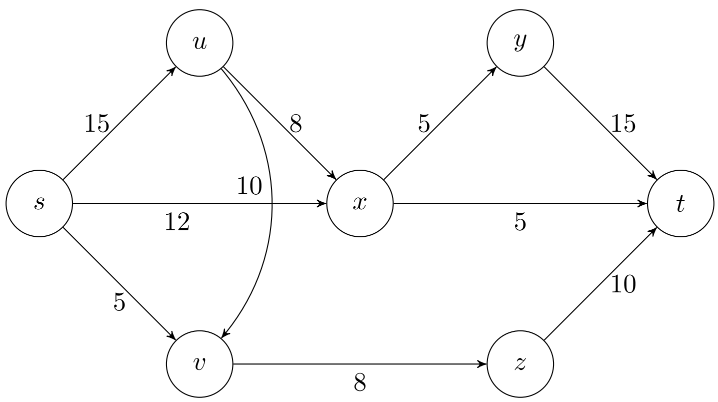
Weekly Assignment 5:
Flow algorithms

October 2023

1. Run the Ford-Fulkerson algorithm on the following flow network. For each iteration, pick the shortest augmenting path from s to t and give the residual network. Finally, give the value of the maximum flow and a minimum cut (S, T).



2. Decide whether the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

> *Let G be an arbitrary flow network, with a source s, a sink t, and positive integer capacity $c(e)$ on every edge e. If f is a maximum $s-t$ flow in G, then f saturates every edge out of s, that is, for every edge e out of s, we have $f(e) = c(e)$.*

3. Consider the following network (the numbers are edge capacities).

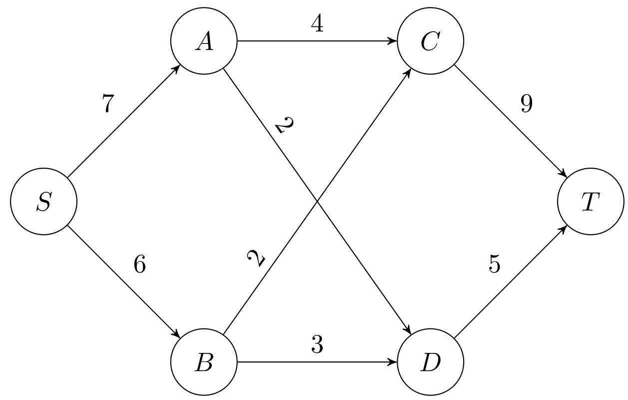


Figure 1: Network

(a) Find the maximum flow f and a minimum cut.

(b) Draw the residual graph G_f (along with its edge capacities). In this residual network, mark the vertices reachable from S and the vertices from which T is reachable.

(c) An edge of a network is called a *bottleneck edge* if increasing its capacity results in an increase in the maximum flow. List all bottleneck edges in the above network.

(d) Give a very simple example (containing at most four nodes) of a network which has no bottleneck edges.

(e) Give an efficient algorithm to identify all bottleneck edges in a network (Hint: Start by running the usual network flow algorithm, and then examine the residual graph.)

4. There are X job applicants and Y jobs. Each applicant has a subset of jobs that they are interested in. Each job opening can only accept one applicant and a job applicant can be appointed for only one job. Find an assignment of jobs to applicants in such that as many applicants as possible get jobs

5. Consider a flow network $G = (V, E)$ with capacity function c and an edge $(u, v) \in E$. Suppose we modify this flow network by adding a new vertex x and replacing (u, v) by edges (u, x) and (x, v) with the same capacity. Show that the value of a max-flow in the modified network is the same as the value of a max-flow in the original network. (By applying this construction repeatedly, we can transform any flow network to one in which there are no u, v such that both (u, v) and (v, u) are edges. This is convenient when we construct residual graphs during the Ford-Fulkerson algorithm.)