Note on Group Extensions

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I try to give some results about maps between group extensions, following mainly [Lan96]. The purpose of this was so that I could construct the Weil group W_F of a class formation (most generally - but mainly for absolute Galois groups of local and global fields!) in order to better understand [Tat79]. In particular, Tate states that we can construct the Weil group as an inverse limit of relative Weil groups, which arise as the group extensions of $\mathrm{Gal}(L/K)$ by C_L.

In particular, to form an inverse system of relative Weil groups $W_{L/K}$, where L/LK is Galois, we need some map $W_{L'/K} \to W_{L/K}$, when $L \subset L'$. Hopefully, the theory explores here will provide detail on how this is constructed! (Currently waiting to find access to [Art67] - will continue once this happens.)

Group extensions

We begin recall some mathematics from group cohomology.

Let G be a group, and M a G-module. An extension of M by G is an exact sequence of groups

$$1 \to M \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

such that the G-action on M is the same as that induced by conjugation by elements in E. That is - given $g \in G$ and $u \in E$ in the fibre $\pi^{-1}(g)$ we have

$$g \cdot m = umu^{-1}.$$

¡Warning! Usually for a module M we write the underlying abelian group operation additively, while here we tend to write this multiplicatively and view i as an inclusion in E!
Ok, we recall that two extensions

$$1 \to M \xrightarrow{i_1} E_1 \xrightarrow{\pi_1} G \to 1$$
$$1 \to M \xrightarrow{i_2} E_2 \xrightarrow{\pi_2} G \to 1$$

of M by G are equivalent if there is an isomorphism ϕ such that we have a commutative diagram;

$$\begin{tikzcd} & & E_1 \arrow[dd, "\phi"] \arrow[dr] & & \\ 1 \arrow[r] & M \arrow[ur] \arrow[dr] & & G \arrow[r] & 1 \\ & & E_2 \arrow[ur] & & \end{tikzcd}$$

Let $E(G, M)$ denote the set of equivalence classes of extensions of M by G, then we have a bijection

$$E(G, M) \overset{\sim}{\leftrightarrow} H^2(G, M).$$

As stated in the introduction, I want to look at maps between group extensions, so consider

$$1 \to M \to E \to G \to 1$$
$$1 \to M' \to E' \to G' \to 1$$

two extensions and homomorphisms $\phi : G \to G'$, $f : E \to E'$.

Question: Does there exist a homomorphism $E \to E'$ such that the diagram

$$\begin{tikzcd}
1 \arrow[r] & M \arrow[r] \arrow[d, "f"'] & E \arrow[r] \arrow[d, "\exists F?"] & G \arrow[r] \arrow[d, "\phi"] & 1 \\
1 \arrow[r] & M' \arrow[r] & E' \arrow[r] & G' \arrow[r] & 1
\end{tikzcd}$$

commutes?

Theorem 1 Such a homomorphism exists if and only if we have

(i) f is a G-module map, where M' has G-action induced by ϕ, [1]

(ii) $\phi^*\alpha' = f_*\alpha \in H^2(G, M')$, where $\phi*, f_*$ are the maps on cohomology induced by the compatible pairs (id_G, f) and $(\phi, \mathrm{id}_{M'})$, and α, α' the classes corresponding to E, E' respectively.

Proof. The proof is quite hands on - we shall use cocycles. Indeed, let $s : G \to E$ and $s' : G' \to E'$ be sections with $s(e_G) = e_E$ and $s'(e_{G'}) = e_{E'}$ (normalised sections). Then the class α is represented by the (normalised) 2-cocycle

$$\chi : G^2 \to M; (\sigma, \tau) \mapsto s(\sigma)s(\tau)s(\sigma\tau)^{-1},$$

and we define χ' similarly.

First lets outline what our two conditions (i), (ii) mean in this language;

(i) this one is simple, it says that $f(\sigma \cdot m) = \phi(\sigma) \cdot f(m)$ for all $m \in M$ and $\sigma \in G$

(ii) This second one is slightly more complicated, it requires that

$$\chi'(\phi(\sigma), \phi(\tau)))f(\chi(\sigma, \tau))^{-1} = d\psi(\sigma, \tau)$$

for some $\psi : G \to M'$. In words we're saying that $f_*\chi$ and $\phi^*\chi'$ differ by a 2- coboundary.

[1]Not to be confused with the very similar condition called *compatibility of maps*, where we have $\phi : G' \to G$ and $f : M \to M'$ are compatible if f is a G'-module homomorphism . By using cocycles we can show that such maps will induce a homomorphism on cohomology $H^r(G, M) \to H^r(G', M')$.

Let's begin by supposing that there exists a homomorphism F such that the diagram above commutes. Then we have for any $\sigma \in G$, that

$$\pi'(F(s(\sigma))) = \phi(\pi(s(\sigma))) = \phi(\sigma),$$

and so $s'(\phi(\sigma))\big(F(s(\sigma))\big)^{-1} \in M'$ for all $\sigma \in G$. Therefore we may define a map $\psi : G \to M'$ by

$$\psi(\sigma) = s'(\phi(\sigma))\big(F(s(\sigma))\big)^{-1},$$

which as you may have guessed, will be the ψ that we required to show that (ii) holds.

To show that (i) holds, note that $f(\sigma \cdot m) = F(s(\sigma)ms(\sigma)^{-1})$, and since F is a homomorphism we can expand this to get $F(s(\sigma))f(m)F(s(\sigma))^{-1}$. Then using ψ to replace the $F(s(\sigma))'s$ by $\psi(\sigma)^{-1}s'(\phi(\sigma))$, we find that

$$f(\sigma \cdot m) = \psi(\sigma)^{-1}\big(s'(\phi(\sigma))f(m)s'(\phi(\sigma))^{-1}\big)\psi(\sigma) = s'(\phi(\sigma))f(m)s'(\phi(\sigma))^{-1} = \phi(\sigma)\cdot f(m)$$

as required.

Now for (ii), which is rather messy. (I suppose I could use different notation to clean this up, but sometimes this just obscures the point. In reality calculations with co-cycles are always a bit messy!)

We want to show that $\chi'(\phi(\sigma), \phi(\tau)))f(\chi(\sigma, \tau))^{-1} = d\psi(\sigma, \tau)$, so lets first expand the right hand side of this equation;

$$\begin{aligned} d\psi(\sigma, \tau) &= (\sigma \cdot \psi(\tau))\psi(\sigma\tau)^{-1}\psi(\sigma) \\ &= s'(\phi(\sigma))s'(\phi(\tau))\big(F(s(\tau))\big)^{-1}s'(\phi(\sigma))^{-1}F(s(\sigma\tau))s'(\phi(\sigma\tau))^{-1}s'(\phi(\sigma))\big(F(s(\sigma))\big)^{-1}, \end{aligned}$$

and we want to show that this is equal to

$$\chi'(\phi(\sigma), \phi(\tau)))f(\chi(\sigma\tau))^{-1} = s'(\phi(\sigma))s'(\phi(\tau))s'(\phi(\sigma\tau))^{-1}F(s(\sigma\tau))F(s(\tau))^{-1}F(s(\sigma))^{-1}.$$

Cool. We can immediately cancel a few of the end terms to reduce this to showing that

$$F(s(\tau))^{-1}s'(\phi(\sigma))^{-1}F(s(\sigma\tau))s'(\phi(\sigma\tau))^{-1}s'(\phi(\sigma)) = s'(\phi(\sigma\tau))^{-1}F(s(\sigma\tau))F(s(\tau))^{-1}$$

which requires some finicky manipulation. Multipling on the left by $s(\phi(\sigma\tau))$ gives

$$s'(\phi(\sigma\tau))F(s(\tau))^{-1}s'(\phi(\sigma))^{-1}F(s(\sigma\tau))s'(\phi(\sigma\tau))^{-1}s'(\phi(\sigma)) = F(s(\sigma\tau))F(s(\tau))^{-1}$$

a nd notice that the first three terms of the left hand side multiply to give an element in M' (To see this, just apply π' and you'll get $e_{G'}$). Similarly the term $F(s(\sigma\tau))s'(\phi(\sigma\tau))^{-1}$ on the left hand side lies in M' (indeed, this is just

$\psi(\sigma\tau)^{-1}$), hence we can swap these terms by commutativity in M'.
This gives

$$F(s(\sigma\tau))s'(\phi(\sigma\tau))^{-1}s'(\phi(\sigma\tau))F(s(\tau))^{-1}s'(\phi(\sigma))^{-1}s'(\phi(\sigma)) = F(s(\sigma\tau))F(s(\tau))^{-1}$$

from which the result follows immediately by cancellation!

Okay, so we've proven that the existence of such an F implies the conditions (i),(ii) must hold.

Conversely, if the conditions hold we can use the fact that any $x \in E$ can be written uniquely as $x = ms(\sigma)$, where $\sigma = \pi(x)$ and $m \in M$, to define F. Indeed, by condition (ii) there exists some $\psi : G \to M'$ such that $\chi'(\phi(\sigma), \phi(\tau)))f(\chi(\sigma,\tau))^{-1} = d\psi(\sigma,\tau)$, then set

$$F(x) = F(ms(\sigma)) = f(m)\psi(\sigma)^{-1}s'(\phi(\sigma))$$

Working backwards through our proof above then shows that this is indeed a homomorphism. We'll explicitly work through this for some peace of mind!

Consider $x = ms(\sigma), y = ns(\tau)$ in E, where $\sigma = \pi(x)$ and $\tau = \pi(y)$, then we have

$$xy = ms(\sigma)ns(\tau) = m(\sigma \cdot n)s(\sigma)s(\tau) = m(\sigma \cdot n)\chi(\sigma,\tau)s(\sigma\tau)$$

so applying F gives

$$\begin{aligned}
F(xy) &= f(m)f(\sigma \cdot n)f(\chi(\sigma,\tau))\psi(\sigma\tau)^{-1}s'(\phi(\sigma\tau)) \\
&= f(m)[\phi(\sigma) \cdot f(n)][d\psi(\sigma,\tau)^{-1}\chi'(\phi(\sigma),\phi(\tau))]\psi(\sigma\tau)^{-1}s'(\phi(\sigma\tau)) \\
&= f(m)[\phi(\sigma) \cdot f(n)]\psi(\sigma)^{-1}\psi(\sigma\tau)[\phi(\sigma) \cdot \psi(\tau)^{-1}]\psi(\sigma\tau)^{-1}\chi'(\phi(\sigma),\phi(\tau))s'(\phi(\sigma\tau)).
\end{aligned}$$

Notice that almost all of these terms - all except $s'(\phi(\sigma\tau))$ - are in M', hence we can swap them around at will. Doing so, and expanding the G-actions as conjugation actions, we do indeed find that

$$F(xy) = [f(n)\psi(\sigma)^{-1}s'(\phi(\sigma))][f(m))\psi(\tau)^{-1}s'(\phi(\tau))] = F(x)F(y),$$

and so F is a homomorphism! The final checks that F makes the diagram are actually far less laborious, so we'll omit them and move on with our lives. $\square$

Notice that in the theorem, F is uniquely determined by how it acts on the values $s(\sigma)$ where $\sigma \in G$, and we have

$$F(s(\sigma)) = \psi(\sigma)^{-1}s'(\phi(\sigma)).$$

Thus - having already chosen s, s' and hence the cocycle representatives $\chi, \chi'$of E, E'- the homomorphism F relies *only* on the 1-cochain ψ that we chose, who's coboundary $d\psi$ is the difference between $f_*\chi$ and $\phi^*\chi'$. We can therefore modify ψ by any *1-cocycle* of $Z^1(G, M')$ and get another homomorphism.

Question: Ss this new homomorphism 'equivalent' to the old one? What is the 'correct' definition for equivalence of such homomorphisms?!

In [Lan96], he states that we should consider two homomorphisms F_1, F_2 as being equivalent if there exists some inner automorphism $c_{m'}$ of E' by $m' \in M'$ such that the diagram

$$\begin{tikzcd}
1 \arrow[r] & M \arrow[r] & E \arrow[rr] \arrow[ddd, "F_1"] \arrow[dr, "F_2"] & & G \arrow[r] & 1 \\
& & & E' \arrow[dr] & & \\
1 \arrow[r] & M' \arrow[uurr] \arrow[dr] & & & G' \arrow[r] & 1 \\
& & E' \arrow[uur, "c_{m'}"] \arrow[urr] & & &
\end{tikzcd}$$

commutes.

But why should we take an *inner* automorphism, and why shouldn't we allow automorphisms of E too? Why do we not allow something more general like considering commutative diagrams;

$$\begin{tikzcd}
& & & E \arrow[dr] \arrow[ddd, "F_2"] & & \\
1 \arrow[r] & M \arrow[urr] \arrow[dr] & & & G \arrow[r] & 1 \\
& & E \arrow[uur, "\beta"] \arrow[urr] \arrow[ddd, "F_1"] & & & \\
& & & E' \arrow[dr] & & \\
1 \arrow[r] & M' \arrow[urr] \arrow[dr] & & & G' \arrow[r] & 1 \\
& & E' \arrow[uur, "\beta'"] \arrow[urr] & & &
\end{tikzcd}$$

where β and β' are any automorphisms such that the diagram commutes? I suppose maybe there could be some 'cancellation' between β and β' which is accounted for in our definition, otherwise the diagram above seems like a more category-theoretic definition (I mean - obviously it depends which category we're working in!).

For now we'll go with what Lang says. Of course, his definition is the 'correct' one - I'm just currently having trouble seeing why, my best guess is that it's defined *such that* the next theorem holds. Indeed we mentioned above that we can get different homomorphisms by varying ψ by elements of $Z^1(G, M')$.

What does it look like when we vary ψ by a 1-coboundary? Given $m' \in M$, let $\beta_{m'}(\sigma) = (\sigma \cdot m')m'^{-1}$? Well, the map becomes

$$\begin{aligned} F_{\text{new}}(ms(\sigma)) &= f(m)\psi(\sigma)^{-1}\beta_{m'}(\sigma)^{-1}s'(\phi(\sigma)) \\ &= f(m)\psi(\sigma)^{-1}m'^{-1}(\sigma \cdot m')s'(\phi(\sigma)) \\ &= m'^{-1}[f(m)\psi(\sigma)^{-1}s'(\phi(\sigma))]m' \\ &= c_m \circ F(ms(\sigma)). \end{aligned}$$

Aha! So homomorphisms are equivalent in Lang's sense if and only if the 1-cocycle that their respective ψ's vary by is a coboundary! This implies that the following theorem holds:

Theorem 2 Let f, ϕ be as in Theorem 1, then the set X of equivalence classes of homomorphisms $F : E \to E'$ (as in Theorem 1) form a 'principal homogeneous space' of $H^1(G, M')$.[2] The action of $H^1(G, M')$ on X is given by

$$(\beta \cdot F)(u) = \beta(\pi(u))F(u).$$

Corollary 3 If $H^1(G, M') = 0$, then all homomorphisms F_1, F_2 as in Theorem 1 are (Lang)-equiavlent.

Construction of the Weil Group

Let $(G, \{G_E\}, A, \text{inv}_E)$ be a class formation, for which the existence theorem holds (so we're assuming the axioms from [Ser79] Ch.XI §5). We'll use language assuming that G is a Galois group, but this needn't necessarily be the case! Note that in keeping with the exposition of [Ser79], we have F/E will be finite unless otherwise stated.

There are fundamental classes $u_{F/E} \in H^2(F/E)$ for all F/E Galois, which are the unique classes such that $\text{inv}_E(u_{F/E}) = \frac{1}{n} \mod \mathbb{Z}$. A natural thing to do - considering we've just talked about group extensions - is to look at the group extensions associated to these classes. With this in mind, define $W_{F/E}$ to be the group extension

$$1 \to A_F \to W_{F/E} \to G(F/E) \to 1$$

corresponding to the fundamental class.

Aim: Show that the groups $W_{F/E}$ form an inverse system for the directed set $\{F/E \text{ Galois extensions}\}$ ordered by inclusion.

[2] This basically just means that X bijects with G, and there is some G-action on X.

We first need to define the transition maps from $W_{F/E} \to W_{F'/E}$ whenever $F \supset F' \supset E$ with F, F' Galois over E. To do so, consider the diagram

$$\begin{tikzcd}
1 \arrow[r] & A_F \arrow[r] \arrow[d, "N"'] & W_{F/E} \arrow[r] \arrow[d, "\exists T?"] & G(F/E) \arrow[r] \arrow[d, "p"] & 1 \\
1 \arrow[r] & A_{F'} \arrow[r] & W_{F'/E} \arrow[r] & G(F'/E) \arrow[r] & 1
\end{tikzcd}$$

and ask if there exists a transition map T such that it commutes. The map p is the projection map by restriction to F', and the map $N = N_{F/F'}$ is the norm map.

This was exactly the problem that motivated Theorem 1, hence it should come as no surprise that we are going to invoke it now to prove that T exists! To do so, we need to check the conditions (i) and (ii) of the theorem.[3]

For condition (i), we need to show that for any $x \in A_F$, and $\sigma \in G(F/E)$ we have

$$N_{F/F'}(\sigma(x)) = \sigma|_{F'}(N_{F/F'}(x).$$

This is not difficult, noting that $G(F/F')$ is a normal subgroup of $G(F/E)$ (with quotient $G(F'/E)$), we have

$$\begin{aligned}
N_{F/F'}(\sigma(x)) &= \prod_{\tau \in G(F/F')} \tau\sigma(x), \\
&= \prod_{\tau \in G(F/F')} \sigma\sigma^{-1}\tau\sigma(x), \\
&= \sigma\Bigg(\prod_{\tau' \in G(F/F')} \tau'(x) \Bigg), \\
&= \sigma(N_{F/F'}(x)),
\end{aligned}$$

as required.

Next we look to prove (ii), which is the statement that

$$p^*(u_{F'/E}) = N_*(u_{F/E}) \in H^2(G(F/E), A'_F).$$

Just for now, I shall assume that this is true.

Having assumed (ii), we have $T : W_{F/E} \to W_{F'/E}$ such that our diagram commutes. We shall now argue that these can be used as the transition maps. So consider $F \supset F' \supset F'' \supset E$, and we have maps $T_{F/F'}, T_{F/F''}$ and $T_{F'/F''}$ as described above. What we'd really like is that the diagram

[3] I currently don't have my hands on the text [Art67], which every *other* textbook assures me is the most useful book on this topic.

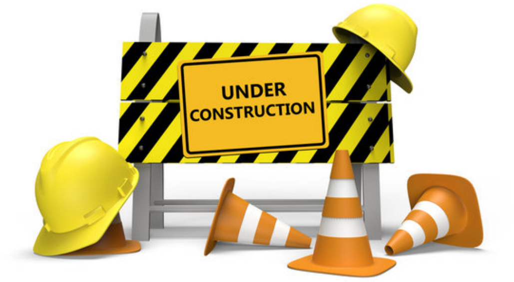

Figure 1: Caption

$$\begin{tikzcd}
1 \arrow[r] & A_F \arrow[r] \arrow[d, "N"'] & W_{F/E} \arrow[r] \arrow[d, "T_{F/F'}"] \arrow[dd, bend right, "T_{F/F''}"'] & G(F/E) \arrow[r] \arrow[d, "p"] & 1 \\
1 \arrow[r] & A_{F'} \arrow[r] \arrow[d, "N"'] & W_{F'/E} \arrow[r] \arrow[d, "T_{F'/F''}"] & G(F'/E) \arrow[r] \arrow[d, "p"] & 1 \\
1 \arrow[r] & A_{F''} \arrow[r] & W_{F''/E} \arrow[r] & G(F''/E) \arrow[r] & 1
\end{tikzcd}$$

commutes. But this needn't be the case due to the non-uniqueness of the maps T.

References

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