \Lambda,\ b \in \overline{U_{\alpha\beta}}.$$

Then there exists a multiplicative Ehresmann connection E *such that* $T\psi_\alpha(\mathrm{E}|_{\psi_\alpha^{-1}(U_\alpha \times S_\alpha)}) = TU_\alpha \times 0_{S_\alpha}$.

The proof of this proposition is almost analogous to the proof of Proposition **??**, where we only need to check for multiplicativity at each point. One can verify the details skipped in this proof by comparing with Proposition **??**.

Sketch of proof. Suppose that $\mathcal{G} \hookrightarrow \mathcal{K} \xrightarrow{\phi} B$ is a family of Lie groupoids with a trivialising cover $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ and let $\{U_\alpha\}_{\alpha\in\Lambda}$ and $\{S_\alpha\}_{\alpha\in\Lambda}$ be as in the statement. The canonically induced connection on $\mathcal{K}|_{U_\alpha}$, given by

$$h_\alpha \colon \phi^* TU_\alpha \to T\mathcal{K}|_{U_\alpha} \colon (x, v) \mapsto T_{\psi_\alpha(x)}\psi_\alpha^{-1}(v, 0),$$

defines a multiplicative connection. Additionally, the sets W_α, defined by

$$W_\alpha = \phi^{-1}(U_\alpha) \backslash \bigcup_{\beta\neq\alpha} \psi_\beta^{-1}(\overline{U_\beta} \times S_\beta),$$

give an open cover of $\mathcal{K}$. Moreover, we can show that these are $\mathcal{K}$-biinvariant sets. Remark that the action of $\mathcal{K}$ on itself can be restricted to actions on the fibred of ϕ. Therefore, it is enough to show that $\phi^{-1}(b) \cap W_\alpha$ is $\mathcal{K}$-biinvariant. Notice that this set can be rewritten as

$$\phi^{-1}(b) \cap W_\alpha = \phi^{-1}(b) \backslash \bigcup_{\beta\neq\alpha} \psi_\beta^{-1}(\overline{U_\beta} \times S_\beta) = \phi^{-1}(b) \backslash \bigcup_{\beta\neq\alpha} \psi_\beta^{-1}(\{b\} \times S_\beta) = \bigcap_{\beta\neq\alpha} \psi_\beta^{-1}(\{b\} \times \mathcal{G}\backslash S_\beta)$$

As S_β is $\mathcal{G}$-biinvariant, so is $\mathcal{G}\backslash S_\beta$, and as ψ_β is a Lie groupoid isomorphism, the sets $\psi_\beta^{-1}(\{b\} \times \mathcal{G}\backslash S_\beta)$ are $\mathcal{K}$-biinvariant. We can conclude that their intersection, and thus $\phi^{-1}(b) \cap W_\alpha$, is $\mathcal{K}$-biinvariant as well.

Due to W_α being $\mathcal{K}$-biinvariant, we can define $V_\alpha = q \circ \boldsymbol{s}(W_\alpha) = q \circ \boldsymbol{t}(W_\alpha) \subset \mathcal{K}_0/\mathcal{K}$. As $\mathcal{K}$ is a proper groupoid, then its orbit space $\mathcal{K}_0/\mathcal{K}$ admits a smooth partition of unity $\{\widetilde{\chi}_\alpha\}_{\alpha\in\Lambda}$, where smoothness is in the sense as described after Definition **??**, subordinate to $\{V_\alpha\}_{\alpha\in\Lambda}$, see [`?Crainic2017`, Prp. 3.9]. We then obtain a $\mathcal{K}$-biinvariant partition of unity subordinate to $\{W_\alpha\}_{\alpha\in\Lambda}$, defined as $\{\chi_\alpha\}_{\alpha\in\Lambda} = \{\widetilde{\chi}_\alpha \circ q \circ \boldsymbol{s}\}_{\alpha\in\Lambda}$. The glueing of the canonical multiplicative connections with respect to this partition of unity, denoted $h = \sum_{\alpha\in\Lambda} \chi_\alpha h_\alpha$, then defines a multiplicative connection, whose Ehresmann connection satisfies

$$T\psi_\alpha(\mathrm{E}|_{\psi_\alpha^{-1}(U_\alpha \times S_\alpha)}) = TU_\alpha \times 0_{S_\alpha}.$$

□

Much like in the proof of Theorem **??**, we want to define the sets S_α as the level sets of some proper function. To have such biinvariant sets, we must require our proper functions to be biinvariant as well. We remark the following proposition relating the properness of $\boldsymbol{s}$ to the existence of biinvariant proper maps.

Proposition 4.3.12. *Let $\mathcal{G}$ be a Lie groupoid, then the following are equivalent:*

i) $\boldsymbol{s}$ *is a proper map.*

ii) *There exists a smooth function* $f\colon \mathcal{G}_0 \to \mathbb{R}_{\geq 0}$ *such that*

- *it is* $\mathcal{G}$*-invariant;*
- $\boldsymbol{s}^* f$ *is proper.*

To prove this, we will use the following lemmas.

Lemma 4.3.13. *Let* $\mathcal{G}$ *be a Lie groupoid such that* $\boldsymbol{s}$ *is proper, then all orbits* $\mathscr{O}_x$ *of* $\mathcal{G}$ *are compact and they admit a precompact neighbourhood which is invariant for action of* $\mathcal{G}$ *on* $\mathcal{G}_0$*.*

Proof. Suppose that $\mathcal{G}$ is a Lie groupoid such that $\boldsymbol{s}$ is proper. Remark that the orbit of x in $\mathcal{G}$ is defined as $\mathscr{O}_x = \boldsymbol{t}(\boldsymbol{s}^{-1}(x))$. Therefore, they are all compact. Next, we let $\widetilde{U}$ be a precompact neighbourhood of $\mathscr{O}_x$. The saturation $U = \boldsymbol{t}(\boldsymbol{s}^{-1}(\widetilde{U}))$ is then precompact neighbourhood as well. Moreover, it is invariant for the left $\mathcal{G}$ action on $\mathcal{G}_0$ as for some $x \in U$, there exists a $g \in \mathcal{G}$ such that $\boldsymbol{t}(g) = x$ and $\boldsymbol{s}(g) \in \widetilde{U}$. If $h \in \mathcal{G}_x$, it follows that $h \cdot x = \boldsymbol{t}(h) = \boldsymbol{t}(hg)$ and $\boldsymbol{s}(hg) = \boldsymbol{s}(g) \in \widetilde{U}$. Therefore, $h \cdot x \in \boldsymbol{t}(\boldsymbol{s}^{-1}(\widetilde{U})) = U$. □

Lemma 4.3.14. *Let* $\mathcal{G}$ *be a Lie groupoid such that* $\boldsymbol{s}$ *is proper, then* $q\colon \mathcal{G}_0 \to \mathcal{G}_0/\mathcal{G}$ *is a proper map.*

Proof. Let $\mathcal{G}$ be a Lie groupoid such that $\boldsymbol{s}$ is proper, and let $q\colon \mathcal{G}_0 \to \mathcal{G}_0/\mathcal{G}$ denote the quotient map with respect to the left action of $\mathcal{G}$. Consider a compact subset $K \subset \mathcal{G}_0/\mathcal{G}$. By Lemma **??**, using the $\boldsymbol{s}$-properness, any $\mathcal{O} \in K$, which is an orbit, admits a precompact neighbourhood $U_{\mathcal{O}}$ of $\mathcal{O} \subset \mathcal{G}_0$ which is invariant for the left $\mathcal{G}$-action. Therefore, $\{U_{\mathcal{O}}/\mathcal{G}_0 | \, \mathcal{O} \in K\}$ defines an open cover of K. By the compactness of K, there must exist an open subcover $\{U_{\mathscr{O}_i}/\mathcal{G}\}_{i=1}^n$. It follows that $q^{-1}(K) \subset \bigcup_{i=1}^n U_{\mathscr{O}_i}$. As $\mathscr{O}_i$ is precompact, it follows that $q^{-1}(K)$ is compact, such that q is a proper map. □

Proof of Prp. **??**. Suppose that $\mathcal{G}$ is a Lie groupoid.

i) $\Longrightarrow$ ii): Let $\boldsymbol{s}$ be a proper map and notice that this in particular implies that $\mathcal{G}$ is a proper groupoid as for a compact $K \subset \mathcal{G}_0 \times \mathcal{G}_0$ we find that $(\boldsymbol{t}, \boldsymbol{s})^{-1}(K) \subset \boldsymbol{s}^{-1}(\mathrm{pr}_2(K))$. Additionally, let $q\colon \mathcal{G}_0 \to \mathcal{G}_0/\mathcal{G}$ denote the quotient map with respect to the left action of $\mathcal{G}$ on $\mathcal{G}_0$. The orbit space of $\mathcal{G}$ admits a proper map $f_0\colon \mathcal{G}_0/\mathcal{G} \to \mathbb{R}_{\geq 0}$ such that $f_0 \circ q\colon \mathcal{G}_0 \to \mathbb{R}_{\geq 0}$ is smooth, see [`?Crainic2017`, Prp. 3.9]. From Lemma **??**, it follows that $f = f_0 \circ q$ is a proper smooth biinvariant map, and thus $\boldsymbol{s}^* f = f \circ \boldsymbol{s}$ is proper a well.

ii) $\Longrightarrow$ i): Let $f\colon \mathcal{G}_0 \to \mathbb{R}_{\geq 0}$ be a map such that $\boldsymbol{s}^* f = \boldsymbol{t}^* f$ and $\boldsymbol{s}^* f$ is proper. Let $K \subset \mathcal{G}_0$ be a compact set, and notice that

$$\boldsymbol{s}^{-1}(K) = \{g \in \mathcal{G} | \, \boldsymbol{s}(g) \in K\} \subset \{g \in \mathcal{G} | \, f \circ \boldsymbol{s}(g) \in f(K)\} \subset (\boldsymbol{s}^* f)^{-1}(f(K)).$$

As $\boldsymbol{s}^* f$ is proper and $f(K)$ is compact, this last set is compact. Moreover, $\boldsymbol{s}^{-1}(K)$ is closed, and as it is contained in a compact set, it is compact itself. Therefore, $\boldsymbol{s}$ is a proper map. □

Combining the above results then lets us prove a multiplicative version of Theorem **??** under some additional compactness conditions.

Theorem 4.3.15. *Let* $p\colon \mathcal{K} \to B$ *be a locally trivial family of Lie groupoids with typical fibre* $\mathcal{G}$*. Suppose that* $\mathcal{G}$ *is a Lie groupoid whose source map is proper, then* p *admits a complete multiplicative connection.*

Sketch of the proof. The proof of this theorem is completely analogous to that of Theorem **??**. However, we will point out some of the modifications here:

Assume that $p\colon \mathcal{K} \to B$ is a locally trivial family of Lie groupoids, with local trivialisations $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$, where we have the same restrictions on this set as in the proof of Theorem **??**, such that $\mathcal{G}$ has a proper source map.

By Proposition **??**, it follows that there exists a map $f\colon \mathcal{G}_0 \to \mathbb{R}_{\geq 0}$ such that $\widetilde{f} = \boldsymbol{s}^* f\colon \mathcal{G} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{G}$-biinvariant proper map. Following the constructions of the proof of Theorem **??**, we can define suitable N_α and $S_\alpha = f^{-1}(N_\alpha) \subset \mathcal{G}$. Because f is $\mathcal{G}$-invariant, these sets are also $\mathcal{G}$-biinvariant and thus they satisfy all the conditions of Proposition **??**, such that we obtain a multiplicative Ehresmann connection E satisfying

$$T\psi_\alpha \left(\mathrm{E}\,|_{\psi_\alpha^{-1}(U_\alpha \times S_\alpha)}\right) = TU_\alpha \times 0_{S_\alpha}.$$

Moreover, the connected components of $F\backslash S_\alpha$ will be precompact again. By Proposition **??**, this defines a complete multiplicative Ehresmann connection. □

This describes the analogue of Theorem **??** for the multiplicative setting. It is even a generalisation if we view a surjective submersion as a fibred Lie groupoid of the identity groupoids by remarking that the source map of the identity groupoid is proper. However, while a fibred Lie groupoid's completeness can be deduced from the kernel bundle, there is an additional, smaller surjective submersion internal to a family of Lie groupoids, namely, its base map. For a fibred Lie groupoid, we know that the completeness of a multiplicative connection implies that the connection on the base is complete, see Proposition **??**. In a particular case, the completeness of a connection on a family of Lie groupoids can be fully recovered from the completeness on the base; in other words, we then have a converse of Proposition **??**.

Proposition 4.3.16. *Let $p\colon \mathcal{K} \to B$ be a family of Lie groupoids which is $\boldsymbol{t}$-connected and suppose that E is a multiplicative Ehresmann connection. If E_0 is a complete connection on p_0, then E is complete.*

Proof. Let E be a multiplicative Ehresmann connection on a $\boldsymbol{t}$-connected family of Lie groupoids $p\colon \mathcal{K} \to B$, for which the induced connection E_0 on the base is complete. Take an arbitrary curve $\gamma\colon\ [0,1] \to B$ and set $x = \gamma(0)$. We will denote the lift through E by $\widetilde{\gamma}^{\mathrm{E}}$ and by E_0 by $\widetilde{\gamma}^{\mathrm{E}_0}$.

In this proof, we want to argue through Proposition **??**. Hence, we want to find a neighbourhood U of $p_0^{-1}(x) \subset p^{-1}(x)$ such that the horizontal lift $\widetilde{\gamma}_y$ is defined on the whole of $[0,1]$ for all $y \in U$. If we then take some $g \in p^{-1}(x)$, then there exist $\{u_i\}_{i=1}^n \subset U$ such that $g = u_1 \cdots u_n$. To lift to g, we can simply lift to each u_i and remark that as γ maps to only units, it is idempotent with respect to the multiplication, such that

$$\widetilde{\gamma}_g^{\mathrm{E}} = \widetilde{\gamma \cdots \gamma}^{\mathrm{E}}_{u_1\cdots u_n} = \widetilde{\gamma}_{u_1}^{\mathrm{E}} \cdots \widetilde{\gamma}_{u_n}^{\mathrm{E}}.$$

Therefore, the lift to g will be complete if it is complete to all u_i.

For some unit $1_y \in p^{-1}(x)$ we can obtain a lift by taking $\widetilde{\gamma}_{1_y}^{\mathrm{E}} = \boldsymbol{u} \circ \widetilde{\gamma}_y^{\mathrm{E}_0}$. As E_0 is complete, this lift will also be complete. Moreover, it is horizontal as $T\boldsymbol{u}(\mathrm{E}_0) \subset \mathrm{E}$. From the tube lemma, it follows that there exists some open neighbourhood U_y of 1_y in $p^{-1}(x)$ such that the horizontal lift exists on the whole of $[0,1]$ to any of the points in U_y. If we set $U = \bigcup_{y\in p^{-1}(x)} U_y$, this will be an open neighbourhood of $p_0^{-1}(x) \subset p^{-1}(x)$ on which the horizontal lifts are complete. □

Again, we can show that we cannot fully drop this assumption by a variation on Example **??**.

Example 4.3.17. Consider $\mathcal{H}$ as in Example **??**, and take the family of groupoids:

$$\begin{tikzcd} \mathcal{G} = (\mathcal{H}\setminus\{(0,\hat{1})\}) \times \mathbb{R} \arrow[d, shift left] \arrow[d, shift right] \arrow[dr, "\mathrm{pr}_2"] & \\ \mathbb{R}\times\mathbb{R} \arrow[r, "\mathrm{pr}_2"] & \mathbb{R} \end{tikzcd}$$

We remark that $T_{(x,\hat{n};y)}\mathcal{G} = T_{(x,\hat{n})}(\mathcal{H}\setminus\{(0,\hat{1})\}) \oplus T_y\mathbb{R}$ and $\ker T\,\mathrm{pr}_2 = T_{(x,\hat{n})}(\mathcal{H}\setminus\{(0,\hat{1})\})$. Consider the multiplicative connection $\mathrm{E}_{(x,\hat{n};y)} = \left\{(\alpha\,\partial_x\big|_p, \alpha\,\partial_y\big|_p) : \alpha\in\mathbb{R}\right\}$, where we identify $T_{(x,\hat{n})}\mathcal{H}\cong T_x\mathbb{R}$ and $T_{(x,\hat{n};y)}\mathcal{G} = T_{(x,\hat{n})}\mathcal{H}\oplus T_y\mathbb{R}$. Then consider the lift of the curve $\gamma : [0,1]\to\mathbb{R} : t\mapsto 1-t$ to the point $(1,1;1)$. Then this lift clearly exists for $t<1$ as it is given by $\widetilde{\gamma}(t) = (1-t,1;1-t)$. Remark that indeed $\mathcal{K}$ is not $\boldsymbol{t}$-connected. //

4.4 Induced maps and Morita Equivalences

The last section of this chapter will involve the construction of a particular functor on fibred Lie groupoids, which will also relate to the existence of a complete connection. Let us denote $\mathbf{LieGrpd}_{\text{weak},0}$ for the collection of Lie groupoids. Additionally, fix some fibred Lie groupoid $\phi\colon \mathcal{G}\to\mathcal{H}$ for this section. Recall that the kernel of a fibred Lie groupoid defines a family of Lie groupoids. In particular, this gives us the map

$$\mathcal{F}_0\colon \mathcal{H}_0\to\mathbf{LieGrpd}\colon x\mapsto(\phi^{-1}(1_x)\rightrightarrows\phi_0^{-1}(x)).$$

By Proposition **??**, this indeed maps into **LieGrpd**. Additionally, we can define a map from the arrows of $\mathcal{H}_0$ to bibundles as

$$\mathcal{F}\colon \mathcal{H}\to\text{Bibundles}\colon \left[y\xleftarrow{h}x\right]\mapsto\left[\phi^{-1}(1_y)\circlearrowright_{\boldsymbol{t}}\phi^{-1}(h)\,{}_{\boldsymbol{s}}\!\circlearrowleft\phi^{-1}(1_x)\right].$$

If we view $\mathcal{H}$ as a category, we see that this almost defines a functor if we view $\phi^{-1}(1_y)\circlearrowright_{\boldsymbol{t}}\phi^{-1}(h)\,{}_{\boldsymbol{s}}\!\circlearrowleft\phi^{-1}(1_x)$ as a morphism from $\phi^{-1}(1_x)$ to $\phi^{-1}(1_y)$. However, the collection of bibundles does not admit a well-defined composition because the construction of Proposition **??** works in the case where the bibundles are right principal. This was needed to solve one problem: the quotient by the middle groupoid. We can argue why this is needed by the following example:

Let $\mathcal{G}$ be a Lie groupoid, then $\mathcal{G}$ is a principal $\mathcal{G}$-$\mathcal{G}$-bibundle as with left and right multiplication. We can view this as the identity morphism on a Lie groupoid in $\mathbf{LieGrpd}_{\text{weak}}$. If we consider "composing" these principal bibundles, we could take the product with the naturally induced left and right action:

$$\mathcal{G}\circlearrowright_{\boldsymbol{t}\circ\mathrm{pr}_1}\mathcal{G}\times\mathcal{G}\,{}_{\boldsymbol{s}\circ\mathrm{pr}_2}\!\circlearrowleft\mathcal{G}.$$

However, this will in general no longer be isomorphic to $\mathcal{G}$ as this will have twice the dimension of $\mathcal{G}$. Therefore, we want to reduce the dimension by taking a quotient by the middle action. However, this implies that the diagonal action on the middle space must exist and that the action needs to be free and proper. This is exactly what is ensured by restricting the bibundles to right principal bibundles.

Let us investigate in which manner we fail to obtain the principal bibundle in this case.

Proposition 4.4.1. *Given a fibred Lie groupoid $\phi\colon \mathcal{G} \to \mathcal{H}$. If $y \xleftarrow{h} x \in \mathcal{H}$ then the trivially induced actions $\phi^{-1}(1_y) \circlearrowright_{\boldsymbol{t}} \phi^{-1}(h)$ and $\phi^{-1}(h)\, {}_{\boldsymbol{s}}\circlearrowleft \phi^{-1}(1_x)$ are free and proper.*

Proof. Take $y \xleftarrow{h} x \in \mathcal{H}$ and consider the induced actions by multiplication. By symmetry we only have to show that $\phi^{-1}(1_y) \circlearrowright_{\boldsymbol{t}} \phi^{-1}(h)$ free and proper action. Note that the action map is given by

$$\alpha\colon \phi^{-1}(1_y)\, {}_{\boldsymbol{s}}\!\times_{\boldsymbol{t}} \phi^{-1}(h) \to \phi^{-1}(h) \times \phi^{-1}(h)\colon (g_1, g_2) \mapsto (g_1 g_2, g_2)$$

We can restrict the image to $\phi^{-1}(h)\, {}_{\boldsymbol{s}}\!\times_{\boldsymbol{s}} \phi^{-1}(h)$, which is an embedded submanifold of the product. Here, we find an inverse given by the map

$$\phi^{-1}(h)\, {}_{\boldsymbol{s}}\!\times_{\boldsymbol{s}} \phi^{-1}(h) \to \phi^{-1}(1_y)\, {}_{\boldsymbol{s}}\!\times_{\boldsymbol{t}} \phi^{-1}(h)\colon (g_1, g_2) \mapsto (g_1 g_2^{-1}, g_2)$$

Remark that this defines a map of Lie groupoids between the submersion groupoid and the action groupoid, cf. Example **??**. This implies that the action is free and proper. □

Therefore, combined with Proposition **??**, the only obstruction to the bundles being principal is the bijectivity on the quotient of the moment maps. Under similar assumptions as before, when proving the completeness of connections, we can show that $\mathcal{F}$ is indeed a better-behaved map.

Theorem 4.4.2. *If $\phi\colon \mathcal{G} \to \mathcal{H}$ is a fibred Lie groupoid that is arrow complete, then*

$$\mathcal{F}\colon \mathcal{H} \to \mathrm{PrincipalBibundles} \subset \mathrm{Bibundles}\,.$$

Moreover, this assignment preserves composition up to equivalence of bibundles.

Proof. From Proposition **??** and **??** and the assumption that $\boldsymbol{s}\colon \phi^{-1}(h) \to \phi_0^{-1}(\boldsymbol{s}(h))$ is always surjective, which implies that $\boldsymbol{t}\colon \phi^{-1}(h) \to \phi_0^{-1}(\boldsymbol{t}(h))$ is surjective, we find that $\mathcal{F}$ maps into principal bibundles.

To see that it is functorial, we need to check that it preserves composition. In other words, we need to find an equivalence any $(h_1, h_2) \in \mathcal{H}^{(2)}$:

$$\phi^{-1}(h_1) \otimes \phi^{-1}(h_2) \cong \phi^{-1}(h_1 h_2).$$

For now, fix some $z \xleftarrow{h_1} y \xleftarrow{h_2} x \in \mathcal{H}^{(2)}$ and recall that this composition of bibundles was defined as:

$$\phi^{-1}(h_1) \otimes \phi^{-1}(h_2) = \frac{\phi^{-1}(h_1)\, {}_{\boldsymbol{s}}\!\times_{\boldsymbol{t}} \phi^{-1}(h_2)}{\phi^{-1}(1_{\boldsymbol{s}(h_1)})}$$

We will define an equivariant map on the fibred product of the two spaces and show that it descends to a diffeomorphism on the quotient. Define the map as follows:

$$\alpha\colon \phi^{-1}(h_1)\, {}_{\boldsymbol{s}}\!\times_{\boldsymbol{t}} \phi^{-1}(h_2) \to \phi^{-1}(h_1 h_2)\colon (g_1, g_2) \mapsto g_1 g_2.$$

This is simply the restriction of the multiplication map to an embedded submanifold, and therefore it is smooth. Remark that this is also clearly equivariant for the left and right actions. To see that it descends to the quotient, we remark that

$$\alpha(g \cdot (g_1, g_2)) = \alpha(g_1 g^{-1}, g g_2) = g_1 g^{-1} g g_2 = g_1 g_2 = \alpha(g_1, g_2).$$

This implies that α is constant on the fibres of the quotient map, and thus it automatically descends to the quotient. Let us denote this map by $\overline{\alpha}$.

To show that it is a diffeomorphism, we will show that it is surjective, injective and an immersion, which is enough by the global rank theorem.

Surjective: Suppose that $g \in \phi^{-1}(h_1 h_2)$, by arrows completeness we can find some $g_1 \in \phi^{-1}(h_1)$ such that $\boldsymbol{s}(g_1) = \boldsymbol{s}(g)$. If we set $g_2 = g_1^{-1} g$, it follows that

$$\phi(g_2) = \phi(g_1^{-1} g) = \phi(g_1^{-1})\phi(g) = h_1^{-1} h_1 h_2 = h_2 \quad \text{and} \quad g_1 g_2 = g_1 g_1^{-1} g = g.$$

Therefore, $\alpha(g_1, g_2) = g$ and $(g_1, g_2) \in \phi^{-1}(h_1) \,{}_{\boldsymbol{s}}\!\times_{\boldsymbol{t}} \phi^{-1}(h_2)$. The map on the quotient is therefore also surjective.

Injective: Suppose that $\alpha(g_1, g_2) = \alpha(g_3, g_4)$, such that $g_1 g_2 = g_3 g_4$. We can then define g as the element $g_3^{-1} g_1 = g_4 g_2^{-1}$. It then satisfies

$$g \cdot (g_1, g_2) = (g_1 g^{-1}, g g_2) = (g_1 g_1^{-2} g_3, g_4 g_2^{-1} g_2) = (g_3, g_4) \quad \text{and} \quad \phi(g) = \phi(g_3^{-1} g_1) = h_1^{-1} h_1 = 1_y.$$

We conclude that $\overline{\alpha}$ is indeed injective.

Immersion: To show that $\overline{\alpha}$ is an immersion, remark that for some $(g_1, g_2) \in \phi^{-1}(h_1) \,{}_{\boldsymbol{s}}\!\times_{\boldsymbol{t}} \phi^{-1}(h_2)$ we have the following short exact sequence

$$0 \to T_{(g_1,g_1)}(\phi^{-1}(1_y) \cdot (g_1, g_2)) \to T_{(g_1,g_2)}(\phi^{-1}(h_1) \,{}_{\boldsymbol{s}}\!\times_{\boldsymbol{t}} \phi^{-1}(h_2)) \to T_{(g_1,g_2)}\phi^{-1}(h_1) \otimes \phi^{-1}(h_2) \to 0$$

Therefore the tangent bundle $\phi^{-1}(h_1) \otimes \phi^{-1}(h_2)$ at (g_1, g_2) is canonically isomorphic to

$$T_{(g_1,g_2)}\phi^{-1}(h_1) \otimes \phi^{-1}(h_2) \cong T_{(g_1,g_2)}\phi^{-1}(h_1) \,{}_{\boldsymbol{s}}\!\times_{\boldsymbol{t}} \phi^{-1}(h_2) / T_{(g_1,g_2)}(\phi^{-1}(1_y) \cdot (g_1, g_2))$$

Therefore it is enough to show that $\ker_{(g_1,g_2)} T\alpha \subset T_{(g_1,g_2)}(\phi^{-1}(1_y) \cdot (g_1, g_2))$.

Set $g = \alpha(g_1, g_2)$ and suppose that $(u_1, u_2) \in \ker T_{(g_1,g_2)}\alpha$. Take $\Gamma\colon (-\epsilon, \epsilon) \to \phi^{-1}(h_1) \,{}_{\boldsymbol{s}}\!\times_{\boldsymbol{t}} \phi^{-1}(h_2)$ to be a curve integrating this tangent vector. We can then project this curve onto the components, denote by $\gamma_i = \mathrm{pr}_i \circ \Gamma$, such that $\gamma_1\colon (-\epsilon, \epsilon) \to \phi^{-1}(h_1)$ and $\gamma_2\colon (-\epsilon, \epsilon) \to \phi^{-1}(h_2)$.

Notice that $\alpha(\gamma_1(t), \gamma_2(t)) = g$ as $T_{(g_1,g_2)}\alpha(u_1, u_2) = 0$. This implies that $g_1 g_2 = \gamma_1(t)\gamma_2(t)$ for all $t \in (-\epsilon, \epsilon)$, and thus we can define $h\colon (-\epsilon, \epsilon) \to \mathcal{G}\colon t \mapsto \gamma_1(t)^{-1} g_1 = \gamma_2(t) g_2^{-1}$. The curve Γ can then be rewritten as $\Gamma(t) = (g_1 h(t)^{-1}, h(t) g_2) = h(t) \cdot (g_1, g_2)$. Applying ϕ to h, we see that

$$\phi(h(t)) = \phi(\gamma_2(t) g_2^{-1}) = \phi(\gamma_2(t))\phi(g_2^{-1}) = h_2 h_2^{-1} = 1_y.$$

Therefore, Γ maps into $(g_1, g_2) \cdot \phi^{-1}(1_y)$ and thus $\ker T_{(g_1,g_2)}\alpha \subset T_{(g_1,g_2)}((g_1, g_2)\phi^{-1}(1_y))$.

We conclude that $\overline{\alpha}$ is a bijective immersion and thus it is a diffeomorphism. In particular, we conclude that the assignment preserves the principal bibundles up to equivalence. □

Corollary 4.4.3. *Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a fibred Lie groupoid with a complete multiplicative connection such that $\mathcal{H}$ is $\boldsymbol{s}$-connected, then $\mathcal{F}$ defines a functor from $\mathcal{G}$ to* $\boldsymbol{\mathit{LieGrpd}}_{\text{weak}}$.

Proof. This follows from Proposition **??** and Theorem **??**. □

Lastly, we remark that any element in $\mathcal{H}$ is invertible and therefore if $\mathcal{F}$ is a functor to $\mathbf{LieGrpd}_{\mathrm{weak}}$, then it maps into Morita equivalences. For such fibred Lie groupoids, we therefore find that the family of Lie groupoids $\ker\phi \rightrightarrows \mathcal{G}_0$ canonically becomes a family of Morita equivalent Lie groupoids.

The most natural condition is the existence of a complete connection, but by Proposition **??** we already know that the fibres are isomorphic as Lie groupoids. However, these isomorphisms are not canonically defined and depend on the choice.

Chapter 5

Symplectic applications

In this last chapter, we will discuss some of the applications of (multiplicative) connections to (multiplicative) symplectic fibrations. While the study of symplectic geometry is relatively old, more recent research in Poisson geometry has led to a specific interest in symplectic Lie groupoids, which integrate Poisson structures, see [?Mackenzie2000]. In a recent paper, [?Fernandes2024], there was a construction of a normal form around Poisson submanifolds, which led to a fibred Lie groupoid construction where the Lie groupoids were symplectic as well. This has sparked a new interest in these specific objects.

We will start with a retelling of the classical theory of symplectic fibrations as can be found in [?Guillemin1996] or [?McDuff2017], where the relation between the existence of symplectic connections and the existence of globally integrating forms is essential. We will then define a context of multiplicative symplectic fibrations and showcase how multiplicative connections may play a similar role as they do in the classical theory.

5.1 Symplectic fibrations

To introduce symplectic fibrations, we will work similarly to our treatment of fibre bundles in Chapter **??**. To add structure to our surjective submersions, but not on the total or base space, we turn to its fibres. Naively, we can define a *family of symplectic forms* on $\pi\colon M \to B$ as a collection $\{\sigma_b\}_{b\in B}$ such that $\sigma_b \in \Omega^2(M_b)$ is a symplectic form. However, as our indexing set is a manifold, we would like to incorporate some smoothness conditions here.

Definition 5.1.1. A family of symplectic forms $\{\sigma_b\}_{b\in B}$ on $\pi\colon M \to B$ is called *smooth* if the glueing to a single vector bundle morphism is smooth, i.e. the following map is smooth:

$$\sigma\colon \bigwedge^2 \mathrm{Ver} \to \mathbb{R}\colon u \wedge v \in \bigwedge^2 \mathrm{Ver}_x \mapsto \sigma_{\pi(x)}(u, v).$$

Notation. We will always denote the "glueing" of a family of forms by dropping the subscript of the basepoint.

We remark that the construction of smooth families of forms is natural in the theory of foliations, in the sense that a family of symplectic forms defines a foliated form on the associated simple foliation. In this setting, we also have an idea of closed forms, which then coincides with forms which are closed when restricted to each fibre. This, in particular, implies that this is a good notion of a smooth family of forms on a surjective submersion. More details on these objects can be found in [?Crainic2021, App. C].

Definition 5.1.2. A *symplectically fibred manifold* is a surjective submersion $\pi\colon M \to B$ with a smooth family of symplectic forms $\{\sigma_b\}_{b\in B}$.

Additionally, we have a notion of isomorphism for such structures. This not only preserves the fibred structure of the surjective submersion, but also the symplectic structure on each fibre.

Definition 5.1.3. Let $\pi_i\colon M_i \to B$ be symplectically fibred manifolds, with $i = 1, 2$, where the families are denoted by $\{\sigma_{i,b}\}_{b\in B}$. An isomorphism between them is a fibred diffeomorphism $\phi\colon M_1 \to M_2$ such that for each $b \in B$ it pulls back the forms on the fibres, i.e. $\phi^*\sigma_{2,b} = \sigma_{1,b}$.

Just like the case of families of Lie groupoids and surjective submersions, we have a trivial notion of symplectically fibred manifolds.

Example 5.1.4. Given a manifold B and symplectic manifold (F, σ), then $\mathrm{pr}_1\colon B \times F \to F$ is symplectically fibred by $\{\sigma_b\colon (u, v) \mapsto \sigma(u, v)\}_{b\in B}$. Clearly this is a smooth as $\overline{\sigma} = \mathrm{pr}_2^*\,\sigma$.

We call a symplectically fibred manifold *trivial* if it is isomorphic to such a trivial example. //

Using the above notion of triviality, we can then again consider a local version of it as well.

Definition 5.1.5. A *symplectic fibre bundle* is a symplectically fibred manifold $\pi\colon M \to B$ with typical fibre (F, σ), a symplectic manifold, such that it admits a trivialising cover $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ consisting of isomorphisms of symplectically fibred manifolds.

If all the fibres are symplectomorphic to (F, σ), we will also denote this by $(F, \sigma) \hookrightarrow M \xrightarrow{\pi} B$.

We remark that, as the name suggests, a symplectic fibre bundle is indeed a fibre bundle. Recall that a fibre bundle is fully characterised, up to isomorphism, by its transition data. Therefore, we would also like to incorporate the geometrical data of a symplectic fibre bundle into this data.

Definition 5.1.6. A fibre bundle $F \hookrightarrow M \xrightarrow{\pi} B$ has *structure group* $G \subset \mathrm{Diff}(F)$ if there exists a trivialising cover $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ whose transition data has values in G.

To translate this back to Definition **??**, given a fibre bundle with structure group $\mathrm{Symp}(F, \sigma)$, we construct forms on the fibres. Let $F \hookrightarrow M \xrightarrow{\pi} B$ be a fibre bundle, with (F, σ) a symplectic manifold, and $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ a trivialising cover with transition data in $\mathrm{Symp}(F, \sigma)$. Due to the transition data mapping into $\mathrm{Symp}(F, \sigma)$ we remark that the following holds:

$$\psi_{\alpha,b}^*\sigma = (\psi_{\beta,b})^*((\psi_{\beta,b})^*)^{-1}\psi_{\alpha,b}^*\sigma = (\psi_{\beta,b})^*(\psi_{\beta\alpha,b})^*\sigma = (\psi_{\beta,b})^*\sigma.$$

Therefore, we can define a symplectic form on M_b as $\sigma_b = \psi_{\alpha,b}^*\sigma$, for $b \in U_\alpha$, and note that it is independent of the choice of α.

Remark that these forms may not be dependent on the choice of local trivialisation within a symplectic trivialising cover, but they do depend on the symplectic trivialising cover itself. In practice, we almost always implicitly choose a symplectic trivialising cover and directly work with the family of forms $\{\sigma_b\}_{b\in B}$ without mention of the cover.

Proposition 5.1.7. *Let $F \hookrightarrow M \xrightarrow{\pi} B$ be a fibre bundle, then it is a symplectic fibre bundle if and only if it has structure group* $\mathrm{Symp}(F, \sigma)$.

Proof. Let $(F, \sigma) \hookrightarrow M \xrightarrow{\pi} B$ be a symplectic fibre bundle, then its transition functions have transition data in $\mathrm{Symp}(F, \sigma)$ as they are by isomorphism of trivial symplectically fibred manifolds.

Conversely, suppose that the transition data is in $\mathrm{Symp}(F, \sigma)$ and let $\{\sigma_b\}_{b \in B}$ be the induced family of forms on the fibres, obtained from a symplectic trivialising cover $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in \Lambda}$. These are symplectic forms as $\psi_{\alpha,b}$ is a diffeomorphism. To show that σ is smooth, we will show that it is locally isomorphic to the trivial model of Example **??**. Consider the restriction of σ to $\pi^{-1}(U_\alpha)$. Here we find that

$$\sigma(u \wedge v) = \sigma_{\pi(x)}(u \wedge v) = (\psi^*_{\alpha,b}\sigma)(u \wedge v) = \psi_\alpha \operatorname{pr}_2^* \sigma(u \wedge v).$$

This latter expression is smoothly defined and therefore $\overline{\sigma}$ is a smooth vector bundle map. This, in particular, implies that it is a symplectic fibre bundle. $\square$

5.1.1 Fibre-compatible forms

We remark that the vertical bundle of a surjective submersion is included in TM, the tangent space of the total space, and thus, we obtain a pullback map $\iota^* \colon \Omega^k(TM) \to \Omega^k(\mathrm{Ver})$ by precomposition in each component. To define an inverse to this map, one can choose a connection and use the splitting of forms as described in Section **??**. In particular, this means that global form $\omega \in \Omega^2(M)$ might define a symplectically fibred manifold structure on $\pi \colon M \to B$ if and only if the fibres are symplectic submanifolds. If we start with a symplectically fibred manifold, we may wonder whether we can find such a global extension.

Definition 5.1.8. On a symplectically fibred manifold $\pi \colon M \to B$ with family of symplectic forms $\{\sigma_b\}_{b \in B}$, a *fibre-compatible form* is a form $\omega \in \Omega^2(M)$ such that $\iota^*\omega = \sigma$.

There is a direct relation between the existence of connections and the existence of fibre-compatible forms, which is why connections are particularly important for the theory of symplectically fibred manifolds.

Proposition 5.1.9. *Let $\pi \colon M \to B$ be a symplectically fibred manifold, then a fibre-compatible form ω uniquely defines a connection E_ω such that they are compatible.*
Conversely, a connection E defines a fibre- and E-compatible form ω_{E}.

Proof. Suppose that $\pi \colon M \to B$ is a symplectically fibres manifold with family of forms $\{\sigma_b\}_{b \in B}$.

Given a fibre-compatible form ω, we can define an Ehresmann connection

$$\mathrm{E}_\omega = \mathrm{Ver}^\omega = \{u \in TM \colon \omega(u, v) = 0 \text{ for all } v \in \mathrm{Ver}\}\,.$$

It is a standard result in linear algebra and differential geometry to check that this is a complementary subbundle to Ver. Clearly, then $\omega_{(1,1)} = 0$ under the induced splitting of forms. Moreover, we can see that this is the only connection for which this holds.

Conversely, if we are given a connection, we obtain a canonical splitting of $\Omega^2(M)$, with an inclusion of $\Gamma(\bigwedge^2 \mathrm{Ver}^*)$. Under this inclusion, we obtain a connection- and fibre-compatible form. $\square$

We remark that there may be multiple fibre-compatible forms corresponding to a single connection, and that this freedom is described exactly by a choice in $\Omega^2(\mathrm{E})$. Therefore, the map $\mathrm{E} \mapsto \omega_{\mathrm{E}}$ is only a right inverse to $\omega \mapsto \mathrm{Ver}^{\omega}$.

5.2 Comments on the existence of complete symplectic connections

In this next section, we want to make a digression on a version of Theorem **??** in the symplectic case, and most importantly, why the proof of this fails. Recall that there is a geometric notion of a symplectic connection in terms of the holonomy maps.

Definition 5.2.1. A connection on a symplectically fibred manifold is called *symplectic* if the holonomy maps are by symplectic maps.

Clearly, given a symplectic connection which is complete on a symplectically fibred manifold, we immediately obtain local trivialisations with transition data in $\mathrm{Symp}(F, \sigma)$. We also remark that on a symplectic fibre bundle, there always exists a symplectic connection by simply glueing together closed fibre-compatible forms on each locally trivial part by some partition of unity on the base space, see [?Guillemin1996]. However, we remark that this is fundamentally different from our proof of Theorem **??**, where we let the form vary over the fibres. Let us therefore discuss why the converse is not necessarily true, and our methods do not apply. First, we will go into some of the properties of symplectic connections, to get a better grip on how to possibly apply the proof of Theorem **??**.

While being symplectic is a statement on the induced parallel transport of the connection, we can translate this to the associated fibre-compatible forms instead. To do this, we need the following standard lemma on the flows of time-dependent vector fields.

Lemma 5.2.2 ([?Lee2013, Prp 22.14])**.** *Let M be a manifold, $X\colon I \times M \to TM$ a time-dependent vector field and $\omega \in \Omega^k(M)$, then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_1} \left[(\phi_X^{t,t_0})^*\omega\right]_p = \left[(\phi_X^{t_1,t_0})^*\mathcal{L}_{X_{t_1}}\omega\right]_p$$

Theorem 5.2.3. *Let $\pi\colon M \to B$ be a symplectically fibred manifold and ω a fibre-compatible form, the following are equivalent:*

i) *The induced connection, $\mathrm{E} = \mathrm{Ver}^{\omega}$ is symplectic.*

ii) *For all $X \in \mathfrak{X}(B)$ and $v_1, v_2 \in \mathrm{Ver}_x$, ranging over all $x \in M$, we have $\mathcal{L}_{h(X)}\omega(v_1, v_2) = 0$.*

iii) *For all $v_1, v_2 \in \mathrm{Ver}_x$ we have $\iota_{v_1 \wedge v_2} d\omega = 0$, ranging over all $x \in M$.*

Proof. Suppose that $\pi\colon M \to B$ is a symplectically fibred manifold and ω a compatible form. We will denote $\mathrm{E} = \mathrm{Ver}^{\omega}$ and $h\colon \mathfrak{X}(B) \to \mathfrak{X}(M)$ as the induced horizontal lift.

i) $\Longrightarrow$ ii) Suppose that the connection is symplectic, i.e. $(\tau_\gamma^{s,t})^*\sigma_{\gamma(t)} = \sigma_{\gamma(s)}$ for all curves $\gamma\colon I \to B$. Fix a $X \in \mathfrak{X}(B)$ and let $\gamma\colon I \to B$ be its integral curve starting at b. By setting $s = 0 = t_0$ in Lemma **??**, we can deduce that on some $v_1, v_2 \in \mathrm{Ver}_x$, wih $x \in M_{\gamma(t_0)}$, the Lie derivative satisfies

$$\begin{aligned}\mathcal{L}_{h(X)}\omega(v_1, v_2) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(\phi^t_{h(X)}\right)^* \omega \right)(v_1, v_2) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \left(\tau^{0,t}_\gamma\right)^* \sigma_{\gamma(t)} \right)(v_1, v_2), \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \sigma_{\gamma(0)} \right)(v_1, v_2) = 0.\end{aligned}$$

Here, we used that $\tau^{0,t}_\gamma$ maps between fibres and thus $T\tau^{0,t}_\gamma$ maps a vertical vector to a vertical vector. Therefore, we can restrict ω to the vertical bundle, where it is given by σ_b.

ii) $\implies$ i) Suppose that for all $X \in \mathfrak{X}(B)$ and $v_1, v_2 \in \mathrm{Ver}_x$, ranging over all $x \in M$, we have $\mathcal{L}_{h(X)}\omega(v_1, v_2) = 0$. Let $\gamma\colon I \to B$ be some regular curve and fix some $a \in I$. Notice that we can assume regularity as the holonomy, which we consider for symplectic connections, is invariant under reparametrisation. For any $b \in I$, we can find an $\epsilon > 0$ such that $\gamma|_{[b-\epsilon,b+\epsilon]}$ is an embedding. In particular, the tangent of $\gamma|_{[b-\epsilon,b+\epsilon]}$ extends to a vector field $X \in \mathfrak{X}(B)$, such that $X_{\gamma(t)} = \dot\gamma(t)$ for $t \in [b-\epsilon, b+\epsilon]$. It follows that for some vertical vectors $v_1, v_2 \in \mathrm{Ver}_{\gamma(a)}$ we have:

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=b} \left(\tau^{a,t}_\gamma\right)^* \sigma_{\gamma(t)} \right)(v_1, v_2) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=b} (\tau^{a,b}_\gamma)^*(\tau^{b,t}_\gamma)^*\sigma_{\gamma(t)} \right)(v_1, v_2), \\ &= \left((\tau^{a,b}_\gamma)^* \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=b} (\phi^{t-b}_{h(X)})^*\sigma_{\gamma(t)} \right)(v_1, v_2), \\ &= (\tau^{a,b}_\gamma)^*(\phi^0_{h(X)})^*\mathcal{L}_{h(X)}\omega(v_1, v_2), \\ &= \mathcal{L}_{h(X)}\omega(T\tau^{a,b}_\gamma v_1, T\tau^{a,b}_\gamma v_2) = 0.\end{aligned}$$

Here, we again used the fact that $T\tau^{a,b}_\gamma$ sends vertical vectors to vertical vectors. This implies that $(\tau^{a,t}_\gamma)^*\sigma_{\gamma(t)}$ is constant and thus the holonomy is by symplectic maps.

ii) $\iff$ iii) Using Cartan's magic formula we have for any $X \in \mathfrak{X}(B)$, we have

$$\iota_b^*\mathcal{L}_{h(X)}\omega = \iota_b^* d(\iota_{h(X)}\omega) + \iota_b^*\iota_{h(X)}(d\omega).$$

Remark that $\iota_b\colon F \to M$ denotes the fibre inclusion and $\iota_{h(X)}$ the interior multiplication. Due to ω being compatible with the connection, it follows that the restriction of $\iota_{h(X)}\omega$ to Ver vanishes. Therefore, the pullback along ι_b, which corresponds to the restriction to $\mathrm{Ver}\,|_{M_b}$, vanishes as well. Using the fact that d commutes with pullbacks, we find that for each $v_1, v_2 \in \mathrm{Ver}_x$ we have

$$d(\iota_{h(X)}\omega)(v_1, v_2) = \iota^*_{\pi(x)} d(\iota_{h(X)}\omega)(v_1, v_2) = d(\iota^*_{\pi(x)}\iota_{h(X)}\omega)(v_1, v_2) = 0.$$

We conclude that $\iota_b^*\mathcal{L}_{h(X)}\omega = \iota_b^*\iota_{h(X)}(d\omega)$. It is clear that if $\iota_{v_1\wedge v_2} d\omega = 0$, then $\mathcal{L}_{h(X)}\omega(v_1, v_2) = 0$. Conversely if $\mathcal{L}_{h(X)}\omega(v_1, v_2) = 0$, we notice that in general

$$d\omega(w, v_1, v_2) = d\omega(w^\top, v_1, v_2) + d\omega(w^\perp, v_1, v_2),$$

where $w^\top \in \mathrm{E}$ and $w^\perp \in \mathrm{Ver}$. It follows that $d\omega(w^\perp, v_1, v_2) = d\sigma_{\pi(x)}(w^\perp, v_1, v_2) = 0$ as $\sigma_{\pi(x)}$ is symplectic. The other term vanishes as $w^\top$ extends to a horizontal vector field of the form $h(X)$, such that per our assumption, it holds. $\square$

Hence, we can capture whether a connection is symplectic completely in terms of the fibre-compatible form it generates. Additionally, the previous description of symplectic connections in terms of the Lie derivatives lets us deduce that on a trivial symplectically fibred manifold, we obtain a description in terms of the fibrewise horizontal lifts. Where these normally map into symplectic vector fields, we can require them to instead preserve the symplectic structure as well, by mapping into the Lie algebra associated to $\mathrm{Symp}(F, \sigma)$.

Corollary 5.2.4. *Let h be a connection on $(F, \sigma) \hookrightarrow B \times F \overset{\mathrm{pr}_1}{\to} B$. It is symplectic if and only if $h_b \colon T_b B \to \mathfrak{X}(F)$ maps into $\mathfrak{symp}(F, \sigma)$.*

Lastly, we remark that being a symplectic connection is a local condition on a symplectic fibre bundle.

Proposition 5.2.5. *Let $M \overset{\pi}{\to} B$ be a symplectic fibre bundle and E is a connection, then the following are equivalent:*

i) *The connection E is symplectic.*

ii) *There exists a symplectic trivialising cover $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ such that $\mathrm{E}|_{M|_{U_\alpha}}$ is symplectic on $M|_{U_\alpha} \overset{\pi}{\to} U_\alpha$.*

Proof. Suppose $M \overset{\pi}{\to} B$ be a symplectic fibre bundle and E is a connection.

i) $\implies$ ii) If E is symplectic, and $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ is a symplectic trivialising cover, the lift of $\gamma \colon I \to U_\alpha$ through $\mathrm{E}|_{M|_{U_\alpha}}$ is the same as the lift through E. In particular, this implies that their holonomies coincide and therefore they are always symplectic.

ii) $\implies$ i) Suppose now we have a symplectic trivialising cover $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ such that $\mathrm{E}|_{M|_{U_\alpha}}$ is symplectic. Let $\gamma \colon I \to B$ be a curve and fix some $s, t \in I$. Pick a finite partition $s = t_0 < t_1 < \cdots < t_n = t$ and a sequence $\{(U_i, \psi_i)\}_{i=1}^n \subset \{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Lambda}$ such that $\gamma\,[t_i, t_{i+1}] \subset U_i$. As the holonomy of $\gamma|_{[t_i,t_{i+1}]}$ is always symplectic, it follows that:

$$\left(\tau_\gamma^{s,t}\right)^* \sigma_{\gamma(t)} = \left(\tau_\gamma^{t_{n-1},t}\right)^* \left(\tau_\gamma^{t_{n-2},t_{n-1}}\right)^* \cdots \left(\tau_\gamma^{s,t_1}\right)^* \sigma_{\gamma(t)} = \sigma_{\gamma(s)}.$$

Here, we again used that the holonomy of paths mapping within U_α is given by the holonomy along the restricted connection. This implies that E is also symplectic. ☐

Let us now discuss why the proof of Theorem **??** does not extend to the symplectic case. Here, we will assume that we have constructed the sets S_α and the associated cover W_α with a partition of unity ϕ_α subordinate to it. We can then consider the glueing of the canonically induced connections h_α, let us denote this glueing by h, and remark that it is complete by construction. To check whether this is a symplectic connection, we need only focus on the local properties of the connection, by Proposition **??**. As this is a trivial symplectically fibred manifold, we can thus check only the maps h_b, as seen from Corollary **??**. However, the image of $h_b(v)$, for some $v \in T_b B$, will be a $C^\infty(F)$-linear combination of symplectic vector fields. However, as symplectic vector fields are defined to be such that $\mathcal{L}_X \omega = 0$, they are not closed under $C^\infty(F)$-linear combinations.

5.3 Symplectic Lie groupoid fibrations

In this last section, we will give a new notion of a symplectic Lie groupoid fibration which incorporates both the symplectic structure of a symplectically fibred manifold and the multiplicative structure of a fibred Lie groupoid, such that it generalises objects like symplectic Lie groupoids. Then, we finish off with a result which is the multiplicative analogue to Proposition **??**. In particular, we remark that for $(h_1, h_2) \in \mathcal{H}^{(2)}$ we obtain some maps which encode the multiplicative data of the total groupoid $\mathcal{G}$:

$$\boldsymbol{m}\colon \phi^{-1}(h_1)\,{}_{\boldsymbol{s}}\times_{\boldsymbol{t}}\phi^{-1}(h_2) \to \phi^{-1}(h_1h_2), \quad \mathrm{pr}_i\colon \phi^{-1}(h_1)\,{}_{\boldsymbol{s}}\times_{\boldsymbol{t}}\phi^{-1}(h_2) \to \phi^{-1}(h_i)$$

These are simply the restrictions of $\boldsymbol{m}, \mathrm{pr}_1, \mathrm{pr}_2$ on $\mathcal{G}^{(2)}$. This then lets us intertwine the multiplicative data between the fibres of the fibred Lie groupoid.

Definition 5.3.1. A *symplectic Lie groupoid fibration* is a Lie groupoid fibration $\phi\colon \mathcal{G} \to \mathcal{H}$ such that it is a symplectic fibration and the induced symplectic forms on the fibres, $\{\sigma_h\}_{h\in\mathcal{H}}$, satisfy

$$\boldsymbol{m}^*\sigma_{hh'} = \mathrm{pr}_1^*\,\sigma_h + \mathrm{pr}_2^*\,\sigma_{h'}$$

Example 5.3.2. Let $(\mathcal{G}, \Omega)$ be a symplectic Lie groupoid, i.e. Ω is a symplectic form on $\mathcal{G}$ which is multiplicative, then it is a symplectic Lie groupoid fibration over a point. //

Example 5.3.3. Let $\pi\colon M \to B$ be a symplectically fibred manifold, then it defines a symplectic Lie groupoid fibration when considered as the identity groupoids. //

Proposition 5.3.4. *Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a symplectic Lie groupoid fibrations, with family of forms $\{\sigma_h\}_{h\in\mathcal{H}}$. For any unit $1_x \in \mathcal{H}$, the fibre $(\phi^{-1}(1_x), \sigma_{1_x})$ is a symplectic groupoid.*

We then show the equivalent of Proposition **??** in the two directions separately.

Proposition 5.3.5. *Let ω be a multiplicative fibre-compatible form on a symplectic Lie groupoid fibration, then Ver^ω is a multiplicative Ehresmann connection.*

Proof. Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a symplectic Lie groupoid fibration. Suppose $\omega \in \Omega^2_{\mathrm{mult}}(\mathcal{G})$ is fibre-compatible, we know that $\mathrm{E} = \mathrm{Ver}^\omega$ is the unique Ehresmann connection such that ω is compatible by Proposition **??**. To check that E is a multiplicative Ehresmann connection, we need to check that it is closed under the multiplication and inversion maps, induced on $T\mathcal{G} \rightrightarrows TM$ as the maps $T\boldsymbol{m}\colon T(\mathcal{G}^{(2)}) \to T\mathcal{G}$ and $T\boldsymbol{i}\colon T\mathcal{G} \to T\mathcal{G}$. Consider some $(u, v) \in T_{(g,h)}\mathcal{G}^{(2)} \cap \mathrm{E}^2_{(g,h)}$ and $w \in \mathrm{Ver}_{gh}$, we then find that

$$\begin{aligned}\omega(T\boldsymbol{m}(u,v), w) &= \omega(T\boldsymbol{m}(u,v), T\boldsymbol{m}(T\boldsymbol{u}\circ T\boldsymbol{t}(w), w)) = \boldsymbol{m}^*\omega((u,v), (T\boldsymbol{u}\circ T\boldsymbol{t}(w), w)) \\ &= \omega(u, T\boldsymbol{u}\circ T\boldsymbol{t}(w)) + \omega(v, w) = \omega(u, T\boldsymbol{u}\circ T\boldsymbol{t}(w))\end{aligned}$$

We remark that the following holds by the groupoid properties of Φ:

$$T\Phi \circ T\boldsymbol{u} \circ T\boldsymbol{t} = T(\Phi \circ \boldsymbol{u} \circ \boldsymbol{t}) = T(\boldsymbol{u} \circ \phi_0 \circ \boldsymbol{t}) = T(\boldsymbol{u} \circ \boldsymbol{t} \circ \Phi) = T\boldsymbol{u} \circ T\boldsymbol{t} \circ T\Phi$$

This shows that $w \in \ker T\Phi$ implies that $T\boldsymbol{u} \circ T\boldsymbol{t}(w) \in \ker T\Phi$, and thus $\omega(u, T\boldsymbol{u} \circ T\boldsymbol{t}(w)) = 0$. This implies that $T\boldsymbol{m}(u, v) \in \mathrm{E}$ for $(u, v) \in T\mathcal{G}^{(2)}$.
To show that the inversion, $T\boldsymbol{i}$, maps horizontal vectors to horizontal vectors, we remark that $\boldsymbol{i}$ is a diffeomorphism (as it is its own inverse) and thus $T\boldsymbol{i}\colon T\mathcal{G} = \mathrm{E} \oplus \mathrm{Ver} \to T\mathcal{G} = \mathrm{E} \oplus \mathrm{Ver}$ is an isomorphism. As Φ is a groupoid map, we see that

$$T\phi \circ T\boldsymbol{i} = T(\phi \circ \boldsymbol{i}) = T(\boldsymbol{i} \circ \phi) = T\boldsymbol{i} \circ T\phi$$

This implies that $T\boldsymbol{i}(\mathrm{Ver}) = T\boldsymbol{i}(\ker T\phi) \subset \ker T\phi = \mathrm{Ver}$. This implies that E is closed under inversions and therefore $\mathrm{E} \subset T\mathcal{G} \rightrightarrows TM$ is a subgroupoid. Therefore, $\mathrm{E} = \mathrm{Ver}^{\omega}$ is a multiplicative Ehresmann connection on $\phi\colon \mathcal{G} \to \mathcal{H}$. □

Proposition 5.3.6. *Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a symplectic Lie groupoid fibration with family of forms $\{\sigma_h\}_{h\in\mathcal{H}}$ and $\mathrm{E} \subset T\mathcal{G}$ be a multiplicative Ehresmann connection, then there exists some fibre-compatible form $\omega \in \Omega^2_{\mathrm{mult}}(\mathcal{G})$ such that $\mathrm{E} = \mathrm{Ver}^{\omega}$.*

Proof. Let $\phi\colon \mathcal{G} \to \mathcal{H}$ be a symplectic Lie groupoid fibration with family of forms $\{\sigma_h\}_{h\in\mathcal{H}}$ and $\mathrm{E} \subset T\mathcal{G}$ be a multiplicative Ehresmann connection. Then there exists a fibre-compatible form, as seen in Proposition **??**. To see that it is multiplicative, we remark that the induced map $\omega\colon \bigoplus^2 T\mathcal{G} \to \mathbb{R}$ factors as follows:

$$\begin{tikzcd}
\bigoplus^2 T\mathcal{G} \arrow[r, "\mathrm{pr}\oplus\mathrm{pr}"] \arrow[rr, bend left, "\omega"] & \bigoplus^2 \mathrm{Ver} \arrow[r, "\sigma"] & \mathbb{R}
\end{tikzcd}$$

As E is a multiplicative connection, the map $\mathrm{pr}\colon T\mathcal{G} \to \mathrm{Ver}$ is a $\mathcal{VB}$-groupoid morphism, and thus its direct sum $\mathrm{pr}\oplus\mathrm{pr}\colon \bigoplus^2 T\mathcal{G} \to \bigoplus^2 \mathrm{Ver}$ is a $\mathcal{VB}$-groupoid morphism as well. The conditions on $\{\sigma_h\}_{h\in\mathcal{H}}$ result in the fact that $\sigma\colon \bigoplus^2 \mathrm{Ver} \to \mathbb{R}$ is a $\mathcal{VB}$-groupoid morphism and therefore so is ω. □

These propositions indicate that the defined notion of a symplectic Lie groupoid fibrations is indeed the multiplicative equivalent of a symplectically fibred manifold, as we obtain an equivalent of Proposition **??**. Other interesting results would be the extensions of theorems like minimal coupling and Thurston's trick to the multiplicative setting.