We need a definition of a set. Try this yourself first! If your approach is to explain it in terms of simpler notions, you might find yourself ending up in a circular argument or even in an infinite descend of finding even simpler sub-definitions. Is a set a list of objects? But then you must define a list first, otherwise you are too vague. Also, can lists be infinite? We will later see that there are sets that contain so many elements that you cannot enumerate them as a list!

Our general approach is to use axioms, which are laws according to which any set behaves. Everything that satisfies these axioms is a set. To reduce the wordyness a bit, I will introduce letters for sets and various symbols to express relations between them. We can later define other symbols from logical sentences. As an example, we can define the equality and subset relation by the following sentence:

A subset B if and only if (for all x, if x \in A then x in B)

A = B iff. (for all x, d in A iff. x in B)

The interesting thing is that the elementhood symbol \in does not have it's own definition in the axioms, it is very elementary in that sense.

The second sentence is actually already the first axiom of set theory, named "Extensionality", "axiom of extension" or "the extension axiom". It expresses \in in terms of = and = in terms of \in, and avoids the "what is a set" question completely. It defines what it means for two sets to be equal.

We will now leave ourselves leave the philosophical interpretations of axioms and theories and state the other axioms of set theory.

We are not yet aware of what sets actually exist, and in fact we should be very cautious about this because it might lead to contradictions to simply assume the existence of certain sets. For example, we might like to take all existing sets and consider this to be a set. A set of all sets, combined with other axioms will however lead to contradictions, the most famous one of which is probably Russel's paradox. We will discuss this later.

First, we postulate that there is a set which has no members. In other words, a set B for which it holds that there is no x such that x \in B. Note that we have just given B a name, as if B were a unique set satisfying the sentence

For all x, x \not \in B

This is allowed, however, since sets are by extensionality uniquely defined by the elementhood relationship.