Weil Groups of Local Fields

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I have come across the Weil Group in the context of local class field theory, with its abelianisation being the image of the Artin map in $\mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}}$. I wanted to investigate this more so have cobbled together some thoughts on the subject. For example; is there a global analogue, and what properties do Weil groups have?

I suppose in wanting some sense of completeness in these notes, I've started with the case of local fields, and reviewed some infinite Galois theory. I then attempt to show that $W(K^{\mathrm{ab}}/K)$ is the image of the Artin map. Finally I try and look further to other instances of Weil groups, namely in global fields. For this final aim, I write some initial thoughts having read through Tate 1979 §1 and Serre Local fields ChXI.

Infinite Galois Theory

Consider L/K an algebraic extension of fields, we say that

- L/K is separable if $\forall \alpha \in L$ the minimal polynomial m_α over K is separable.
- L/K is normal if $\forall \alpha \in L$, the minimal polynomial m_α splits in L
- L/K is Galois if it is both normal and separable.

One of the most satisfying parts of finite Galois theory is the Galois correspondence between subgroups of $\mathrm{Gal}(L/K)$ and the the intermediate fields of L/K. Naturally, we seek an analogue of this result that extends to the infinite case. Before doing so, we must take an excursion into the topological aspects of $\mathrm{Gal}(L/K)$.

Consider that $I := \{F/K \text{ finite Galois extensions } : F \subset L\}$ is a directed set with respect to inclusion. Indeed, given F_1 and F_2 in I, we can take their composite, which gives a finite Galois extension of K contained in L, and which contains both F_1 and F_2.
Moreover, given $F \subset M$ finite Galois extensions of K, the restriction maps

$$\mathrm{Res} : \mathrm{Gal}(M/K) \to \mathrm{Gal}(F/K)$$

make $\big(\mathrm{Gal}(F/K)\big)_{F \in I}$ into an inverse system. We can then take the inverse limit, and consider the natural map

$$\begin{aligned} \mathrm{Gal}(L/K) &\to \varprojlim_{F} \mathrm{Gal}(F/K) \\ \sigma &\mapsto (\sigma|_F)_{F\in I}. \end{aligned}$$

Proposition 1 This map is an isomorphism

Proof. **Injective**: If $\sigma \in \mathrm{Gal}(L/K)$ maps to the identity, then $\sigma|_F = \mathrm{id}$ for all F/K finite Galois. Given $x \in L$, we can take the normal closure M of $K(x)$, which is finite Galois over K and contained in L, so $\sigma(x) = \sigma|_M(x) = x$. Hence $\sigma = \mathrm{id}$.
Surjective: Consider $(\sigma|_F)_{F\in I}$ in the inverse limit. Then we define σ as follows;

$$\text{Given } x \in F, F/K \text{ finite Galois, let } \sigma(x) = \sigma|_F(x).$$

We need to show that this is well defined. Indeed, suppose $x \in F, F'$ distinct fields in I, then we have FF' contains both of these, and is a finite Galois extension, hence

$$\sigma|_F(x) = \sigma|_{FF'}(x) = \sigma|_{F'}(x).$$

Therefore σ is a well defined map from $L \to L$. It is not hard to check that σ is then a K-automorphism, and therefore defines an element of $\mathrm{Gal}(L/K)$, which completes the proof. $\square$

The inverse limit $G = \lim_{\leftarrow} \mathrm{Gal}(F/K)$ can be endowed with the profinite topology - that is, the weakest topology on the group such that all projection maps $\pi_F : G \to \mathrm{Gal}(F/K)$ are continuous when $\mathrm{Gal}(F/K)$ are given the discrete topology. Then the isomorphism in Proposition 1 induces the profinite[1] topology on $\mathrm{Gal}(L/K)$.

Equivalently, one can define the profinite topology on $\mathrm{Gal}(L/K)$ by declaring the subsets of the form $\mathrm{Gal}(L/F)$, where F/K is finite and Galois form a fundamental basis of neighbourhoods about $\mathrm{id} \in \mathrm{Gal}(L/K)$.

Now, a *profinite group* is defined as an inverse limit of finite discrete groups, like G above. Such groups are pervasive in number theory, with examples being Galois groups (as above) and the valuation rings of complete discretely valued field. Here we give a few fundamental facts about profinite groups, that aren't too hard to prove:

Facts: Let G be a profinite group, then

1. G is a topological group- that is $m : G \times G \to G$ and $i : G \to G$ are continuous maps, where $G \times G$ has the product topology.

[1] Also refered to as the Krull topology

2. G is compact and Hausdorff

3. Given $H \subset G$ a subgroup,

   (a) If H open then it has finite index in G.

   (b) If H is closed and of finite index, then H is open.

Remark
- In fact (3.b) holds for any top. group, and (3.a) holds for any *compact* top. group.
- All of the above results are still valid for L/K a finite Galois extension, in which case L just happens to be in I, so the profinite topology is simply the discrete topology.

We shall omit the proof of this for now (did on LF ex sheet 4 Q2), and get on to the meat of this section.

Theorem 2 Galois Correspondence
Let L/K be a Galois extension, and endow $\mathrm{Gal}(L/K)$ with the profinite topology. Then there is an order reversing bijection

$$\begin{aligned} \{\text{Intermediate fields } K \subset F \subset L\} &\leftrightarrow \{\text{Closed subgroups } H \subset \mathrm{Gal}(L/K)\} \\ F &\mapsto \mathrm{Gal}(L/F), \\ L^H &\leftarrow\!\shortmid H, \end{aligned}$$

such that

- F/K is finite if and only if $\mathrm{Gal}(L/F)$ is open, and $\big(\mathrm{Gal}(L/K) : \mathrm{Gal}(L/F)\big) = [F : K]$,
- F/K is Galois if and only if $\mathrm{Gal}(L/F)$ is normal.

This is the most general form of Galois correspondence!

Proof. The idea is that we like the *finite* Galois correspondence, and will try our best to reduce to this case wherever we can!
Consider F/K a finite Galois extension, let π_F denote the restriction map (ie. projection from the inverse limit, if you want to think of it like that) from $\mathrm{Gal}(L/K)$ to $\mathrm{Gal}(F/K)$.

Claim 0: Let F/K be a finite Galois extension, and let H be a subgroup of $\mathrm{Gal}(L/K)$, then

$$F^{\pi_F(H)} = L^H \cap F$$

Proof of claim 0 :

$$\begin{aligned} x \in L^H \cap F &\iff x \in F \text{ and } \sigma(x) = x \text{ for all } \sigma \in H \\ &\iff x \in F \text{ and } \sigma|_F(x) = x \text{ for all } \sigma \in H \\ &\iff x \in F \text{ and } \tau(x) = x \text{ for all } \tau \in \pi_F(H) \\ &\iff x \in F^{\pi_F(H)} \end{aligned}$$

□

Claim 1: Let F/K be a sub-extension of L/K, then $\mathrm{Gal}(L/F)$ is closed in $\mathrm{Gal}(L/K)$.

Proof of claim 1 We'll break this down into subcases!

First suppose F is a finite extension of K, then we have $\mathrm{Gal}(L/F) = \pi|_F^{-1}(\mathrm{id}_F)$, hence is closed because π_F are continuous.

If F is any olde finite extension of K, then M/K its normal closure is a finite Galois extension of K. Then we have $\mathrm{Gal}(L/F)$ is a finite union of cosets of $\mathrm{Gal}(L/M)$, and is therefore closed.

Lastly, if F is any intermediate extension of L/K, then F is the union of all F' finite contained in F. Hence $\mathrm{Gal}(L/F) = \bigcap \mathrm{Gal}(L/F')$ is an intersection of closed sets, and so is itself closed. □

Claim 2: If $H \leq \mathrm{Gal}(L/K)$ is closed[2], then $\mathrm{Gal}(L/L^H) = H$.

Proof of claim 2: Clearly we have $H \subset \mathrm{Gal}(L/L^H)$, and since the subsets $\pi_F^{-1}(H_F)$ of $\mathrm{Gal}(L/K)$ - where F/K is finite Galois and H_F is any subset of $\mathrm{Gal}(F/K)$- form a basis of closed sets, we have

$$H = \bigcap_{F \in I} \pi_F^{-1}(H_F), \text{ some } H_F \subset \mathrm{Gal}(F/K)$$

with $I = \{F/K \text{ finite Galois extensions contained in } L\}$.

Suppose $\sigma \in H$, then $\pi_F(\sigma) \in H_F$ for all $F \in I$. That is

$$\pi_F(H) \subset H_F \text{ for all } F \in I.$$

and so by taking intersections over all $F \in I$, we deduce that

$$\bigcap_F \pi_F^{-1}(\pi_F(H)) \subset \bigcap_F \pi_F^{-1}(H_F) = H.$$

[2] I suppose one could with a little more work show that for general H, $\mathrm{Gal}(L/L^H) = \overline{H}$

Conversely, if $\sigma \in H$ then $\pi_F(\sigma) \in \pi_F(H)$, which means we have $\sigma \in \pi_F^{-1}(\pi_F(H))$ and so $H \subset \bigcap_F \pi_F^{-1}(\pi_F(H))$. Combining these two results, we deduce that

$$\bigcap_F \pi_F^{-1}(\pi_F(H)) = H.$$

Then consider

$$\begin{aligned}
\pi_F(H) &= \mathrm{Gal}(F/F^{\pi_F(H)}) \text{ (by finite Galois theory)} \\
&= \mathrm{Gal}(F/L^H \cap F) \text{ (by Claim 0)} \\
&= \{\sigma \in \mathrm{Gal}(F/K) : \sigma \text{ fixes } L^H \cap F\} \\
&= \pi_F(\mathrm{Gal}(L/L^H))
\end{aligned}$$

hence we have finally that

$$H = \bigcap_F \pi_F^{-1}(\pi_F(\mathrm{Gal}(L/L^H))) = \mathrm{Gal}(L/L^H).$$

□

(I'm sure there are elegant ways to do this, and I need to double check my proof here, it seems a bit strange to me.)

Proof of Galois correspondence: Firstly, by Claim 1 we have the map $\mapsto$ lands in the correct set, and claim 2 implies that the maps are inverses on one side.
To show that this is a bijection, we must show for any F/K a subextension of L/K that

$$F = L^{\mathrm{Gal}(L/F)}.$$

Clearly we have $F \subset L^{\mathrm{Gal}(L/F)}$, and for the reverse inclusion we shall reduce to the finite case.
Indeed, any $x \in L^{\mathrm{Gal}(L/F)}$ is contained in some finite extension F'/K, and so

$$x \in F' \cap L^{\mathrm{Gal}(L/F)} = (F')^{\pi_{F'}(\mathrm{Gal}(L/F))}$$

by Claim 0. Then we just note that $\pi_{F'}(\mathrm{Gal}(L/F)) = \mathrm{Gal}(F'/F \cap F')$, which implies that $x \in F \cap F' \subset F$ by finite Galois correspondence.

As for the rest of the theorem, note that the bijection is clearly order reversing, and so we look to prove the statments concerning open and normal subgroups.

Let F be an intermediate field, then since $\mathrm{Gal}(L/F)$ is closed, it is open if and only if it has finite index in $\mathrm{Gal}(L/K)$. Consider M a normal closure of F/K in L, so M is Galois. Then we have $\mathrm{Gal}(L/M) \subset \mathrm{Gal}(L/F)$, and the exact sequences

$$1 \to \mathrm{Gal}(L/M) \to \mathrm{Gal}(L/K) \to \mathrm{Gal}(M/K) \to 1$$
$$1 \to \mathrm{Gal}(L/M) \to \mathrm{Gal}(L/F) \to \mathrm{Gal}(M/F) \to 1$$

Suppose that F is finite over K, then M is also finite over K and the exact sequences imply that

$$[\mathrm{Gal}(L/K) : \mathrm{Gal}(L/M)] = |\mathrm{Gal}(M/K)| < \infty$$
$$[\mathrm{Gal}(L/F) : \mathrm{Gal}(L/M)] = |\mathrm{Gal}(M/F)| < \infty$$

which combine to give

$$[\mathrm{Gal}(L/K) : \mathrm{Gal}(L/F)] = [\mathrm{Gal}(M/K) : \mathrm{Gal}(M/F)] = [F : K].$$

Conversely, if $\mathrm{Gal}(L/F)$ has finite index in $\mathrm{Gal}(L/K)$, then by our inclusion $\mathrm{Gal}(L/M)$ must also have finite index, hence the first exact sequence implies that M - and hence F - is finite over K.

Lastly, we must show that F/K is Galois if and only if $\mathrm{Gal}(L/F)$ is normal in $\mathrm{Gal}(L/K)$. Indeed, the first exact sequence above proves one direction. For the converse, suppose $\mathrm{Gal}(L/F)$ is normal, we must show that F/K is normal.

This is equivalent to proving that for any $x \in F$ and $\sigma \in \mathrm{Gal}(L/K)$, we have $\sigma(x) \in F$. Let $N = \mathrm{Gal}(L/F)$, then $\sigma(x) \in L$ if and only if $\tau(\sigma(x)) = \sigma(x)$ for all $\tau \in N$. Noting that N is a normal subgroup gives $\tau\sigma = \sigma\tau'$ for some $\tau' \in N$, and since x lies in $F = L^N$ we deduce that

$$\tau\sigma(x) = \sigma\tau'(x) = \sigma(x).$$

Hence F is indeed normal - and thus Galois - over K. □

This is nice! We end this section with a simple result about Galois extensions.

Lemma 3 Let L/K be an extension of fields, this is Galois if and only if L is a union of finite Galois extensions over K.

Proof. Suppose that L is Galois, and let $\alpha \in L$. This is algebraic, and so the splitting field of it's minimal polynomial is a finite Galois extension contained in L. Then taking union of all such fields gives the $\implies$ direction.

Conversely, suppose that L is a union of Galois extensions and let $\alpha \in L$. Then α lies in some finite Galois extension F/K contained in L, hence α is separable and the minimal polynomial of α over K splits in F, and therefore in L also. □

Okay, enough of reviewing Galois theory, let us talk about local fields!

Weil Groups of Non-Archimedean Local Fields

Let K be a non-Archimedean local field, L/K a separable algebraic extension of K and l/k the respective extension of residue fields. We say that

- L/K is unramified if every finite sub-extension F/K is unramified
- L/K is totally ramified if every finite sub-extension F/K is totally ramified.

One should note that an infinite extension of a complete discretely valued field is not complete, and not necessarily discretely valued, so L may not be a local field here! However, the absolute value on K still extends uniquely to any separable algebraic extension by patching together the unique absolute values extending $|\ |_K$ in the finite intermediate fields.
Recall that finite unramified extensions are Galois, and the natural map

$$\mathrm{Res} : \mathrm{Gal}(L/K) \to \mathrm{Gal}(l/k),$$

is an isomorphism. We see also that this holds for the infinite case.

Proposition 4 Let L/K be an unramified extension, then L/K is Galois and $\mathrm{Gal}(L/K)$ is isomorphic to $\mathrm{Gal}(l/k)$ via passing to the quotient.

Proof. Noting that an extension is Galois if and only if it is a union of Galois extensions, we deduce immediately that L/K is Galois. Next consider that

$$\{F/K \text{ finite subextensions of } L/K\} \leftrightarrow \{k_F/k \text{ finite subextensions of } l/k\}$$

and for any finite $F \subset F'$ over K contained in L the diagram

$$\begin{tikzcd}
\mathrm{Gal}(F'/K) \arrow[r, "\cong"] \arrow[d, "\mathrm{res}"] & \mathrm{Gal}(k_{F'}/k) \arrow[d, "\mathrm{res}"] \\
\mathrm{Gal}(F/K) \arrow[r, "\cong"] & \mathrm{Gal}(k_F/k)
\end{tikzcd}$$

commutes. Hence the inverse limits are isomorphic and we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Gal}(L/K) \arrow[r] \arrow[dd, "\cong"] & \mathrm{Gal}(l/k) \arrow[d, "\cong"] \\
 & \lim_{\leftarrow l'} \mathrm{Gal}(l'/k) \arrow[d, leftrightarrow, "="] \\
\lim_{\leftarrow F} \mathrm{Gal}(F/K) \arrow[r, "\cong"] & \lim_{\leftarrow F} \mathrm{Gal}(k_F/k)
\end{tikzcd}$$

□

Now, consider L_1 and L_2 two separable extensions of K (in some separable closure $\overline{K}$ of K). In the finite case if L_1 and L_2 are unramified, then L_1L_2 is also unramified. This property carries over into the general definitions where L_i/K are allowed to be infinite, indeed the definition is posed in terms of finite subextensions so the proof is rather simple.

Therefore, given a separable algebraic extension L/K of local fields, there is a unique maximal unramified subextension K_0/K which is obtained by taking the union of all unramified subextensions of L/K.
If we assume further that L/K is a Galois extension, then we have a surjection

$$\mathrm{res} : \mathrm{Gal}(L/K) \to \mathrm{Gal}(K_0/K) \cong \mathrm{Gal}(l/k)$$

where the kernel $I_{L/K}$ is called the *inertia subgroup*.

Let $\mathrm{Frob}_{l/k} \in \mathrm{Gal}(l/k)$ be the Frobenius map $x \mapsto x^q$, where $q = \#k$. This generates a subgroup $\langle \mathrm{Frob}_{l/k} \rangle$ in $\mathrm{Gal}(l/k)$, which is not necessarily equal to the whole group - unlike the case where l/k is finite. We shall see that the preimage of this group in $\mathrm{Gal}(L/K)$ is an important object called the *Weil group* of L/K.

Interlude: Infinite Extensions of Finite Fields

Consider a characteristic p finite field $\mathbb{F}_q$, and let $\overline{\mathbb{F}}_q$ denote an algebraic closure.

Question: What is the Galois group of $\overline{\mathbb{F}}_q/\mathbb{F}_q$, and what does the cyclic group generated by Frob_q look like in this?

Well, we know what all finite Galois extensions of $\mathbb{F}_q$ look like. Indeed given any $m \geq 1$, there is a unique finite extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ which turns out to be a cyclic Galois extension of degree m generated by Frob_q.
Given $m, n \geq 1$, we have $\mathbb{F}_{q^m}$ subset $\mathbb{F}_{q^n}$ if and only if $m \mid n$, and so we have a topological isomorphism

$$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \lim_{\leftarrow (\mathbb{N},|)} \frac{\mathbb{Z}}{n\mathbb{Z}} := \hat{\mathbb{Z}},$$

and the Frobenius element Frob_q corresponds to $(1, 1, 1, \ldots)$.
There is actually another description of this group as follows;

Proposition 5 There is an isomorphism of groups

$$\hat{\mathbb{Z}} \cong \prod_p \mathbb{Z}_p$$

given by $(x_n)_n \mapsto \left((x_{p^m})_{m=1}^{\infty}\right)_p$.

Proof omitted for now (did on LF ex sheet 3 Q1, Question to self, is this a top. iso?).

Anyway, by Galois correspondence, the Galois groups of infinte extensions of $\mathbb{F}_q$ look like quotients of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ by closed normal subgroups of infinite index. A natural question is whether there exists an infinite extension of $\mathbb{F}_q$ with cyclic Galois group $\langle \mathrm{Frob}_q \rangle$. The answer is no!
Indeed, consider the group $\prod_p \mathbb{Z}_p$ is a direct product of groups hence quotients look like $\prod_p \mathbb{Z}_p/N_p$ for some normal subgroups N_p of $\mathbb{Z}_p$ and so the question reduces to asking whether $\mathbb{Z}_p$ has a quotient group isomorphic to $\mathbb{Z}$.
(I dont think this does but will come back to. There is surely an easy explanation for this. I have read that a topological argument implies that there are no countably infinite Galois groups, so this is a strong way to prove it I suppose! Below I have a stab just looking at inverse limits and trying a diagonal argument to investigate the question of the cardinality of an infinite Galois group.)

Claim: There are no countably infinite Galois groups.

Sketch proof: Let K be a field and L/K an infinite Galois extension with Galois group G. Now

$$G \cong \varprojlim \mathrm{Gal}(F/K) = \left\{ (\sigma_F) \in \prod_F \mathrm{Gal}(F/K) : \text{ consistent} \right\}.$$

Suppose that this is countable, say the elements are listed $\tau_1, \tau_{2,\ldots}$. Begin by choosing F_1 , with some $\sigma_1 \in \mathrm{Gal}(F_1/K)$ not equal to τ_1 on F_1. Next, choose F_2/K such that $F_1 \subset F_2$ and there exists $\sigma_2 \in \mathrm{Gal}(F_2/K)$ with $\sigma_2|_{F_1} = \sigma_1$ *and* $\sigma_2 \neq \tau_2$ on F_2. Then we repeat this to get a tower $F_1 \subset F_2 \subset \ldots$ of Galois extensions of K contained in L, and elements $\sigma_i \in \mathrm{Gal}(F_i/K)$ which restrict to each other and such that $\tau_i \neq \sigma_i$ when restricted to F_i.

Now, let $F = \bigcup F_i$ be the union of these sub-extensions, then this is Galois and contained in L, and $\mathrm{Gal}(F/K) \cong \lim_{\leftarrow i} \mathrm{Gal}(F_i/K)$. Hence our σ_i's patch together to give $\sigma \in \mathrm{Gal}(F/K)$. We can then extend this to an element of $\mathrm{Gal}(L/K)$, which cannot be equal to any of the τ_i's by construction, thus completing the proof. (Double check this, wrote as I was going along so bound to be mistakes in here).

That interlude went on a bit, the point of this was to say that when l/k is infinite, we have $\langle \mathrm{Frob}_l \rangle$ in $\mathrm{Gal}(l/k)$ is a *proper* subset.

The Weil group

Definition 6 Let L/K be a Galois extension of local fields, the Weil group $W(L/K)$ is defined to be the set of elements of $\mathrm{Gal}(L/K)$ that map to $\langle \mathrm{Frob}_{l/k} \rangle$

in $\mathrm{Gal}(l/k)$. That is

$$W(L/K) = \mathrm{res}^{-1}(\langle \mathrm{Frob}_{l/k} \rangle).$$

If $\mathrm{Gal}(l/k)$ is finite then we have $\langle \mathrm{Frob}_{l/k} \rangle = \mathrm{Gal}(l/k)$, so $W(L/K) = \mathrm{Gal}(L/K)$. Whereas if $\mathrm{Gal}(l/k)$ is infinite, our interlude shows that $\langle \mathrm{Frob}_{l/k} \rangle \subsetneq \mathrm{Gal}(l/k)$, and so $W(L/K) \subsetneq \mathrm{Gal}(L/K)$.

Now, since $I(L/K)$ is the set of elements that map to the identity in $\mathrm{Gal}(l/k)$, it is clear that $I(L/K) \subset W(L/K)$, and the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & I(L/K) \arrow[r] \arrow[d, "="'] & W(L/K) \arrow[r] \arrow[d, hook] & \langle \mathrm{Frob}_{l/k} \rangle \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & I(L/K) \arrow[r] & \mathrm{Gal}(L/K) \arrow[r] & \mathrm{Gal}(l/k) \arrow[r] & 0.
\end{tikzcd}$$

Let's explore some properties of Weil groups.

Proposition 7 Let L/K be a Galois extension, then

(i) $W(L/K)$ is dense in $\mathrm{Gal}(L/K)$,

(ii) Given F/K a finite subextension, $W(L/F) = \mathrm{Gal}(L/F) \cap W(L/K)$,

(iii) Given F/K a finite Galois subextension, $W(L/F)$ is normal in $W(L/K)$, and $\mathrm{Gal}(F/K) \cong \frac{W(L/K)}{W(L/F)}$

Proof. (i) Recall that the cosets of $\mathrm{Gal}(L/F)$, where F/K is a finite Galois extension, form a basis of open sets of $\mathrm{Gal}(L/K)$. Hence we have

$$\begin{aligned}
W(L/K) \text{ dense } &\iff W(L/K) \text{ intersects all cosets of } \mathrm{Gal}(L/F)\ \forall F/K \text{ finite Galois} \\
&\iff W(L/K) \text{ surjects onto } \mathrm{Gal}(F/K)\ \forall F/K \text{ finite Galois}
\end{aligned}$$

To show this, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I(L/K) \arrow[r] \arrow[d, "a"'] & W(L/K) \arrow[r] \arrow[d, "b"'] & \langle \mathrm{Frob}_{k_F/k} \rangle \arrow[r] \arrow[d, equal, "c"'] & 0 \\
0 \arrow[r] & I(F/K) \arrow[r] & \mathrm{Gal}(F/K) \arrow[r] & \mathrm{Gal}(k_F/k) \arrow[r] & 0
\end{tikzcd}$$

The five lemma states that if a, c are both surjective, then b is also surjective. It is clear that c is surjective, so let's consider a.

Let K_0/K be the maximal unramified extensions in L/K, then $F \cap K_0$ is the maximal unramified extension in F/K and we have

$$I(F/K) = \mathrm{Gal}(F/F \cap K_0) \text{ and } I(L/K) = \mathrm{Gal}(L/K).$$

Surjectivity then follows from the corresponding result for Galois groups, by considering the diagram

$$\begin{tikzcd}
I(L/K) = \mathrm{Gal}(L/K_0) \arrow[r, two heads] \arrow[dr] & \mathrm{Gal}(FK_0/K_0) \arrow[d, "\cong"] \\
& I(F/K) = \mathrm{Gal}(F/F \cap K_0)
\end{tikzcd}$$

Hence we have proven (i).

(ii) This is an exercise in saying what's what. Indeed for $\sigma \in \mathrm{Gal}(L/F)$, we have

$$\begin{aligned}
\sigma \in W(L/F) &\iff \sigma|_l \in \langle \mathrm{Frob}_{l/k_F} \rangle \\
&\iff \sigma|_l \in \langle \mathrm{Frob}_{l/k} \rangle \cap \mathrm{Gal}(l/k_F)
\end{aligned}$$

so, viewing σ as an element of $\mathrm{Gal}(l/k)$, it lies in $W(L/F)$ if and only if it is in $\mathrm{Gal}(L/F)$ and restricts to an element of $\langle \mathrm{Frob}_{l/k} \rangle \cap \mathrm{Gal}(l/k_F)$ in the residue field l. Hence done.

(iii) This follows immediately from the second isomorphism theorem.

□

This is all well and good, but why do we *care* about the Weil group? Well, recall local Artin reciprocity

Theorem 8 (Local Artin Reciprocity)
Let K be a non-Archimedean local field, then there exists a unique homomorphism

$$\phi_K : K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$$

such that

- for any uniformiser $\pi \in K$, we have $\phi_K(\pi) \mid_{K^{\mathrm{ur}}} = \mathrm{Frob}_{K^{\mathrm{ur}}}$
- for any finite Abelian extension L/K, we have ϕ_K induces an isomorphism

$$\phi_{L/K} : \frac{K^\times}{N_{L/K}(L^\times)} \xrightarrow{\sim} \mathrm{Gal}(L/K).$$

One method of proving this is to use Lubin-Tate theory to construct totally ramified extensions $K_{\pi,n}$ with norm groups $U_K^{(n)}\langle\pi\rangle$, and take their union to get K_π with Galois group isomorphic to $\mathcal{O}_K^\times$. Then using Hasse-Arf Theorem[3], we can prove

[3]This note cant cover everything about LFT else would be quite long, I refer to Milne's CFT notes as a reference.

Theorem 9 (Local Kronecker Weber)

$$K^{\text{ab}} = K_\pi K^{\text{ur}}$$

and use this to construct our Artin map as follows.
We have isomorphisms $\psi : \text{Gal}(K_\pi/K) \xrightarrow{\sim} \mathcal{O}_K^\times$ and $\chi : \text{Gal}(K^{\text{ur}}/K) \xrightarrow{\sim} \hat{\mathbb{Z}}$ where $\chi(\text{Frob}_{K^{\text{ur}}}) = (1, 1, \ldots)$. Then consider that Kronecker-Weber, along with the fact that $K_\pi \cap K^{\text{ur}} = K$, imply that

$$\text{Gal}(K^{\text{ab}}/K) \cong \text{Gal}(K_\pi/K) \times \text{Gal}(K^{\text{ur}}/K) \cong \mathcal{O}_K^\times \times \hat{\mathbb{Z}}. \tag{1}$$

Under this isomorphism, the Weil group of K^{ab}/K is

$$W(K^{\text{ab}}/K) \cong \mathcal{O}_K^\times \times \mathbb{Z} \subset \mathcal{O}_K^\times \times \hat{\mathbb{Z}}. \tag{2}$$

he Artin map is then given by

$$K^\times \xrightarrow{\sim} \mathcal{O}_K^\times \times \mathbb{Z} \xrightarrow{(1)} \text{Gal}(K^{\text{ab}}/K)$$

which, by equation (2), has image equal to the Weil group $W(K^{\text{ab}}/K)$!

An alternative method of proving the existence of ϕ_K is via Galois cohomology and Tate's theorem. First construct an isomorphism for L/K a finite Galois extension:

$$\phi_{L/K}^{-1} : \text{Gal}(L/K)^{\text{ab}} \cong H_T^{-2}(\text{Gal}(L/K), \mathbb{Z}) \xrightarrow{\sim} H_T^0(\text{Gal}((L/K), L^\times) = \frac{K^\times}{N(L^\times)}$$

which can be described either as a cup product with a fundamental class $u_{L/K} \in H^2(K, L^\times)$, or by two consecutive boundary homomorphisms arising form long exact sequences.

It can then be shown that the homomorphisms $\phi_{L/K} : K^\times \to \text{Gal}(L/K)^{\text{ab}}$ behave 'nicely' with respect to inflation maps, and patch together to give a local Artin map ϕ_K. The Artin map is *unique*[4], hence our two Artin maps above do in fact agree. While the second of these proofs is less constructive, however it has far more scope for generalisation to the global case and beyond.

Topology of the Weil Group

Something that we didn't prove in my Local Fields lectures (though we did state it) is that the Artin map is continuous into $\text{Gal}(K^{\text{ab}}/K)$, so I'll quickly do this here.

Proposition 10 The Artin map $\phi_K : K^\times \to \text{Gal}(K^{\text{ab}}/K)$ is continuous, but not open.

[4]The existence theorem - that every open subgroup of finite index in $K^\times$ is a norm subgroup - implies this, cf Milne CFT page 24

Proof. The definition of the profinite topology implies that the subsets $\pi_F^{-1}(\sigma)$, for F/K finite Abelian, $\sigma \in \mathrm{Gal}(F/K)$ and $\pi_F : \mathrm{Gal}(K^{\mathrm{ab}}/K) \to \mathrm{Gal}(F/K)$ the restriction map. are a basis of open subsets. Hence it suffices that their preimages under ϕ_K are open in $K^\times$.

Notice that $\pi_F \circ \phi_K = \phi_{F/K}$, and $\phi_{F/K} : K^\times \to \mathrm{Gal}(F/K)$ is surjective with kernel $N(F^\times)$. This implies that $\phi_K^{-1}(\pi_F^{-1}(\sigma)) = \phi_{F/K}^{-1}(\sigma)$ is a coset of $N(F^\times)$, and it is a well known fact that norm groups of finite Abelian groups are open (eg. Milne p.21) , hence ϕ_K is continuous.
To show that it is not open consider the lemma after this Proposition says that the Weil group is not open in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$. □

Lemma 11 Let L/K be an infinite Galois extension, then the Weil group $W(L/K)$ is not an open subgroup of $\mathrm{Gal}(L/K)$

Proof. In any compact topological group, an open subgroup is closed. Therefore if $W(L/K)$ were open, then it would also be closed, and so its closure would be $W(L/K)$ itself.
However by Proposition 7, we have $W(L/K)$ is dense in $\mathrm{Gal}(L/K)$, and recall that if L/K is infinite, then $W(L/K)$ is a *proper* subgroup of $\mathrm{Gal}(L/K)$, giving a contradiction. □

So, what about when we restrict to looking at the Weil group? Is the Artin map a homeomorphism (and hence an isomorphism of *topological groups*)? Well, this depends on how we topologise the Weil group.

One idea would be to give the Weil group the subspace topology from the Galois group. Indeed, with the subspace topology we automatically get that the Artin map is continuous into $W(K^{\mathrm{ab}}/K)$, however we shall see that it is still not open, by showing that $I(K^{\mathrm{ab}}/K)$ is not open in $W(K^{\mathrm{ab}}/K)$ for the subspace topology.

For myself, the following diagram of groups is quite useful to bear in mind:

$$\begin{tikzcd}
\mathrm{Gal}(K^{\mathrm{ab}}/K) \arrow[r, leftrightarrow, "\sim"] & \mathcal{O}_K^\times \times \hat{\mathbb{Z}} & \\
W(K^{\mathrm{ab}}/K) \arrow[u, hook] \arrow[r, leftrightarrow, "\sim"] & \mathcal{O}_K^\times \times \mathbb{Z} \arrow[u, hook] \arrow[r, leftrightarrow, "\sim"] & K\times \\
I(K^{\mathrm{ab}}/K) \arrow[u, hook] \arrow[r, leftrightarrow, "\sim"] & \mathcal{O}_K^\times \arrow[u, hook] \arrow[r, leftrightarrow, "\sim"] & \mathcal{O}_K^\times \arrow[u, hook]
\end{tikzcd}$$

Anyway, the next result gives us a result about open subgroups of $W(L/K)$ similar to that for profinite groups (Note if L/K is infinite then $W(L/K)$ cannot be profinite, since this would imply it to be compact, and hence closed, as $\mathrm{Gal}(L/K)$ is Hausdorff).

Lemma 12 Suppose $H \subset W(L/K)$ is an open subgroup in the subspace topology, and that $H \leq \text{Gal}(L/K)$ is normal. Then H has finite index in $W(L/K)$

Proof. Consider H a subgroup of $W(L/K)$ that is open in the subspace topology, and is normal in $\text{Gal}(L/K)$. By definition there exists some open $V \subset \text{Gal}(L/K)$ such that $H = V \cap W(L/K)$. Now, since $1 \in H \subset V$, and the normal subgroups of finite index in $\text{Gal}(L/K)$ are a fundamental system of neighbourhoods about 1, we deduce that there is some normal open subgroup N that is contained in V. We then have HN is also open and normal in $\text{Gal}(L/K)$, since for any element $\sigma \in \text{Gal}(L/K)$, $\sigma HN = H\sigma N = HN\sigma$.
Next, we claim that $HN \cap W(L/K) = H$. Indeed, the intersection clearly contains H, and if we have $xy : x \in H, y \in N$ such that $xy \in W(L/K)$, then $H \subset W(L/K)$ implies $y \in N \cap W(L/K) \subset V \cap W(L/K) = H$, and so $HN \cap W(L/K) \subset H$.

Finally, consider

$$W(L/K) \hookrightarrow \text{Gal}(L/K) \to \frac{\text{Gal}(L/K)}{GN}$$

has kernel H. Thus $\big(W(L/K) : H\big) \leq \big(\text{Gal}(L/K/) : HN\big) < \infty$ as required. □

Corollary 13 The inertia subgroup $I(K^{\text{ab}}/K)$ is not an open subgroup of $W(K^{\text{ab}})/K$ when it is endowed with the subspace topology. In particular this implies that, the Artin map $\phi_K : K^\times \to W(K^{\text{ab}}/K)$ is not open.

Proof. The inertia subgroup $I(K^{\text{ab}}/K)$ is a normal subgroup of $\text{Gal}(K^{\text{ab}}/K)$, as it is the kernel of res: $\text{Gal}(K^{\text{ab}}/K) \to \text{Gal}(\overline{k}/k)$, however it does not have finite index in $W(K^{\text{ab}}/K)$ and so by Lemma 12 cannot be open. For the second statement, simply note that $\mathcal{O}_K^\times \subset K^\times$ is open, and $\phi_K(\mathcal{O}_K^\times) = I(K^{\text{ab}}/K)$. □

Therefore we endow $W(K^{\text{ab}}/K)$ with a the topology such that the Artin map *is* a homeomorphism.
More generally, given a Galois extension L/K, the topology on $W(L/K)$ is defined to be the weakest topology such that $W(L/K)$ is a topological group, and $I(L/K)$ with the subspace topology from $\text{Gal}(L/K)$ is an open subspace. In the case where L/K is infinite, this is finer than the subspace topology on $W(L/K)$.
(This topology is 'locally profinite', something which I may come back to, indeed the topology of $K^\times$ is also locally profinite, but this is clear since $\mathcal{O}_K$ is a profinite group!)

Beyond Local fields

The natural question to ask having considered Weil groups for local fields, is whether there is an analogue for Global fields? Indeed, class field theory of global fields is posed in terms of ideles, and the Artin map ϕ_K is the unique map that restricts correctly to local fields

Proposition 14 Let K be a global field, then there is a unique continuous homomorphism $\Phi_K : \mathbb{I}_K \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ such that for any $L \subset K^{\mathrm{ab}}$ and any prime w of L lying above v a prime of K, the following diagram commutes,

$$\begin{array}{ccc} K_v^\times & \xrightarrow{\phi_{K_v}} & \mathrm{Gal}(L_w/K_v) \\ \big\downarrow & & \big\downarrow \\ \mathbb{I}_K & \xrightarrow[x \to \phi_K(x)|_L]{} & \mathrm{Gal}(L/K) \end{array}$$

This map then satisfies the main theorems of global class field theory, namely Global Artin reciprocity and the existence theorem.

Theorem 15 (Global Artin reciprocity) Let K be a global field. The Artin map ϕ_K as given in Proposition 14 has the following properties;

(a) the diagonal embedding of $K^\times$ in $\mathbb{I}_K$ is in the kernel of ϕ_K,

(b) for L/K any finite Abelian extension, the Artin map induces an isomorphism
$$\phi_{L/K} : \mathbb{I}_K/\big(K^\times N(\mathbb{I}_L)\big) \to \mathrm{Gal}(L/K)$$

Upon replacing the idele group by the idele class group $C_K = \mathbb{I}_K/(K^\times)$, the Artin map can be written as a homomorphism from C_K to the Galois group. Then for any L/K we have an isomorphism

$$\phi_{L/K} : C_K/N(C_L) \to \mathrm{Gal}(L/K).$$

One should note that in contrast to the local case, ϕ_K may not be injective. In fact;

- if K is a number field, then ϕ_K is surjective but not injective,
- if K is a global function field, then ϕ_K is injective but not surjective.

So perhaps we cannot always view the Weil group simply as a subgroup of the Galois group!

Abstracting the Weil group

Let K be a local field with separable closure $\overline{K}$ and given L/K a finite field extension set $G_L = \mathrm{Gal}(\overline{K}/L)$. The Weil group $W_K := W(\overline{K}/K)$ is a topological group, with a map $\psi : W_K \to G_K$ which has dense image by Proposition 7(i). Moreover, consider that $\psi^{-1}(G_L) \subset W_K$ will be $W_L = W(\overline{K}/L)$ by Proposition 7(ii), and we have for L/K a Galois extension that $W_K/W_L \cong \mathrm{Gal}(L/K)$.

The final point was that the abelianisation of the Weil group is isomorphic to $K^\times$ as a topological group *via* the local Artin map.
That is, for all L/K finite, we have isomorphisms of topological groups $r_L : L^\times \to W_L^{\mathrm{ab}}$ such that

$$L^\times \xrightarrow{r_L} W_L^{\mathrm{ab}} \xrightarrow{\psi} G_L^{\mathrm{ab}}$$

is the Artin map. We therefore seek groups W_K - or more precisely, triples $(W_K, \psi, \{r_E\})$ - that generalise these ideas to the global case.

Definitions

Let K be a local or global field and $\overline{K}$ a fixed separable closure. For any finite extension L/K we let $G_E := \mathrm{Gal}(\overline{K}/L)$ and define C_L to be

$$C_L = \begin{cases} L^\times & \text{in local case,} \\ \mathbb{I}_L/L^\times & \text{in the global case.} \end{cases}$$

A Weil group for $\overline{K}/K$ consists of a triple $(W_K, \psi, \{r_L\})$:

- W_K a topological group,
- $\psi : W_K \to G_K$ a continuous homomorphism with dense image,
- for each L/K finite, an isomorphism of topological groups $r_L : C_L \to W_L^{\mathrm{ab}}$, where $W_L := \psi^{-1}(G_L) \subset W_K$.

Notice that when L/K is Galois we have

$$\frac{W_K}{W_L} \xrightarrow{\sim} \frac{G_K}{G_L} \xrightarrow{\sim} \mathrm{Gal}(L/K)$$

is an isomorphism of groups, by denseness of the image of the Weil group.

In analogy to the local case discussed above, we'd like W_K to have the key property

(i) For any L/K finite, we have

$$W_L \to W_L^{\mathrm{ab}} \xrightarrow{\psi} G_L^{\mathrm{ab}}$$

is the Artin map.

In addition to this W_K must satisfy the following properties:

(ii) if $w \in W_K$ and $\sigma = \psi(w) \in G_K$, the following diagram commutes;

$$\begin{tikzcd} C_L \arrow[r, "r_L"] \arrow[d, "\sigma"'] & W_L^{\mathrm{ab}} \arrow[d, "\text{conjugation by } w"] \\ C_{\sigma L} \arrow[r, "r_{\sigma L}"] & W_{\sigma L}^{\mathrm{ab}} \end{tikzcd}$$

(iii) if $L' \subset L$ are finite extensions of K, we have the diagram

$$\begin{array}{ccc} C_{L'} & \xrightarrow{r_{L'}} & W_L^{\mathrm{ab}} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \text{transfer map}} \\ C_L & \xrightarrow{r_L} & W_{\sigma L}^{\mathrm{ab}} \end{array}$$

commutes, where i is the map induced by the inclusion $L' \subset L$.

(iv) There is an topological group isomorphism given by the natural map,

$$W_K \to \lim_{\leftarrow L} W_{L/K},^5$$

where $W_{L/K} := \frac{W_K}{W_L^c}$, with W_L^c the closure of the commutator subgroup of W_L.

So, why do we want conditions (ii) - (iv)?

Well, often mathematical definitions arise when we have constructed some object, and discovered some interesting properties of this object. Having discovered these, we are lead to ask whether they are the *defining* properties of the object, and if so we can take them as the definition itself. Below we shall construct such an interesting object, and then show that it is the Weil group.

First, recall that given a group G and M a G-module, there is a bijection between $H^2(G, M)$, and extensions of G by M up to equivalence. In the cohomological proofs of class field theory, we find that for L/K a finite Galois extension, there is a 'fundamental class' $u_{L/K} \in H^2(\mathrm{Gal}(L/K), C_L)$ for which the inverse of the Artin map corresponds to taking the cup product with $u_{L/K}$,

$$\phi_{L/K}^{-1} : \mathrm{Gal}(L/K)^{\mathrm{ab}} \cong H_T^{-2}(\mathrm{Gal}(L/K), \mathbb{Z}) \xrightarrow{\sim} H_T^0(\mathrm{Gal}((L/K), C_L) = \frac{C_K}{N(C_L)}.$$

We can therefore consider the extension $E_{L/K}$ of $\mathrm{Gal}(L/K)$ by C_L corresponding to this fundamental class $u_{L/K}$. One can show[6] that these extensions form an inverse system, where L ranges over fintie Galois extension of K. Taking limits gives an object E_K which will turn out to be the Weil group.

I feel as though this construction makes the conditions become a bit more apparent. We have (ii) is a reflection of the fact that $W_{L/K}$ will be a group extension of $\mathrm{Gal}(L/K)$ by C_L, while (iii) tells us that the Weil groups behave well with respect to restriction, and finally (iv) is just saying that we arrive at

[5] This scared me because it seemed to imply that W_K must have a profinite topology - which from our local field section we know it doesn't. But all is well once we realise that $W_{L/K}$ may be infinite groups.

[6] This was bugging me, I have investigated this in the note ".

the Weil group for K as a projective limit, as was done in our mock construction.

We then find that these are the defining conditions by proving uniqueness of the Weil groups up to isomorphism. To prove this, we work in the other direction.

By (ii), $W_{L/K}$ is indeed a group extension of $\mathrm{Gal}(L/K)$ and so there is some corresponding class $\alpha_{L/K} \in H^2(L/K, C_L)$. Then one uses (iii) to show that the cup product with $\alpha_{L/K}$ on Tate cohomology is an isomorphism - from which we can deduce that the classes $\alpha_{L/K}$ are equal to the fundamental classes of class field theory. Therefore we have that $W_{L/K}$ and $E_{L/K}$ constructed above are equivalent extensions! Finally, we invoke condition (iv) of the definition to find that $W_K \cong E_K$, in some canonical way.

This is all quite hand-wavy at the minute. I would like to read Artin-Tate's coverage of the topic, however dont currently have access.

I should note that we can *further* abstract these concepts. The triple $(G, \{G_F\}, A)$ consisting of the absolute Galois group $G = \mathrm{Gal}(\overline{K}/K)$ of a field K, the Galois groups $G_F = \mathrm{Gal}(\overline{K}/F)$ for F/K finite, and the G-module $A = \lim_{\rightarrow} C_F$, form what is called a *formation.*

When we introduce the invariant maps $\mathrm{inv}_K : H^2(\mathrm{Gal}(F/K), C_F) \to \mathbb{Q}/\mathbb{Z}$ obeying certain properties - or equivalently we can introduce some 'fundamental' classes $u_{L/K} \in H^2(\mathrm{Gal}(F/K), C_F)$ obeying certain properties - we get what is called a *class formation.*

Essentially this is the abstract structure we need to get reciprocity maps which act like the Artin map! By abstracting this idea, we simply have to check that certain axioms are satisfied to discern the existence of a reciprocity map. Indeed, we could approach local and global reciprocity by first proving results about class formations, then showing that $(G, \{G_F\}, A, \mathrm{inv}_F)$ defined above form a class formation! As a reference, see Serre - Local Fields chapter XI.