Analysis 2, Chapter 13

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1 Lebesgue Integration

Notationwise, we again use $\overline{\mathbb{R}}$ to denote the extended real line $\mathbb{R} \cup \{\infty\}$

The Darboux (equivalently, Riemann) integral looks at all finite "partitions" (actually, covers in rectangles that are pairwise disjoint but for their boundary) of the *domain*, say $\{E_i\}_{i=1}^k$, and at all upper and lower Riemann sums:

$$\sum_{i=1}^{k}[\sup f(E_i)]\mathcal{PJ}(E_i) \quad \sum_{i=1}^{k}[\inf f(E_i)]\mathcal{PJ}(E_i)$$

If the inf over all upper sums coincides with the sup over all lower sums, the integral is defined.

Looking forward, Lebesgue's method allows non-finite partitions of the domain, by first partitioning the target space into intervals and then looking at approximating functions that are constant on the inverse images under f on these intervals.

This gives rise to the notion of a **simple function**, which is an extension of the piecewise constant function: it is, again, a function of the following form:

$$f(x) = \sum_{i=1}^{k} y_i \mathbb{1}_{E_i}(x)$$

But now, the E_i are arbitrary sets rather than (pluri)-rectangles. To measure the volume of these sets, we use the Lebesgue measure on rather than the Peano-Jordan content.

Then, the integral considers all simple functions that majorize f and looks at the simple Lebesgue integral on these functions. Taking the infinimum over all these simple integrals, it turns out we can integrate *more* than the Darboux integral can.

In the above, we can thus far only argue for functions that admit approximations by simple functions

$$h(x) = \sum_{i=1}^{M} y_i \mathbb{1}_{E_i}(x)$$

But if we can write h in a different way, say

$$h(x) = \sum_{i=1}^{M} y_i \mathbb{1}_{E_i}(x), \quad h(x) = \sum_{i=1}^{K} z_i \mathbb{1}_{F_i}(x)$$

We have to show that the simple integrals with respect to each partition are equal. Only then can we have a unique and well-defined simple integral Where, w.l.o.g. we assume that $E := \{E_i\}$ is a **refinement** of the partition $F := \{F_i\}$ (which we can assume w.l.o.g. since otherwise we can consider the **mutual refinement** $E\#F := \{E_i \cap F_j\}$, which refines both E and F, and conclude the equality via $E\#F$), then we want to show that the simple integral

$$\int_D hd\mathcal{L}(x)(x) := \sum_{i=1}^{M} y_i \mathcal{L}(E_i)$$

Equals

$$\sum_{i=1}^{K} z_i \mathcal{L}(F_i)$$

Otherwise, the simple integral cannot be well-defined. We could alternatively require in the definition of **simple functions** that they be of the form

$$\sum_{i=1}^{K} z_i \mathcal{L}(F_i) \quad \text{where } F_i \text{ is measurable for all } i$$

Why is that equivalent? Well, if there is a F_i that is not measurable, say F_1 up to renumbering, then we can consider the split of $F_1 = F_1^1 \cup F_1^2$ in disjoint sets such that $\mathcal{L}(F_1^1) + \mathcal{L}(F_1^2) > \mathcal{L}(F_1)$ and show that the equality fails to hold for the alternative writing of h as a simple function:

$$\sum_{i=2}^{K} z_i \mathbb{1}_{F_i} + z_1 \mathbb{1}_{F_1^1} + z_1 \mathbb{1}_{F_1^2}$$

By failure of measurability to hold for the split $F_1^1 \cup F_1^2 = E_i$, we can conclude that this writing integrates to a different value! This is why we require measurability of the partition sets E_i.

As a final remark, we only need to show how to approximate these measurable functions with simple functions ***from above***, arguing for signed functions f that we can write

$$f = f^+ - f^-, \quad \text{where } f^+ := \max\{f, 0\} \text{ and } f^- := -\min 0, f$$

And both of these functions f^+, f^- can be proven to be Lebesgue measurable.

1.1 Lebesgue Measurable Functions

Definition 1. *Let $A \subset \mathbb{R}^N$ and $f : \Omega \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. We say f is **$\mathcal{L}$-measurable** if for all $a \in \mathbb{R}$, the set*

$$f^{-1}((a, \infty)) = \{x \in A : f(x) > a\}$$

is $\mathcal{L}$-measurable.

Lemma 1. *The following are equivalent:*

(i) for all $y \in \mathbb{R}$, $f^{-1}((-\infty, y))$ is measurable.

(ii) for all $y \in \mathbb{R}$, $f^{-1}((y, \infty))$ is measurable.

(iii) for all $y \in \mathbb{R}$, $f^{-1}((-\infty, y])$ is measurable.

(iv) for all $y \in \mathbb{R}$, $f^{-1}([y, \infty))$ is measurable.

(v) $f^{-1}(B)$ is measurable for all Borel sets B.

Proof. The equivalence of (i) to (iv) follows simply because we can take unions and complements through the preimage function, and because

$$(-\infty, a) = \bigcup_{n \in \mathbb{N}} (-\infty, a - \frac{1}{n}]$$

With a countable union on the right hand side, and we already know that measurability of sets is closed under countable union.

That (v) implies (i) to (iv) is also clear, since the sets mentioned here are Borel sets.

To prove (v) from (i), we note that we can assume (i) to (iv) simultaneously, without loss of generality since one premise implies all other premises. Then, since we can write any borel set B in disjunctive normal form,

$$B = \bigcup_{i \in \mathbb{N}} \bigcap_{j \in \mathbb{N}} \Omega_{ij} \quad \text{with } \Omega_{ij} \text{ open}$$

we only need to prove that $\Omega \subset \mathbb{R}$ open can be written as a countable union of open intervals. To see this, consider for every $q \in \mathbb{Q} \cap \Omega$ the maximal interval $(q - \epsilon_q, q + \epsilon_q) \subset \Omega$ that is contained in Ω (by openness of Ω). Then, for any $x \in \Omega$, let $(x - \epsilon_x, x + \epsilon_x) \subset \Omega$ be the maximal interval surrounding x that is contained in Ω, and let $q \in \mathbb{Q} \cap \Omega$ be the fraction within $\frac{1}{3}\epsilon_x$ of x. Then we see that $(q - \frac{2}{3}\epsilon_x, q + \frac{2}{3}\epsilon_x)$ should fall withing Ω, so $\epsilon_q \geq \frac{2}{3}\epsilon_x$, and therefore $x \in (q - \epsilon_q, q + \epsilon_q)$ for some $q \in \mathbb{Q} \cap \Omega$, so we conclude that we can write Ω as the countable union

$$\bigcup_{q \in \mathbb{Q} \cap \Omega} (q - \epsilon_q, q + \epsilon_q)$$

This makes that we can write

$$B = \bigcup_{i\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} \bigcup_{q\in\mathbb{Q}\cap\Omega_{ij}} (q-\epsilon_q,\infty)\cap(-\infty,q+\epsilon_q)$$

Which is a countable union and intersection of sets whose preimage is measurable. Apply de Morgan's laws to write this as a countable union entirely, and conclude by taking the union through the preimage function. $\square$

Since continuous functions have open preimages of all open sets in their codomain, we have the following lemma:

Lemma 2. *Continuous functions $f : A \to \mathbb{R}$ are measurable.*

Lemma 3. *If $A \subset \mathbb{R}$ is measurable and $f : A \to \mathbb{R}$ is monotone, then f is measurable.*

Proof. Assume w.l.o.g. (as we will soon see, f is measurable if and only if $-f$ is) that f is ascending, that is

$$x \geq y \implies f(x) \geq f(y)$$

We therefore have $f^{-1}((a,\infty)) = (x,\infty)\cap A$ or $[x,\infty)\cap A$. These sets are finite intersections of measurable sets (namely an interval and A), therefore measurable. $\square$

Lemma 4. *If we define*

$$f^+ := \max\{f,0\},\quad f^- := -\min\{f,0\},\quad |f|,\quad \min\{f,g\},\quad \max\{f,g\},\quad f+g,\quad fg,\quad cf,\quad -f$$

pointwise, then these are all $\mathcal{L}$-measurable functions.

Proof. One of the more interesting ones to prove is $f+g$. We show it for $f-g$, since then we also know that it holds for $-(f-g) = 0-(f-g)$, as 0 is $\mathcal{L}$-measurable (trivially), and this makes $f--g$ also measurable.

Note, for this, that

$$\begin{aligned}\{f-g<a\} &= \bigcup_{q\in\mathbb{Q}}\{f<q<a+g\}\\ &= \bigcup_{q\in\mathbb{Q}}(\{f<q\}\cap\{q<a+g\})\end{aligned}$$

And by the fact that measurability is preserved under countable intersections and unions, this set is measurable for all $a\in\mathbb{R}$, giving the required result. As another result,

$$\{cf<a\} = \{f<a/c\}$$

and

$$\{f^2<a\} = \{f>-\sqrt{a}\}\cap\{f<\sqrt{a}\}$$

and since

$$fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$$

It follows these are measurable too. $\square$

Lemma 5. *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence of measurable functions* $A \to \mathbb{R}$*. Then also*

$$\inf_{n\in\mathbb{N}} f_n, \quad \sup_{n\in\mathbb{N}} f_n, \quad \liminf_{n\to\infty} f_n, \quad \limsup_{n\to\infty} f_n$$

are measurable. Note that we can only say this for countable index sets (compare this to preservance continuity, which held for arbitrary index sets, Chapter 3).

Proof. Write:

$$\left(\inf_{n\geq k} f_n\right)^{-1}((a,\infty)) \quad = \{x \in A \mid \inf_{n\geq k} f_n(x) > a\} = \bigcup_{n\geq k} f_n^{-1} \quad ((x,\infty))$$

$$\left(\sup_{n\geq k} f_n\right)^{-1}((-\infty,a)) \quad = \{x \in A \mid \sup_{n\geq k} f_n(x) < a\} = \bigcap_{n\geq k} f_n^{-1} \quad ((-\infty,a))$$

So we can conclude it for $k = 1$ already, and since we have $\sup_{n\geq k} f_n$, $\inf_{n\geq k} f_n$ measurable for **every** k, we can also conclude it for:

$$\liminf_{n\to\infty} f_n = \sup_{n\in\mathbb{N}} \inf_{n\geq k} f_n, \quad \limsup_{n\to\infty} f_n = \inf_{n\in\mathbb{N}} \sup_{n\geq k} f_n$$

□

All continuous functions are Lebesgue measurable, and form a strict subset of all Lebesgue measurable functions. If $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, $B \subset f(A)$ are both Lebesgue measurable, this does not guarantee anything about $f^{-1}(g^{-1}((a,\infty)))$ for any $a \in \mathbb{R}$, since $g^{-1}((a,\infty))$ may fail to be Borel. Therefore, we need a stronger condition on g:

Lemma 6. *If* $f : A \to \mathbb{R}$ *and* $g : B \to \mathbb{R}$, $B \subset f(A)$*, where* g *is continuous (in the topological sense) and* f *is Lebesgue measurable, then* $g \circ fLA \to \mathbb{R}$ *is Lebesgue measurable.*

Proof. This follows since $g^{-1}((a,\infty))$ is open, hence Borel, and f is measurable, so that

$$(g \circ f)^{-1}((a,\infty) = f(g^{-1}((a,\infty)))$$

is Lebesgue measurable for any $a \in \mathbb{R}$. □

Definition 2. *A Lebesgue-measurable function* $f : A \to \mathbb{R}$ *where* $A \subset \mathbb{R}^N$ *is called* ***simple*** *if its image* $f(A)$ *is* ***finite****. This means it is possible to write*

$$f = \sum_{i=1}^{k} y_i \mathbb{1}_{E_i}$$

Where $E_1, ...E_k$ *are measurable. We can w.l.o.g. assume* $E_1, ..., E_m$ ***disjoint****, by defining a new block where these sets overlap (these intersecting blocks are measurable again because they are finite unions of measurable sets).*

Note that while in our definition for functions, we allowed values in $\overline{\mathbb{R}}$, we do not allow this for simple functions: they only admit finite values. By this, they are also bounded (since their range is a finite set).

We will now give the main lemma of this section, which is that we can approximate any Lebesgue-measurable function pointwise with an increasing sequence of simple functions:

Lemma 7. *Let $f : A \to \mathbb{R}$ with $A \subset \mathbb{R}$. Then, the following are equivalent:*

(i) f is $\mathcal{L}$-measurable.

(ii) It is possible to find a sequence $(f_n)_{n\in\mathbb{N}} \subset (A \to \mathbb{R})$ with $f_n \leq f_{n+1}$ and $f_n \leq f$ for all $n \in \mathbb{N}$ and $f_n \to f$ pointwise.

Proof. To prove (ii) $\implies$ (i), we simply note that the limsup of a sequence of $\mathcal{L}$-measurable functions is measurable, and the pointwise limit of a sequence equals the pointwise limsup:

$$f = \limsup_{n\to\infty} f_n \text{ is therefore measurable}$$

To prove (i) $\implies$ (ii), we will find an explicit sequence of simple, measurable functions that converges pointwise to f. Of course, the construction of these functions f_n hinges on the fact that f is itself measurable. Namely, we let f_n assume a constant value in the interval $\left[\frac{k}{n}, \frac{k+1}{n}\right)$, everywhere on the measurable set $E_k := f^{-1}(\left[\frac{k^2}{n}, \frac{(k+1)^2}{n}\right))$:

To make this precise, first assume w.l.o.g. that $f \geq 0$. Otherwise, construct sequences $(f_n^+)_{n\in\mathbb{N}}$ and $(f_n^-)_{n\in\mathbb{N}}$ for f^+ and f^- separately, then we also have

$$f_n^+ - f_n^- \to f^+ - f_n^- = f \text{ pointwise}$$

and we conclude.

This means $f(A) \subset [0, \infty)$. Therefore, define, for $n \in \mathbb{N}$ and $k \in \{0, ..., n+1\}$:

$$E_k := \left[\frac{k}{n}, \frac{k+1}{n}\right), \qquad \text{for } k = 0, n^2 E_{n^2+1} := [n, \infty)$$

This is a partition of $[0, \infty)$. Next, let, for $k = 0, ..., n^2 + 1$:

$$F_k := f^{-1}(E_k) \subset A$$

This is clearly a partition of A, and $f(F_k) \subset \left[\frac{k}{n}, \frac{k+1}{n}\right)$. Define $f_n : A \to f$ as follows:

$$f_n := \sum_{k=0}^{n^2+1} \frac{k}{n} \mathbb{1}_{F_k}$$

Then, we see that

$$\begin{cases} \text{if } f(x) \in [n, n+1): & f_n(x) = 0 \leq (n+1)\lfloor \frac{f(x)}{n+1} \rfloor = f_{n+1}(x), \\ & f_{n+1}(x) = \lfloor \frac{f(x)}{n+1} \leq f(x) \\ \text{if } f(x) \in [0, n): & f_n(x) = n\lfloor \frac{f(x)}{n} \rfloor \leq (n+1)\lfloor \frac{f(x)}{n+1} \rfloor = f_{n+1}(x), \\ & f_{n+1}(x) = \lfloor \frac{f(x)}{n+1} \rfloor \leq f(x) \\ \text{if } f(x) \in [n+1, \infty): & f_n(x) = 0 \leq 0 = f_{n+1}(x), \\ & f_{n+1}(x) = 0 \leq f(x) \end{cases}$$

Therefore, $f_1 \leq f_2 \leq ... \leq f$.

Moreover, if we fix $x \in A$, and pick $\epsilon > 0$ arbitrarily, then if we pick N such that $N > f(x)$ and $\frac{1}{N} < \epsilon$, we have for all $n \geq N$ that

$$f(x) \in \left[n\lfloor \frac{f(x)}{n} \rfloor, (n+1)\lfloor \frac{f(x)}{n} \rfloor \right)$$

And therefore

$$\left| n\lfloor \frac{f(x)}{n} \rfloor - f(x) \right| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

while $N > f(x)$ guarantees that for $n \geq N$, $f_n(x) = n\lfloor \frac{f(x)}{n} \rfloor$, and therefore we conclude

$$|f_n(x) - f(x)| < \epsilon, \quad \forall n \geq N$$

So that $f_n \to f$ pointwise. □

Remark 1. *As a consequence of three facts, namely:*

1. *The simple Lebesgue integral of simple functions equals their Lebesgue integral;*
2. *The above defined simple functions increase monotonically to f.*
3. *Lebesgue's Monotone Convergence Theorem.*

We even have that $\int_{\mathbb{R}^N} f_n(x) d\mathcal{L}(x) \to \int_{\mathbb{R}^N} f(x) d\mathcal{L}(x)$ as $n \to \infty$ for this specific sequence of simple functions.

1.2 Lebesgue Integral, basic properties

Definition 3. ***Simple Lebesgue Integral***

Let $h : \mathbb{R}^N \to [0, \infty)$ be a simple Lebesgue measurable function, which can be written as

$$h = \sum_{i=1}^{k} y_i \mathbb{1}_{E_i}$$

*Then we define the **simple Lebesgue integral** of h as*

$$\int_{\mathbb{R}^N} h(x) d\mathcal{L}(x) := \sum_{i=1}^{k} y_i \mathcal{L}(E_i)$$

Note that this definition does not depend on the choice of sets E_i. Indeed, if there was another partition $\{F_j\}_{j=1}^m$ for $\mathbb{R}^N$ by which we would define h, i.e.

$$h = \sum_{i=1}^{k} y_i \mathbb{1}_(E_i) = \sum_{j=1}^{k} z_i \mathbb{1}_(E_i)$$

then whenever $x \in F_j \cap E_i$, we must have $h(x) = y_i = z_j$, and it follows that:

$$h = \sum_{j=1}^{m} \sum_{i=1}^{k} y_i \mathbb{1}_{E_i \cap F_j} = \sum_{j=1}^{m} \sum_{i=1}^{k} z_j \mathbb{1}_{E_i \cap F_j}$$

From which it easily follows

$$\sum_{i=1}^{k} y_i \mathcal{L}(E_i) = \sum_{j=1}^{m} \sum_{i=1}^{k} y_i \mathcal{L}(E_i \cap F_j) = \sum_{j=1}^{m} \sum_{i=1}^{k} z_j \mathcal{L}(E_i \cap F_j) = \sum_{j=1}^{k} z_i \mathcal{L}(E_i)$$

Where we can liberally take intersections when needed; this is fine by Lebesgue-measurability of the sets E_i, F_j and $E_i \cap F_j$.

Remark 2. *We add the convention that if $y_i = \infty$ and $\mathcal{L}(E_i) = 0$, then $y_i \mathcal{L}(E_i) = 0$.*

Definition 4. ***Lebesgue integral of a general $\mathcal{L}$-measurable function***

Let $f : \mathbb{R}^N \to \mathbb{R}$ be $\mathcal{L}$-measurable. Assume $f \geq 0$. Then, we define its Lebesgue integral as:

$$\int_{\mathbb{R}^N} f(x) d\mathcal{L}(x) = \sup \left\{ \int_{\mathbb{R}^N} h(x) d\mathcal{L}(x) : h \leq f, h \text{ simple } \mathbb{R}^N \to \mathbb{R} \right\}$$

Lemma 8. *The general Lebesgue integral of a simple function $h \geq 0$, equals its simple Lebesgue integral.*

Proof. We obviously have $\sum_{i=1}^k y_i \mathcal{L}(E_i) \leq \int_{\mathbb{R}^N} h(x) d\mathcal{L}(x)$, by the *supremum* in the definition. The other way around, we have to look at any $g \leq h$ with g simple. For such a g, assume w.l.o.g. that it is defined in terms of the same simple sets as h:

$$h = \sum_{i=1}^{k} y_i \mathbb{1}_{E_i} \quad g = \sum_{i=1}^{k} z_i \mathbb{1}_{E_i}$$

Then $z_i \le y_i$ for all $i = 1, ..., k$, and from this it follows

$$\sum_{i=1}^{k} y_i \mathcal{L}(E_i) \ge g = \sum_{i=1}^{k} z_i \mathcal{L}(E_i)$$

Which proves the opposite inequality by taking the supremum on the right. $\square$

Definition 5. ***For signed functions***
Let $f : \mathbb{R}^N \to \overline{\mathbb{R}}$ be $\mathcal{L}$-measurable. Assume at least one of the following hold:

$$\int_{\mathbb{R}^N} f^+(x) d\mathcal{L}(x) < \infty, \text{ or } \quad \int_{\mathbb{R}^N} f^-(x) d\mathcal{L}(x) < \infty$$

Then, we say f is $\mathcal{L}$-integrable and define its integral as

$$\int_{\mathbb{R}^N} f(x) d\mathcal{L}(x) := \int_{\mathbb{R}^N} f^+(x) d\mathcal{L}(x) - \int_{\mathbb{R}^N} f^-(x) d\mathcal{L}(x)$$

Most proofs below will be done for functions of a positive sign; this does not harm their generality, as we can see that the definition of an arbitrary integral is given in terms of positively signed functions anyway, and as it will soon follow, the Lebesgue integral has linearity.

Definition 6. *If $f : \mathbb{R}^N \to \mathbb{R}$ is measurable, and $E \subset \mathbb{R}^N$ is measurable, we define*

$$\int_E f(x) d\mathcal{L}(x) = \int_{\mathbb{R}^N} f(x) \mathbb{1}_E d\mathcal{L}(x)$$

Note that indeed the product $f\mathbb{1}_E$ is measurable, as $\mathbb{1}_E$ is.

A major property of the Lebesgue integral is that we can at least integrate all Riemann-integrable functions with the same result:

Lemma 9. *If $f : R \to \mathbb{R}$ be Riemann integrable, implying $R \subset \mathbb{R}^N$ is a pluri-rectangle. Then, f is Lebesgue integrable, and the two integrals have the same value.*

Lemma 10. *If $\subset \mathbb{R}^N$ is Lebesgue-negligible, then for any Lebesgue-measurable $f : \mathbb{R}^N \to \mathbb{R}$:*

$$\int_E f(x) d\mathcal{L}(x) = 0$$

Conversely, if $f : \mathbb{R}^N \to \mathbb{R}$ has a Lebesgue integral of 0 over $\mathbb{R}^N$, then it must be 0 $\mathcal{L}$-almost everywhere.

Lemma 11. *If $g, f : \mathbb{R}^N \to \mathbb{R}$ are Lebesgue measurable and*

$$\int_{\mathbb{R}^N} g(x) d\mathcal{L}(x) = \infty \implies \int_{\mathbb{R}^N} f(x) d\mathcal{L}(x) > -\infty$$
$$\int_{\mathbb{R}^N} g(x) d\mathcal{L}(x) = -\infty \implies \int_{\mathbb{R}^N} f(x) d\mathcal{L}(x) < \infty$$

Then, we already know $\lambda g + f$ is Lebesgue measurable, for any $\lambda \in \mathbb{R}$. But moreover, we have

$$\int_{\mathbb{R}^N} [\lambda g(x) + f(x)] d\mathcal{L}(x) = \lambda \int_{\mathbb{R}^N} g(x) d\mathcal{L}(x) + \int_{\mathbb{R}^N} f(x) d\mathcal{L}(x)$$

Proof. Note that if we let $\lambda \geq 0$ and $g, f \geq 0$ (without loss of generality), then the

$$\int_{\mathbb{R}^N} [\lambda \tilde{g}(x) + \tilde{f}(x)] d\mathcal{L}(x) = \lambda \int_{\mathbb{R}^N} \tilde{g}(x) d\mathcal{L}(x) + \int_{\mathbb{R}^N} \tilde{f}(x) d\mathcal{L}(x)$$

holds for any simple $\tilde{g} \leq g$, $\tilde{f} \leq f$, by homogeneity of finite sums (on both sides, there is just a finite sum). Since for ever such $\tilde{f}$, $\tilde{g}$, we have that $\lambda\tilde{g} + \tilde{f}$ is a simple function below $\lambda g + f$, we can at least conclude one inequality by taking sup on the right:

$$\left\{ \int_{\mathbb{R}^N} [\lambda g(x) + f(x)] d\mathcal{L}(x) \mid \tilde{g} \leq g, \tilde{f} \leq f, \text{simple} \right\} \subset$$

$$\left\{ \int_{\mathbb{R}^N} \tilde{h}(x) d\mathcal{L}(x) \mid \tilde{h} \leq \lambda g + f, \text{simple} \right\}, \text{therefore}$$

$$\int_{\mathbb{R}^N} [\lambda g(x) + f(x)] d\mathcal{L}(x) \leq \lambda \int_{\mathbb{R}^N} g(x) d\mathcal{L}(x) + \int_{\mathbb{R}^N} f(x) d\mathcal{L}(x)$$

For the other, let's take sequences of simple functions $\tilde{f_n} \to f$ and $\tilde{g_n} \to g$. The pointwise convergence $\lambda\tilde{g_n} + \tilde{f_n} \to \lambda g + f$ is evident, therefore if the sequences of the simple integrals are at least not growing unboundedly to opposite infinities (the second premise), then the other inequality follows by Fatou's Lemma. Now, a proof of Fatou's Lemma relies on homogeneity of the Lebesgue integral, so we cannot use it yet. Luckily, we don't need homogeneity of the Lebesgue integral, but only of the simple integral (which is trivial, finite summations are homogenous):

$$\sum_{i=1}^{m} \lambda y_i \mathcal{L}(E_i) = \lambda \sum_{i=1}^{m} \lambda y_i \mathcal{L}(E_i)$$

So, using this tacitly assumed weak version of Fatou's lemma, where homogeneity of the Lebesgue integral is not needed (in fact, the lecture notes contain a proof), the opposite inequality is evident. We will postpone a rigorous proof of Fatou's Lemma to Chapter 14. $\square$