Adeles, Ideles, and such

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This is a note to help me revise the ideas of adeles and ideles introduced to me in an algebraic number theory course taught by Hanneke Wiersema, I generally use [Cas10], [Neu99] and [Lan94] as references.

1 Restricted products

We introduce restricted products here, because we are going to use them to define the adele ring and the idele group.

Definition 1 Let $(X_i)_{i\in I}$ be a family of topological spaces, and $(W_i)_{i\in I}$ open subsets. We define the restricted product to be

$$A = \prod_i{}'(X_i, W_i) = \left\{(x_i) \in \prod_i X_i : x_i \in W_i \text{ for almost all } i\right\}.$$

with the restricted product topology having a basis of open sets

$$\mathcal{B} = \left\{\prod V_i : V_i \text{ are open in } X_i, \text{ and } V_i = W_i \text{ for almost all } i\right\}.$$

Remark

- The phrase 'for almost all' is a marginally shorter way of saying 'for all but finitely many',
- Often we simply write $\prod_i'(X_i, W_i)$ as $\prod_i' X_i$ where the W_i are implicitly known,
- We only actually need almost all of the W_i to be open – but we can without loss of generality say that they are – see the lemma below.

The restricted product topology is in general finer than that induced by considering $\prod_i' X_i$ as a subspace of the product space $\prod_i X_i$.

Indeed, recall that the products topology is the coarsest topology such that projection maps are continuous, which is a natural definition in both a category theory sense (is the category theoretic product[1]) and in a practical sense, since it allows us to verify whether a map $Y \to \prod X_i$ is continuous by simply checking the projection onto the components. The product topology then has the basis of products of opens $\prod V_i$ where almost all V_i are equal to X_i.

Consider that the subspace topology for $A = \prod' X_i$ will have basis

$$\mathcal{B}_{sub} = \left\{(\prod V_i) \cap A : V_i \text{ are open in } X_i, \text{ and } V_i = X_i \text{ for almost all } i\right\},$$

[1] Question for later me: is there a category theoretical restricted product?

and we wish to show that this topology is coarser than the restricted product topology.

Consider an element $B = (\prod V_i) \cap A$ of $\mathcal{B}'$, and let $S \subset I$ be the finite set of i such that $V_i \neq X_i$. Given an element $x \in B$, we have a finite set $J_0 \subset I$, where $x_i \in W_i$ for all $i \in I \setminus J_0$, and let $J = J_0 \setminus S$.
Then $U_J = \prod_{i \in J} X_i \prod_{i \in S} V_i \prod_{i \notin J \cup S} W_i$ is a basic open of the restricted product topology, and $x \in U_J \subset B$. Taking unions over *all* finite subsets $J_0 \subset I$ then implies that B is open in the restricted product topology.

In general, when the set $\{i \in I : W_i \neq X_i\}$ is not finite, the restricted product topology is strictly finer than the subspace topology. Indeed, supposing that I is infinite and $W_i \subsetneq X_i$ for all i, we claim the open set $\prod W_i$ is not open in the subspace topology.

To show this, it suffices to show that no element of $\mathcal{B}_{sub}$ is contained in $\prod W_i$, so consider $B = \left(\prod_{i \in S} V_i \prod_{i \notin S} X_i\right) \cap A \in \mathcal{B}_{sub}$, where S is a finite set. Choose any index $i_0 \notin S$, and any element $(x_i) \in A$ with $x_i \in V_i$ for all $i \in S$, $x_i \in W_i$ for all $i_0 \neq i \notin S$ and $x_{i_0} \in X_{i_0} \setminus W_{i_0}$. By construction this is contained in B but not in $\prod W_i$, which completes our argument.

Definition 2 Let $S \subset I$ be finite, then we define

$$A_S = \prod_{i \in S} X_i \prod_{i \notin S} W_i$$

This is clearly open in the restricted product topology, and we have

$$A = \bigcup_{S \text{ finite } \subset I} A_S$$

Moreover, the product topology on A_S is equal to the subspace topology inherited from A, which is easy to show by comparing the bases.

A useful point is that due to this 'almost all' condition, we can change certain things for a finite number of $i \in I$, and not change anything about the topology of A. For example:

Lemma 1 If W_i and $\hat{W}_i$ are two collections of subsets of X_i which are equal for almost all i, then the restricted products are identical, with the same topology.

Proof. This is trivial by looking at the definition. □

We now come to a useful theorem for our applications to number theory:

Lemma 2 Let X_i be a collection of *locally compact* spaces, and W_i open subsets which are almost all compact. Then the restricted product $\prod' X_i$ is locally compact.

Proof. By our lemma we may assume that all W_i are compact. Now, we have $A = \cup A_S$ for S finite subsets of I, hence it suffices to prove that these are compact. As stated in the discussion proceeding the lemma, the topology that A_S inherits from A is equal to the product topology. This is key since it allows us to use Tychnoff's theorem.

Let $x \in A_S$, then since X_i are locally compact we may choose compact neighbourhoods K_i of $x_i \in X_i$ for all $i \in S$. Then consider the set $\prod_{i\in S} K_i \prod_{i\notin S} W_i$ is a compact neighbourhood of x in A_S, which completes the proof. $\square$

We shall be particularly interested in the cases where $X_i = G_i$ are topological groups with open subspaces $W_i = H_i$. Then the restricted product A is also a topological group.

Claim: A is indeed a topological group where the operation is defined componentwise.

Proof. First we need to check that this is well defined. Indeed, if $(x_i) \in A$ and $(y_i) \in A$, let S_x and S_y denote the finite set of indices $i \in I$ such that $x_i \notin H_i$ and $y_i \notin H_i$ respectively, then for all $i \notin S_x \cup S_y$ we have $x_i y_i \in H_i$.

Next we must show that multiplication $m : A\times A \to A$ and inversion $i : A \to A$ are both continuous, which can be combined into showing that $\phi : (x, y) \mapsto x^{-1}y$ is continuous. Let $B = \prod_{i\in S} V_i \prod_{i\notin S} H_i$ be a basic open of A, then we have

$$\phi^{-1}(B) = \{(x, y) \in A^2 : x^{-1}y \in B\} = \bigcup_{\text{finite } J\subset I} \{(x, y) \in A_J \times A : x^{-1}y \in B\}.$$

It suffices to show that of the sets in the union is open, call them Λ_J. Consider that Λ_J is the set of $(x, y) \in A_J \times A$ such that

$$(x_i, y_i) \in \begin{cases} \phi_i^{-1}(V_i) = U_i & \text{if } i \in S, \\ \phi_i^{-1}(H_i) = U_i & \text{if } i \in J \setminus S, \\ \phi_i^{-1}(H_i) = H_i^2 & \text{if } i \notin J \cup S, \end{cases}$$

where ϕ_i is the i component of ϕ. By the fact that G_i are all topological groups we have $\phi_i{-}1(V_i)$ and $\phi_i^{-1}(H_i)$ are all open in $G_i \times G_i$. Moreover, in the final case, since we know that $x_i \in H_i$, we simply have $\phi_i^{-1}(H_i) = H_i^2$.

When $i \in J \cup S$, the open set U_i in G_i^2 is some union of Cartesian products $X_i \times Y_i$ with $X_i, Y_i \subset G_i$ open. Therefore Λ_J is a union of sets of the form

$$\Lambda_J = (\prod_{i\in S\cup J} X_i \prod_{i\notin S\cup J} H_i) \times (\prod_{i\in S\cup J} Y_i \prod_{i\notin S\cup J} H_i)$$

which is open in A^2, hence we are done. $\square$

(I feel like there must be some more elegant way to do this - it feels quite brute force. I think perhaps can consider the union $\bigcup\{(x,y) \in A_J \times A_J : x^{-1}y \in B\}$ over all finite J index sets (note that A^2 is in fact equal to the union $\bigcup A_J^2$, since $J \subset J'$ implies $A_J \subset A_{J'}$ so $A^2 = \bigcup A_J \times A_{J'} = \bigcup A_{J\cup J'}^2$), then just reduce to the same fact for usual products of topological groups!)

Similarly, we can show that if R_i are topological rings with S_i open subrings then the restricted product is a topological ring.

2 Adeles and Ideles

Adeles

The adele ring of a global field is an important object of study in number theory. It bundles together *all* local completions of the global fields allowing us to work with them at the same time, and using the restricted product allows us to preserve niceties of these local fields such as local compactness. This is important for getting a unique Haar measure on a topological group and doing Fourier analysis. (I'd at some point like to get onto this and Tate's thesis.)

Definition 3 The Adele ring is defined to be the restricted product

$$\mathbb{A}_K = \prod_{v \in M_K}{}' (K_v, \mathcal{O}_{K_v})$$

where M_K is the set of all primes of K, and for infinite places we take $\mathcal{O}_{K_v}$ to be K_v (although by Lemma 1 this really doesn't matter).

In particular we have shown that $\mathbb{A}_K$ is a topological group additively, and in fact it is also a topological ring, where multiplication is also defined componentwise. Moreover, from our work in the previous section we have

Proposition 3 $\mathbb{A}_K$ is locally compact and Hausdorff

Proof. The local compactness follows from Lemma 2, and Hausdorff follows because the restricted product topology is finer than the subspace topology from $\prod_v K_v$, which is itself Hausdorff since all K_v are metric spaces. □

The open subset $\mathbb{A}_{K,S} = \prod_{v\in S} K_v \prod_{v\notin S} \mathcal{O}_{K_v}$ of $\mathbb{A}_K$, for S a finite set of primes is called the set of S-adeles. As discussed in the last section, the S-adeles have the same topology as the usual product topology.

Next, note that there is a diagonal embedding $K \hookrightarrow \mathbb{A}_K$ given by $x \mapsto (x, x, \ldots)$, because each element of x is has non-zero factorisation for only finitely many finite primes, which itself follows by unique prime factorisation of fractional ideals. We tend to just write K to mean the image of this embedding of K in $\mathbb{A}_K$, and call these the *principal adeles*.

In fact, adeles can be defined as rings without referencing the restricted product, by taking $\mathbb{A}_{\mathbb{K}} = (\mathbb{R} \otimes_{\mathbb{Z}} K) \times (\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} K)$. Notice that this first term is the product of completions of K at infinite places, while the second term is $(\prod_p \mathbb{Q}_p) \otimes_{\mathbb{Z}} K$ accounts for finite primes (Note: tensor products distribute over direct sums of modules, not necessarily direct products!).

Ideles

Definition 4 The group of ideles $\mathbb{I}_K$ is the set of units $\mathbb{A}_K^\times$ in $\mathbb{A}_K$, although not topologised with the subspace topology. Instead, notice that

$$\mathbb{I}_K = \prod_{v \in M_K}{}' (K_v^\times, \mathcal{O}_{K_v}^\times)$$

and give this group the restricted product topology.

The notation $\mathbb{I}_K$ as opposed to $\mathbb{A}_K^\times$ is used to emphasise the difference in choice of topology.

Remark

- For any topological ring R, the units $R^\times$ may not be a *topological* group when given the subspace topology, because inversion may not be continuous! Consider the embedding

  $$R^\times \to R \times R; x \mapsto (x, x^{-1})$$

  and give the image of this embedding the subspace topology from $R \times R$. This in turn gives a topology on $R^\times$ which makes it into a topological group. Moreover in the case of adeles, this induced topology on $\mathbb{A}_K^\times$ is equal to the restricted product topology! This gives a neat proof Proposition 7, that $K^\times$ is discrete in $\mathbb{I}_K$ if we already know $K \subset \mathbb{A}_K$ is discrete. See [Cas10] Chapter 2 §14 for details.

- The restricted product topology is finer than the subspace topology on $\mathbb{A}_K^\times$. Checking this can be done by using similar methods to those in section 1, and noticing that $\prod_v \mathcal{O}_{K_v}^\times$ is open in the restricted product topology, but not in the subspace topology.

Once again, the from the results of the previous section we have

Proposition 4 $\mathbb{I}_K$ is locally compact and Hausdorff.

Proof. Same as Proposition 3. □

The S-ideles are defined in the obvious way to be $\mathbb{I}_{K,S} = \prod_{v \in S} K_v^\times \prod_{v \notin S} \mathcal{O}_{K_v}^\times$, and $K^\times$ embeds into $\mathbb{I}_K$ via the diagonal embedding with image called the *principal ideles*.

Definition 5 The quotient $C_K = \mathbb{I}_K / K^\times$ is called the *idele class group* of K.

This group has interesting ties to the *ideal* class group, although unlike Cl_K it is not finite.

Proposition 5 Let S_∞ be the set of infinite primes of K, then

$$\frac{\mathbb{I}_K}{\mathbb{I}_{K,S_\infty}} \cong I_K, \text{ and } \frac{\mathbb{I}_K}{K^\times \mathbb{I}_{K,S_\infty}} \cong Cl_K$$

via the map $\lambda : x = (x_v) \mapsto \prod_{\mathfrak{p}\nmid\infty} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(\alpha_\mathfrak{p})}$

Proof. Clear by just looking at the kernels of the maps. □

Moreover, since $\mathbb{I}_{K,S_\infty}$ is open the homomorphism $\mathbb{I}_K \to I_K$ is continuous when I_K is given the discrete topology.

Now, because the ideal class group is finite, we can use the result $\mathbb{I}_K/K^\times\mathbb{I}_{K,S_\infty} \cong Cl_K$ to show that by taking S large enough (but still finite) we have $\mathbb{I}_K = K^\times\mathbb{I}_{K,S}$.

Proposition 6 Given any finite set S' of primes, we may extend S' to a larger finite set S such that $\mathbb{I}_K = K^\times\mathbb{I}_{K,S}$

Proof. Begin by noting that $S_0 \subset S_1 \implies \mathbb{I}_{K,S_0} \subset \mathbb{I}_{K,S_1}$, hence we can without loss of generality set $S' = \emptyset$.

Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be representatives in I_K of Cl_K, and let S be the set of all infinite primes, and finite primes that divide any of the $\mathfrak{a}_i$. For any $x \in \mathbb{I}_K$ we have $\lambda(x) = (a)\mathfrak{a}_i \in I_K$ for some $a \in K^\times$, so $\lambda(xa^{-1}) = \mathfrak{a}_i$. For any finite prime $\mathfrak{p} \notin S$ this implies that $(xa^{-1})_\mathfrak{p} \in \mathcal{O}_{K_v}^\times$, so $xa^{-1} \in \mathbb{I}_{K,S}$ and in turn $x \in K^\times\mathbb{I}_{K,S}$. □

Let's continue to get a picture of what the idele class group looks like. It is given the quotient topology of a locally compact space, and is therefore itself locally compact.
Is it Hausdorff? Well, this holds if and only if $K^\times$ is a closed subgroup of $\mathbb{I}_K$, which the next proposition shows is true!

Proposition 7 $K^\times$ is a discrete subgroup of $\mathbb{I}_K$

Proof. We are dealing with topological groups, which is nice because this means $K^\times$ is discrete in $\mathbb{I}_K$ if and only if 1 is an isolated point, hence we seek a neighbourhood U of 1 in $\mathbb{I}_K$ such that $U \cap K^\times = \{1\}$.

Let U be the (basic) open

$$U = \{x \in \mathbb{I}_K : |x_\mathfrak{p}|_\mathfrak{p} = 1 \text{ for } \mathfrak{p}\nmid\infty, |x_v - 1|_v < 1 \text{ for } v \mid \infty\}.$$

Clearly 1 lies in U, and suppose $\alpha \in K^\times$ also lies in U, then by the product formula we have

$$1 = \prod_v |x-1|_v \leq \prod_{v|\infty} |1-x|_v < 1$$

which is a contradiction. Hence we conclude that $U \cap K^\times = \{1\}$ and are done. □

Corollary 8 The idele class group C_K is locally compact and Hausdorff.

To round out this subsection, we introduce the absolute norm on ideles (note that this takes numerous other names such as the 'content' or the 'idele norm').

Definition 6 The absolute norm is the homomorphism $|\cdot| : \mathbb{I}_K \to \mathbb{R}_{>0}$ defined by

$$|x| = \prod_v |x_v|_v$$

and let $\mathbb{I}_K^0$ be the kernel of this map. Notice that by the product theorem, the group of principal ideles is contained in the kernel, hence we may also define the group

$$C_K^0 = \{[x] \in C_K : |x| = 1\} = \mathbb{I}_K^0 / K^\times$$

It is not hard to verify that the absolute norm is a continuous homomorphism, which implies that $\mathbb{I}_K^0$ is a closed subspace of the idele group, so C_K^0 is Hausdorff, and locally compact. However we can say more!

Theorem 9 C_K^0 is compact.

The proof of this in each reference requires a series of decently lengthy preliminary results. So for now I shall just state the theorem, and perhaps in a short while come back and fill in a proof.

Now, for each $v \in M_K$ there is an injection $n_v : K_v^\times \to \mathbb{I}_K$ where

$$(n_v(\alpha))_w = \begin{cases} \alpha & \text{for } w = v \\ 1 & \text{otherwise.} \end{cases}$$

The composition of this with $\mathbb{I}_K \to C_K$ is denoted $\overline{n}_v$.

Proposition 10 There is a non-canonical isomorphism $C_K \cong C_K^0 \times \mathbb{R}_{>0}$

Proof. Consider that we have a short exact sequence

$$1 \to C_K^0 \to C_K \to \mathbb{R}_{>0} \to 1. \tag{1}$$

If we can show that this splits, i.e there exists a section $s : \mathbb{R}_{>0} \to C_K$ that is a homomorphism, then we're done.

Pick some infinite prime v of K, then we can define the section to be

$$s(x) = \begin{cases} \overline{n}_v(x) & \text{if } v \text{ is real} \\ \overline{n}_v(\sqrt{x}) & \text{if } v \text{ is complex.} \end{cases}$$

□

Some Fun Applications

Cool, so we've defined the ideles and proven some basic facts about their structure. We now show some interesting applications, by proving Dirichlet's unit theorem and that the ideal class group is finite. We begin with a generalisation of the former.

Definition 7 Let S be a finite set of primes of K containing S_∞. The S-*integers* are the ring

$$\mathcal{O}_{K,S} = \{x \in K : |x|_v \in \mathcal{O}_{K_v} \text{ for all } v \notin S\}.$$

In other words this is $K \cap \mathbb{A}_{K,S}$.
The group of units $\mathcal{O}_{K,S}^\times$ is called the S-*units* of K, which we can write in terms of ideles as $\mathbb{I}_{K,S} \cap K^\times$.

Recall that proving Dirichlet's unit theorem without using ideles involved some log-embedding[2] $K^\times \to R^{r+s}$, where r, s are the number of real and complex primes of K respectively. We shall generalise this to a log-embedding of the ideles, and use this to prove a generalised version of the unit theorem.

Theorem 11 The group of S-units $\mathcal{O}_{K,S}$ is a finitely generated abelian group of rank $|S| - 1$.

Proof. Consider the continuous log mapping $\mathbb{I}_{\mathbb{K},\mathbb{S}} \to \mathbb{R}^s$, given by

$$x \mapsto (\log |x_v|_v)_{v \in S}$$

where $s = |S|$. Then the image of $\mathbb{I}_{K,S}^0$ maps into the hyperplane

$$\mathcal{H}_S : t_1 + \ldots + t_s = 0.$$

The principal ideles are discrete in $\mathbb{I}_K$ hence the S-units are discrete in $\mathbb{I}_{K,S}$, and so their image Λ in R^s is also discrete. We have the following picture

$$\begin{tikzcd}
\mathcal{O}_{K,S}^\times \arrow[r, hook] \arrow[d] & \mathbb{I}_{K,S}^0 \arrow[r, hook] \arrow[d] & \mathbb{I}_{K,S} \arrow[d] \\
\Lambda \arrow[r, hook] & \mathcal{H}_S \arrow[r, hook] & \mathbb{R}^s
\end{tikzcd}$$

Okay, so $\Lambda \subset \mathcal{H}_S$ is a discrete subgroup in $\mathbb{R}^s$, and therefore is a free Abelian group of rank m for some $m \leq s - 1$.

First note that $\mathcal{H}_S$ is generated over $\mathbb{R}$ by the image of $\mathbb{I}_{K,S}^0$. To show this consider ordering S such that the first prime v_1 is infinite, then for $i = 2, ..., s$ we can choose $x \in \mathbb{I}_{K,S}^0$ as follows:

$$x_v = 1 \text{ for } v \notin \{v_1, v_i\}, x_{v_i} = \alpha, x_{v_1} = \beta,$$

[2]Note that although we oft call this an embedding, it's not injective!

where $\alpha \in K_{v_i}^\times$ is any element with $|\alpha|_{v_i} = u \neq 1$ and $\beta \in K_{v_1}^\times$ is any element[3] with $|\beta|_{v_1} = u^{-1}$. Then the image of x under our log map is

$$(-\log u, ..., 0, \log u, 0, ...),$$

so by varying i we get $s-1$ independent vectors in $\mathcal{H}_S$, which must therefore generate $\mathcal{H}_S$. Now, let V be the subspace generated by Λ, which is contained in $\mathcal{H}_S$. Then the log map induces a continuous homomorphism

$$\frac{\mathbb{I}_{K,S}^0}{\mathcal{O}_{K,S}^\times} \to \frac{\mathcal{H}_S}{V}.$$

We claim that $\frac{\mathbb{I}_{K,S}^0}{\mathcal{O}_{K,S}^\times}$ is compact, and that this map is surjective, in which case we are done because $\mathcal{H}_S/V$ is therefore 0, meaning Λ has rank $s-1$.

The subgroup $\mathbb{I}_{K,S}^0 \subset \mathbb{I}_K^0$ is open, and hence closed since $\mathbb{I}_K^0$ is a topological group. It's image $\frac{\mathbb{I}_{K,S}^0}{\mathcal{O}_{K,S}^\times}$ in C_K^0 is then closed, and therefore compact.

For surjectivity, we decompose S into $S_\infty \cup S_0$ the subsets of infinite and finite primes respectively, with $|S_\infty| = a$ and $|S_0| = b$. Given $\mathbf{t} = (t_1, ...t_s) \in \mathcal{H}_S$, we must show that it lies in $\mathrm{im}(\mathbb{I}_{K,S}^0) + V$. Let $\mathfrak{p}$ be a finite prime in S_0, then by the finiteness of the ideal class group $\mathfrak{p}^n = (x)$ for some $x \in K^\times$, $n \in \mathbb{N}$. Hence $\log |x|_\mathfrak{p} \neq 0$ while $\log |x|_\mathfrak{q} = 0$ for all other finite primes $\mathfrak{q}$ of K. Since V is the subspace generated over $\mathbb{R}$ by $\Lambda = \mathrm{im}(\mathcal{O}_{K,S}^\times)$, we may use a multiple of this x to clear the $t_\mathfrak{p}$ component of our target vector $\mathbf{t}$.
Repeating for all $\mathfrak{p} \in S_0$ reduces the task to showing that $(t_1, ..., t_a, 0, ...0) \in \mathrm{im}(\mathbb{I}_{K,S}^0) + V$, where $\sum t_i = 1$. However this is easy, because $\log |\cdot|_v : K_v^\times \to \mathbb{R}$ is surjective for all infinite primes.

Hence we are done! Moreover, the kernel of the map $\mathcal{O}_{K,S}^\times \to \Lambda$ is finite as it is the preimage of $\{0\}$, a compact set, in a discrete space! It is then rather easy to show that the kernel is just μ_K, the (cyclic) group of roots of unity in K. □

Remark Taking $S = S_\infty$, we have proven Dirichlet's unit theorem!

This was quite a fun application! Another consequence of the compactness of C_K^0 is that the ideal class group is finite.

Proposition 12 The ideal class group Cl_K is finite.

Proof. Recall from Proposition 5 we have an isomorphism of topological groups $\mathbb{I}_K/\mathbb{I}_{K,S_\infty} \cong I_K$ where I_K has the discrete topology. This remains surjective when we restrict to $\mathbb{I}_K^0$ because $\mathbb{I}_K = \mathbb{I}_K^0 \mathbb{I}_{K,S_\infty}$. The kernel of the composition $\mathbb{I}_K^0 \to I_K \to Cl_K$ contains $K^\times$, therefore $C_K^0 \to Cl_K$ is a continuous surjective homomorphism. So Cl_K is discrete and compact, and therefore finite. □

[3] We specified that v_1 is an infinite prime exactly such that a β can always be chosen!

Ideles and Moduli

Recall that when working with global class field theory in terms of ideals, we often consider moduli $\mathfrak{m}$ of K, the ideals coprime to $\mathfrak{m}$ and such. We shall similarly work with ideles, and try and bridge the gap between the two viewpoints.
Let $\mathfrak{m} = \prod_{\mathfrak{p}} \mathfrak{p}^{n_{\mathfrak{p}}}$ be a modulus of K, then we define the unit groups to be

$$U_{\mathfrak{p}}^{(n_{\mathfrak{p}})} = \begin{cases} \mathcal{O}_{K_{\mathfrak{p}}}^{\times} & \text{if } \mathfrak{p} \text{ is finite and } n_{\mathfrak{p}} = 0, \\ 1 + \pi_{\mathfrak{p}}^{n_{\mathfrak{p}}} \mathcal{O}_{K_p} & \text{if } \mathfrak{p} \text{ is finite and } n_{\mathfrak{p}} > 0, \\ \mathbb{R}^{\times} & \text{if } \mathfrak{p} \text{ is real and } n_{\mathfrak{p}} = 0, \\ \mathbb{R}_{>0}^{\times} & \text{if } \mathfrak{p} \text{ is real and } n_{\mathfrak{p}} = 1, \\ \mathbb{C}^{\times} & \text{if } \mathfrak{p} \text{ is complex,} \end{cases}$$

and given $\alpha \in K_{\mathfrak{p}}^{\times}$ we shall write $\alpha \equiv 1 \mod \mathfrak{p}^{n_{\mathfrak{p}}}$ if $\alpha \in U_{\mathfrak{p}}^{(n)}$ (Note that for $\mathfrak{p}$ finite, this is exactly what it says on the tin, while for $\mathfrak{p}$ infinite we are giving a definition).
For $x \in \mathbb{I}_K$ an idele, we say $x \equiv 1 \mod \mathfrak{m}$ if $x_v \equiv 1 \mod \mathfrak{p}^{n_{\mathfrak{p}}}$ for all $\mathfrak{p} \in M_K$, then

Definition 8 The set $\mathbb{I}_K(\mathfrak{m})$ is defined to be

$$\mathbb{I}_K(\mathfrak{m}) = \{x \in \mathbb{I}_K : x \equiv 1 \mod \mathfrak{m}\} = \prod_{\mathfrak{p}} U_{\mathfrak{p}}^{(n_{\mathfrak{p}})}$$

which is open, because a modulus is a finite product.

Similarly, we descend to the idele class group and define $C_K(\mathfrak{m})$ to be the image of $\mathbb{I}_K(\mathfrak{m})$. That is,

Definition 9 The group

$$C_K(\mathfrak{m}) = \frac{\mathbb{I}_K(\mathfrak{m}) K^{\times}}{K^{\times}}$$

is called the congruence subgroup modulo $\mathfrak{m}$ in C_K. The quotient $C_K / C_K(\mathfrak{m})$ is called the ray class group modulo $\mathfrak{m}$.

As the name suggests, this is isomorphic to the ideal theoretic ray class group $I_K(\mathfrak{m}) / P_K(\mathfrak{m})$

Theorem 13 There is an isomorphism

$$\frac{C_K}{C_K(\mathfrak{m})} \to \frac{I_K(\mathfrak{m})}{P_K(\mathfrak{m})}$$

Proof. Consider the usual map from the ideles to fraction ideals

$$\chi : x \mapsto \prod_{\mathfrak{p} \nmid \infty} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(x_{\mathfrak{p}})}.$$

Given $x = (x_v) \in \mathbb{I}_K$, we may have $x_\mathfrak{p} \notin \mathcal{O}_{K_\mathfrak{p}}^\times$ for some finite primes $\mathfrak{p} \mid \mathfrak{m}$. The (weak) approximation theorem[4] implies the existence of $\alpha \in K^\times$ such that $\alpha^{-1}x_\mathfrak{p} \equiv 1 \mod \mathfrak{p}^{n_\mathfrak{p}}$ for all finite $\mathfrak{p} \mid \mathfrak{m}$, *and* $\mathrm{sign}(x_\sigma) = \mathrm{sign}(\sigma(\alpha))$ for all real primes σ dividing $\mathfrak{m}$. Therefore we have $\chi(x\alpha^{-1}) \in I_K(\mathfrak{m})$, but this in general depends on the choice of normalising element[5] $\alpha \in K^\times$.

Now, notice that $C_K/C_K(\mathfrak{m}) = (\mathbb{I}_K/K^\times)/(\mathbb{I}_K(\mathfrak{m})K^\times/K^\times) = \mathbb{I}_K/(\mathbb{I}_K(\mathfrak{m})K^\times)$, so we begin by defining a map $\phi : \mathbb{I}_K \to I_K(\mathfrak{m})/P_K(\mathfrak{m})$. Given $x \in \mathbb{I}_K$, choose a normalising element α as above, then let $\phi(x) = \chi(x\alpha^{-1}) \in I_K(\mathfrak{m})$. We would like to show is well defined, ie. independent of α, so consider two possible choices α, α' as above. Then $\chi(x\alpha^{-1})^{-1}\chi(x\alpha'^{-1}) = \chi(\alpha\alpha'^{-1}) = (\alpha\alpha^{-1})$, which lies in $P_K(\mathfrak{m})$ by using the definition of our normalising elements. Moreover, it is clear that ϕ is a homomorphism, because if x, y have normalising elements α, β respectively, then $\alpha\beta$ is a normalising element for xy.

Clearly ϕ is surjective, so it remains to show that the kerned is $I_K(\mathfrak{m})K^\times$. We have $\phi(x) \in P_K(\mathfrak{m})$ if and only if we have α a normalising element of x and $\beta \in K^\times$ with $\beta \equiv 1 \mod \mathfrak{p}^{n_\mathfrak{p}}$ for all finite $\mathfrak{p} \mid \mathfrak{m}$, and $\sigma(\beta) > 0$ for all real $\sigma \mid \mathfrak{m}$, such that $\chi(x\alpha^{-1}) = \chi(\beta)$. The kernel of χ is just $\mathbb{I}_{K,S_\infty}$ so we deduce that $x \in \ker(\phi) \iff x = \alpha\beta y$ for some $y \in \mathbb{I}_{K,S_\infty}$ and β with the properties above.

Thus $\ker(\phi) = K^\times(\prod_{\mathfrak{p}\mid\infty} K_\mathfrak{p}^\times \prod_{\mathfrak{p}\nmid\infty} U_\mathfrak{p}^{n_\mathfrak{p}}) = K^\times\mathbb{I}_K(\mathfrak{m})$, where the final inequality again follows the approximation theorem. Indeed, given any element u of $(\prod_{\mathfrak{p}\mid\infty} K_\mathfrak{p}^\times \prod_{\mathfrak{p}\nmid\infty} U_\mathfrak{p}^{n_\mathfrak{p}})$, there exists some $t \in K^\times$ such that for all real $\mathfrak{p} \mid \mathfrak{m}$, $\sigma(t)$ and u_σ have the same sign. Thus $t^{-1}u \in K^\times\mathbb{I}_K(\mathfrak{m})$. $\square$

(I feel like I may have made a slight meal out of this. Alas, anything goes in the name of certitude.)

Now, recall in the ideal-theoretic formulation of global class field theory, we were interested in the subgroups of the ray class group, since the existence theorem in terms of ideals stated that every such subgroup admitted a class field! Under the isomorphism above, they correspond to subgroups of the idele class group containing a congruence subgroup. So, can we say anything about these subgroups? (Spoiler: yes)

Proposition 14 The closed subgroups of finite index in C_K are exactly those which containing a congruence subgroup $C_K(\mathfrak{m})$ for some $\mathfrak{m}$.

Proof. It is clear that the $\Longleftarrow$ direction shall be easier to prove that the opposite one. Indeed, for the $\Longrightarrow$ direction we have to conjure up some modulus that works! So let's begin with the former.

[4]Recall that is the statement that for $|\cdot|_1, ..., |\cdot|_n$ non-equivalent absolute values on K, the diagonal image of K in the product $\prod K_i$ – where K_i is just K with the topology induced by $|\cdot|_i$ – is dense.

[5]This is not a conventional name for such an element, I just had to call them something so that I can refer to them below.

A subgroup of a topological group is closed of finite index if and only if it is open of finite index, so if we can show that congruence subgroups are themselves open of finite index, then we're done because $C_K(\mathfrak{m}) \subset H$ will imply that H is finite index and open.
Now, Theorem 13 implies that $C_K(\mathfrak{m})$ is finite index, because the ray class group has finite order (by using ideals), and $C_K(\mathfrak{m}) = \mathbb{I}_K(\mathfrak{m})K^\times/K^\times$ is the image of the open subset $\mathbb{I}_K(\mathfrak{m}) \subset \mathbb{I}_K$ in the quotient, so is itself open.

Conversely, suppose H is a closed subgroup of finite index in $C_K(\mathfrak{m})$, then it is also open. The preimage $J \subset \mathbb{I}_K$ of H is therefore an open subgroup, and must contain some open neighbourhood of 1, say

$$V = \prod_{\mathfrak{p}\in S} V_\mathfrak{p} \prod_{\mathfrak{p}\notin S} \mathcal{O}_{K_\mathfrak{p}}^\times,$$

where S is a finite set of primes. For any finite prime, the sets $U_\mathfrak{p}^{(n)}$ form a fundamental system of neighbourhoods of 1 in $K_\mathfrak{p}^\times$, so we can shrink V and let $V_\mathfrak{p} = U_\mathfrak{p}^{(n_\mathfrak{p})}$ for some $n_\mathfrak{p} \in \mathbb{N}$. For a complex primes $\mathfrak{p} \in S$, any neighbourhood of 1 in $\mathbb{C}^\times = K_\mathfrak{p}^\times$ generates all of $\mathbb{C}^\times$ as a multiplicative group, while if $\mathfrak{p}$ is a real prime, then we may shrink $V_\mathfrak{p}$ to be contained in $\mathbb{R}_{>0}$, then $V_\mathfrak{p}$ generates $\mathbb{R}_{>0}$ as a multiplicative group. Therefore the subgroup of J generated by V is equal to $\mathbb{I}_K(\mathfrak{m})$, where $m = \prod_{\mathfrak{p}\in S \text{ finite}} \mathfrak{p}^{n_\mathfrak{p}} \prod_{\mathfrak{p}\in S \text{ real}} \mathfrak{p}$. This completes the proof, as we have $\mathbb{I}_K(\mathfrak{m}) \subset J$, and so $C_K(\mathfrak{m}) \subset H$.

□

In fact, notice that the second part of our proof states that any open subgroup J of $\mathbb{I}_K$ contains some $\mathbb{I}_K(\mathfrak{m})$, and implies that all open subgroups of C_K have finite index.

This theorem feels rather existence formula-esque. Indeed, the existence theorem states that closed finite index subgroups are exactly the *norm groups* of C_K, so the above theorem and proposition will provide the *link* between the ideal and idele versions of class field theory!

Ideles and Field extensions

Our next task is to investigate how the ideles of field extensions relate. Looking at field extension L/K in general, the inclusion map $K^\times \hookrightarrow L^\times$ and the norm map $L^\times \to K^\times$ play an important role in the study of the multiplicative groups, so let look at the corresponding maps on ideles.
Suppose L/K is a finite extension of number fields, then we embed

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