Analysis 2, Chapter 14

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1 Limiting Theorems

1.1 Introduction

In this section, the context is a sequence of functions $(f_n)_{n\in\mathbb{N}}$ where $f_n : A \to \mathbb{R}$, and a function f where $f_n \to f$ either **pointwise** or **uniformly**. We look at sufficient conditions that allow the exchange of the integral with the limit, limsup or liminf that is, situations where one of the following equality holds:

$$\lim_{n\to\infty} \int_A f_n(x)dx = \int_A [\lim_{n\to\infty} f_n(x)]dx$$
$$\limsup_{n\to\infty} \int_A f_n(x)dx = \int_A [\limsup_{n\to\infty} f_n(x)]dx$$
$$\liminf_{n\to\infty} \int_A f_n(x)dx = \int_A [\liminf_{n\to\infty} f_n(x)]dx$$

We will state three limiting theorems for the Lebesgue integral, namely Fatou's Lemma, Lebesgue's Monotone Convergence Theorem and Lebesgue's Dominated Convergence theorem. First, a classification of the situations where we cannot relate the limit of the integral and the integral of the limit. For this, consider the following three convergent sequences:

$$f_n := \mathbb{1}_{[n,n+1]}, \quad h_n := \frac{1}{n}\mathbb{1}_{[0,n]}, \quad g_n := n\mathbb{1}_{[0,\frac{1}{n}]}$$

These functions all converge to $f = 0$, where g_n and f_n pointwise, h_n uniformly. However,

$$\int_{\mathbb{R}^N} f_n d\mathcal{L}^N(x) = 1, \quad \forall n \in \mathbb{N}, \quad \text{but} \int_{\mathbb{R}^N} f d\mathcal{L}^N(x) = 0$$

And this also holds for h_n and g_n.

Why is the volume under the graph "lost" in the limit? It has to do with

1. f_n : Volume *walking off to infinity.*
2. h_n : Volume *spreading out, becoming infinitely flat.*

3. g_n : Volume *converging to a spire, becoming infinitely narrow.*

However, we seem to be able to guarantee the inequality:

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} f_n d\mathcal{L}^N(x) \geq \int_{\mathbb{R}^N} \liminf_{n\to\infty} f_n d\mathcal{L}^N(x)$$

This result will be the first theorem to be proven. It is called Fatou's Lemma. First, we will recall the best we can do for the Riemann integral.

1.2 Riemann integral

For the Riemann integral, one can exchange the limit of a sequence f_n of functions with the integration **only** when the integral is taken over a **compact set** and $f_n \to f$ **uniformly**:

Theorem 1. *(exchange of limit with integral for the Riemann integral) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $f_n : K \to \mathbb{R}$, Riemann-integrable, $K \subset \mathbb{R}$ compact, and $f_n \to f$ uniformly. Then:*

$$\lim_{n\to\infty} \int_K f_n(x)dx = \int_K f(x)dx$$

Proof. We know that f is uniformly continuous on K, as is f_n for every n, so we can integrate all functions that are shown in the display above.

Moreover, since K is bounded, let $[a, b]$ a bounded interval such that $K \subset [a, b]$. Then:

$$\begin{aligned}\int_K f_n(x)dx - \int_K f(x)dx &= \int_K (f_n(x) - f(x))dx \\ &=\leq |a-b||f_n - f|_{C^0(K)} \\ &< \epsilon, \text{for } n \geq N,\ N \text{ sufficiently large.}\end{aligned}$$

□

1.3 Fatou's Lemma

Just for completeness' sake, here is the definition of L^p (Lebesgue) spaces.

Definition 1. L^p***-spaces***

Let $p \in \mathbb{R}$. We say $f : A \to \mathbb{R}$, $A \subset \mathbb{R}^N$ is in the space $\mathcal{L}^p(A)$ if

$$\int_A |f_n(x)|^p d\mathcal{L}^N(x)$$

exists and is finite.

We then go on to define

$$|f|_p := \left(\int_A |f_n(x)|^p d\mathcal{L}^N(x)\right)^{\frac{1}{p}}$$

This is not a normed vector space yet: in particular, there are many pathological functions that are 0 on a $\mathcal{L}$-negligible subset of A, so that $f \neq 0$ yet $|f|_p = 0$. In order to do functional analysis, we need to consider equivalence classes of such functions modulo $\mathcal{L}$-negligible differences, or formally, modulo additions in the set

$$\mathcal{N}(A) := \{f \in \mathcal{L}^p(A) : |f|_p = 0\}$$

Then it is provable that $\mathcal{L}^p(A)/\mathcal{N}(A)$ is a normed vector space, and it is called $L^p(A)$. For this chapter, the definition of $\mathcal{L}^p$ is sufficient. We only need it to polish up some notation.

Lemma 1. *Fatou's Lemma*

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $f_n : A \to \mathbb{R}$, each $\mathcal{L}^N$-measurable functions. Then:

(i) if for all $n \in \mathbb{N}$, $f_n \geq g$ with $\int_{\mathbb{R}^N} |g| d\mathcal{L}^N(x) < \infty$, then

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} f_n(x) \geq \int_{\mathbb{R}^N} \liminf_{n\to\infty} f_n(x)$$

(ii) equivalently, if for all $n \in \mathbb{N}$, $f_n \leq g$ with $\int_{\mathbb{R}^N} |g| d\mathcal{L}(x) < \infty$, then

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} f_n(x) \leq \int_{\mathbb{R}^N} \limsup_{n\to\infty} f_n(x)$$

Proof. The proof is quite similar to Dini's theorem. Note that it is sufficient to show (i). Then, if we have the premises of (ii), $f_n \leq g$, then $-f_n \geq 0$, so we can argue via (i):

$$\begin{aligned}
\limsup_{n\to\infty} \int_{\mathbb{R}^N} f_n(x) &= -\liminf_{n\to\infty} \left(-\int_{\mathbb{R}^N} f_n(x) \right) \\
&= -\liminf_{n\to\infty} \int_{\mathbb{R}^N} (-f_n)(x) \\
&\leq -\int_{\mathbb{R}^N} \liminf_{n\to\infty} (-f_n)(x) \\
&= -\int_{\mathbb{R}^N} [-\limsup_{n\to\infty} f_n](x) \\
&= \int_{\mathbb{R}^N} \limsup_{n\to\infty} f_n(x)
\end{aligned}$$

Therefore, we will only have to show (i) holds.

let $f = \liminf_{n\to\infty} f_n$. Define $\tilde{f}_n := f_n - g$, and $\tilde{f} = \liminf_{n\to\infty} \tilde{f}_n$. It is clear $\tilde{f} = f - g$, by linearity of $\liminf$, and that $\tilde{f}_n \geq 0$, so also $\tilde{f} \geq 0$. Finally, we define $\psi_n := \inf_{k\geq n} \tilde{f}_k$, which is a sequence that monotonically increases to $h = \sup_{n\in\mathbb{N}} \psi_n$.

Next, let s be a **simple function**

$$s(x) := \sum_{i=1}^{k} \mathbb{1}_{E_i} y_i$$

such that $s \leq h$ with (using the supremum property on the definition of the integral):

$$\int_{\mathbb{R}^N} s(x) d\mathcal{L}^N(x) > \int_{\mathbb{R}^N} h(x) d\mathcal{L}^N(x) - \epsilon$$

If we can show there is a $N \in \mathbb{N}$ such that $\psi_N \geq s$, then we can proceed, as $\psi_n \geq s$ for all $n \geq N$ by monotonicity. However, this may fail to hold, for any N, at points x where $h(x) = s(x)$. Therefore, we first consider a $\lambda \in (0, 1)$ and look at λs, for which we have $\lambda s(x) = h(x)$ only if $s(x) = h(x) = 0$. Now, whenever $s(x) = h(x) = 0$, note that we always have $\psi_n(x) \geq 0$, so this is fine.

Define:

$$F_n := \{x \in \mathbb{R}^N : \psi_n(x) \geq \lambda s(x)\}$$

Then, since for all $x \in \mathbb{R}^N$, $\psi_n(x) \to h(x)$ we first have for all $x \in \mathbb{R}^N$ a $n \in \mathbb{N}$ such that $x \in F_n$, and by $\psi_{n+1} \geq \psi_n$ we have $F_n \subset F_{n+1}$. Therefore, $F_n \uparrow \mathbb{R}^N$, making $E_i \cap F_n \uparrow E_i$ for all $i = 1, ..., k$. Note that $E_i \cap F_n$ is measurable, in particular because F_n is, which is due to $\psi_n - \lambda s$ being a measurable function.

By upward continuity of the Lebesgue measure, we therefore have:

$$\lim_{n\to\infty} \mathcal{L}^N(E_i \cap F_n) = \mathcal{L}(E_i), \text{ for all } i = 1, ..., k$$

We can now conclude:

$$\begin{aligned}
\lambda \int_{\mathbb{R}^N} h(x) d\mathcal{L}^N(x) &= \int_{\mathbb{R}^N} \lambda h(x) d\mathcal{L}^N(x) \\
&= \sum_{i=1}^{k} \mathcal{L}^N(E_i) \\
&= \lim_{m\to\infty} \lambda y_i \mathcal{L}(E_i \cap F_m) \\
&= \lim_{m\to\infty} \int_{\mathbb{R}^N} \sum_{i=1}^{k} \mathbb{1}_{E_i \cap F_m} \lambda y_i d\mathcal{L}^N(x)
\end{aligned}$$

So far, only equalities. But how to proceed? We now define the simple function on the right hand side:

$$s_m(x) := \sum_{i=1}^{k} \mathbb{1}_{E_i \cap F_m} \lambda y_i$$

And we note $s_m \leq \psi_n$ for all $n \geq m$, precisely by definition of F_m, and monotonicity of $(\psi_n)_{n\in\mathbb{N}}$.

Also note that therefore, by monotonicity of the Lebesgue integral:

$$\forall n \geq m : \int_{\mathbb{R}^N} s_m(s) d\mathcal{L}^N(x) \leq \int_{\mathbb{R}^N} \psi_n(x) d\mathcal{L}^N(x)$$

Therefore, taking the inf on the right, then limits on both sides:

$$\lim_{m\to\infty}\int_{\mathbb{R}^N} s_m(s)d\mathcal{L}^N(x) \le \liminf_{n\to\infty}\int_{\mathbb{R}^N}\psi_n(x)d\mathcal{L}^N(x)$$

And this is all we need to know: we can substitute this in the equality:

$$\lambda\int_{\mathbb{R}^N} h(x)d\mathcal{L}^N(x) = \int_{\mathbb{R}^N}\lambda h(x)d\mathcal{L}^N(x) \le \lim_{m\to\infty}\int_{\mathbb{R}^N} s_m(x)d\mathcal{L}^N(x)$$
$$\le \liminf_{n\to\infty}\int_{\mathbb{R}^N}\psi_n(x)d\mathcal{L}^N(x)$$

Next, we note $\psi_n \le \tilde{f_m}$ for all $m \ge n$, so by monotonicity of the integral:

$$\int_{\mathbb{R}^N}\psi_n(x)d\mathcal{L}^N(x) \le \int_{\mathbb{R}^N}\tilde{f_m}(x)d\mathcal{L}^N(x), \quad m \ge n$$

Taking the liminf in m first, then in n, we have

$$\liminf_{n\to\infty} Lint\psi_n(x) \le \liminf_{m\to\infty}\int_{\mathbb{R}^N}\tilde{f_m}(x)d\mathcal{L}^N(x), \quad m \ge n$$

Therefore, combining with the previous inequality for s:

$$\lambda\int_{\mathbb{R}^N} s(x)d\mathcal{L}^N(x) \le \liminf_{m\to\infty}\int_{\mathbb{R}^N}\tilde{f_m}(x)d\mathcal{L}^N(x)$$

For any $\lambda \in (0,1)$, so take $\lambda \uparrow 1$, and conclude (by continuity of scalar multiplication):

$$\int_{\mathbb{R}^N} s(x)d\mathcal{L}^N(x) = \lim_{\lambda\uparrow 1}\lambda\int_{\mathbb{R}^N} s(x)d\mathcal{L}^N(x) \le \liminf_{m\to\infty}\int_{\mathbb{R}^N}\tilde{f_m}(x)d\mathcal{L}^N(x)$$

Finally, we had $\int_{\mathbb{R}^N} s(x)d\mathcal{L}^N(x) > \int_{\mathbb{R}^N}\tilde{f}(x)d\mathcal{L}^N(x) - \epsilon$. For any $\epsilon > 0$. This is just a more explicit formulation of "and now we take the supremum on the left hand side over all such s":

$$\int_{\mathbb{R}^N}\tilde{f}(x)d\mathcal{L}^N(x) \le \liminf_{m\to\infty}\int_{\mathbb{R}^N}\tilde{f_m}(x)d\mathcal{L}^N(x)$$

Next, linearity (the following is **legal** since $\int_{\mathbb{R}^N}|g(x)|d\mathcal{L}^N(x) < \infty$):

$$\int_{\mathbb{R}^N} f(x)d\mathcal{L}^N(x) - \int_{\mathbb{R}^N} g(x)d\mathcal{L}^N(x) \le \liminf_{m\to\infty}\int_{\mathbb{R}^N} f_m(x)d\mathcal{L}^N(x) - \int_{\mathbb{R}^N} g(x)d\mathcal{L}^N(x)$$

And now, we really see why we would need $f \ge g$, but more particularly $g \in \mathcal{L}^1(\mathbb{R}^N)$ (how easy it would have been without to just pick $g = f$...). Without this, we would not be allowed to just split up the integral into a f and $(-g)$ part, since we could be doing ill-defined arithmetic on infinities here, and the proof would fail . But since g integrates to a finite real, we can conclude by adding the value $\int_{\mathbb{R}^N} g(x)d\mathcal{L}^N(x)$ on both sides:

$$\int_{\mathbb{R}^N} f(x)d\mathcal{L}^N(x) \le \liminf_{m\to\infty}\int_{\mathbb{R}^N} f_m(x)d\mathcal{L}^N(x)$$

$\square$

1.4 Lebesgue's Monotone Convergence Theorem

Theorem 2. ***Lebesgue's Monotone Convergence Theorem***
*Let $(f_n)_{n\in\mathbb{N}}$ be an **increasing sequence** ($f_{n+1} \geq f_n$) of $\mathcal{L}$ -**measurable** functions $f : \mathbb{R}^N \to \mathbb{R}$, such that $f_n \geq g$, for some $g \in \mathcal{L}^1(\mathbb{R}^N)$. Then, we define the limiting function $f : \mathbb{R}^N \to \mathbb{R} \cup \{\infty\}$ (to the extended real line):*

$$f(x) := \lim_{n\to\infty} f_n(x) = \sup_{n\in\mathbb{N}} f_n(x)$$

*Which may be infinite if $\forall R \in \mathbb{R} : \exists N \in \mathbb{N} : \forall n \geq N : f_n(x) > R$, i.e. $f_n(x)$ grows unboundedly. In this way, we can at least speak of a (pointwise) limit of a **monotone increasing** sequence of functions. It then holds:*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f_n(x) d\mathcal{L}^N(x) = \int_{\mathbb{R}^N} f(x) d\mathcal{L}^N(x)$$

*The same result holds in the case of a **decreasing sequence** $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{L}^N$-**measurable** functions such that $f_n \leq g$ for some $g \in \mathcal{L}^1(\mathbb{R}^N)$.*

Proof. We show both inequalities. One is trivial; the other is done from Fatou's Lemma.

First, by monotonicity of the Lebesgue integral, we always have:

$$\int_{\mathbb{R}^N} f_n(x) d\mathcal{L}^N(x) \leq \int_{\mathbb{R}^N} \sup_{n\in\mathbb{N}} f_n(x) d\mathcal{L}^N(x) = \int_{\mathbb{R}^N} f(x) d\mathcal{L}^N(x)$$

From which we conclude, by taking the limit as $n \to \infty$ on the right (which is also legal if $\int_{\mathbb{R}^N} f_n(x) d\mathcal{L}^N(x) \to \infty$, since in those cases we already had $\int_{\mathbb{R}^N} f(x) d\mathcal{L}^N(x) = \infty$):

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f_n(x) d\mathcal{L}^N(x) \leq \int_{\mathbb{R}^N} f(x) d\mathcal{L}^N(x)$$

The other inequality now follows because $\lim_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)$ for real sequences (also if such a limit is defined as ∞, check this!). Therefore, by the fact that $f_n \geq g$, for a $g \in \mathcal{L}^1(\mathbb{R}^N)$, we can use Fatou's Lemma:

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}^N} f_n(x) d\mathcal{L}^N(x) &= \liminf_{n\to\infty} \int_{\mathbb{R}^N} f_n(x) d\mathcal{L}^N(x) \\
&\geq \int_{\mathbb{R}^N} \liminf_{n\to\infty} f_n(x) d\mathcal{L}^N(x) \\
&= \int_{\mathbb{R}^N} \lim_{n\to\infty} f_n(x) d\mathcal{L}^N(x)
\end{aligned}$$

So we can conclude!

□

1.5 Lebesgue's Dominated Convergence Theorem

This convergence theorem is again about a poinwise limiting function $f_n(x) \to f(x)$. It does not only relate $\lim_{n\to\infty} \int_{\mathbb{R}^N} f_n(x) d\mathcal{L}^N(x)$ to $\int_{\mathbb{R}^N} \lim_{n\to\infty} f_n(x) d\mathcal{L}^N(x)$, but is able to say something stronger, namely:

$$\lim_{m\to\infty} \int_{\mathbb{R}^N} \Big|[\lim_{n\to\infty} f_n(x)] - f_m(x)\Big| d\mathcal{L}^N(x) = 0$$

This obviously implies that

$$\left|\int_{\mathbb{R}^N} f_m(x) d\mathcal{L}^N(x) - \int_{\mathbb{R}^N} f(x) d\mathcal{L}^N(x)\right| \leq \int_{\mathbb{R}^N} |f(x) - f_m(x)| d\mathcal{L}^N(x) \to 0$$

Which by definition means

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f_n(x) d\mathcal{L}^N(x) = \int_{\mathbb{R}^N} \lim_{n\to\infty} f_n(x) d\mathcal{L}^N(x)$$

Note that this only works if $f(x) := \lim_{n\to\infty} f_n(x)$ is well-defined, meaning the right-hand side is indeed a convergent sequence, or is **possibly** divergent to either $-\infty$ or ∞, but **only on a $\mathcal{L}$-negligible set**, since then the integral ignores this hickup. Apart from this, the assumptions are rather mild.

The proof can be made very short again, now with the help of Lebesgue's Monotone Convergence Theorem.

Theorem 3. ***Lebesgue's Dominated Convergence Theorem***

*Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{L}^N$-**measurable** functions such that $f_n(x) \to f(x)$, $\mathcal{L}^N$-**almost everywhere**. Assume that $|f_n(x)| \leq g(x)$, $\mathcal{L}^N$-**almost everywhere**, where $g \in \mathcal{L}^1(\mathbb{R}^N)$. Then, $f \in \mathcal{L}^1(\mathbb{R}^N)$, and*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |f_n(x) - f(x)| d\mathcal{L}^N(x) = 0$$

In particular,

$$\int_{\mathbb{R}^N} f_n(x) d\mathcal{L}^N(x) = \int_{\mathbb{R}^N} f(x) d\mathcal{L}^N(x)$$

Proof. Since $f(x)$ is well-defined by the premises of the theorem, let's also define

$$a_n(x) := |f_n(x) - f(x)|$$

Since $|f_n(x)| \leq |g(x)|$ $\mathcal{L}^N$-a.e., we have that $|f(x)| \leq |g(x)|$, $\mathcal{L}^N$-a.e., and therefore a_n is bounded $\mathcal{L}^N$-a.e. by $|f(x)| + |f_n(x)| \leq 2|g(x)|$, and clearly $2|g| \in \mathcal{L}^1(\mathbb{R}^N)$. Moreover, $a_n(x) \to 0$ for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$. It is

Now, we cannot apply the monotone convergence theorem yet, since we would need a_n to be monotonically **decreasing** (by $a_n \leq 2|g|$). Therefore, we just define $b_n := \sup_{k\geq n} a_k$, which also converges to 0 and is bounded by $2|g|$ almost everywhere, but now with monotonic convergence. Conclude:

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \sup_{k\geq n} |f_n(x) - f(x)| d\mathcal{L}^N(x) = \int_{\mathbb{R}^N} 0 d\mathcal{L}^N(x) = 0$$

By monotonicity of the Lebesgue integral, we also have, for all $m \geq n$:

$$\int_{\mathbb{R}^N} |f_m(x) - f(x)| d\mathcal{L}^N(x) \leq \int_{\mathbb{R}^N} \sup_{k \geq n} |f_n(x) - f(x)| d\mathcal{L}^N(x)$$

Therefore, for any $\epsilon > 0$ we can find a $N \in \mathbb{N}$ such that for $n \geq N$, we have

$$\int_{\mathbb{R}^N} \sup_{k \geq n} |f_n(x) - f(x)| d\mathcal{L}^N(x) < \epsilon$$

Then, for $m \geq N$, we have

$$0 \leq \int_{\mathbb{R}^N} |f_m(x) - f(x)| d\mathcal{L}^N(x) \leq \int_{\mathbb{R}^N} \sup_{k \geq n} |f_n(x) - f(x)| d\mathcal{L}^N(x) < \epsilon$$

And this is the neatest I could do; at least it completes the proof:

$$\lim_{m \to \infty} \leq \int_{\mathbb{R}^N} |f_m(x) - f(x)| d\mathcal{L}^N(x) = 0$$

$\square$